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HÖLDER-LIKENESS AND FIRST (SECOND)-ORDER CONTINGENT DERIVATIVES OF AN IMPLICIT MULTIFUNCTION

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Dedicated to Francis Clarke on the occasion of his 75th birthday

Abstract. In this paper, by introducing a key assumption, we discuss the Hölder-likeness and the q-order metric regularity of an implicit multifunction. Firstly, we prove that the key assumption is equivalent to the Robinson metric regularity of the implicit multifunction and that under some suitable conditions the key assumption is sufficient for the Hölder-likeness (metric regularity) of the implicit multifunction. Then, by the Robinson metric regularity we establish the contingent derivative and the second-order contingent derivative for the implicit multifunction. Finally, we apply the results obtained to the solution mapping of a parametric vector equilibrium problem.

Keywords. Implicit multifunction; Hölder-likeness; Robinson metric regularity.

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1. Introduction

In this paper we study the Hölder-likeness and $q-$order metric regularity of the following implicit multifunction (which is also called a parametric variational system)

$$S(\mu) = \{x \in K(\mu) | f(\mu, x) = 0\}, \tag{1.1}$$

where $f : \Lambda \times X \to R$ is a real-valued mapping and $K : \Lambda \rightrightarrows X$ is a set-valued mapping, Λ and X are two normed spaces whose norms will both be denoted by $||\cdot||$.

Many optimal solution mappings of parametric optimization problems can be written as the form (1.1). For example, the parametric mathematical programming problem (see [15, 29]), the parametric variational inequality (see [13]), the parametric (set-valued) vector variational inequality (see [18, 27, 30]), and the parametric (generalized) vector equilibrium problem (see [17, 20]), etc. Zhao [29] made the optimal solution mapping of a parametric mathematical programming problem write as an implicit multifunction which is similar to the form (1.1), and then

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he introduced a key assumption which is equivalent to the lower-semicontinuity of the optimal solution mapping on certain conditions. Recently, Kien [15] proved that the same results with those of Zhao [29] are still valid under weaker assumptions. Motivated by the work reported in [15, 29], Li and Chen [18] made use of a gap function for a weak vector variational inequality and introduced a similar key assumption with that of [29]. Then under the key assumption they obtained the lower-semicontinuity of the solution mapping for the problem. Chen et al. [4] further extended the main results of Li and Chen [18] to the parametric weak vector quasivariational inequality of Stampacchia type in Hausdorff topological vector spaces. Lately, Zhong and Huang [30] introduced a key assumption similar to that of Li and Chen [18] by virtue of a gap function for a parametric set-valued weak vector variational inequalities, and under some conditions they got that the key assumption is equivalent to the lower-semicontinuity of the optimal solution mapping for the problem.

Recently, the study of Hölder-likeness and q-order metric regularity has attracted a lot of interest from many researchers due to their wide applications to different areas of variational analysis and optimization theory, see e.g. [8, 12, 16, 28] and the references therein. Especially, for an implicit multifunction, Chuong and Kim [8] gave some sufficient conditions of q-order Robinson metric regularity in terms of the Fréchet/Mordukhovich coderivative.

Motivated by the papers [4, 8, 15, 18, 29, 30], we discuss the Hölder-likeness and the q-order metric regularity of the implicit multifunction (1.1). Firstly, we introduce a new key assumption, and then we prove that the key assumption is equivalent to the Robinson metric regularity of (1.1) and that under some suitable conditions the key assumption is sufficient for the Hölder-likeness (metric regularity) of (1.1). Then, by the Robinson metric regularity we establish the contingent derivative and the second-order contingent derivative for the implicit multifunction. Finally, we apply the results obtained to the solution mapping of a parametric vector equilibrium problem.

The rest of this paper is organized as follows. In Sect. 2, we recall some important definitions related with this paper. In Sect. 3, by introducing a key assumption, we study the Hölder-likeness and the q-order metric regularity of the implicit multifunction (1.1). In Sect. 4, we apply the results obtained to the solution mapping of a parametric vector equilibrium problem.

2. Preliminaries

In this paper, let X and Λ denote normed spaces, and let $F : X \rightrightarrows \Lambda$ be a set-valued mapping. The effective domain, graph and inverse of F are defined by $\text{dom}F := \{\mu \in \Lambda | F(\mu) \neq \emptyset\}$, $\text{gph}F := \{(\mu,x) \in \Lambda \times X | x \in F(\mu)\}$ and $F^{-1}(x) := \{\mu \in \Lambda | x \in F(\mu)\}$, respectively. Let $g : \Lambda \to R$ be a real-valued mapping.

Definition 2.1. Given a point $(\hat{\mu},\hat{x}) \in \text{gph}F$,

(i): F is said to be Hölder continuous around $\hat{\mu}$ with constant $c \geq 0$ and exponent $q > 0$ if and only if there exists a neighborhood U of $\hat{\mu}$ such that

$$F(\mu_1) \subseteq F(\mu_2) + c||\mu_1 - \mu_2||^q \mathbb{B},\ \forall \mu_1, \mu_2 \in U,$$

where $\mathbb{B} \subset \Lambda$ indicates the open unit ball.

(ii): F is said to be Hölder-like around $(\hat{\mu},\hat{x})$ with constant $l > 0$ and exponent $q > 0$ if and only if there exist neighborhoods U of $\hat{\mu}$ and W of $\hat{x}$ such that

$$F(\mu_1) \cap W \subseteq F(\mu_2) + l||\mu_1 - \mu_2||^q \mathbb{B},\ \forall \mu_1, \mu_2 \in U.$$

In the above definition, when $q=1$, Hölder continuity and Hölder-likeness reduce to Lipschitz continuous and Lipschitz-likeness, respectively. The later properties have been extensively investigated for set-valued mappings and single-valued mappings, see [2, 9, 25].

