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A PROBLEM OF PURSUIT WITH ELLIPTICAL ODOGRAPH DOMAINS WITH COMPLETE AND PARTIAL INFORMATION

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In memory of Professor Josef Shinar

Abstract. In this paper, a 2-dimensional pursuit-evasion problem is solved, first with complete state information, then when the pursuer knows only one evader's coordinate. A method is proposed to compute the optimal pursuit strategy in real time.
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1. Introduction

This problem was initially formulated in 2014 as one in economics, to modelize the competition between a new start firm trying to overcome a larger incumbent. Each was interested in two criteria: margin and total sales, possibly sacrificing the former to increase the latter. However, it can be seen as a pursuit problem per se, much in the spirit of the many such problems solved yesteryear by my old friend Josef Shinar, who knew better than any of us mathematicians what pursuit-evasion means, and to whose memory this article is dedicated.

1.1. The model.

The pursuit takes place in the plane (x,y). The parameters of the model are:

- A positive definite 2×2 matrix A, with $A^{-1}=B$, defining the shape of the odograph domains (Isaacs' *vectorgrams*) of the firms,
- two positive numbers r_1 and r_2 (for *rate* of change) defining the size of these domains, with $\delta := r_2 - r_1 > 0$,
- a positive definite 2×2 matrix C, with $C^{-1}=D$ defining the shape of the target set C,
- a positive number c defining the size of the target set.

For any 2×2 positive definite matrix M, and 2-vector h, we denote by h^t the transpose of h, and $\|h\|_M = \langle h, Mh\rangle^{1/2} = (h^tMh)^{1/2}$. (It is indeed a norm.)

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Evader's and Pursuer's states z_1 and z_2 respectively, are of the form

$$z_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \qquad i = 1,2.$$

Each is able to have these quantities move according to a velocity

$$\dot z_i = w_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \qquad i = 1,2,$$

constrained to stay in the ellipses W_i defined by

$$\|w_i\|_A \le r_i .$$

Clearly, we may use coordinates centered on Pursuer, and set

$$z = z_1 - z_2 , \qquad \dot z = w_1 - w_2 , \qquad z(0) = z_0 := z_1(0) - z_2(0) .$$

The aim of Pursuer is to reach as quickly as possible the capture set $\mathsf{C} = \{z \mid \|z\|_C \le c\}$. More precisely, it wishes to minimize the *guaranteed time to capture*, by using a *nonanticipative strategy*. We make these concepts precise now.

1.2. **Strategies and optimality.** Let $\mathscr{W}_i$, $i = 1,2$ be the set of measurable time functions $w_i(\cdot)$ from $\mathbb{R}^+$ into W_i. The following concept is classically used to mean that Pursuer, player 2, may choose its control w_2 as a function of past actions of Evader, player 1.

Definition 1.1. A nonanticipative strategy ϕ_2 of Pursuer is a function from $\mathscr{W}_1$ into $\mathscr{W}_2$ satisfying the property: $\forall t \in \mathbb{R}_+$, $\forall w_1(\cdot), w_1'(\cdot) \in \mathscr{W}_1$,

$$[\forall s < t, w_1(s) = w_1'(s)] \Rightarrow [\phi_2(w_1(\cdot))(t) = \phi_2(w_1'(\cdot))(t)].$$

We denote by Φ_2 the set of nonanticipative strategies of Pursuer.

A state feedback $w_2(t) = \varphi_2(z(t))$ generates a nonanticipative strategy if it ensures existence of a unique solution to the differential equation in $\mathbb{R}^2$:

$$\dot z = w_1 - \varphi_2(z) , \qquad z(0) = z_0$$

from any initial condition z_0 and for all $w_1(\cdot) \in \mathscr{W}_1$.

Concerning the aim of Pursuer, we set:

Definition 1.2. We define the function *capture time* T_c from $\mathbb{R}^2 \times \mathscr{W}_1 \times \Phi_2$ into $\mathbb{R}_+$ as

$$T_c(z_0; w_1(\cdot), \phi_2) = \sup\{t \mid \forall s \in (0,t), \|z(s)\|_C > c\}$$

where $z(\cdot)$ is the unique solution of the differential equation

$$\dot z(t) = w_1(t) - \phi_2(w_1(\cdot))(t) , \qquad z(0) = z_0 .$$

Definition 1.3. We define the *guaranteed capture time* $G : \mathbb{R}^2 \times \Phi_2 \to \mathbb{R}_+$ as

$$G(z_0; \phi_2) = \sup_{w_1(\cdot) \in \mathscr{W}_1} T_c(z_0; w_1(\cdot), \phi_2) .$$

The aim of Pursuer is to find, if it exists, the strategy $\phi_2^\star$ minimizing G:

$$V(z_0) = G(z_0; \phi_2^\star) = \min_{\phi_2 \in \Phi_2} G(z_0, \phi_2) = \min_{\phi_2 \in \Phi_2} \sup_{w_1(\cdot) \in \mathscr{W}_1} T_c(z_0; w_1(\cdot), \phi_2) .$$

We are therefore confronted with a classic pursuit-evasion differential game.

