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ALTERNATING MINIMIZATION FOR SIMULTANEOUS ESTIMATION OF A LATENT VARIABLE AND IDENTIFICATION OF A LINEAR CONTINUOUS-TIME DYNAMIC SYSTEM

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Dedicated to the memory of Roland Glowinski

Abstract. We propose an optimization formulation for the simultaneous estimation of a latent variable and the identification of a linear continuous-time dynamic system, given a single input-output pair. We justify this approach based on Bayesian maximum a posteriori estimators. Our scheme takes the form of a convex alternating minimization, over the trajectories and the dynamic model respectively. We prove its convergence to a local minimum which verifies a two point-boundary problem for the (latent) state variable and a tensor product expression for the optimal dynamics.

Keywords. Alternating minimization; Continuous-time linear dynamic system; Latent variable; System identification.

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1. Introduction

The theory of latent variables in Data Science has been progressing very fast in the recent years, with the objective of reducing the dimension of the dataset. Since data is often associated with dynamic systems, it is natural to consider in this context the framework of identification and estimation of dynamic systems. We refer to [10] for a survey of the main ideas in this direction and to [11] for details, in connection with the Kalman filter in discrete time. The general idea is to consider the latent variable as described by a dynamic system in state space representation. The difficulty is that we need to identify the system while estimating it. The maximum likelihood approach is a natural way to proceed. A discrete time version for a simpler

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model can be found in [3]. While most of the papers on this specific topic are indeed in discrete time, there is a huge swath of literature dealing with dynamic systems in continuous time, justifying tackling this setting as well. To simplify the theory and the algorithm, we consider that some aspects of the linear dynamic system are known, in particular the covariance matrices of the noises and the observation matrix.

2. The model

We consider an input-output problem $(v(t),y(t))_{t\in[0,T]}$ in state space representation, where we want from a single trajectory to reconstruct the state and its dynamic equation, in other words to perform both estimation and system identification at the same time. We assume that the linear dynamic systems with noise are described by two stochastic differential equations, *with all the underlined quantities being known*,

$$dx = (Ax + B\underline{v})dt + \underline{G}dw, \qquad x(0) = \xi \sim \mathcal{N}(\underline{x}_0, \underline{\Pi}_0), \tag{2.1}$$

$$dy = \underline{C}x(t)dt + db(t), \qquad y(0) = 0, \tag{2.2}$$

where we assume for simplicity that the dimensions of the operators are fixed, for instance chosen minimal through realization theory, see e.g. [3]. More precisely we take $x(t) \in \mathbb{R}^N$, $v(t) \in \mathbb{R}^d$, $A \in \mathscr{L}(\mathbb{R}^N;\mathbb{R}^N)$, $B \in \mathscr{L}(\mathbb{R}^d;\mathbb{R}^N)$; $v(t)$ being a given deterministic control. There is no optimal control in this setup, but $v(.)$ is an input decided by the controller. The process $w(t)$ is a Wiener process in $\mathbb{R}^m$, with correlation matrix Q, and $G \in \mathscr{L}(\mathbb{R}^m;\mathbb{R}^N)$. The random variable ξ is Gaussian with mean $x_0 \in \mathbb{R}^N$ and covariance matrix Π_0. It is independent of the Wiener process $w(t)$. The matrices A and B are not known, although they are in the vicinity of known matrices A_0, B_0 used as priors. The state of the system $x(t)$ is not observable. We observe instead the process $y(t) \in \mathbb{R}^p$ where $C \in \mathscr{L}(\mathbb{R}^N;\mathbb{R}^p)$ is known. The process $b(t)$ is a Wiener process in $\mathbb{R}^p$, with covariance matrix R, also independent from ξ and $w(.)$. For simplicity, we assume in the sequel that the matrices Π_0, Q, R are invertible (inverses would then be replaced with pseudo-inverses).