Definition 2.2. Given a point $(\hat{\mu},\hat{x}) \in \text{gph}F$, F is said to be q-order metrically regular around $(\hat{\mu},\hat{x})$ with constant $k>0$ if and only if there exist neighborhoods U of $\hat{\mu}$ and W of $\hat{x}$ such that

$$d(\mu,F^{-1}(x)) \le kd^q(x,F(\mu)),\ \forall \mu \in U, x \in W,$$

where the distance from $x \in X$ to $A \subset X$ is defined by $d(x,A) := \inf_{a\in A} ||x-a||$ with the convention that $d(x,\emptyset)=\infty$.

It is well known that q-order metrically regularity of F is equivalent to Hölder-likeness with exponent $q>0$ of F^{-1}, see [8, Theorem 7].

Definition 2.3. The implicit multifunction S defined by (1.1) is called q-order Robinson metric regularity around $(\hat{\mu},\hat{x}) \in \text{gph}S$ if there exist $\gamma>0$ and neighborhoods U of $\hat{\mu}$, V of $\hat{x}$ such that

$$d(x,S(\mu)) \le \gamma |f(\mu,x)|^q,\ \forall \mu \in U, x \in K(\mu)\cap V.$$

Remark 2.1 (i) Note that Robinson metric regularity is originated by Robinson [21, 22, 23]. (ii) If $f(\mu,x)=f(x)$ and $K(\mu)=K, \forall \mu \in U$ and $x \in K(\mu)\cap V$, then the Robinson metric regularity reduces to the classical error bound.

Definition 2.4. g is said to be Hadamard directionally differentiable at $\hat{\mu}$ in the direction μ if the following limit exists:

$$\lim_{t\downarrow 0,\mu'\to\mu} \frac{g(\hat{\mu}+t\mu')-g(\hat{\mu})}{t},$$

and the directional derivative is denoted by $g'(\hat{\mu})(\mu)$. If g is Hadamard directionally differentiable at $\hat{\mu}$ in all direction, then g is said to be Hadamard directionally differentiable at $\hat{u}$.

Definition 2.5. ([1, 2]) Let K be a subset of X and $\hat{x} \in \text{cl}\, K$, where $\text{cl}\, K$ denotes the closure of K. The contingent cone of K at $\hat{x}$ is the set

$$T(K,\hat{x}) := \{x | \exists h_n \downarrow 0 \text{ and } \{x_n\} \subset X \text{ with } x_n \to x \text{ and } \hat{x}+h_n x_n \in K, \forall n\}.$$

The adjacent cone of K at $\hat{x}$ is the set

$$T^b(K,\hat{x}) := \{x | \forall h_n \downarrow 0, \exists \{x_n\} \subset X \text{ with } x_n \to x \text{ and } \hat{x}+h_n x_n \in K, \forall n\}.$$

K is said to be derivable at $\hat{x}$ if $T(K,\hat{x}) = T^b(K,\hat{x})$. The second-order contingent set of K at $\hat{x}$ in the direction $w \in X$ is the set

$$T^{(2)}(K,\hat{x},w) := \{x | \exists h_n \downarrow 0 \text{ and } \{x_n\} \subset X \text{ with } x_n \to x \text{ and } \hat{x}+h_n w+\frac{1}{2}h_n^2 x_n \in K,\ \forall n\}.$$

Let $(\hat{\mu},\hat{x}) \in \text{gph}F$. The contingent derivative of F at $(\hat{\mu},\hat{x})$ is the set-valued mapping $DF(\hat{\mu},\hat{x}) : \Lambda \rightrightarrows X$ whose graph is $T(\text{gph}F,(\hat{\mu},\hat{x}))$. The adjacent derivative of F at $(\hat{\mu},\hat{x})$ is the set-valued mapping $D^bF(\hat{\mu},\hat{x}) : \Lambda \rightrightarrows X$ whose graph is $T^b(\text{gph}F,(\hat{\mu},\hat{x}))$. F is said to be proto-differentiable at $(\hat{\mu},\hat{x})$ if and only if $T(\text{gph}F,(\hat{\mu},\hat{x})) = T^b(\text{gph}F,(\hat{\mu},\hat{x}))$ (see [24, 25]). Let $(\hat{w},\hat{v}) \in T(\text{gph}F,(\hat{\mu},\hat{x}))$. The second-order contingent derivative of F at $(\hat{\mu},\hat{x})$ in the direction $(\hat{w},\hat{v})$ is the set-valued map $D^{(2)}F(\hat{\mu},\hat{x},\hat{w},\hat{v}) : \Lambda \rightrightarrows X$ whose graph is $T^{(2)}(\text{gph}F,(\hat{\mu},\hat{x}),(\hat{w},\hat{v}))$.

3. Main results

In this section, we establish several implicit multifunction theorems for (1.1).

It follows from (1.1) that S has close relationship with f. Motivated by the papers [15, 18, 29, 30], we introduce a key assumption as follows:

$$(A_1) \quad \exists \alpha > 0, \delta > 0, q > 0 \text{ such that } \forall \mu \in \mathbb{B}(\hat{\mu}, \delta), \forall \varepsilon > 0 \text{ and } \forall x \in K(\mu) \backslash U(S(\mu), \varepsilon) \text{ with } x \in \mathbb{B}(\hat{x}, \delta), \text{ one has } |f(\mu, x)|^q \geq \alpha\varepsilon,$$

where $\hat{x} \in S(\hat{\mu})$, $\mathbb{B}(\hat{\mu}, \delta)$ indicates the open ball centered at $\hat{\mu}$ with radius δ and $U(S(\mu), \varepsilon) = \{x \in K(\mu) | d(x, S(\mu)) < \varepsilon\}$.