1.3. **Parametrization.** One may obviously multiply A and r by the same positive number without changing the odograph set W_i. Using this feauture, we may normalize the matrix A in the following way: let a be a positive number, b and d be positive numbers whose squares add up to 1 : $b^2+d^2=1$, and

$$A=\frac{1}{a^2d^2}\begin{pmatrix} a^2 & ab \\ ab & 1\end{pmatrix}, \qquad A^{-1}=B=\begin{pmatrix} 1 & -ab \\ -ab & a^2\end{pmatrix}.$$

We rely heavily on the following simple result:

Lemma 1.4. *Let A be a positive definite $n\times n$ matrix, h a vector of $\mathbb{R}^n$, r a positive number, and*

$$\mathsf{W}=\{w\in\mathbb{R}^n \mid \|w\|_A\le r\},$$

then

$$\max_{w\in\mathsf{W}}\langle h,w\rangle = r\|h\|_{A^{-1}}, \quad \textit{reached at } w=\hat{w}=\frac{r}{\|h\|_{A^{-1}}}A^{-1}h.$$

As a consequence, with our model, we have, for $i=1,2$:

$$\max_{w_i\in\mathsf{W}_i} u_i = r_i, \qquad \text{reached at } w_i = r_i\begin{pmatrix}1\\-ab\end{pmatrix},$$

$$\max_{w_i\in\mathsf{W}_i} v_i = ar_i, \qquad \text{reached at } w_i = r_i\begin{pmatrix}-b\\a\end{pmatrix}.$$

and symmetrically with the opposite signs for the minima. Therefore, W_i is an ellipse centered at the origin, inscribed in the rectangle $u=\pm r_i$, $v=\pm ar_i$, which it touches at the points w_i as above and at the opposites. A further hint concerning the role of the parameters is that the area of W_i is πadr_i^2. As d approaches 0, and thus b approaches 1, the points of contact with the rectangle collapse at the vertices of the rectangle in the first and third quadrants, and the ellipse shrinks toward the line segment joining them.

A further useful result is that given v_i such that $|v_i|\le ar_i$, we have

$$u_i\in[\Upsilon_i^-(v_i),\Upsilon_i^+(v_i)], \tag{1.1}$$

with

$$\Upsilon_i^-(v)=\frac{1}{a}\left(-bv-d\sqrt{a^2r_i^2-v^2}\right), \quad \Upsilon_i^+(v)=\frac{1}{a}\left(-bv+d\sqrt{a^2r_i^2-v^2}\right). \tag{1.2}$$

2. Complete information game

In this section, states and velocities may live in $\mathbb{R}^n$ for any integer n.

2.1. **General case.** Isaacs' equation for this game is

$$\begin{array}{l}\forall z\mid \|z\|_C>c, \quad 1+\max_{w_1\in\mathsf{W}_1}\langle\nabla V(z),w_1\rangle-\max_{w_2\in\mathsf{W}_2}\langle\nabla V(z),w_2\rangle=0,\\ \forall z\mid \|z\|_C=c, \quad V(z)=0.\end{array}$$

hence, according to Lemma 1.4,

$$\forall z\mid \|z\|_C>c, \quad 1-\delta\|\nabla V(z)\|_B=0, \tag{2.1}$$

$$\forall z\mid \|z\|_C=c, \quad V(z)=0. \tag{2.2}$$

And the minimizing strategy of Pursuer is

$$w_2(t)=\varphi_2^\star(z(t)):=\frac{r_2}{\|\nabla V(z(t))\|_B}B\nabla V(z(t)).$$

(We may also notice that the worst control $w_1(\cdot)$ when Pursuer uses $\varphi_2^\star$ is generated by the same strategy, mutatis mutandis.)

It follows from (2.2) that for any $Z \in \mathbb{R}^2$ such that $\|Z\|_C = c$, $\nabla V(Z)$ is orthogonal to the boundary of the capture set C, i.e. parallel to CZ. Using further (2.1) (or rather its limit as $z \to \partial\mathsf{C}$), we obtain

$$\nabla V(Z) = \frac{1}{\delta\|CZ\|_B} CZ\,.$$

The equations of the characteristics, or adjoint equations, simply read

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla V(z(t)) = 0\,,$$

hence $\nabla V(z(t))$ is constant along an extremal trajectory. Thus, the extremal trajectory reaching C in Z is given by

$$z(t) = Z + \frac{\delta}{\|CZ\|_B} BCZ(T_c - t)\,.$$

Moreover, $T_c - t = V(z(t))$ is the minimum guaranteed capture time. Let

$$p(t) := \frac{\delta}{\|CZ\|_B}(T_c - t)\,,$$

we therefore have, for the extremal rajectory reaching C at Z:

$$z(t) = (I + p(t)BC)Z\,.$$

Notice that the product of two positive definite matrices has all its eigenvalues strictly positive. Hence $(I + pBC)$ is always invertible:

$$Z = (I + pBC)^{-1} z\,,$$

and for a given z, p can be found by solving

$$\|(I + pBC)^{-1} z\|_C = c\,. \tag{2.3}$$

(It may be noticed that Z is therefore the projection, in A-norm, of z on C.) A more appealing, and numerically easier, form of this equation is

$$\|(D + pB)^{-1} z\|_D = c\,. \tag{2.4}$$

However, it has to be solved numerically for p, yielding

$$V(z) = \frac{p}{\delta}\|(D + pB)^{-1} z\|_B\,, \tag{2.5}$$

and

$$\hat{w}_i = r_i\omega(z)\,, \qquad \omega(z) = \frac{1}{\|(D + pB)^{-1} z\|_B} B(D + pB)^{-1} z\,. \tag{2.6}$$

Hence, both players should run in straight line in the direction ω, which may not be the pursuer's "line of sight" z.

It should be emphasized that, whatever Evader does, Pursuer should continuously solve equation (2.4) yielding $p = \hat{p}(z)$, and play

$$w_2(t) = \varphi_2^\star(z(t))$$

with

$$\varphi_2^\star(z) = \frac{r_2}{\|(D + \hat{p}(z)B)^{-1}\|_B} B(D + \hat{p}(z)B)^{-1} z\,.$$

By so doing, at each instant of time, it is guaranteed to "capture" Evader in no more time than $V(z(t))$, which will be constant and equal to $V(z(0))$ if Evader plays optimally, and decreasing with time otherwise. (It is further known that this is a saddle-point, so that the symmetric statement holds true also.)