The only information is $v(t), y(t), t \in [0,T]$. If A, B were known, the problem would reduce to estimating the evolution of the state $x(t)$. This is the classical Kalman smoothing problem. There are many equivalent ways to solve it. For instance, it is well known that the maximum likelihood is equivalent to the following least square deterministic control problem

$$\frac{dx}{dt} = Ax + Bv + Gw, \quad x(0) = \xi, \tag{2.3}$$

in which the control is the pair $(\xi, w(.))$ minimizing the payoff

$$\frac{1}{2}\Pi_0^{-1}(\xi - x_0).(\xi - x_0) + \frac{1}{2}\int_0^T Q^{-1}w(t).w(t)dt + \frac{1}{2}\int_0^T R^{-1}(y(t) - Cx(t)).(y(t) - Cx(t))dt. \tag{2.4}$$

When A, B are not known, one way to proceed is to approach (2.3) as a constraint. This leads to the following formulation. Introduce the argument $Z = (A, B, x(.).w(.))$ where $x(.) \in$

In continuous time, $y(t) \in L^2(\mathscr{T}, \mathbb{R}^m)$ "is reminiscent of the observation process, in fact rather the derivative of the observation process (which, as we know, does not exist)" [2, p180]. Thus it is as if we observed this derivative to do the reconstruction, which is justified since only integrals of it appear in (2.4).

$H^1(0,T;\mathbb{R}^N)$, $w(.) \in L^2(0,T;\mathbb{R}^m)$, and define the norm

$$||Z||^2 = \mathrm{tr}(AA^* + BB^*) + |x(0)|^2 + \int_0^T |\frac{dx}{dt}|^2 dt + \int_0^T |w(t)|^2 dt, \tag{2.5}$$

thus Z belongs to a Hilbert space, denoted by $\mathscr{Z}$. Based on (2.4), we then introduce the following functional J over $\mathscr{Z}$ to perform the reconstruction. It will be justified in Section 3 through Bayesian arguments.

$$\begin{aligned} \min_{(A,B,x(.),w(.))} J((A,B,x(.),w(.)) &:= \frac{\alpha}{2} \mathrm{tr}\left((A-\underline{A}_0)(A-\underline{A}_0)^* + (B-\underline{B}_0)(B-\underline{B}_0)^*\right) \\ &+ \frac{\beta}{2} \int_0^T \left| \frac{dx}{dt} - Ax(t) - B\underline{v}(t) - \underline{G}w(t) \right|^2 dt + \frac{1}{2} \underline{\Pi_0}^{-1}(x(0)-\underline{x}_0).(x(0)-\underline{x}_0) \\ &+ \frac{1}{2} \int_0^T \underline{Q}^{-1} w(t).w(t) dt + \frac{1}{2} \int_0^T \underline{R}^{-1}(\underline{y}(t) - \underline{C}x(t)).(\underline{y}(t) - \underline{C}x(t)) dt, \end{aligned} \tag{2.6}$$

for given parameters $\alpha, \beta > 0$. To alleviate notations, *from now on we do not underline the known quantities* and we refer the reader to this section. The first term in α is a regularizing term in (A,B) using our prior. The second term in β is a penalty term if the constraint (2.3) is not satisfied. It can be seen as a regularization of (2.4) (see Theorem 3.1 below). The other terms penalize deviations of $(x(0), w(t), Cx(t))$ from their references $(x_0, 0, y(t))$. So the problem amounts to minimizing the functional $J(Z)$ on the Hilbert space $\mathscr{Z}$. Note that (A,B) only appear in the two first terms, while $(x(.), w(.))$ appear in all terms but the two first. This suggests to do an alternating minimization of J as in Section 5 below. However J is non convex due to term $Ax(t)$, so we cannot hope to reach a global minimum for every initialization. We will first justify the choice of J in Section 3 and then prove the existence of a minimum in Section 4, giving also the first-order optimality conditions that it satisfies. Note that a similar methodology can be replicated if we consider some other matrices to be unknown (e.g. C or G). In the discrete time case, the problem of estimating simultaneously two matrices while minimizing an expression of their product has been considered in [9].

3. Bayesian justification of the model

We follow here the presentation of [6, Section 2] for the classical derivation of a least squares problem from a maximum a posteriori (MAP) estimator of system based on a model (M) and observations operator (O). To avoid technicalities, we do the justification for random variables over a finite set, thus not for the stochastic differential equations (2.1)-(2.2) with Brownian motions which we considered. However the ideas and results extend to infinite dimensions [5]. To obtain the objective function (2.6), we consider that (2.2) is an equation of the form

$$Y = O(x(.), w(.)) + \eta_{obs}$$

The key reason for the methodological difference when moving to continuous time is reminded in [5] "While in the finite-dimensional setting, the prior and posterior distribution of such statistical problems can typically be described by densities w.r.t. the Lebesgue measure, such a characterisation is no longer possible in the infinite dimensional spaces [...] no analogue of the Lebesgue measure exists in infinite dimensional spaces." However Gaussian measures can still serve as a replacement in our case [4]. They correspond here to the Wiener processes we consider. Radon–Nikodym derivatives are then obtained through Girsanov's theorem [2, Chapter 6.5].

