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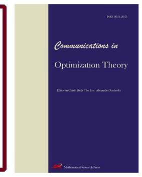

HIGHER-ORDER TANGENT CONES AND THEIR APPLICATIONS TO CONSTRAINED OPTIMIZATION AND FLOW-INVARIANCE

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Dedicated to the memory of Professor Ram Verma

Abstract. This is an expository article on higher-order optimality conditions for extremum problems via the technique of higher-order tangent cones to a nonempty subset of a Banach space X. Such a technique is also used in the study of the flow-invariance of a set with respect to a higher-order differential equation (motion on a given orbit in a force field). Many of the existing results in these areas are included here.

Keywords. Flow-invariant sets with respect to a differential equation; Flight Mechanics; Higher-order optimality conditions; Set constrained optimization; Tangential cones.

2020 Mathematics Subject Classification. 90C30, 34G20, 47N20, 90C29, 49K27.

1. Introduction

The goal of this survey paper is twofold:

1) to present some of the main existing results obtained via the higher-order tangent cones in Pavel-Ursescu sense, results that concern constrained optimization problems, flow-invariance problems and some of their applications,

2) to point out the unifying effect of the theory of tangent cones in the areas of differential equations and optimization.

We deal with the following scalar set constrained minimization problem

$$\text{Minimize } F(x) \text{ subject to } x \in D, \tag{P}$$

where X is a linear normed space of norm $\|\cdot\|$, $\bar{x} \in D \subseteq U$, and $F : U \subseteq X \to \mathbb{R}$ is a function of class C^n on the open set U, n positive integer.

Also, we are concerned with the vector optimization problem

$$\text{Minimize } F(x) \text{ subject to } x \in M, \tag{VP}$$

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where $F : X \to Y$ is sufficiently often Gâteaux differentiable at the weak minimum point, X and Y are real linear normed spaces, $M \subseteq X$, and Y is ordered via a closed convex cone $Q \subset Y$ with nonempty interior.

We consider the multiobjective optimization problem (MOP), which is a particular case of (VP).

$$\text{Minimize } F(x) \text{ subject to } x \in M, \tag{MOP}$$

where X is a real linear normed spaces, $M \subseteq X$, $F = (F_1, \dots, F_r) : X \to \mathbb{R}^r$ has a component that is sufficiently often Gâteaux differentiable at the minimum point, and $Q = \mathbb{R}^r_+ = \{x = (x_1, \dots, x_r) \in \mathbb{R}^r : \ x_s \geq 0,\ s = 1, \dots, r\}$, r nonnegative integer.

We present conditions for a closed subset D of a Banach space X to be flow-invariant with respect to the n-th order autonomous differential equation

$$u^{(n)}(t) = F(u(t)), t \geq 0,$$

where $F : U \to X$ is a locally Lipschitz mapping on an open subset U of X.

The paper is organized as follows. In Section 1, that is based on [5, 6, 13, 12, 21, 24, 25, 26], we include the definitions of the higher-order tangent cones and some of their characterizations. In Section 2, we collect from [5, 8, 9, 10, 14] optimality conditions formulated via the tangent cones of order $n \geq 1$. In Section 3, we gather from [6, 7, 21] some of the main results concerning flow-invariance problems.

2. Higher-Order Tangent Cones

The tangent cones are the main tools for formulating the results of this paper.

Definition 2.1. Let D be a nonempty subset of X and let $x \in D$ be a given point.

i) (Ursescu, [27]) An element $v_1 \in X$ is called a tangent vector to D at x, if

$$\lim_{t \downarrow 0} \frac{1}{t} d(x + tv_1; D) = 0. \tag{2.1}$$

ii) ([6]) An element $v_n \in X$ is called a n-th order tangent vector to D at $x \in D$, $n \geq 2$, if there are $v_1, \dots, v_{n-1} \in X$, such that

$$\lim_{t \downarrow 0} \frac{1}{t^n} d(x + tv_1 + \frac{t^2}{2!} v_2 + \frac{t^3}{3!} v_3 + \dots + \frac{t^n}{n!} v_n; D) = 0. \tag{2.2}$$

where $d(x; D) = \inf\{\|x - y\| : y \in D\}$.

The sets of all first-order tangent vectors to D at $x \in D$ are denoted by $T_x D$. The set of all n-th order tangent vectors to D at $x \in D$ is denoted by $T^n_x D$, $n \geq 1$. Here $T^1_x D = T_x D$. For a given $v_n \in T^n_x D$, the vectors $v_1, v_2, \dots, v_{n-1}$ as in (2.2) are said to be associate vectors of v_n, or associated with v_n, or correspondent vectors of v_n.

The cone $T_x D$ is also known as the cone of attainable directions [17] or the adjacent cone [2].

The definition of $T^2_x D$ was suggested by a formula of Pavel from 1975 (see [24] with $f(x)$ in place of v_2).

Definition 2.1, ii) is a modification of the definition of a n-th order tangent vector v_n given in [26] where the associated vectors v_i, $i = 1, \dots, n-1$ are required to belong to $T^i_x S$, respectively. We showed in Proposition 1.2.3, [6] that the assumption $v_i \in T^i_x S$, $i = 1, \dots, n-1$ is redundant.