2.2. **Two particular cases.**

2.2.1. *Case $A = C$.* The above calculation simplifies greatly in case $A = C$, hence $B = D$, or $BC = I$. In that case, equation (2.3) is simply $(1+p)^{-1}\|z\|_A = c$, or

$$\hat{p}(z) = \frac{\|z\|_A}{c} - 1$$

and, with an elementary calculation

$$V(z) = \frac{1}{\delta}(\|z\|_A - c),$$

and

$$\hat{w}_i = \frac{r_i}{\|z\|_A} z,$$

i.e. a simple line chase where both players should run along the "line of sight" $z = (z_1 - z_2)$, regardless for the fact that it may not be a direction of maximum velocity for them.

It is an elementary calculation to check that indeed, $\nabla V(z) = (1/\delta\|z\|_A)Az$ does satisfy Isaacs' equation (2.1)(2.2).

2.2.2. *Point capture: $c = 0$.* If capture is defined by exact coincidence of the two players' states, i.e. $z = 0$, there is no C matrix anymore, but the above formulas remains correct with $c = 0$:

$$V(z) = \frac{\|z\|_A}{\delta}, \qquad \hat{w}_i = \frac{r_i}{\|z\|_A} z.$$

This can be seen using, e.g. Tonelli's construction in calculus of variations which tells us that we do synthesize a valid Value function by integrating the adjoint equations backward from $z = 0$ with all possible vectors λ such that $\|\lambda\|_B = 1/\delta$ as final value of ∇V.

Of course, as previously, the optimal chase is directly along the line of sight, and the optimal escape as well.

3. Partial information game

We now turn to the problem where, as in PLato's allegory of the cave, Pursuer only observes Evader's shade on the wall of the cave, say the y coordinate, but not the x coordinate. However, it knows an interval $[X_1^-(0), X_1^+(0)]$ containing $x_1(0)$.

3.1. **Fundamentals.**

We aim to apply the theory developed in [1, 3, 4] and more specifically [5]. We consider a min-sup problem for the dynamics

$$\dot{z} = f(z, w_1, w_2), \quad z(0) = 0, \quad w_i \in \mathsf{W}_i,\ i = 1, 2,$$

and performance index

$$J = U(z(t_c)) + \int_0^{t_c} L(z(t), w_1(t), w_2(t))\,\mathrm{d}t. \tag{3.1}$$

The unknown quantity is $\omega = (z(0), w_1(\cdot))$. The available measurement at time t is some $y(t) \in \mathsf{Y}$. The available information at time t is therefore $y(s)$, $s \in [0,t]$. It defines a subset Ω_t, satisfying the axioms proposed in [3], the set of past ω compatible with the past observations.

One should compute the function "worst conditional cost to come"[2] $W_t(\zeta)$, maximum possible running cost from initial time to t for a trajectory compatible with the available information at time t, and ending in $z(t) = \zeta$:

$$W_t(\zeta) = \sup_{\omega\in\Omega_t \mid z(t)=\zeta} \int_0^t L(z(s), w_1(s), w_2(s))\,\mathrm{d}s\,.$$

The first step, thus, is to find a "filter" equation of the form

$$\frac{\partial}{\partial t} W_t = F(W_t, y(t), w_2(t))\,, \tag{3.2}$$

initialized at zero with the a priori information on the initial state $z(0)$. This plays the role of an (infinite dimensional) state in a Hamilton-Jacobi-Isaacs equation for the Value function $V(t, W_t)$:

$$\frac{\partial V(t,W_t)}{\partial t} + \min_{w_1\in \mathsf{W}_1} \sup_{y\in\mathsf{Y}} \mathrm{D}_W V(t,W_t) F(W_t, y(t), w_2(t)) = 0\,, \tag{3.3}$$

$$V(t_c, W_{t_c}) = \max_z [U(z) + W_{t_c}(z)]\,. \tag{3.4}$$

where $\mathrm{D}_W V$ is a *chain derivative* (see [2])[3].

It yields both the optimum pursuit strategies as the maximizing and minimizing strategies in the Isaacs equation (3.3), and the min-sup value as $V(0, W_0)$.

3.2. **Point capture.**

3.2.1. *Information.* We particularize the theory sketched above to our pursuit problem. Notice first that, with the notation (3.1), we have $L = 0$. Therefore,

$$W_t(\zeta) = \begin{cases} 0 & \text{if } \zeta \text{ is a possible current state given the information}\,, \\ -\infty & \text{if } \zeta \text{ is not a possible current state} \end{cases}$$

Moreover, the set of current states compatible with the information is a line interval $y = y_2(t) - y_1(t)$ directly observed, and $x_1(t) \in [X_1^-(t), X_1^+(t)]$, or $x(t) = x_1(t) - x_2(t) \in [X^-(t), X^+(t)]$. Thus, the function W_t is completely described by a 3-dimensional data. We assume that Evader can observe $\dot y$, thus v_1. And the filter equation is now, using (1.1) and (1.2):

$$\left.\begin{aligned} \dot X^+ &= \frac{1}{a}\left[-bv_1 + d\sqrt{a^2r_1^2 - v_1^2}\right] - u_2\,, \\ \dot X^- &= \frac{1}{a}\left[-bv_1 - d\sqrt{a^2r_1^2 - v_1^2}\right] - u_2\,, \\ \dot y &= v_1 - v_2\,. \end{aligned}\right\} \tag{3.5}$$

[2] called *informational state* in [4]

[3] and not just a Gateaux derivative as stated in [5].