where $Y = (y_t)_{t\in\mathscr{T}}$, $\eta_{obs} \sim \mathscr{N}(0,\mathscr{R})$, $\mathscr{R}(s,t) = \delta_{s=t}R(t)$. Similarly we relax (2.3) by introducing a model error

$$0 = M(x(.),w(.),A,B) + \eta_{model}$$

where $\eta_{model} \sim \mathscr{N}(0,\mathrm{Id}/\beta)$. We put a Gaussian prior μ_0 on $(x(.),w(.),A,B)$ of the form $x(0) \sim \mathscr{N}(x_0,\Pi_0)$, $w(.) \sim \mathscr{N}(0,\mathscr{Q})$ with $\mathscr{Q}(s,t) = \delta_{s=t}Q(t)$, $A \sim \mathscr{N}(A_0,\mathrm{Id}/\alpha)$ and $B \sim \mathscr{N}(B_0,\mathrm{Id}/\alpha)$. Thus, by Bayes' theorem, the posterior distribution is given by

$$\mu^*(dZ) \propto \exp(-\frac{\beta}{2}\|M(x(.),w(.),A,B)\|^2 - \frac{1}{2}\|y(.)-O(x(.),w(.))\|_{\mathscr{R}})^2\mu_0(dZ).$$

where $\|y(.)\|^2_{\mathscr{R}} = (\mathscr{R}^{-1}y(.)).y(.)$. The MAP estimator is then given by $\mathrm{argmax}_{Z\in\mathscr{Z}}\,\mu^*(dZ)$, and thus equivalently by minimizing the log-density, $\mathrm{argmin}_{Z\in\mathscr{Z}} -\log\mu^*(dZ)$ which is precisely (2.6). More formally, in continuous time, to derive (2.6) as the problem solved by the MAP, one can just apply [5, Corollary 3.10] to identify J as an Onsager-Machlup functional.

Remark 3.1 (RKHS constraint in the limit case)**.** Interestingly [6, Proposition 1] recalls that for $\beta \to \infty$ (vanishing model noise case), we have that the accumulation points of the optimum $\hat{Z}_\beta$ all satisfy (2.3). For given A,B, (2.3) says that $x(.)$ belongs to the affine vector space of functions $\mathscr{H} = \{x(.)\,|\,\exists w(.),\, {}^{dx}/_{dt} = Ax+Bv+Gw,\, \int_0^T Q^{-1}w(t).w(t)dt < \infty\}$. This space can be equipped with a quadratic norm based on $\Pi_0^{-1}(x(0)).(x(0)) + \int_0^T Q^{-1}w(t).w(t)dt + \int_0^T C^*R^{-1}Cx(t).x(t)dt$ and, once the affine term is removed, has a reproducing kernel Hilbert space (RKHS) structure. We refer to [1] for more on this topic. Since we consider $\beta \neq \infty$, we authorize $x(.)$ to live beyond this RKHS. In other words the "noise" term $w(.)$ can be understood as a control, and by introducing β we assume implicitly some extra noise on the model that was not present in (2.1). Moreover $\beta \to \infty$ implies that the minimizer $\hat{w}(.)$ is equal to $G^{\ominus}({}^{dx}/_{dt} - Ax(t) - Bv(t))$, with $G^{\ominus}$ the pseudo-inverse of G for the Euclidean norm. Consequently the objective simplifies to

$$\begin{aligned}\tilde{J}(A,B,x(.)) &= \frac{\alpha}{2}\operatorname{tr}((A-A_0)(A-A_0)^* + (B-B_0)(B-B_0)^*) + \frac{1}{2}\Pi_0^{-1}(x(0)-x_0).(x(0)-x_0)\\ &+ \frac{1}{2}\int_0^T G^{\ominus,\top}Q^{-1}G^{\ominus}\left(\frac{dx}{dt} - Ax(t) - Bv(t)\right).\left(\frac{dx}{dt} - Ax(t) - Bv(t)\right)dt\\ &+ \frac{1}{2}\int_0^T R^{-1}(y(t)-Cx(t)).(y(t)-Cx(t))dt, \end{aligned}\tag{3.1}$$

where it is effectively the noise $w(.)$ of (2.1) that is penalized, and $\tilde{J}$ corresponds to the traditional least square estimator used in Kalman smoothing [1, 7].