Geometrically, the hypothesis (A_1) means that, there exist positive numbers α and δ, such that for all problems in the δ-neighborhood of the parameter $\hat{\mu}$ and for all $\varepsilon > 0$, if a feasible point $x \in \mathbb{B}(\hat{x}, \delta)$ is away from the solution set by a distance of at least ε, then a "gap" by an amount with ε (i.e., $\alpha\varepsilon$) will be yielded.

Throughout this paper, we always assume that $S(\mu) \neq \emptyset$, for all μ in a neighborhood of $\hat{\mu} \in \Lambda$. At first, we need the following important lemma.

Lemma 3.1. *The assumption (A_1) is equivalent to the q-order Robinson metric regularity of S around $(\hat{\mu}, \hat{x})$ in (1.1).*

Proof. Let (A_1) hold. Then, for each $\mu \in \mathbb{B}(\hat{\mu}, \delta)$ and $x \in K(\mu) \cap \mathbb{B}(\hat{x}, \delta)$, there are two cases: (i) $x \in S(\mu)$; (ii) $x \notin S(\mu)$. If the former case is true, then we need not to prove. If the latter case holds, then by (A_1), for any $\varepsilon \leq d(x, S(\mu))$, we have that $|f(\mu, x)|^q \geq \alpha\varepsilon$. Thus, $|f(\mu, x)|^q \geq \alpha d(x, S(\mu))$.

Let S be q-order Robinson metrically regular around $\hat{\mu} \in \Lambda$, i.e., there exist $\gamma > 0$ and $\delta > 0$ such that

$$d(x, S(\mu)) \leq \gamma |f(\mu, x)|^q, \ \forall \mu \in \mathbb{B}(\hat{\mu}, \delta), x \in K(\mu) \cap \mathbb{B}(\hat{x}, \delta).$$

Set $\alpha := \frac{1}{\gamma}$. For each $\mu \in \mathbb{B}(\hat{\mu}, \delta)$, each $\varepsilon > 0$ and $x \in K(\mu) \backslash U(S(\mu), \varepsilon)$ with $x \in \mathbb{B}(\hat{x}, \delta)$, one has

$$\varepsilon \leq d(x, S(\mu)) \leq \gamma |f(\mu, x)|^q.$$

Thus, $|f(\mu, x)|^q \geq \alpha\varepsilon$. This completes the proof. □

Remark 3.2. (a) It follows from Theorem 3.2 in [26] that sufficient conditions for (A_1) with $q = 1$ are as follows:

- **(i):** f is metrically regular around $(\hat{\mu}, \hat{x}, 0)$ with respect to x uniformly in μ with a constant $\gamma > 0$, i.e., there exist neighborhoods U of $\hat{\mu}$, V of $\hat{x}$ and W of 0 such that

$$d(x, f_\mu^{-1}(z)) \leq \gamma |z - f(\mu, x)|, \ \forall \mu \in U, x \in K(\mu) \cap V, z \in W,$$

  where $f_\mu^{-1}(z) = \{x \in K(\mu) | z = f(\mu, x)\}$;
- **(ii):** f is continuous around $(\hat{\mu}, \hat{x})$ with respect to x uniformly in μ;
- **(iii):** f is calm at $(\hat{\mu}, \hat{x})$ with respect to μ uniformly in x, i.e., there exist a constant $\alpha > 0$ and neighborhoods $V_{\hat{\mu}}$, $V_{\hat{x}}$ such that

$$||f(\mu, x) - f(\hat{\mu}, x)|| \leq \alpha ||\mu - \hat{\mu}|| \ \forall \mu \in V_{\hat{\mu}}, x \in V_{\hat{x}};$$

- **(iv):** X is a complete normed space.

(b) If (1.1) does not relate to the parametric μ, then it follows from Lemma 3.1 that the following statements are equivalent:

(i): $\exists\alpha>0,\delta>0,q>0$ such that $\forall\varepsilon>0$ and $\forall x\in K\backslash U(S,\varepsilon)$ with $x\in\mathbb{B}(\hat{x},\delta)$, one has
$$|f(x)|^q\geq\alpha\varepsilon;$$

(ii): f has a q-order error bound $\alpha>0$ at $\hat{x}$, i.e., $|f(x)|^q\geq\alpha d(x,S),\forall x\in K\cap\mathbb{B}(\hat{x},\delta)$.

Next, we give the first implicit multifunction theorem of this paper.

Theorem 3.3. *Let (A_1) hold. If K is Hölder-like around $(\hat{\mu},\hat{x})$ with constant $l_1>0$ and exponent $q_1>0$ and f is Hölder continuous around $(\hat{\mu},\hat{x})$ with constant $l_2>0$ and exponent $q_2>0$, then S is Hölder-like around $(\hat{\mu},\hat{x})$ with exponent* $\min\{q_1,qq_2,qq_1q_2\}$.