3.2.2. *Final phase.* We state the following proposition:

Proposition 3.1. *A min-sup chase necessarily ends with a line chase at $y = 0$.*

Proof Capture implies $y(t_c) = 0$. But moreover, in a min-sup sense, it cannot happen the first time that $y(t) = 0$. Indeed, if it did, a different Evader's strategy yielding the same information would, against the same Pursuer's strategy, avoid capture at that time instant, a contradiction with the hypothesis that this is supremum of the capture time against Evader's strategy. ∎

We therefore investigate a final pursuit along $y = 0$. Let t_0 be the time when it begins, and $[X_0^-, X_0^+]$ be the interval of possible x at that time. We investigate only the case where the middle point $(X_0^+ + X_0^-)/2$ is positive. The other case is symmetrical, as, as we will see, all happens at $v_1 = v_2 = 0$.

Case $X_0^- \geq 0$. All the possible positions of Evader are "to the right" of Pursuer. It follows from the same type of argument as above that a min-sup chase necessarily has Pursuer traverse all of the set of possible Evader's positions, hence reach $X^+(t_c) = 0$. During this chase, Pursuer must keep $v_2 = v_1$. Its relative x-speed, or Excess speed $\mathscr{E}$, as a function of v_1 is

$$\mathscr{E}(v_1) = \Upsilon_2^+(v_1) - \Upsilon_1^+(v_1) = \frac{d}{a}\left(\sqrt{a^2r_2^2 - v_1^2} - \sqrt{a^2r_1^2 - v_1^2}\right).$$

Its derivative is easily seen to have the sign opposite of v_1. It is therefore minimum at $v_1 = 0$ and yields $\mathscr{E} = d\delta$.

The min-sup capture time is therefore

$$t_c - t_0 = \frac{X_0^+}{d\delta}.$$

Case $X_0^- \leq 0 \leq X_0^+$. Pursuer has reached $y = 0$ in the set of possible Evader's positions. We investigate the sub-case where this happens "left" of the middle point: $X_0^+ + X_0^- < 0$. The other case is symmetrical.

The min-sup chase is easily seen to include first an excursion of Pursuer to X_1^- and from there, traverse all the segment to $X_1^+(t_c)$. During that chase, the width $X^+(t) - X^-(t)$ increases at a rate $\mathscr{F}$:

$$\mathscr{F}(v_1) = \Upsilon_1^+(v_1) - \Upsilon_1^-(v_1) = \frac{2d}{a}\sqrt{a^2r_1^2 - v_1^2}.$$

This is also maximized by $v_1 = 0$, yielding $\mathscr{F} = 2dr_1$. So that there is no dilemma for Evader: it should choose $v_1 = 0$, simultaneously minimizing Pursuer's speed superiority and maximizing the rate of widening of the set of its possible positions.

Taking into account the two phases, the min-sup chase lasts

$$t_c - t_0 = \frac{-X_0^-}{\mathscr{E}} + \frac{1}{\mathscr{E}}\left(X_0^+ - X_0^- + \frac{-X_0^-}{\mathscr{E}}\mathscr{F}\right).$$

Replacing $\mathscr{E}$ and $\mathscr{F}$ by their above values yields

$$t_c - t_0 = \frac{1}{d\delta^2}(\delta X_0^+ - 2r_2X_0^-).$$

Summary of final phase. The final phase happens at $y=0$. Pursuer should adapt its velocity v_2 to stay equal to the observed v_1, and choose its velocity u_2 to reach as quickly as compatible with v_2 the closest of the two points $X_1^-(t)$ or $X_1^+(t)$, then the other one. The guaranteed (min-sup) capture time is reached if Evader happens to be at the farthest end of the segment $[X_1^-,X_1^+]$ and chooses $v_1=0$, and is $U(X^+(t),X^-(t))$ given by

$$U(X^+,X^-)=\begin{cases}\dfrac{X^+}{d\delta} & \text{if } 0\le X^-\,,\\ \dfrac{1}{d\delta^2}(\delta X^+-2r_2X^-) & \text{if } -X^+\le X^-\le 0\,,\\ \dfrac{1}{d\delta^2}(-\delta X^-+2r_2X^+) & \text{if } 0\le X^+\le -X^-\,,\\ \dfrac{-X^-}{d\delta} & \text{if } X^+\le 0\,.\end{cases}\tag{3.6}$$

3.2.3. *First phase.* We may now consider the pursuit problem with dynamics (3.5) ending at $t_0=\min\{t\mid y(t)=0\}$, with the final cost U as in (3.6) above. We are in the set up of a classical 3-D differential game. We call V its Value function. The problem is globally symmetrical with respect to the origin. We will therefore investigate the half-space $y\ge 0$. The other half-space follows by symmetry.

Let

$$Z:=\begin{pmatrix}X^+\\X^-\\y\end{pmatrix},\qquad \nabla V=\begin{pmatrix}\partial V/\partial X^+\\ \partial V/\partial X^-\\ \partial V/\partial y\end{pmatrix}=\begin{pmatrix}\lambda\\ \mu\\ \nu\end{pmatrix},\qquad \begin{array}{l}\lambda+\mu=\phi\,,\\ \lambda-\mu=\psi\,.\end{array}$$

ϕ and ψ are the partial derivatives of V with respect to, respectively, the mid point and the half width of the segment $[X^-,X^+]$. Let also

$$\begin{aligned}H_1(\nabla V,v_1)&=\left(\nu-\frac{b\phi}{a}\right)v_1+\frac{d\psi}{a}\sqrt{a^2r_1^2-v_1^2}\,,\\ H_2(\nabla V,w_2)&=\phi u_2+\nu v_2\,.\end{aligned}$$

Isaacs' equation reads

$$\begin{cases}\forall t<t_0\,, & 1+\displaystyle\max_{v_1\in[-ar_1,ar_1]}H_1(\nabla V,v_1)-\max_{w_2\in\mathsf{W}_2}H_2(\nabla V,w_2)=0\,,\\ t=t_0\,, & V(t_0,Z)=U(X^+,X^-)\,.\end{cases}$$