Remark 3.2 (Relation with EM)**.** It is well-known that the Expectation-Maximization (EM) algorithm is an alternating minimization of a log-likelihood [8], which is the form of algorithm we propose in Section 5. More precisely, given a probability space $(\mathscr{U},\bar{\mu})$, the relative entropy (Kullback–Leibler divergence) is defined as

$$\mathrm{KL}(\mu|\bar{\mu}) = \int_{\mathscr{U}} \ln\left({}^{d\mu}/_{d\bar{\mu}}(u)\right)d\mu(u) \tag{3.2}$$

for μ absolutely continuous w.r.t. $\bar{\mu}$ and $+\infty$ otherwise. In our case, we assume our observations Y to be sampled according to $\bar{\nu}$ and X serves as a latent, hidden random variable $X \in (\mathscr{X},\bar{\mu})$.

We posit a joint distribution $p_\theta(dx,dy)$ parametrized by an element $\theta=(A,B)$ of the set $\Theta=\mathscr{L}(\mathbb{R}^N;\mathbb{R}^N)\times\mathscr{L}(\mathbb{R}^d;\mathbb{R}^N)$. As presented in [8,], the goal is to infer θ by solving

$$\min_{\theta\in\Theta}\mathrm{KL}(\bar{\nu}|p_Y p_\theta), \tag{3.3}$$

where $p_Y p_\theta(dy)=\int_{\mathscr{X}} p_\theta(dx,dy)$ is the marginal in $\mathscr{Y}$. The EM approach starts by minimizing a surrogate function of θ upperbounding $\mathrm{KL}(\bar{\nu}|p_Y p_\theta)$. For any $\pi\in\Pi(*,\bar{\nu})=\{\pi\,|\,p_Y\pi=\bar{\nu}\}$, by the data processing inequality, i.e. KL of the marginals is smaller than KL of the plans,

$$\mathrm{KL}(\bar{\nu}|p_Y p_\theta)\le \mathrm{KL}(\pi|p_\theta)=:L(\pi,\theta).$$

EM then proceeds by alternating minimizations of $L(\pi,\theta)$ [8, Theorem 1]:

$$\theta_n=\operatorname*{argmin}_{\theta\in\Theta}\mathrm{KL}(\pi_n|p_\theta), \tag{3.4}$$

$$\pi_{n+1}=\operatorname*{argmin}_{\pi\in\Pi(*,\bar{\nu})}\mathrm{KL}(\pi|p_{\theta_n}). \tag{3.5}$$

The above formulation consists in (3.4), optimizing the parameters θ_n at step n (M-step), and then (3.5), optimizing the joint distribution π_{n+1} at step $n+1$ (E-step). This is actually what we propose as algorithm to minimize J by minimizing alternatively in $X=(x(.),w(.))$ and $\theta=(A,B)$, J being obtained as previously as the KL divergence of Gaussian measures. However making explicit the (Gaussian) measures underlying our parametrization goes beyond our scope and we now move to the study of our specific least-squares J.

4. Existence of a minimum and necessary condition

Before searching for a minimum, we prove that J has indeed one.

Proposition 4.1. *The functional $J(Z)$ attains its infimum.*

Proof. The functional $J(Z)$ is continuous on $\mathscr{Z}$. It is also weakly lower semicontinuous. Indeed if $Z_n\rightharpoonup Z$ (weakly), then $A_n\to A$ in $\mathscr{L}(\mathbb{R}^N;\mathbb{R}^N)$, $B_n\to B$ in $\mathscr{L}(\mathbb{R}^d;\mathbb{R}^N)$, $x_n(.)\rightharpoonup x(.)$ in $H^1(0,T;\mathbb{R}^N)$, $w_n(.)\rightharpoonup w(.)$ in $L^2(0,T;\mathbb{R}^m)$. We deduce that $(x_n(.))_n$ is equicontinuous, hence, by Ascoli's theorem, $x_n(.)\to x(.)$ in $C^0([0,T];\mathbb{R}^N)$. It follows that $A_nx_n(.)\to Ax(.)$ in $C^0([0,T];\mathbb{R}^N)$ and $\dfrac{dx_n}{dt}\rightharpoonup\dfrac{dx}{dt}$ in $L^2(0,T;\mathbb{R}^N)$. Consequently we have