Proof. Fix $\mu,\mu'\in\mathbb{B}(\hat{\mu},\delta)$ and $x\in S(\mu)\cap\mathbb{B}(\hat{x},\delta)$. Since K is Hölder-like around $(\hat{\mu},\hat{x})$ with constant $l_1>0$ and exponent $q_1>0$, there exists $x'\in K(\mu')$ such that $||x-x'||\leq l_1||\mu-\mu'||^{q_1}$. By Lemma 3.1, we have
$$\begin{aligned}
d(x,S(\mu')) &\leq ||x-x'||+d(x',S(\mu'))\\
&\leq l_1||\mu-\mu'||^{q_1}+\frac{1}{\alpha}|f(\mu',x')|^q\\
&= l_1||\mu-\mu'||^{q_1}+\frac{1}{\alpha}|f(\mu',x')-f(\mu,x)|^q\\
&\leq l_1||\mu-\mu'||^{q_1}+\frac{1}{\alpha}l_2(||\mu-\mu'||+||x-x'||)^{qq_2}\\
&\leq l_1||\mu-\mu'||^{q_1}+\frac{1}{\alpha}l_2(l_1+1)||\mu-\mu'||^{qq_2\min\{q_1,1\}}\\
&\leq \frac{l_1l_2+l_2+l_1\alpha}{\alpha}||\mu-\mu'||^{\min\{q_1,qq_2,qq_1q_2\}}.
\end{aligned}$$
This completes the proof. □

The assumption (A_1) is very important for the above theorem and the following examples illustrate that it is essential.

Example 3.4. Let $X=R,\Lambda=R^2$,
$$K(\mu)=\begin{cases}[-\mu_2,\mu_2] & \text{if } \mu_2\geq 0,\\ \emptyset & \text{if } \mu_2<0\end{cases}$$
and
$$f(\mu,x)=\begin{cases}\mu_1x^4 & \text{if } \mu_1>0,\\ 0 & \text{if } \mu_1=0,\\ \mu_1(x^4-\mu_2^4) & \text{if } \mu_1<0.\end{cases}$$
Then we easily verify that
$$S(\mu)=\begin{cases}\{0\} & \text{if } \mu_1>0,\mu_2\geq 0,\\ [-\mu_2,\mu_2] & \text{if } \mu_1=0,\mu_2\geq 0,\\ \{-\mu_2,\mu_2\} & \text{if } \mu_1<0,\mu_2\geq 0.\end{cases}$$
Set $\hat{\mu}=(0,1)^T$ and $\bar{x}=0$. It is easy to verify that the assumption (A_1) does not hold when $\mu_1\downarrow 0$, $\mu_2>0$ and $x\to 0$. Thus, Theorem 3.3 is not applicable. Moreover, it is easy to verify that S is not Hölder-like around $(\hat{\mu},\hat{x})$.

Example 3.5. Let $X = R, \Lambda = R^2$, $K(\mu) = [-|\mu_2|, \mu_2]$ and

$$h(\mu,x) = \begin{cases} \mu_1(x+|\mu_2|) & \text{if } \mu_1 > 0, \\ 0 & \text{if } \mu_1 = 0, \\ \mu_1(x-\mu_2) & \text{if } \mu_1 < 0. \end{cases}$$

Then we easily verify that

$$S(\mu) = \begin{cases} \{-|\mu_2|\} & \text{if } \mu_1 > 0, \\ [-|\mu_2|, \mu_2] & \text{if } \mu_1 = 0, \\ \{\mu_2\} & \text{if } \mu_1 < 0 \end{cases}$$

Set $\hat{\mu} = (1,1)^T$ and $\bar{x} = -1$. It is easy to verify that all conditions of Theorem 3.3 hold. Thus, Theorem 3.3 is applicable.

To establish the metric regularity of the implicit multifunction (1.1), we introduce the following assumption:

(A_1') $\exists \alpha > 0, \delta > 0, q > 0$ such that $\forall x \in \mathbb{B}(\hat{x}, \delta), \forall \varepsilon > 0$ and $\forall \mu \in K^{-1}(x) \backslash U(S^{-1}(x), \varepsilon)$ with $\mu \in \mathbb{B}(\hat{\mu}, \delta)$, one has $|f(\mu,x)|^q \geq \alpha \varepsilon$,

where $\hat{\mu} \in S^{-1}(\hat{x})$ and $U(S^{-1}(x), \varepsilon) = \{\mu \in K^{-1}(x) | d(\mu, S^{-1}(x)) < \varepsilon\}$.

Theorem 3.6. *Let (A_1') hold. If K is q_1-order metrically regular around $(\hat{\mu}, \hat{x})$ with constant $l_1 > 0$ and f is Hölder continuous around $(\hat{\mu}, \hat{x})$ with constant $l_2 > 0$ and exponent $q_2 > 0$, then S is $\min\{q_1, qq_2, qq_1q_2\}$-order metrically regular around $(\hat{\mu}, \hat{x})$.*

Proof. Let $Q(x) = \{\mu \in K^{-1}(x) | f(\mu,x) = 0\}$. It follows from Theorem 3.3 and the equivalence between the metric regularity and the Hölder-likeness that we easily get Q is Hölder-likeness around $(\hat{\mu}, \hat{x})$. Since $Q(x) = S^{-1}(x)$, S is $\min\{q_1, qq_2, qq_1q_2\}$-order metrically regular around $(\hat{\mu}, \hat{x})$. This completes the proof. □

Now we establish the third implicit multifunction theorem.

Theorem 3.7. *Let $\hat{x} \in S(\hat{\mu})$. Suppose (A_1) holds with $q = 1$ and f is Hadamard directionally differentiable at $(\hat{\mu}, \hat{x})$. Then for each $\mu \in dom(DS(\hat{\mu}, \hat{x}))$ one has*

$$DS(\hat{\mu}, \hat{x})(\mu) = \{x \in DK(\hat{\mu}, \hat{x})(\mu) | f'(\hat{\mu}, \hat{x})(\mu, x) = 0\}. \tag{3.1}$$

When K is proto-differentiable at $(\hat{\mu}, \hat{x})$, S is also proto-differentiable at $(\hat{\mu}, \hat{x})$.