It is convenient to introduce the quantity

$$\chi:=a\nu-b\phi\,.$$

One easily sees that the maximum of H_1 is reached at

$$v_1=\hat v_1=r_1\frac{a\chi}{\sqrt{\chi^2+d^2\psi^2}}\qquad\Rightarrow\qquad \Upsilon_1^\pm(\hat v_1)=r_1\frac{-b\chi\pm d^2\psi}{\sqrt{\chi^2+d^2\psi^2}}\tag{3.7}$$

and is equal to

$$\bar H_1=r_1\sqrt{\chi^2+d^2\psi^2}\,.$$

Using lemma 1.4 and $b^2+d^2=1$, we see that the maximum of H_2 is reached at

$$w_2 = \hat{w}_2 = \frac{r_2}{\sqrt{\chi^2+d^2\phi^2}}\begin{pmatrix} -b\chi+d^2\phi \\ a\chi \end{pmatrix} \tag{3.8}$$

and is equal to

$$\bar{H}_2 = r_2\sqrt{\chi^2+d^2\phi^2}\,.$$

Moreover, the equations of the characteristics (or *adjoint equations*) of Isaacs' equation are just $\mathrm{d}\nabla V/\mathrm{d}t=0$, so that λ, μ, ν, ϕ, ψ, and χ are constant along each characteristic line, which also is a min-max chase. It follows from equations (3.5), (3.7), (3.8), that these are straight lines ending at $y=0$.

We are left with the task of determining these directions as a function of the initial state $Z(0)$. To do this, we have the transversality conditions to find λ and μ, hence ϕ and ψ, at the end time t_0, and Isaacs' equation

$$1+r_1\sqrt{\chi^2+d^2\psi^2}-r_2\sqrt{\chi^2+d^2\phi^2}=0\,. \tag{3.9}$$

to find ν at that time. And since this is an equation for χ^2, we shall retain its (positive) square root for χ, corresponding to $v_2-v_1>0$, i.e. $y>0$. We therefore investigate the characteristic lines from final time backward, as a function of $Z(t_0)$.
Case $X_0^-<0<X_0^+$. Two sub-cases occur depending on whether $X_0^+ + X_0^-$ is positive or negative. however, it follows from equation (3.6) that switching from the first sub-case to the second changes λ in $-\mu$, μ in $-\lambda$, and therefore ϕ in $-\phi$ and leaves ψ unchanged. Only ϕ^2 and ψ^2 appear in equation (3.9), so that we may consider the first sub-case only.

Equation (3.6) now yields

$$\begin{pmatrix}\lambda\\ \mu\end{pmatrix}=\begin{pmatrix}\frac{1}{d\delta}\\ \frac{-2r_2}{d\delta^2}\end{pmatrix},\qquad \begin{pmatrix}d\phi\\ d\psi\end{pmatrix}=\frac{1}{\delta^2}\begin{pmatrix}-r_1-r_2\\ 3r_2-r_1\end{pmatrix}.$$

Observe first that $\chi=0$ is again a solution of (3.9) with these values of ϕ and ψ. This again corresponds to the final phase and do not provide characteristic lines to compute V and the strategies for positive y. And it follows from the analysis given in the appendix that this is the only solution of equation (3.9).

Thus there is no characteristic line, or min-max trajectory arriving at a point having a neighborhood where U is differentiable. One must therefore look at trajectories arriving at one of the points where U is not differentiable: the mid point and the extreme points of the segment.
Case $X_0^-+X_0^+=0$. This is a relative maximum of U, where it has a nontrivial superdifferential. However, Pursuer has a global maneuverability superiority over its opponent. It could chose to reach $y=0$ at a small value to the left or to the right. At worst, this would produce a second order increase in t_0, but a first order decrease in U, therefore a better result for it. Thus, no optimal pursuit can end there.

The reader is reminded that such symbols as $d\delta$, $d\phi$, $d\psi$ are not differentials but the products of $d=\sqrt{1-b^2}$ with δ, ϕ and ψ respectively.

Case $X_0^- = 0$. At $X_0^- = 0$, the function U has a local minimum, with a nontrivial subdifferential, obtained as all the convex combinations of the two gradients, i.e.

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \frac{1}{d\delta^2}\begin{pmatrix} \delta \\ -2\theta r_2 \end{pmatrix}, \qquad \begin{pmatrix} d\phi \\ d\psi \end{pmatrix} = \frac{1}{\delta^2}\begin{pmatrix} \delta - 2\theta r_2 \\ \delta + 2\theta r_2 \end{pmatrix}, \qquad \theta \in [0,1].$$

It follows from the general theory of differential games (or from Tonelli's construction in the calculus of variations) that each subdifferential, together with a ν value satisfying equation (3.9), can generate backward a valid characteristic line. And the analysis of that equation in the appendix shows that there is indeed one and only one (positive) solution for χ, hence for ν.

This generates a field of characteristic straight lines, obtained by placing $\hat{v}_1$ and $\hat{w}_2$ as given by equations (3.7) and (3.8) in equations (3.5) and integrating backward from $Z(t_0) = (X_0^+, 0, 0)$, and on each of them, the optimal pursuit strategy $\hat{w}_2$ and a candidate Value of the game as a function of the end parameters and integration time:

$$V^-(Z) = t_0 - t + \frac{X_0^+}{d\delta}. \tag{3.10}$$

The region of $Z \in \mathbb{R}^3$ thus covered is indeed three dimensional, as the state $Z(t)$ reached depends on three degrees of freedom: X_0^+, θ defining the slope of the characteristic line, and the length of time of (backward) integration $t_0 - t$. It should be noticed that the influence of X_0^+ is just a translation of the characteristic line parallel to that axis.