$$\begin{aligned}&\frac{\alpha}{2}\operatorname{tr}\left((A_n-A_0)(A_n-A_0)^*+(B_n-B_0)(B_n-B_0)^*\right)\\&\qquad+\frac{1}{2}\Pi_0^{-1}(x_n(0)-x_0).(x_n(0)-x_0)+\frac{1}{2}\int_0^T R^{-1}(y(t)-Cx_n(t)).(y(t)-Cx_n(t))dt\\&\qquad\qquad\xrightarrow{n\to\infty}\frac{\alpha}{2}\operatorname{tr}\left((A-A_0)(A-A_0)^*+(B-B_0)(B-B_0)^*\right)\\&\qquad\qquad+\frac{1}{2}\Pi_0^{-1}(x(0)-x_0).(x(0)-x_0)+\frac{1}{2}\int_0^T R^{-1}(y(t)-Cx(t)).(y(t)-Cx(t))dt.\end{aligned}$$

From the weak lower semicontinuity of the norm in the spaces $L^2(0,T;\mathbb{R}^N)$ and $L^2(0,T;\mathbb{R}^m)$, we conclude easily that $J(Z)\le\liminf J(Z_n)$. If we consider a minimizing sequence Z_n, namely

$$J(Z_n)\to\inf J(Z)\ge 0.$$

Then, since $J(Z_n) \leq J(0)$ for n sufficiently large, it follows easily that the sequence Z_n is bounded in $\mathscr{Z}$. Since weakly closed bounded sets are weakly compact, we can extract a subsequence, still denoted $Z_n \rightharpoonup \widehat{Z}$ in $\mathscr{Z}$ weakly. From weak lower semicontinuity of J, we obtain $J(\widehat{Z}) \leq \liminf J(Z_n) = \inf J(Z)$. This implies that $\widehat{Z}$ is a minimum of $J(Z)$, which concludes the proof.■ □

We now check that $J(Z)$ has a Gâteaux differential in Z.

Proposition 4.2. *The gradient $DJ(Z) \in \mathscr{Z}$ is given by the formula*

$$
\begin{aligned}
((DJ(Z),\widetilde{Z})) = {} & \operatorname{tr}\left(\left[\alpha(A-A_0)+\int_0^T q(t)x^*(t)dt\right]\widetilde{A}^*\right)+\operatorname{tr}\left(\left[\alpha(B-B_0)+\int_0^T q(t)v^*(t)dt\right]\widetilde{B}^*\right)\\
& +\left(\Pi_0^{-1}(x(0)-x_0)+\int_0^T A^*q(t)-C^*R^{-1}(y(t)-Cx(t))dt\right).\widetilde{x}(0)\\
& +\int_0^T\left[-q(t)+\int_t^T A^*q(s)-C^*R^{-1}(y(s)-Cx(s))ds\right].\frac{d\widetilde{x}}{dt}(t)dt\\
& +\int_0^T (G^*q(t)+Q^{-1}w(t)).\widetilde{w}(t)dt
\end{aligned}
\tag{4.1}
$$

in which $Z=(A,B,x(.),w(.))$, $\widetilde{Z}=(\widetilde{A},\widetilde{B},\widetilde{x}(.),\widetilde{w}(.))$ and $q(t)$ is defined by

$$
q(t) = -\beta\left(\frac{dx}{dt}(t)-Ax(t)-Bv(t)-Gw(t)\right). \tag{4.2}
$$

Proof. From the definition of the Gâteaux differential in Z, we must check that

$$
\frac{d}{d\theta}J(Z+\theta\widetilde{Z})|_{\theta=0} = ((DJ(Z),\widetilde{Z})) \tag{4.3}
$$

is equal to the right hand side of (4.1). We fix Z, $\widetilde{Z}$ with $q(t)$ defined by (4.2). As easily checked we can write

$$
\begin{aligned}
\frac{d}{d\theta}J(Z+\theta\widetilde{Z})|_{\theta=0} = {} & \operatorname{tr}\left(\left[\alpha(A-A_0)+\int_0^T q(t)x^*(t)dt\right]\widetilde{A}^*\right)\\
& +\operatorname{tr}\left(\left[\alpha(B-B_0)+\int_0^T q(t)v^*(t)dt\right]\widetilde{B}^*\right)\\
& +\Pi_0^{-1}(x(0)-x_0).\widetilde{x}(0)-\int_0^T q(t).\left(\frac{d}{dt}\widetilde{x}-A\widetilde{x}(t)\right)dt\\
& -\int_0^T C^*R^{-1}(y(t)-Cx(t)).\widetilde{x}(t)dt+\int_0^T (G^*q(t)+Q^{-1}w(t)).\widetilde{w}(t)dt.
\end{aligned}
\tag{4.4}
$$