Proof. Let $\mu \in dom(DS(\hat{\mu}, \hat{x}))$ and $x \in DS(\hat{\mu}, \hat{x})(\mu)$. Then there exist sequences $t_n \downarrow 0$ and $(\mu_n, x_n) \to (\mu, x)$ such that

$$\hat{x} + t_n x_n \in S(\hat{\mu} + t_n \mu_n),$$

which implies that $\hat{x} + t_n x_n \in K(\hat{\mu} + t_n \mu_n)$ and $f(\hat{\mu} + t_n \mu_n, \hat{x} + t_n x_n) = 0$. Thus, $x \in DK(\hat{\mu}, \hat{x})(\mu)$. It follows from Hadamard directional differentiability of f at $(\hat{\mu}, \hat{x})$ that

$$f(\hat{\mu} + t_n \mu_n, \hat{x} + t_n x_n) = f(\hat{\mu}, \hat{x}) + t_n f'(\hat{\mu}, \hat{x})(\mu_n, x_n) + o(||(t_n \mu_n, t_n x_n)||). \tag{3.2}$$

Thus, we easily get that $f'(\hat{\mu}, \hat{x})(\mu, x) = 0$. Consequently, the right set of (3.1) includes the left set.

Let $x \in \{x \in DK(\hat{\mu}, \hat{x})(\mu) | f'(\hat{\mu}, \hat{x})(\mu, x) = 0\}$. Then there exist sequences $t_n \downarrow 0$ and $(\mu_n, x_n) \to (\mu, x)$ such that

$$\hat{x} + t_n x_n \in K(\hat{\mu} + t_n \mu_n).$$

By Lemma 3.1 and by (3.2), we have that S is Robinson metrically regular around $(\hat{\mu},\hat{x})$. Then for sufficiently large n we get

$$\alpha d(\hat{x}+t_n x_n, S(\hat{\mu}+t_n\mu_n)) \leq |f(\hat{\mu}+t_n\mu_n,\hat{x}+t_n x_n)| = t_n\beta_n,$$

where $\beta_n = |f'(\hat{\mu},\hat{x})(\mu_n,x_n)+\frac{o(||(t_n\mu_n,t_nx_n)||)}{t_n}|$ and it is clear that $\beta_n \downarrow 0$. Hence,

$$\hat{x}+t_n x_n \in S(\hat{\mu}+t_n\mu_n)+\frac{1}{\alpha}(t_n\beta_n+t_n^2)\bar{\mathbb{B}},$$

where $\bar{\mathbb{B}} \subset X$ denotes the closed unit ball, i.e., there exists $b_n \in \bar{\mathbb{B}}$ such that

$$\hat{x}+t_n[x_n-\frac{1}{\alpha}(\beta_n+t_n)b_n] \in S(\hat{\mu}+t_n\mu_n).$$

Since $\frac{1}{\alpha}(\beta_n+t_n)||b_n|| \to 0$, we get $x \in DS(\hat{\mu},\hat{x})(\mu)$. Hence, (3.1) is valid.

To prove that S is proto-differentiable at $(\hat{\mu},\hat{x})$, it follows from the above proof procedure that we only need to verify

$$D^bS(\hat{\mu},\hat{x})(\mu) \supset \{x \in D^bK(\hat{\mu},\hat{x})(\mu) | f'(\hat{\mu},\hat{x})(\mu,x)=0\},$$

whose proof procedure is similar to that of the above process. So we omit it. This completes the proof. □

To obtain the final implicit multifunction theorem in this paper, we need the following interesting result.

Lemma 3.8. *Let $(\hat{\mu},\hat{x}) \in gphS$ and f be Hadamard directionally differentiable at $(\hat{\mu},\hat{x})$. Consider the following statements:*

- **(i):** *There exists $\alpha > 0$ such that $\forall \mu \in domDS(\hat{\mu},\hat{x}), \forall \varepsilon > 0$ and $\forall x \in DK(\hat{\mu},\hat{x})(\mu)$ with $x \notin U(DS(\hat{\mu},\hat{x})(\mu),\varepsilon)$, one has*

$$|f'(\hat{\mu},\hat{x})(\mu,x)| \geq \alpha\varepsilon.$$

- **(ii):** *There exists $\alpha > 0$ such that*

$$\alpha d(x, DS(\hat{\mu},\hat{x})(\mu)) \leq |f'(\hat{\mu},\hat{x})(\mu,x)|,\ \forall \mu \in domDS(\hat{\mu},\hat{x}), x \in DK(\hat{\mu},\hat{x})(\mu).$$

- **(iii):** *S is Robinson metrically regular around $(\hat{\mu},\hat{x})$.*

Then (i) and (ii) are equivalent. If X is finite dimensional, then (iii) implies (i) and (ii).