Case $X_0^+ = 0$. This case is similar to the previous one, except that now, due to the symmetry of U, we have

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \frac{1}{d\delta^2}\begin{pmatrix} 2\theta r_2 \\ -\delta \end{pmatrix}, \qquad \begin{pmatrix} d\phi \\ d\psi \end{pmatrix} = \frac{1}{\delta^2}\begin{pmatrix} -\delta + 2\theta r_2 \\ \delta + 2\theta r_2 \end{pmatrix}, \qquad \theta \in [0,1].$$

Only $\hat{u}_2$ is modified as a consequence. The candidate Value is

$$V^+(Z) = t_0 - t - \frac{X_0^-}{d\delta}. \tag{3.11}$$

The two fields just constructed overlap. They must be curtailed at a *decision surface* (in technical terms an *evader dispersal surface*) characterized by the equality of the candidates Value (3.10) and (3.11) in both fields. Giving a more explicit formula for this surface is difficult, but we show hereafter how to compute the correct optimal pursuit strategy in real time.

3.2.4. *Synthesis of he min-sup pursuit strategy.* The optimal pursuit strategy is therefore obtained by finding the characteristic lines through the current state in both fields, choosing the one with the smallest Value, and applying the control w_2 given by equation (3.8). And these characteristic lines are completely determined by the direction from the current state to the origin in the (X^-, y) and (X^+, y) planes.

We need to notice that, in both fields, $t_0 - t = y(t)/(\hat{v}_2 - \hat{v}_1)$, and that in the field toward $X_0^- = 0$,

$$X_0^+ = X^+(t) + \frac{y(t)}{\hat{v}_2 - \hat{v}_1}\left(\Upsilon_1^+(\hat{v}_1) - \hat{u}_2\right)$$

and hence

$$V^-(Z) = \alpha^- y + \frac{X^+}{d\delta} \quad \text{with} \quad \alpha^- = \frac{1}{\hat{v}_2 - \hat{v}_1}\left[1 + \frac{\Upsilon^+(\hat{v}_1) - \hat{u}_2}{d\delta}\right]. \tag{3.12}$$

And a similar calculation shows that in the field toward $X_0^+ = 0$,

$$V^+(Z) = \alpha^+ y - \frac{X^-}{d\delta} \quad \text{with} \quad \alpha^+ = \frac{1}{\hat{v}_2 - \hat{v}_1}\left[1 - \frac{\Upsilon_1^-(\hat{v}_1) - \hat{u}_2}{d\delta}\right]. \tag{3.13}$$

Numerical procedure. We propose the following procedure for Pursuer:

(1) Off-line, for both fields of characteristic lines, sample θ with a small enough step, and draw charts showing, as a function of the (inverse) slopes

$$\frac{\hat{u}_2 - \Upsilon_1^-(\hat{v}_1)}{\hat{v}_2 - \hat{v}_1} = \frac{X^-(t)}{y(t)} \quad \text{respectively} \quad \frac{\hat{u}_2 - \Upsilon_1^+(\hat{v}_1)}{\hat{v}_2 - \hat{v}_1} = \frac{X^+(t)}{y(t)}$$

of the characteristic lines, the values of α^- or α^+ according to equations (3.12) and (3.13) respectively, and the inverse slope of the optimal chase strategy $\hat{u}_2/\hat{v}_2$. (We choose the inverse slopes and not the slopes to avoid having to deal with an infinite slope in the interior of the range of interest.)

(2) On-line, continuously integrate equations (3.5) for X^+ and X^- with the observed v_1 and its controls w_2.

(3) On-line, continuously measure the inverse slopes X^-/y and X^+/y, check on the two charts the values of α^- and α^+ and evaluate $V^-(Z)$ and $V^+(Z)$ according to equations (3.12) and (3.13) respectively.

(4) On-line, choose the field with the smallest candidate Value, and apply maximum speed in the direction determined by the slope $\hat{u}_2/\hat{v}_2$ as specified by the corresponding chart.

(5) Upon reaching $y = 0$ at one end of the segment $[X^-, X^+]$, keep $v_2 = v_1$ (i.e. $y = 0$) and move at maximum compatible speed in the direction of the opposite end of the segment until capture.

The fourth step involves checking in real time on which side of the decision surface the state lies, without having ever explicitly determined that surface.

Figure 3.2.4 shows such charts. (Careful check shows that the different "branches" are not linear, nor do the different graphs go through the origin.)

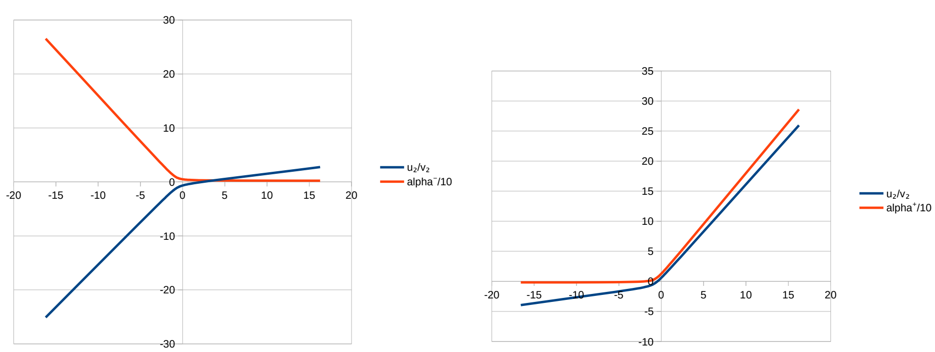

FIGURE 1. The charts giving α^- and $\hat{u}_2/\hat{v}_2$ as a function of X^-/y (top) or α^+ and $\hat{u}_2/\hat{v}_2$ as a function of X^+/y, (bottom) for $a = 1$, $b = d = 1/\sqrt{2}$, $r_1 = 1$, $r_2 = 1.5$.