We then replace $\widetilde{x}(t)$ with $\widetilde{x}(0)+\int_0^t \frac{d}{ds}\widetilde{x}(s)ds$. We perform a change of integration and some rearrangements to obtain the relation (4.1). □

If $\widehat{Z}=(\widehat{A},\widehat{B},\widehat{x}(.),\widehat{w}(.))$ is a point of minimum for $J(Z)$, it follows from formula (4.1) that the corresponding $\widehat{q}(t)$ defined by

$$
\widehat{q}(t) = -\beta\left(\frac{d\widehat{x}}{dt}(t)-\widehat{A}\widehat{x}(t)-\widehat{B}v(t)-G\widehat{w}(t)\right) \tag{4.5}
$$

satisfies

$$\Pi_0^{-1}(\widehat{x}(0)-x_0)+\int_0^T(A^*\widehat{q}(t)-C^*R^{-1}(y(t)-C\widehat{x}(t))dt=0$$
$$-\widehat{q}(t)+\int_t^T(A^*\widehat{q}(s)-C^*R^{-1}(y(s)-C\widehat{x}(s))ds=0,\forall t.$$

It follows that $\frac{d\widehat{q}}{dt}$ is well defined. Differentiating the previous equation, and reordering (4.5), we obtain the following system of optimality conditions for (4.1)

$$\frac{d\widehat{x}}{dt}=\widehat{A}\widehat{x}(t)+\widehat{B}v(t)-(GQ^{-1}G^*+\frac{I}{\beta})\widehat{q}(t), \qquad \widehat{x}(0)=x_0-\Pi_0\widehat{q}(0), \tag{4.6}$$
$$-\frac{d\widehat{q}}{dt}=\widehat{A}^*\widehat{q}(t)-C^*R^{-1}C(y(t)-C\widehat{x}(t)), \qquad \widehat{q}(T)=0,$$

$$\alpha(\widehat{A}-A_0)+\int_0^T\widehat{q}(t)\widehat{x}^*(t)dt=0, \tag{4.7}$$
$$\alpha(\widehat{B}-A_0)+\int_0^T\widehat{q}(t)v^*(t)dt=0,$$

with $\widehat{w}(t)$ given by

$$\widehat{w}(t)=-QG^*\widehat{q}(t). \tag{4.8}$$

Note that (4.7) has an interesting structure, decomposing the optimal $\widehat{A}$ (resp. $\widehat{B}$) as a sum of rank 1 tensor products between the covector q and the trajectory x (resp. covector q and control v).

5. Alternating minimization algorithm

The relations (4.6), (4.7) can be interpreted as a fixed point problem for the pair $(\widehat{A},\widehat{B})$. If we fix the pair $(\widehat{A},\widehat{B})$ then we obtain the pair $(\widehat{x}(.),\widehat{q}(.))$ by solving the system of forward backward equations (4.6). Next for fixed $(\widehat{x}(.),\widehat{q}(.))$ we obtain $(\widehat{A},\widehat{B})$ by the formulas (4.7). This corresponds also to an alternating minimization of J, which happens in two steps.

The first part is associated to a control problem, formulated as a calculus of variations problem

$$\min_{x(.),w(.)}K(\widehat{A},\widehat{B};x(.),w(.)):=\frac{1}{2}\Pi_0^{-1}(x(0)-x_0).(x(0)-x_0)$$
$$+\frac{\beta}{2}\int_0^T\left|\frac{dx}{dt}-\widehat{A}x(t)-\widehat{B}v(t)-Gw(t)\right|^2dt$$
$$+\frac{1}{2}\int_0^TQ^{-1}w(t).w(t)dt+\frac{1}{2}\int_0^TR^{-1}(y(t)-Cx(t)).(y(t)-Cx(t))dt. \tag{5.1}$$

The second part is associated to an optimization problem

$$\min_{A,B}L(A,B;\widehat{x}(.),\widehat{w}(.)):=\frac{\alpha}{2}\operatorname{tr}((A-A_0)(A-A_0)^*+(B-B_0)(B-B_0)^*)$$
$$+\frac{\beta}{2}\int_0^T\left|\frac{d\widehat{x}}{dt}-A\widehat{x}(t)-Bv(t)-G\widehat{w}(t)\right|^2dt. \tag{5.2}$$

It is important to notice that the two problems (5.1), (5.2) are convex quadratic and have a unique solution, whereas the original problem (2.6) is not convex. This highlights the usefulness of algorithm that we propose to find a local optimum $\widehat{Z}$ of $J(Z)$.