Proof. The proof of the equivalence between (i) and (ii) is similar to that of Lemma 3.1. So we omit it. Now, we only need to prove that (iii) implies (ii). Let (iii) hold. For each $\mu \in$ dom$DS(\hat{\mu},\hat{x})$ and $x \in DK(\hat{\mu},\hat{x})(\mu)$, there exist sequences $t_n \downarrow 0$ and $(\mu_n,x_n) \to (\mu,x)$ such that $\hat{x}+t_nx_n \in K(\hat{\mu}+t_n\mu_n)$. For sufficiently large n, by (iii) for some fixed constant $\gamma > 0$ one has

$$d(\hat{x}+t_nx_n, S(\hat{\mu}+t_n\mu_n)) \leq \gamma|f(\hat{\mu}+t_n\mu_n,\hat{x}+t_nx_n)|.$$

By Hadamard directional differentiability of f at $(\hat{\mu},\hat{x})$, one has

$$f(\hat{\mu}+t_n\mu_n,\hat{x}+t_nx_n) = f(\hat{\mu},\hat{x})+t_nf'(\hat{\mu},\hat{x})(\mu_n,x_n)+o(||(t_n\mu_n,t_nx_n)||).$$

Thus, there exists $b_n \in \bar{\mathbb{B}}$ such that

$$\hat{x}+t_n[x_n-\gamma|f'(\hat{\mu},\hat{x})(\mu_n,x_n)+\frac{o(||(t_n\mu_n,t_nx_n)||)}{t_n}|b_n+t_nb_n] \in S(\hat{\mu}+t_n\mu_n).$$

Since X is finite dimensional, b_n has a convergent subsequence. We might as well assume that $b_n \to b \in \bar{\mathbb{B}}$. Hence,

$$x - \gamma |f'(\hat{\mu},\hat{x})(\mu,x)| b \in DS(\hat{\mu},\hat{x})(\mu).$$

Then,

$$d(x, DS(\hat{\mu},\hat{x})(\mu)) \leq ||x - (x - \gamma|f'(\hat{\mu},\hat{x})(\mu,x)|b)|| \leq \gamma |f'(\hat{\mu},\hat{x})(\mu,x)|.$$

This completes the proof. □

Theorem 3.9. *Let $\hat{x} \in S(\hat{\mu})$, $(\hat{w},\hat{v}) \in T(gphS,(\hat{\mu},\hat{x}))$ and X be finite dimensional. Suppose (A_1) holds with $q = 1$ and f is Hadamard directionally differentiable at $(\hat{\mu},\hat{x})$, and $f'(\hat{\mu},\hat{x})$ is Hadamard directionally differentiable at $(\hat{w},\hat{v})$. Then*

$$D(DS(\hat{\mu},\hat{x}))(\hat{w},\hat{v})(\mu) = \{x \in D(DK(\hat{\mu},\hat{x}))(\hat{w},\hat{v})(\mu) | [f'(\hat{\mu},\hat{x})]'(\hat{w},\hat{v})(\mu,x) = 0\}. \tag{3.3}$$

Moreover, if $0 \in T^{(2)}(gphS,(\hat{\mu},\hat{x}),(\hat{w},\hat{v}))$ and $gphS$ is convex, then

$$D^{(2)}S(\hat{\mu},\hat{x},\hat{w},\hat{v})(\mu) = \{x \in D^{(2)}K(\hat{\mu},\hat{x},\hat{w},\hat{v})(\mu) | [f'(\hat{\mu},\hat{x})]'(\hat{w},\hat{v})(\mu,x) = 0\} \tag{3.4}$$

and there exists $\alpha > 0$ such that $\forall \mu \in domD^{(2)}S(\hat{\mu},\hat{x},\hat{w},\hat{v}), x \in D^{(2)}K(\hat{\mu},\hat{x},\hat{w},\hat{v})(\mu)$

$$\alpha d(x, D^{(2)}S(\hat{\mu},\hat{x},\hat{w},\hat{v})(\mu)) \leq |[f'(\hat{\mu},\hat{x})]'(\hat{w},\hat{v})(\mu,x)|. \tag{3.5}$$

Proof. By Theorem 3.7 we have

$$DS(\hat{\mu},\hat{x})(\mu) = \{x \in DK(\hat{\mu},\hat{x})(\mu) | f'(\hat{\mu},\hat{x})(\mu,x) = 0\}.$$

It follows from Lemmas 3.1, 3.8 and Theorem 3.7 that (3.3) holds. Since gph$S$ is convex and $0 \in T^{(2)}(\text{gph}S,(\hat{\mu},\hat{x}),(\hat{w},\hat{v}))$, it follows from Proposition 3.34 in [3] that

$$T^{(2)}(\text{gph}S,(\hat{\mu},\hat{x}),(\hat{w},\hat{v})) = T(\text{gph}DS(\hat{\mu},\hat{x}),(\hat{w},\hat{v})).$$

By the definitions, we have

$$\text{gph}D(DS(\hat{\mu},\hat{x}))(\hat{w},\hat{v}) = T(\text{gph}DS(\hat{\mu},\hat{x}),(\hat{w},\hat{v}))$$

and

$$\text{gph}D^{(2)}S(\hat{\mu},\hat{x},\hat{w},\hat{v}) = T^{(2)}(\text{gph}S,(\hat{\mu},\hat{x}),(\hat{w},\hat{v})).$$

Thus, $D(DS(\hat{\mu},\hat{x}))(\hat{w},\hat{v}) = D^{(2)}S(\hat{\mu},\hat{x},\hat{w},\hat{v})$. Similarly, $D(DK(\hat{\mu},\hat{x}))(\hat{w},\hat{v}) = D^{(2)}K(\hat{\mu},\hat{x},\hat{w},\hat{v})$. Thus, (3.4) holds. The proof of (3.5) is similar to that of Lemma 3.8, so we omit it. This completes the proof. □

Remark 3.10. (i) It follows from Theorems 3.7 and 3.9 that there is an interesting result: the contingent derivative and the second-order contingent derivative of the implicit multifunction are also implicit multifunctions, respectively. Moreover, from Theorem 3.7 to Theorem 3.9 we could deduce that the high-order contingent derivative of the implicit multifunction (1.1) is easily obtained under some similar conditions. (ii) The convexity of gph$S$ can be easily obtained by the convexities of gph$K$ and gph$f$.