3.2.5. *Asymptotic analysis.* One may compute the asymptotic behavior of $\hat{u}_2/\hat{v}_2$ and $\alpha^\pm$ as functions of $X^\pm/y$ in both fields as $\theta\to 0$ and, respectively as $\theta\to 1$. In both cases $\chi\to 0_+$ and the quantities considered diverge to infinity. With straightforward algebra, one finds:
Field toward $X_0^- = 0$.

- $\theta \to 0$

$$\begin{aligned}
\frac{X^-}{y} &\simeq d\frac{r_1+r_2}{a\delta^2\chi}-\frac{b}{a},\\
\frac{\hat{u}_2}{\hat{v}_2} &\simeq \frac{r_2-r_1}{r_2+r_1}\frac{X^-}{y}-\frac{b}{a}\frac{2r_1}{r_2+r_1},\\
\alpha^- &\simeq \frac{b}{ad\delta}
\end{aligned}$$

- $\theta \to 1$

$$\begin{aligned}
\frac{X^-}{y} &\simeq -d\frac{(r_2+r_1)(3r_2-r_1)}{(3r_2+r_1)a\delta^2\chi}-\frac{b}{a},\\
\frac{\hat{u}_2}{\hat{v}_2} &\simeq \frac{3r_2+r_1}{3r_2-r_1}\frac{X^-}{y}+\frac{b}{a}\frac{2r_1}{3r_2-r_1},\\
\alpha^- &\simeq -2\frac{r_2}{d\delta^2}\frac{X^-}{y}-\frac{b}{a}\frac{r_2+r_1}{d\delta^2}.
\end{aligned}$$

Field toward $X_0^+ = 0$.

- $\theta \to 0$

$$\begin{aligned}
\frac{X^+}{y} &\simeq -d\frac{r_1+r_2}{a\delta^2\chi}-\frac{b}{a},\\
\frac{\hat{u}_2}{\hat{v}_2} &\simeq \frac{r_2-r_1}{r_2+r_1}\frac{X^+}{y}-\frac{b}{a}\frac{2r_1}{r_2+r_1},\\
\alpha^+ &\simeq -\frac{b}{ad\delta}
\end{aligned}$$

- $\theta \to 1$

$$\begin{aligned}
\frac{X^+}{y} &\simeq d\frac{(r_2+r_1)(3r_2-r_1)}{(3r_2+r_1)a\delta^2\chi}-\frac{b}{a},\\
\frac{\hat{u}_2}{\hat{v}_2} &\simeq \frac{3r_2+r_1}{3r_2-r_1}\frac{X^+}{y}+\frac{b}{a}\frac{2r_1}{3r_2-r_1},\\
\alpha^+ &\simeq 2\frac{r_2}{d\delta^2}\frac{X^+}{y}+\frac{b}{a}\frac{r_2+r_1}{d\delta^2}.
\end{aligned}$$

Hence, as a function of $X^\pm/y$, the asymptotic formulas for $\hat{u}_2/\hat{v}_2$ are identical in both fields, while those for $\alpha^\pm$ are just opposite. It follows that, as $y\to 0$ with $X^- = -X^+$, both V^- and V^+ converge towards

$$V(X^+,-X^+,0)=\frac{3r_2-r_1}{d\delta^2}X^+ = U(X^+,-X^+).$$

It follows also that the decision surface separating the two optimal fields reaches the axis $y=0$ at $(X^+ + X^-)/2$. However, looking at the formulas (3.7) and (3.8), one easily sees that the solution of the pursuit problem is symmetrical with respect to the plane $X^+ + X^- = 0$ if and only if $b = 0$. Indeed, in that case, the ellipses W_1 and W_2 have their symmetry axes aligned with the axes x and y, and therefore, the whole problem is symmetrical. In that case, the decision surface is just the plane $X^+ + X^- = 0$.

We may also notice that the asymptotes of the curve $\hat{u}_2/\hat{v}_2$ as a function of $X^\pm/y$ meet at the point $(-b/a\ -b/a)$. Therefore, while the slopes of these asymptotes only depend on the ratio r_2/r_1, their positions in translation, as characterized by their intersection point, depend only on the coefficients a and b of the matrix A. The asymptotes of the curves giving $\alpha^\pm$ also meet at $X^\pm/y = -b/a$.

3.3. **Set capture.** We consider now that Pursuer has achieved its goal when $z \in \mathsf{C}$ where C is a bounded convex set of $\mathbb{R}^2$, not necessarily the ellipse of the previous section.

Assumption There exists no $(\xi, \eta) \in \mathbb{R}^2$ such that the set inclusion

$$\begin{pmatrix} [\xi + X^-(0), \xi + X^+(0)] \\ \eta \end{pmatrix} \subset \mathsf{C}$$

holds. Typically, if C is the ellipse

$$\|z\|_C \le c \quad \text{with} \quad C = \begin{pmatrix} 1 & \alpha\beta \\ \alpha\beta & \alpha^2 \end{pmatrix}, \tag{3.14}$$

this means that $X^+(0) - X^-(0) > 2c$.

The same reasoning as in Proposition 3.1 shows that the min-sup capture time has to be when $[X^-, X^+] \subset \mathsf{C}$. If no translation of the initial uncertainty interval holds in C, and consequently for no later such interval either, the only way to make it shrink is by reaching $y = 0$ and traversing that axis as in subsection 3.2.2 above.

Let $[\gamma^-, \gamma^+]$ be the intersection of the x axis with C. Typically, if C is the ellipse (3.14), $-\gamma^- = \gamma^+ = c$. Let $\widetilde{X}^- = X^- - \gamma^-$, and $\widetilde{X}^+ = X^+ - \gamma^+$. They obey the same dynamical equations as X^- and X^+, with translated initial conditions. Then, all the above analysis holds with these new variables replacing X^- and X^+.