We initialize the algorithm with $A_0 = A_0, B_0 = B_0$. For $n \geq 0$, knowing A_n, B_n we define uniquely the pair $(x_n(.), w_n(.))$ which minimizes $K(A_n, B_n; x(.), w(.))$. This leads immediately to the existence and uniqueness of the pair $x_n(.), q_n(.)$ solution of the system of forward-backward relations

$$\frac{dx_n}{dt} = A_n x_n(t) + B_n v(t) - \left(GQ^{-1}G^* + \frac{I}{\beta}\right) q_n(t), \qquad x_n(0) = x_0 - \Pi_0 q_n(0) \tag{5.3}$$

$$-\frac{dq_n}{dt} = (A_n)^* q_n(t) - C^* R^{-1} C(y(t) - Cx_n(t)), \qquad q_n(T) = 0, \tag{5.4}$$

with $w_n(t)$ given by

$$w_n(t)) = -QG^* q_n(t). \tag{5.5}$$

We then define A_{n+1}, B_{n+1} by minimizing $L(A, B; x_n(.), w_n(.))$. We obtain

$$\alpha(A_{n+1} - A_0) - \beta \int_0^T \left(\frac{dx_n}{dt} - A_{n+1} x_n(t) - B_{n+1} v(t) - G w_n(t)\right)(x_n(t))^* dt = 0, \tag{5.6}$$

$$\alpha(B_{n+1} - B_0) - \beta \int_0^T \left(\frac{dx_n}{dt} - A_{n+1} x_n(t) - B_{n+1} v(t) - G w_n(t)\right)(v(t))^* dt = 0.$$

Using (5.3)-(5.5), the term in w_n canceling out with one of those in q_n, we can rewrite the equation (5.6) as follows by factorizing

$$\begin{aligned} A_{n+1}\left(\alpha I + \beta \int_0^T x_n(t)(x_n(t))^* dt\right) + \beta B_{n+1} \int_0^T v(t)(x_n(t))^* dt \\ = \alpha A_0 + \beta A_n \int_0^T x_n(t)(x_n(t))^* dt + \beta B_n \int_0^T v(t)(x_n(t))^* dt - \int_0^T q_n(t)(x_n(t))^* dt, \end{aligned} \tag{5.7}$$

$$\begin{aligned} \beta A_{n+1} \int_0^T x_n(t)(v(t))^* dt + B_{n+1}(\alpha I + \beta \int_0^T v(t)(v(t))^* dt) \\ = \alpha B_0 + \beta A_n \int_0^T x_n(t)(v(t))^* dt + \beta B_n \int_0^T v(t)(v(t))^* dt - \int_0^T q_n(t)(v(t))^* dt. \end{aligned}$$

Our main result is the convergence of the alternating minimization scheme to extremal points for the first-order optimality conditions.

Theorem 5.1. *The sequence $J(Z^n)$ is decreasing. The sequence Z^n is bounded and $Z^{n+1} - Z^n \to 0$. Limits of converging subsequences of Z^n are solutions of the set of necessary conditions (4.6), (4.7).*

Proof. We compute the two differences $K(A_{n+1}, B_{n+1}; x_n(.), w_n(.)) - K(A_{n+1}, B_{n+1}; x_{n+1}(.), w_{n+1}(.))$ and $L(A_n, B_n; x_n(.), w_n(.)) - L(A_{n+1}, B_{n+1}; x_n(.), w_n(.))$ which are nonnegative numbers, since $(x_{n+1}(.), w_{n+1}(.))$ minimizes $K(A_{n+1}, B_{n+1}; x(.), w(.))$ and (A_{n+1}, B_{n+1}) minimizes $L(A, B; x_n(.), w_n(.))$. Since we know the minimizers, we can use the fact that a quadratic function $f(z)$, with Hessian