In the case $n=2$, in the theory of motion on a given orbit in a force field in Flight Mechanics, the vectors v_1 and v_2 above are the initial velocity at x and the initial acceleration at x, respectively.

It is obvious that if x belongs to the interior of D, then $T_x^n D = X$, $n \geq 1$.

The set $T_x^n D$, $n \geq 2$, is a cone in X (Proposition 1.2.1, [6]).

Proposition 2.2. *(Proposition 1.2.3, [6]) If $v_n \in T_x^n D$, then its associated vectors v_i, $1 \leq i \leq n-1$, belong to $T_x^i D$, respectively.*

Proposition 2.3. *i) (Lemma 3.1, [25]) The fact that v_1 belongs to $T_x D$ is equivalent to the existence of a function $\gamma_1 : (0,\infty) \to X$ with $\gamma_1(t) \to 0$ as $t \downarrow 0$, and*

$$x + t(v_1 + \gamma_1(t)) \in D, \ \forall t > 0. \tag{2.3}$$

ii)(Proposition 1.2.2, [6] The fact that v_n belongs to $T_x^n D$ with the corespondent vectors $v_i \in X$, $i = 1,2,...,n-1$, $n \geq 2$, as in (2.2), is equivalent to the existence of a function $\gamma_n : (0,\infty) \to X$ with $\gamma_n(t) \to 0$ as $t \downarrow 0$ and

$$x + tv_1 + \frac{t^2}{2}v_2 + ... + \frac{t^n}{n!}(v_n + \gamma_n(t)) \in D, \ \forall t > 0. \tag{2.4}$$

It can easily be seen that $0 \in T_x D$ (take $\gamma_1 \equiv 0$), and $0 \in T_x^n D$ (take $\gamma_n \equiv 0$, $v_i = 0$, $1 \leq i \leq n-1$), $n \geq 2$.

Definition 2.4. ([2]) Let D be a nonempty subset of X and let $x \in D$ be a given point. An element $v \in X$ is called a contingent vector to D at x if

$$\liminf_{t \downarrow 0} \frac{1}{t} d(x + tv; D) = 0.$$

The set of contingent vectors is a cone denoted by $\Gamma(D,x)$. It follows from the definitions that $T_x D \subseteq \Gamma(D,x)$. For a convex set D, $\Gamma(D,x) = T_x D$, for all $x \in D$.

Throughout the paper, if a function G is n-times differentiable at x, then $G^{'}(x)$, $G^{''}(x)$, $G^{'''}(x)$, $G^{(n)}(x)$, $n \geq 4$ denote its first, second, third and n-th order derivatives at x and $G^{(n)}(x)[y]^n = G^{(n)}(x)\underbrace{(y)\cdots(y)}_{n \text{ times}}$.

There are known characterizations of the first and higher-order tangent cones to the null-set of a mapping $G : X \to Y$, i.e., $D_G = G^{-1}(0) = \{u \in X : \ G(u) = 0, G : X \to Y\}$, where Y is a linear normed space.

Theorem 2.5. *(Pavel-Ursescu, Corollary 3.1, [25]) Assume that X is a linear normed space, Y is a finite dimensional normed space, $G : X \to Y$ is continuous in a neighborhood of x and Fréchet differentiable at x. If $G'(x)$ is onto, then*

$$T_x D_G = Ker G'(x). \tag{2.5}$$

Here $Ker G'(x)$ denotes the null space of $G'(x)$, i.e., $Ker G'(x) = \{v \in X; \ G'(x)(v) = 0\}$.

Theorem 2.6. *(Pavel-Ursescu, Corollary 3.2, [25]) Assume that X is a linear normed space, Y is a finite dimensional normed space, $G : X \to Y$ is twice Fréchet differentiable at x and continuous near x, $G(x) = 0$, and $G^{'}(x)$ is onto.*

Then, $v_2 \in T_x^2 D_G$ with associated vector $v_1 \in T_x D_G$, if and only if

$$G'(x)(v_1) = 0,$$
$$G'(x)(v_2) + G''(x)(v_1)(v_1) = 0.$$

Remark 2.7. If $G'(x)$ is not onto, then in general the strict inclusion $T_xD_G \subset KerG'(x)$ holds.

Definition 2.8. ([28]) Let $G : X \to Y$ be twice Fréchet differentiable at $x \in X$. Then G is said to be 2-regular at x if, given any $v \in X$, $v \neq 0$ with $G''(x)(v)(v) = 0$, we have $G''(x)(v)X = Y$.

In the above two theorems, N.H. Pavel and C. Ursescu characterized the second-order tangent vectors to D_G at $x \in D_G$ when $G : X \to \mathbb{R}^s$ is twice Fréchet differentiable at x and $G^{'}(x)$ is onto. In our 2019 paper [13], we described the second-order tangent cone to D_G at $x \in D_G$ in the degenerate case $G^{'}(x) = 0$.