It follows from Theorems 3.7, 3.9, Lemma 3.1 and Remark 3.2 (b) that we easily get the following conclusion.

Corollary 3.11. *Let $S = \{x \in K | f(x) = 0\}$, $\hat{x} \in S$ and $\hat{v} \in T(S,\hat{x})$. Suppose (i) or (ii) of Remark 3.2 (b) holds and assume that f is Hadamard directionally differentiable at $\hat{x}$. Then*

$$T(S,\hat{x}) = \{x \in T(K,\hat{x}) | f'(\hat{x})(x) = 0\}$$

and there exists $\alpha > 0$ such that $\forall x \in T(K,\hat{x})$

$$\alpha d(x, T(S,\hat{x})) \leq |f'(\hat{x})(x)|.$$

Moreover, if $f'(\hat{x})$ is Hadamard directionally differentiable at $\hat{v}$, X is finite dimensional, $0 \in T^{(2)}(S,\hat{x},\hat{v})$ and gphS is convex, then one has

$$T^{(2)}(S,\hat{x},\hat{v}) = \{x \in T^{(2)}(K,\hat{x},\hat{v}) | [f'(\hat{x})]'(\hat{v})(x) = 0\}$$

and there exists $\alpha > 0$ such that $\forall x \in T^{(2)}(K,\hat{x},\hat{v})$

$$\alpha d(x, T^{(2)}(S,\hat{x},\hat{v})) \leq |[f'(\hat{x})]'(\hat{v})(x)|.$$

Remark 3.12. (a) A more general model than (1.1) is the following implicit multifunction

$$M(\mu) := \{x \in K(\mu) | 0 \in G(\mu, x)\},$$

where $G : \Lambda \times X \rightrightarrows Z$ is a set-valued mapping and Z is a normed space. Being similar to this paper, we can easily discuss the Hölder-likeness and metric regularity of M. Indeed, by some methods of variational analysis which are different from our viewpoints, Durea and Strugariu [11] and Uderzo [26] have discussed the Lipschitz-likeness and metric regularity of M, respectively. One important reason discussing the simple model (1.1) is that our results can be easily applied to the solution mapping of a parametric vector equilibrium problem (see Section 4).

(b) It follows from Theorems 3.7 and 3.9 that we obtain some necessary conditions of Robinson metric regularity. For M, we could easily parallel get similar results. Recently, by using of a contingent derivative criterion, Dontchev et al. [10] got a sufficient condition for Robinson metric regularity. Unfortunately, our necessary conditions are different from their sufficient conditions. Looking for a contingent derivative criterion being equivalent to Robinson metric regularity will be considered in our future work.

4. Applications

In this section, we consider the following parametric vector equilibrium problem (for short, PWVEP): find $x \in K(\mu)$ for the parameter $\mu \in \Lambda$ such that

$$f(\mu, x, y) \notin -\text{int}C, \ \forall y \in K(\mu),$$

where $f : \Lambda \times X \times X \to Z$ is a vector-valued mapping, $K : \Lambda \rightrightarrows X$ is a set-valued mapping, Z is a Banach space and $C \subseteq Z$ is a closed convex and pointed cone with nonempty interior int$C$.

For each $\mu \in \Lambda$, let $S(\mu)$ denote the solution mapping of (PWVEP), i.e.,

$$S(\mu) := \{x \in K(\mu) | f(\mu, x, y) \notin -\text{int}C, \ \forall y \in K(\mu)\}.$$

Now we recall the definition of a real-valued gap function for (PWVEP).

Definition 4.1. A mapping $h : \Lambda \times X \to R$ is said to be a gap function for (PWVEP) with respect to the parameter $\mu \in \Lambda$ if

$$(i)\ h(\mu, x) \geq 0, \ \ \forall x \in K(\mu); \ (ii)\ h(\mu, x^*) = 0 \text{ and } x^* \in K(\mu) \text{ iff } x^* \in S(\mu).$$

For each $\mu \in \Lambda$ and each $x \in K(\mu)$, set

$$h(\mu,x) := \sup_{y\in K(\mu)} -\xi_e(f(\mu,x,y)),$$

where $e \in \mathrm{int}C$ and $\xi_e : Z \to R$ is a nonlinear scalarization function defined by

$$\xi_e(z) = \min\{t \in R | z \in te - C\}, \ \forall z \in Z.$$

The function ξ_e is a continuous, positively homogeneous, subadditive and convex function on Z, and it is monotone (i.e., $z^1 \in z^2 - C \Longrightarrow \xi_e(z^1) \le \xi_e(z^2)$) and strictly monotone (i.e., $z^1 \in z^2 - \mathrm{int}C \Longrightarrow \xi_e(z^1) < \xi_e(z^2)$), and is Lipschitz continuous on Z (see [5, 6, 7, 14, 19]). Moreover, it follows from [7, Proposition 1.43] that for any $z \in Z$ and $r \in R$, the following statements hold:

(i): $\xi_e(z) < r \Longleftrightarrow z \in re - \mathrm{int}C$ (i.e., $\xi_e(z) \ge r \Longleftrightarrow z \notin re - \mathrm{int}C$);
(ii): $\xi_e(z) \le r \Longleftrightarrow z \in re - C$ (i.e., $\xi_e(z) > r \Longleftrightarrow z \notin re - C$);
(iii): $\xi_e(z) = r \Longleftrightarrow z \in re - \partial C$, where ∂C denotes the boundary of C.