H and minimum $\bar{z}$, satisfies $f(z)-f(\bar{z})=\frac{1}{2}(z-\bar{z})^*H(z-\bar{z})$. This is a completion of square argument. Consequently, we have

$$\begin{aligned}
&K(A_{n+1},B_{n+1};x_n(.),w_n(.))-K(A_{n+1},B_{n+1};x_{n+1}(.),w_{n+1}(.))\\
&=\frac{1}{2}\Pi_0^{-1}(x_n(0)-x_{n+1}(0)).(x_n(0)-x_{n+1}(0))\\
&+\frac{\beta}{2}\int_0^T\left|\frac{d}{dt}(x_n-x_{n+1})-A_{n+1}(x_n(t)-x_{n+1}(t))-G(w_n(t)-w_{n+1}(t))\right|^2dt\\
&+\frac{1}{2}\int_0^T Q^{-1}(w_n(t)-w_{n+1}(t)).(w_n(t)-w_{n+1}(t))dt\\
&+\frac{1}{2}\int_0^T R^{-1}C(x_n(t)-x_{n+1}(t)).C(x_n(t)-x_{n+1}(t))dt.
\end{aligned}\tag{5.8}$$

Similarly we can write also

$$\begin{aligned}
&L(A_n,B_n;x_n(.),w_n(.))-L(A_{n+1},B_{n+1};x_n(.),w_n(.))\\
&=\frac{\alpha}{2}\text{tr}\,((A_n-A_{n+1})(A_n-A_{n+1})^*+(B_n-B_{n+1})(B_n-B_{n+1})^*)\\
&+\frac{\beta}{2}\int_0^T|(A_n-A_{n+1})x_n(t)+(B_n-B_{n+1})v(t)|^2dt.
\end{aligned}\tag{5.9}$$

Adding up, we see that the left hand side is $J(Z^n)-J(Z^{n+1})$, so we obtain

$$\begin{aligned}
J(Z^n)-J(Z^{n+1})&=\frac{1}{2}\Pi_0^{-1}(x_n(0)-x_{n+1}(0)).(x_n(0)-x_{n+1}(0))\\
&+\frac{\beta}{2}\int_0^T\left|\frac{d}{dt}(x_n-x_{n+1})-A_{n+1}(x_n(t)-x_{n+1}(t))-G(w_n(t)-w_{n+1}(t))\right|^2dt\\
&+\frac{1}{2}\int_0^T Q^{-1}(w_n(t)-w_{n+1}(t)).(w_n(t)-w_{n+1}(t))dt\\
&+\frac{1}{2}\int_0^T R^{-1}C(x_n(t)-x_{n+1}(t)).C(x_n(t)-x_{n+1}(t))dt\\
&+\frac{\alpha}{2}\text{tr}\,((A_n-A_{n+1})(A_n-A_{n+1})^*+(B_n-B_{n+1})(B_n-B_{n+1})^*)\\
&+\frac{\beta}{2}\int_0^T|(A_n-A_{n+1})x_n(t)+(B_n-B_{n+1})v(t)|^2dt,
\end{aligned}\tag{5.10}$$

which is a nonnegative quantity. It follows that the sequence $J(Z^n)$ is decreasing. Since it is nonnegative, it converges. From the relation (5.10) we see that $Z^n-Z^{n+1}\to 0$ in $\|.\|_{\mathscr{Z}}$. Since $J(Z^n)\leq J(Z^0)$, the sequence Z^n is bounded in $\mathscr{Z}$. If we extract a subsequence which converges weakly to $\widehat{Z}$, also noted Z^n without loss of generality, then $A_n\to\widehat{A},B_n\to\widehat{B}$ and $x_n(.)\to\widehat{x}(.)$ in $H^1(0,T;\mathbb{R}^N)$ weakly, hence strongly in $C^0([0,T];\mathbb{R}^N)$. From (5.3), $q_n(.)\to\widehat{q}(,)$ in $H^1(0,T;\mathbb{R}^N)$ weakly and strongly in $C^0([0,T];\mathbb{R}^N)$. Therefore, from (5.5) $w_n(.)\to\widehat{w}(.)$ in $L^2(0,T;\mathbb{R}^N)$. Finally $Z_n\to\widehat{Z}$. We can thus take the limit in equations (5.3), (5.7) and obtain that $\widehat{Z}$ is solution of the set of equations (4.6), (4.7). This concludes the proof. □

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