Theorem 2.9. *(Theorem 1, [13]) Let X be a linear normed space, $G = (G_1, \dots, G_s) : X \to \mathbb{R}^s$ be three times Fréchet differentiable at x and continuous near $x \in D_G = \{u \in X : \ G(u) = 0\}$. Assume that $G'(x) = 0$ and G is 2-regular at x.*

Then $v_2 \in T_x^2 D_G$ with associated vector $v_1 \neq 0$, if and only if

$$G''(x)(v_1)(v_1) = 0,\ v_1 \neq 0,\ and \tag{2.6}$$

$$G^{(3)}(x)(v_1)(v_1)(v_1) + 3G''(x)(v_1)(v_2) = 0, \tag{2.7}$$

i.e., $G_j''(x)(v_1)(v_1) = 0$, for all $j \in J := \{1, \dots, s\}$ and
${G_j}^{(3)}(x)(v_1)(v_1)(v_1) + 3{G_j}''(x)(v_1)(v_2) = 0$, for all $j \in J$.

In the case where $D = D_G$, $G : X \to Y$, X linear normed spaces, Y finite dimensional normed space, we described in 2004 the higher-order tangent cones to D_G in the regular case, that is, in the case where G has onto Fréchet derivative $G'(x)$.

Theorem 2.10. *(Corollary 2.1, [6], Corollary 1.1, [5])*

Assume that $G : X \to \mathbb{R}^s$ is three times Fréchet differentiable at $x \in X$ with $G(x) = 0$, G is continuous near x, and $G'(x) : X \to \mathbb{R}^s$ is onto.

Then $v_3 \in T_x^3 D_G$ with associated vectors $v_i \in T_x^i D_G$, $i = 1, 2$ if and only if

$$G'(x)(v_1) = 0, G''(x)(v_1)(v_1) + G'(x)(v_2) = 0,$$

$$G'''(x)(v_1)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + G'(x)(v_3) = 0.$$

Furthermore, assume that $G : X \to \mathbb{R}^s$ is n times Fréchet differentiable at $x \in X$ with $G(x) = 0$, G is continuous near x, and $G^{'}(x) : X \to \mathbb{R}^s$ is onto, $n \geq 1$.

Then, $v_n \in T_x^n D_G$ with the associated vectors $v_m \in T_x^m D_G$, $m = 1, \dots, n-1$, if and only if

$$S_m^G(x; v_1, \dots, v_m) = 0,\ \forall 1 \leq m \leq n, \tag{2.8}$$

where $D_G = \{u \in X : \ G(u) = 0\}$ is nonempty.

Here, for a positive integer m and vectors $v_1, \dots, v_m$, $S_m^G(x; v_1, \dots, v_m)$ denotes the expression

$$S_m^G(x; v_1, \dots, v_m) = \sum_{k=1}^{m} \frac{m!}{k!} [\sum_{\substack{i_1, \dots, i_k \in \{1, \dots, m\} \\ i_1 + \dots + i_k = m}} \frac{1}{i_1! i_2! \dots i_k!} G^{(k)}(x)(v_{i_1}) \cdots (v_{i_k})]. \tag{2.9}$$

In particular,
for $m = 2$, $S_2^G(x; v_1, v_2) = G^{'}(x)(v_2) + G^{''}(x)(v_1)(v_1)$,
for $m = 3$, $S_3^G(x; v_1, v_2, v_3) = G^{'''}(x)(v_1)(v_1)(v_1) + 3G^{''}(x)(v_1)(v_2) + G^{'}(x)(v_3)$,

for $m=4$, $S_4^G(x;v_1,v_2,v_3,v_4) = G^{(4)}(x)(v_1)(v_1)(v_1)(v_1) + 6G'''(x)(v_1)(v_1)(v_2) + $
$+4G''(x)(v_1)(v_3) + 3G''(x)(v_2)(v_2) + G'(x)(v_4).$

Definition 2.11. A mapping $G : X \to Z$ is said to be strictly differentiable at a point x, if there exists a continuous linear operator $\Lambda : X \to Z$ with the property that for any $\varepsilon > 0$ there is $\delta > 0$ such that for all x_1 and x_2 satisfying the inequalities $\|x_1 - \bar{x}\| < \delta$ and $\|x_2 - \bar{x}\| < \delta$, the following inequality holds

$$\|G(x_1) - G(x_2) - \Lambda(x_1 - x_2)\| \le \varepsilon \|x_1 - x_2\|.$$

A strictly differentiable function G at a point x is Fréchet differentiable at x and $\Lambda = G'(x)$. A continuously differentiable function near x is strictly differentiable at x (Corollary 2, Section 2.2.3, [1]).

As noted in [16], second-order Fréchet differentiability at a point is not a local property and it does not imply that the function is strictly differentiable at the point, or continuously differentiable around the point, or Lipschitz around the point.

Theorem 2.12. *(Lyusternik's Theorem,* [1]*) Let* X, Z *be Banach spaces,* U *be a neighborhood of a point* $x \in X$, $G : U \to Z$, $G(x) = 0$ *and* $D_G = \{u \in U : \ G(u) = 0\}$.