4. Conclusion

We have therefore proposed a complete solution of the min-sup pursuit problem, or best guaranteed capture time problem, with an interval information on the initial Evader's abscissa and no further information on this coordinate, except the one that Pursuer can infer from the continuous and instantaneous knowledge of Evader's other coordinate. And this, assuming that the model, i.e. the cinematic possibilities (the odograph domain) of both players is common knowledge, and as simple as we have assumed.

Yet this is the correct solution assuming that "capture", i.e. the coincidence of both coordinates of both players, can be observed when it occurs. This is implied by the last two words of the synthesis: "until capture". One may easily think that for some aplications of this model, this is not the case. In an application in industrial strategy, for instance, one could then modify the problem as "best guaranteed time to a Pareto superior position than the pursuer". The solution is to use only the field of extremals toward $X_0^+ = 0$, and once there, use any control such that $v_2 \ge v_1$, and $u_2 = \Upsilon_2^+(v_2)$.

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APPENDIX A. SOLVING EQUATION (3.9)

A formula for χ^2. We need to solve equation (3.9) for χ^2. We rewrite it as

$$r_1\sqrt{\chi^2+d^2\psi^2}+1=r_2\sqrt{\chi^2+d^2\phi^2}\,,$$

and we know that, if it gives a positive χ^2, then $\chi=\sqrt{\chi^2}$ will generate the characteristic lines in the half space $y>0$, and $\chi=-\sqrt{\chi^2}$ in the half space $y<0$. Both sides above are positive. So we introduce no spurious solution by equating the squares. We rearrange as

$$(r_2^2-r_1^2)\chi^2+r_2^2d^2\phi^2-r_1^2d^2\psi^2-1=2r_1\sqrt{\chi^2+d^2\psi^2}\,.$$

We shall again equate the squares of both sides. But now this introduces a spurious "solution", solving

$$(r_2^2-r_1^2)\chi^2+r_2^2d^2\phi^2-r_1^2d^2\psi^2-1=-2r_1\sqrt{\chi^2+d^2\psi^2}$$

The left hand side is positive for the true solution and negative for the spurious one. Therefore, the true solution will be the largest of the two roots in χ^2. Squaring both sides, we arrive at an equation of the form

$$P\chi^4+2Q\chi^2+R=0\,. \tag{A.1}$$

with

$$\begin{aligned} P&=(r_2^2-r_1^2)^2\,,\\ Q&=(r_2^2-r_1^2)(r_2^2d^2\phi^2-r_1^2d^2\psi^2)-r_2^2-r_1^2\\ R&=r_2^4d^4\phi^4+r_1^4d^4\psi^4-2r_1^2r_2^2d^4\phi^2\psi^2-2r_2^2d^2\phi^2-2r_1^2d^2\psi^2+1\,. \end{aligned} \tag{A.2}$$

This is a second degree algebraic equation for χ^2. Clearly, P is positive. It follows from the above analysis that the required root is

$$\chi^2=\frac{1}{P}(-Q+\sqrt{Q^2-PR})\,. \tag{A.3}$$

We already know that for $\theta=0$ and $\theta=1$, the solution is $\chi^2=0$. Indeed, it can be directly checked that for these particular values, we get $R=0$. The rest of the appendix is devoted to proving the following:

Proposition A.1. *Formulas (A.2)(A.3) provide a real positive χ^2 for every $\theta\in(0,1)$.*

A particular value of θ. A particular case arises when

$$\theta=\frac{\delta}{2r_2}=\frac{1}{2}-\frac{r_1}{2r_2}\in\left(0,\frac{1}{4}\right)\,,$$

giving $\phi=0$. Equation (3.9) then reads

$$r_1\sqrt{\chi^2+\frac{4}{\delta^2}}+1=r_2|\chi|\,,$$

Squaring both sides, this is equivalent to

$$\chi^2(r_2^2-r_1^2)-2r_2\chi+1-\frac{4r_1^2}{\delta^2}=0\,.$$

The discriminant of this second degree algebraic equation is

$$\Delta = r_2^2 - (r_2^2 - r_1^2)\left(1 - 4\frac{r_1^2}{\delta^2}\right) = r_1^2\frac{5r_2 + 3r_1}{\delta} > 0\,.$$

Therefore this equation has two roots, whose sum is positive. Thus the maximum root is positive. By continuity, there is a range of θ for which formula (A.3) yields a positive value.

Proof of the proposition. If formula (A.3) gives a negative value for some $\theta \in (0,1)$, by continuity there is to be a value of θ in that range where the same formula yields $\chi^2 = 0$. However, if this solves equation (3.9), then we must have

$$1 + r_1 d\psi - r_2 d|\phi| = 0\,.$$

If $\phi > 0$, this reads

$$\delta^2 + r_1(\delta + 2r_2\theta) - r_2(\delta - 2r_2\theta) = 2r_2\delta\theta = 0\,,$$

i.e. $\theta = 0$, and if $\phi < 0$,

$$\delta^2 + r_1(\delta + 2r_2\theta) - r_2(2r_2\theta - \delta) = 2r_2\delta(1-\theta) = 0\,,$$

i.e. $\theta = 1$. Therefore, formula (A.3) is never negative in the range $\theta \in (0,1)$.

The other way the proposition could be false is if the discriminant $Q^2 - PR$ becomes negative. But by continuity, it has to be zero for some value of $\theta \in (0,1)$. At that value, both solutions of (A.1) coincide. But the spurious solution yields

$$1 + r_1\sqrt{\chi^2 + d^2\psi^2} - r_2\sqrt{\chi^2 + \mathrm{d}^2\phi^2} = 2$$

while the correct one yields zero. Therefore these two values of χ^2 cannot coincide. The proposition is therefore proved.