Under some suitable conditions, h is a gap function for (PWVEP). The reader can refer to Proposition 3.2 in [19]. In the rest of this section, we always assume that h is a gap function of (PWVEP). Then, the solution mapping of (PWVEP) can be written as

$$S(\mu) = \{x \in K(\mu) | h(\mu,x) = 0\}, \ \forall \mu \in \Lambda.$$

Thus, we could apply the results obtained in Section 3 to the solution mapping of (PWVEP). At first, we introduce the following assumption which is similar to (A_1):

(B_1) $\exists \alpha > 0, \delta > 0, q > 0$ such that $\forall \mu \in \mathbb{B}(\hat{\mu},\delta), \forall \varepsilon > 0$ and $\forall x \in K(\mu)\backslash U(S(\mu),\varepsilon)$ with $x \in \mathbb{B}(\hat{x},\delta)$, one has, for some $y \in K(\mu), f(\mu,x,y) \in -(\alpha\varepsilon)e - C$,

where $\hat{x} \in S(\hat{\mu})$.

Theorem 4.2. *Let (B_1) hold. If K is Lipschitz continuous around $(\hat{\mu},\hat{x})$ and f is Lipschitz continuous on $U(\hat{\mu}) \times U(\hat{x}) \times K(U(\hat{\mu}))$, where $U(\hat{\mu})$ and $U(\hat{x})$ are neighborhoods of $\hat{\mu}$ and $\hat{x}$, respectively, then S is Lipschitz-like around $(\hat{\mu},\hat{x})$.*

Proof. It follows from the properties of ξ_e and the definition of h that (B_1) is equivalent to the following statement:

$$\exists \alpha > 0, \delta > 0 \text{ such that } \forall \mu \in \mathbb{B}(\hat{\mu},\delta), \forall \varepsilon > 0 \text{ and } \forall x \in K(\mu)\backslash U(S(\mu),\varepsilon) \text{ with } x \in \mathbb{B}(\hat{x},\delta),$$
$$\text{one has } h(\mu,x) \ge \alpha\varepsilon.$$

Then by Theorem 3.3 we only need verify the Lipschitz continuity of h. By Proposition 3.4 in [19] and by the conditions, we easily get that h is Lipschtiz continuous around $(\hat{\mu},\hat{x})$. This completes the proof. □

In the rest of this section, we set $\hat{\mu} \in \Lambda$ and

$$M(\hat{\mu},\hat{x}) := \{y \in K(\hat{\mu}) | h(\hat{\mu},\hat{x}) = -\xi_e(f(\hat{\mu},\hat{x},y))\}.$$

For $\hat{y} \in M(\hat{\mu},\hat{x})$ and $\hat{z} = f(\hat{\mu},\hat{x},\hat{y})$, set

$$\Omega(\hat{y}) := \{y | \xi_e'(\hat{z})(f'(\hat{\mu},\hat{x},\hat{y})(0_\Lambda,0_X,y)) = 0\}.$$

Theorem 4.3. *Let $\hat{x} \in S(\hat{\mu})$. Suppose that X is a finite dimensional space. Moreover, assume that*

(i): *f is Hadamard directionally differentiable at $(\hat{\mu},\hat{x},\hat{y})$ for each $\hat{y}\in M(\hat{\mu},\hat{x})$;*
(ii): *all conditions of Theorem 4.2 hold;*
(iii): *K is compact-valued and closed ($gphK$ is a closed set);*
(iv): *for each $\hat{y}\in M(\hat{\mu},\hat{x})$, $\Omega(\hat{y})\cap DK(\hat{\mu},\hat{y})(0_\Lambda)=\{0_X\}$.*

Then for each $\mu\in dom(DS(\hat{\mu},\hat{x}))$ one has

$$DS(\hat{\mu},\hat{x})(\mu)=\{x\in DK(\hat{\mu},\hat{x})(\mu)|\sup_{\hat{y}\in M(\hat{\mu},\hat{x})}\sup_{y\in DK(\hat{\mu},\hat{y})(\mu)}\min_{\xi\in B_e^*(\hat{z})}\langle -\xi, f'(\hat{\mu},\hat{x},\hat{y})(\mu,x,y)\rangle=0\}.$$

When K is proto-differentiable at $(\hat{\mu},\hat{y})$ for any $\hat{y}\in M(\hat{\mu},\hat{x})$, one has that S is proto-differentiable at $(\hat{\mu},\hat{x})$.

Proof. It follows from the conditions and Proposition 3.6 in [19] that h is Hadamard directionally differentiable at $(\hat{\mu},\hat{x})$ and that

$$h'(\hat{\mu},\hat{x})(\mu,x)=\sup_{\hat{y}\in M(\hat{\mu},\hat{x})}\sup_{y\in DK(\hat{\mu},\hat{y})(\mu)}\min_{\xi\in B_e^*(\hat{z})}\langle -\xi, f'(\hat{\mu},\hat{x},\hat{y})(\mu,x,y)\rangle.$$

Then, by the similar proof to that of Theorem 3.7, we easily get

$$DS(\hat{\mu},\hat{x})(\mu)=\{x\in DK(\hat{\mu},\hat{x})(\mu)|h'(\hat{\mu},\hat{x})(\mu,x)=0\}.$$

This completes the proof. □

Remark 4.4. In the above theorem, $B_e^*(z)=\{\xi\in B_e^*|\langle\xi,z\rangle=\xi_e(z)\}$, where $B_e^*:=\{\xi\in C^*|\langle\xi,e\rangle=1\}$ which is a weak* compact convex base of C^*, C^* is the dual cone of C defined by $C^*=\{\xi\in Z^*|\langle\xi,z\rangle\geq 0,\forall z\in C\}$, and Z^* is the topological dual space of Z and $\langle\xi,z\rangle$ denotes the value of ξ at z.

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