If G *is strictly differentiable at* x *and* $G'(x)$ *is onto, then*

$$T_x D_G = \{v \in X; G'(x)(v) = 0\}. \tag{2.10}$$

In 2016, we extended the characterizations we established in Corollary 2.1, [6] for the higher-order tangent cones in Pavel sense to the null-set of a mapping taking values in a finite dimensional normed space. In our result below, the mapping takes values into an arbitrary linear normed space.

Theorem 2.13. *(Theorem 3.5,* [12] *Let* X *and* Z *be Banach spaces, let* U *be a neighborhood of a point* x *in* X.

Assume that $G : U \to Z$ *is strictly differentiable at* $x \in U$ *with* $G(x) = 0$, *its derivative* $G'(x) :$ $X \to Z$ *is onto, and* G *is* n *times Fréchet differentiable at* x.

Let $D_G = \{u \in U : \ G(u) = 0\}$.

For any $n \ge 2$, *it holds*

$$T_x^n D_G = \{v_n \in X \textit{ for which there exist } v_1, \ldots, v_{n-1} \in X$$
$$\textit{such that } S_m^G(x, v_1, \ldots, v_m) = 0, \ 1 \le m \le n\}.$$

Here, for every positive integer m, $S_m^G(x, v_1, \ldots, v_m)$ *denotes the expression*

$$S_m^G(x, v_1, \ldots, v_m) = \sum_{k=1}^{m} \frac{m!}{k!} [\sum_{\substack{i_1, \ldots, i_k \in \{1, \ldots, m\} \\ i_1 + \ldots + i_k = m}} \frac{1}{i_1! i_2! \ldots i_k!} G^{(k)}(x)(v_{i_1}) \cdots (v_{i_k})].$$

3. Higher-Order Optimality Conditions via Tangent Cones

3.1. **Higher-Order Optimality Conditions for Scalar Optimization Problems.** In this section we deal with the scalar constrained minimization problem

$$\text{Minimize } F(x) \text{ subject to } x \in D, \tag{P}$$

where X is a real linear normed space of norm $\|\cdot\|$, $\bar{x} \in D \subseteq U$, and $F : U \subseteq X \to \mathbb{R}$ is a function of class C^p on the open set U, p positive integer.

Recall that a point $\bar{x} \in D$ is said to be a local minimum of a function $F : U \to \mathbb{R}$ on $D \subseteq X$, X Banach space, if there exists $\delta > 0$ such that $F(x) \geq F(\bar{x})$, for all $x \in U \cap D$ satisfying $0 < ||x - \bar{x}|| < \delta$. If the defining inequality is strict, then $\bar{x}$ is said to be a strict local minimum of F on D.

A point $\bar{x} \in D$ is said to be an isolated local minimum of order p (p positive integer) of $F : X \to \mathbb{R}$ on D, if there exists a neighborhood V of $\bar{x}$ and a constant $c > 0$ such that

$$F(x) - F(\bar{x}) \geq c\|x - \bar{x}\|^p, \text{ for all } x \in D \cap V \setminus \{\bar{x}\}.$$

In this section we recall our higher-order necessary conditions of extremum for problem (P) with smooth data and an arbitrary constraint set. We generalized the second-order necessary conditions of Theorem 3.1, [26] established for a functional constraint set.

Theorem 3.1. *(Theorem 2.2, [5]) Let $\bar{x}$ be a local minimum of $F : X \to \mathbb{R}$ on D, where D is a nonempty subset of the Banach space X.*

Then

i) If F is of class C^1 near $\bar{x}$, then

$$F^{'}(\bar{x})(v_1) \geq 0, \; \forall v_1 \in T_{\bar{x}}D.$$

ii) If F is of class C^2 near $\bar{x}$, then

$$F^{'}(\bar{x})(v_2) + F^{''}(\bar{x})[v_1]^2 \geq 0,$$

$\forall v_2 \in T^2_{\bar{x}}D$ with the associated vector $v_1 \in T_{\bar{x}}D$ such that $F^{'}(\bar{x})(v_1) = 0$.

iii) If F is of class C^3 near $\bar{x}$, then

$$F^{'''}(\bar{x})[v_1]^3 + 3F^{''}(\bar{x})(v_1)(v_2) + F^{'}(\bar{x})(v_3) \geq 0, \tag{3.1}$$

$\forall v_3 \in T^3_{\bar{x}}D$ with associated vectors $v_1 \in T_{\bar{x}}D$ and $v_2 \in T^2_{\bar{x}}D$ such that

$$F^{'}(\bar{x})(v_1) = 0, \text{ and } F^{'}(\bar{x})(v_2) + F^{''}(\bar{x})[v_1]^2 = 0. \tag{3.2}$$

In general, if F is of class C^p near $\bar{x}$, there must exist a positive integer p with the property that

$$S^F_p(\bar{x}; v_1, \ldots, v_p) \geq 0, \; \forall v_1, v_2, ..., v_p \in X \text{ such that}$$

$$\bar{x} + tv_1 + \frac{t^2}{2}v_2 + ... + \frac{t^p}{p!}(v_p + \gamma(t)) \in D, \; \forall t > 0, \; \gamma(t) \to 0 \text{ as } t \downarrow 0, \text{ and}$$

$$S^F_n(\bar{x}; v_1, \ldots, v_n) = 0, \text{ for every } n, \; 0 < n \leq p-1.$$

We can combine Theorem 3.1 with either one of Theorems 2.10 and 2.13 to formulate the following result. Under hypotheses i), Theorem 3.2 was obtained in Corollary 3.1.2, [6], and in Corollary 2.1, [5].

Theorem 3.2. *Let F and G be two functions on a normed space X, $F : X \to \mathbb{R}$, $G : X \to Y$, X and Y be linear normed spaces, and $\bar{x}$ be a local minimum of F on $D_G = \{x \in X : \; G(x) = 0\}$.*

Assume that either one of the hypotheses i) and ii) hold.

i) Y is finite dimensional and G is continuous near $\bar{x}$.

ii) G is strictly differentiable at $\bar{x}$.

If F is of class C^4 in a neighborhood of $\bar{x}$, G is four times Frechét differentiable at $\bar{x}$, $G^{'}(\bar{x})$ is onto, and

$$F^{'}(\bar{x})(v_1)=0,\ F^{'}(\bar{x})(v_2)+F^{''}(\bar{x})(v_1)(v_1)=0, \tag{3.3}$$

$$F^{'''}(\bar{x})(v_1)(v_1)(v_1)+3F^{''}(\bar{x})(v_1)(v_2)+F^{'}(\bar{x})(v_3)=0, \tag{3.4}$$

for all v_1, v_2 and v_3 for which

$$G^{'}(\bar{x})(v_1)=0,\ G^{'}(\bar{x})(v_2)+G^{''}(\bar{x})(v_1)(v_1)=0,\ \textit{and} \tag{3.5}$$

$$G^{'''}(\bar{x})(v_1)(v_1)(v_1)+3G^{''}(\bar{x})(v_1)(v_2)+G^{'}(\bar{x})(v_3)=0, \tag{3.6}$$

then

$$F^{(4)}(\bar{x})(v_1)(v_1)(v_1)(v_1)+6F^{'''}(\bar{x})(v_1)(v_1)(v_2)+4F^{''}(\bar{x})(v_1)(v_3)+ \\ +3F^{''}(\bar{x})(v_2)(v_2)+F^{'}(\bar{x})(v_4)\geq 0$$

whenever v_1, v_2, v_3, v_4 satisfy

$$G^{(4)}(\bar{x})(v_1)(v_1)(v_1)(v_1)+6G^{'''}(\bar{x})(v_1)(v_1)(v_2)+4G^{''}(\bar{x})(v_1)(v_3)+ \\ +3G^{''}(\bar{x})(v_2)(v_2)+G^{'}(\bar{x})(v_4)=0,$$

besides (3.5) and (3.6).

In general, if F is of class C^p, $p>1$, in a neighborhood of $\bar{x}$, G is p times Frêchet differentiable at $\bar{x}$, and $G^{'}(\bar{x})$ is onto, then

$$S_p^F(\bar{x},v_1,\ldots,v_p)\geq 0,$$

v_1, v_2, $\ldots$,$v_p\in X$ such that

$$S_i^F(\bar{x},v_1,\ldots,v_i)=0, \textit{ for every } i,\ 1\leq i\leq p-1, \textit{ and} \\ S_i^G(\bar{x},v_1,\ldots,v_i)=0, \textit{ for all } i,\ 1\leq i\leq p.$$

Example 3.3. Let us minimize the function $F(x_1,x_2)=x_1^4+x_1x_2^2+x_2^3$ subject to $G(x_1,x_2)=x_1^{\frac{8}{3}}+x_1x_2+x_2^2=0$, $F,G:\mathbb{R}^2\to\mathbb{R}$.

The only critical point, i.e., the only solution of the equation $F^{'}(x_1,x_2)=0$, which satisfies the constraint is $(0,0)$.

The method of Lagrange multipliers and the classical second-order optimality conditions (Propositions 3.1.1 and 3.2.1, [4]) can not be applied to this example because $G'(x)=0$, for all $x\in\mathbb{R}^2$.

We have $T_{(0,0)}D_G\subseteq Ker\ G^{''}(0,0)=\left\{(v_{11},v_{12})\in\mathbb{R}^2,\ v_{11}v_{12}+v_{12}^2=0\right\}$.

Indeed, if $(v_{11},v_{12})\in T_{(0,0)}D_G$ then there exists a mapping $r(t)=(r_1(t),r_2(t))$, $r(t)\to 0$ as $t\downarrow 0$ such that $tv_1+tr(t)\in D_G$, $\forall t>0$, which means that

$$(tv_{11}+tr_1(t))^3+(tv_{12}+tr_2(t))^2+(tv_{11}+tr_1(t))(tv_{12}+tr_2(t))=0,\ \forall t>0.$$

After dividing by $t^2$and letting t go to 0, we get $v_{11}v_{12}+v_{12}^2=0$.

In this example, the first-order derivatives of the functions F and G at $(0,0)$ are identically zero. For any $v_1\in T_{(0,0)}D_G$, $G^{''}(0,0)(v_1)$ is onto. Thus, the second-order tangent cone can be characterized with the aid of our Theorem 2.9 (Theorem 1, [13])

$$T^2_{(0,0)}D_G=\{(v_{21},v_{22})\in\mathbb{R}^2, \text{ for which there exists } (v_{11},v_{12})\neq(0,0) \text{ with } v_{11}v_{12}+v_{12}^2=0, \\ \text{such that } 2v_{11}^3+v_{11}v_{22}+v_{12}v_{21}+2v_{12}v_{22}=0\}.$$

Obviously, $F^{'}(0,0)=F^{''}(0,0)=0$, so $F^{'}(0,0)(v_1)=0$, for all $v_1\in T_{(0,0)}D_G$, and

$$F^{'}(0,0)(v_2)+F^{''}(0,0)(v_1)(v_1)=0,$$

for any $v_2\in T^2_{(0,0)}D_G$ with associated vector $v_1\in T_{(0,0)}D_G$.

Furthermore,

$$F^{'''}(0,0)(v_1)(v_1)(v_1)+3F^{''}(0,0)(v_1)(v_2)+F^{'}(0,0)(v_3)=6v_{12}^2(v_{12}+v_{11})=0,$$

for every $v_1\in T_{(0,0)}D_G$ and

$$F^{(4)}(0,0)(v_1)(v_1)(v_1)(v_1)+6F^{'''}(0,0)(v_1)(v_1)(v_2)+4F^{''}(0,0)(v_1)(v_3)+3F^{''}(0,0)(v_2)(v_2)+\\ F^{'}(0,0)(v_4)=24v_{11}^4+6\left(2v_{12}^2v_{21}+4v_{11}v_{12}v_{22}\right),$$

whenever $v_4\in T^4_{(0,0)}D_G$ with correspondent vectors $v_3\in T^3_{(0,0)}D_G$, $v_2\in T^2_{(0,0)}D_G$ and $v_1\in T_{(0,0)}D_G$.

If $v_1\in T_{(0,0)}D_G$ and $v_1\neq 0$, then one possibility is $v_{12}=0$, in which case the above expression is equal to $24v_{11}^4$, so it is strictly positive because $v_{11}\neq 0$. Another possibility is $v_{12}+v_{11}=0$ ($v_{12}=-v_{11}\neq 0$), when the above expression becomes equal to $12v_{11}^2\left(3v_{21}-2v_{11}^2\right)$, after using the fact that $v_2\in T^2_{(0,0)}D_G$ with associated $v_1\in T_{(0,0)}D_G$, so $2v_{11}^3+v_{11}v_{22}+v_{12}v_{21}+2v_{12}v_{22}=0$, which reduces to $v_{22}=2v_{11}^2-v_{21}$.

The relations that hold between the components of that vectors v_1 and v_2 allow us to conclude that the fourth order expression does not have constant sign for all $v_2\in T^2_{(0,0)}D_G$ with correspondent $v_1\in T_{(0,0)}D_G$. Therefore, by Theorem 3.1 (Theorem 2.2, [5]), $(0,0)$ is neither a local minimum nor a local maximum of F on D_G.

Theorem 3.4. *Suppose that*
a) $F:X\to\mathbb{R}$ is p-times differentiable at $\bar{x}\in D_G\cap S$, where $D_G=\{x\in X,\, G(x)=0\}$, $G:X\to\mathbb{R}^k$, $p\geq 2$, and S is an arbitrary subset of a finite dimensional normed space X.

b) There is a positive integer m such that G is m-times differentiable at $\bar{x}$, $G^{(j)}(\bar{x})=0$, $0\leq j\leq m-1$, and $G^{(m)}(\bar{x})$ is not identically zero.

c) $F^{(j)}(\bar{x})[y]^j\geq 0$, $\forall y\in\mathbb{R}^n$, $1\leq j\leq p-1$, and $F^{(p)}(\bar{x})[y]^p>0$, $\forall y\neq 0$ with $G^{(m)}(\bar{x})[y]^m=0$ and $y\in\Gamma_{\bar{x}}S$.

Then $\bar{x}$ is an isolated local minimum of order p of F on $D_G\cap S$.

Remark 3.5. The above theorem follows from Theorem 3.3, [14] as a p-times differentiable function is p-times Gâteaux differentiable. If the set S is convex, then in the above theorem, the contingent cone $\Gamma(\bar{x},S)$ can be replaced by the tangent cone $T_{\bar{x}}S$ as these cones coincide.

In [11] we obtained second-order sufficient conditions via the tangent cone for an isolated local minimum of order two for problem (P) with a locally Lipschitz objective function and a convex constraint set.

Example 3.6. Let us consider the objective function $F(x_1,x_2)=x_1^4+5x_2^7$, subject to $G(x_1,x_2)=x_1^{8/3}+x_1x_2+x_2^2=0$, $F,G:\mathbb{R}^2\to\mathbb{R}$.

The method of Lagrange multipliers and the classical second-order optimality conditions (Propositions 3.1.1 and 3.2.1, [4]) can not be applied to this example because $G'(x)=0$, for all $x\in\mathbb{R}^2$.

We notice that $\bar{x}=(0,0)$ belongs to the constraint set D_G. Obviously, $\bar{x}$ verifies the well-known first-order necessary optimality conditions $F'(\bar{x})(v)\geq 0$, for all $v\in T_{\bar{x}}D_G$ as $F^{'}(0,0)=0$.

Moreover, $\bar{x}$ satisfies our second-order necessary optimality conditions of [9, Theorem 2.1] as $F^{'}(\bar{x}) = F^{''}(\bar{x}) = 0$.

The conditions of Theorem 3.4 are fulfilled with $m = 2$, $p = 4$ since $F^{'}(\bar{x}) = F^{''}(\bar{x}) = F^{^{(3)}}(\bar{x}) = 0$, and $F^{^{(4)}}(\bar{x})[y]^4 = 24y_1^4 > 0$, $\forall y = (y_1, y_2)$, $y \neq 0$ such that $G^{''}(\bar{x})[y]^2 = 0$, i.e., for all $y \neq 0$ with $y_1 y_2 + y_2^2 = 0$, as $y_1 = 0$ implies $y_2 = 0$ too, which contradicts $y \neq 0$.

By Theorem 3.4, we conclude that $\bar{x} = (0,0)$ is a strict local minimum of F on D_G, it is an isolated local minimum of order four of F on D_G.

In this example, Theorem 3.2, [9] is applicable with $D = D_G$ as $F'(\bar{x}) = 0$. The second-order sufficient optimality conditions of [9, Theorem 3.2] are not verified because $F^{''}(\bar{x}) = 0$ and therefore $F^{''}(\bar{x})(y)(y)$ is not strictly positive in any nonzero direction $y \in T_{\bar{x}} D_G$.

Theorem 2.2, [9] is not applicable to this example as $G'(\bar{x})$ is not onto.

Theorem 4.2, [18] is applicable with $f_s = F$, $h = G$, $g(x) = 0$, $\forall x \in \mathbb{R}^2$, and $C = \mathbb{R}^2$. The higher-order sufficient optimality conditions of D.V. Luu's result are not verified at $(0,0)$ as $F^{^{(4)}}(0,0)[y]^4$ is not necessarily strictly positive for any direction y with $\|y\| = 1$, for example $F^{^{(4)}}(0,0)[(0,1)]^4 = 0$. Thus the origin can not be recognized as a higher-order isolated local minimum of F on $G^{-1}(0)$ by means of [18, Theorem 4.2].

3.2. **Higher-Order Optimality Conditions for Vector Optimization Problems.** In this section we deal with the vector optimization problem

$$\text{Minimize } F(x) \text{ subject to } x \in M, \qquad (VP)$$

where $F : X \to Y$, X and Y are real linear normed spaces, $M \subseteq X$. Let $Q \subset Y$ be a closed convex cone with its interior $int\, Q \neq \emptyset$.

A point $\bar{x} \in M$ is said to be a local weak minimum of F on M, if there exists a number $\delta > 0$ such that

$$F(x) - F(\bar{x}) \notin -int\, Q, \; \forall x \in M \cap B(\bar{x}, \delta), \qquad (3.7)$$

where $B(\bar{x}, \delta)$ stands for the open ball of radius δ centered at $\bar{x}$.

The notion of local weak minimum is the concept of local minimum when $F : X \to \mathbb{R}$ in problem (VP).

A point $\bar{x}$ is called a strict local Pareto minimum of order p of F on M, if there exist numbers $\delta > 0$ and $\alpha > 0$ such that

$$(f(x) + Q) \cap B(f(\bar{x}), \alpha\|x - \bar{x}\|^p) = \emptyset, \; \forall x \in M \cap B(\bar{x}, \delta) \setminus \{\bar{x}\}. \qquad (3.8)$$

In the case $Y = \mathbb{R}$ and $Q = \mathbb{R}_+$, this notion becomes the usual notion of isolated local minimum of order p, since then (3.8) is equivalent to

$$f(x)) > f(\bar{x}) + \alpha\|x - \bar{x}\|^p, \; \forall x \in M \cap B(\bar{x}, \delta) \setminus \{\bar{x}\}.$$

We are also concerned with the multiobjective problem (MOP), which is a special case of (VP) obtained for $X = \mathbb{R}^r$, $Q = \mathbb{R}^r_+$ and $F = (F_1, \ldots, F_r)$.

In this case the above definition of a weak local minimum becomes: a point $\bar{x} \in M$ is a weak local minimum to problem (MOP), if there exists a neighborhood V of $\bar{x}$ such that no $x \in V \cap M$ satisfies $F_i(x) < F_i(\bar{x})$ for all $i = 1, \ldots, r$.

Let G be a mapping from X into Y, where X and Y are real normed linear spaces. Recall that G is Gâteaux differentiable at $\bar{x}$, if there exists a continuous linear mapping Λ_1 from X into Y such that

$$G(\bar{x} + tv) = G(\bar{x}) + t\Lambda_1(v) + o(t), \; \forall v \in X,$$

where $\|o(t)\|/|t| \to 0$ as $t \to 0$. The mapping Λ_1 is said to be Gâteaux derivative of G at $\bar{x}$ and is denoted by $G'_G(\bar{x})$. Note that a mapping which is Gâteaux differentiable at $\bar{x}$ may not be continuous at $\bar{x}$.

The mapping $G : X \to Y$ is p-times Gâteaux differentiable at $\bar{x}$ ($p \geq 2$), if G is Gâteaux differentiable at $\bar{x}$ and there exist continuous multilinear symmetric mappings Λ_k from X^k into Y(continuous linear symmetric in k variables), $k = 2, \dots, p$, such that

$$G(\bar{x}+tv) = G(\bar{x}) + t\Lambda_1(v) + \frac{t^2}{2!}\Lambda_2[v]^2 + \cdots + \frac{t^p}{p!}\Lambda_p[v]^p + o(t^p),\ \forall v \in X,$$

where $\Lambda_1 = G'_G(\bar{x})$, $\|o(t^p)\|/|t|^p \to 0$ as $t \to 0$ (see Luu, [18]). Note that symmetric means it does not change under permutation of variables. For the correctness of this definition, the symmetric multilinear mapping Λ_p should be uniquely determined by the respective form $v \to \Lambda_p(v)^p$ (see, for example, [19]). The continuous multilinear symmetric mapping Λ_k is the k^{th} order Gâteaux derivative of G at $\bar{x}$ and is denoted by $G_G^{(k)}(\bar{x})$. Thus, for a function G which is p-times Gâteaux differentiable at $\bar{x}$, G can be expanded as

$$G(\bar{x}+tv) = G(\bar{x}) + tG'_G(\bar{x})(v) + \frac{t^2}{2!}G_G^{(2)}(\bar{x})[v]^2 + \cdots + \frac{t^p}{p!}G_G^{(p)}(\bar{x})[v]^p + o(t^p),$$

for all $v \in X$, where $\|o(t^p)\|/|t|^p \to 0$ as $t \to 0$.

In 2014, Luu, [18] established higher-order necessary conditions via higher-order tangent cones for (VP) with regular equality constraints, that is, with an equality constraint function that has onto Fréchet derivative at the minimum point.

For a m-times Gâteaux differentiable function $G : X \to Y$, Luu, [18] replaced the Fréchet derivatives by the corresponding Gâteaux derivatives in our expression (2.9) obtained in [6, 5, 10].

$$(S_m^G)_G(\bar{x}, v_1, \dots, v_m) = \sum_{k=1}^{m} \frac{m!}{k!}[\sum_{\substack{i_1,\dots,i_k \in \{1,\dots,m\} \\ i_1+\dots+i_k=m}} \frac{1}{i_1!i_2!\dots i_k!} G_G^{(k)}(\bar{x})(v_{i_1})\cdots(v_{i_k})].$$

Theorem 3.7. *(Theorem 3.3, [18]) Let $\bar{x}$ be a local weak minimum for problem (VP) with $M = \{x \in X :\ -H(x) \in S,\ x \in D_G\}$, where $H : X \to Z$, Z is a linear normed space, S is a closed convex cone in Z, $G : X \to \mathbb{R}^l$, l nonnegative integer, $D_G = \{x \in X :\ G(x) = 0\}$. Assume that F is n-times Gâteaux differentiable at $\bar{x}$, H is p-times Gâteaux differentiable at $\bar{x}$, $p \leq n$, G is of class C^n in a neighborhood of $\bar{x}$ with $G'(\bar{x})$ onto.*

Then, for $v_1, v_2, \dots, v_n$ satisfying
$(S_j^H)_G(\bar{x}; v_1, \dots, v_j) \in -S$, $j = 1, \dots, p-1$,
$(S_p^H)_G(\bar{x}; v_1, \dots, v_p) \in -int\, S$,
$(S_i^G)_G(\bar{x}; v_1, \dots, v_i) = 0$, $i = 1, \dots, n$,
$(S_i^F)_G(\bar{x}; v_1, \dots, v_i) = 0$, $i = 1, \dots, n-1$,
we have
$(S_n^F)_G(\bar{x}; v_1, \dots, v_n) \notin -int\, Q$.

We recall the higher-order sufficient conditions derived in 2014 by Luu, [18], and then the higher-order sufficient conditions we formulated in [14] in 2019 for multiobjective optimization problems.

Denote by $(F_s)_G^{(i)}$ the i-th order Gâteaux derivative of F_i at $\bar{x}$, and $(F_s)_G^{(1)} = (F_s)'_G$.

Theorem 3.8. *(Theorem 4.2, [18]) Consider the problem (MOP) with the feasible set $M = \{x \in C:\ -H(x) \in S,\ x \in D_G\}$, $Y = \mathbb{R}^r$, $Q = \mathbb{R}^r_+$, and $F = (F_1, \ldots, F_r) : X \to \mathbb{R}^r$. Let $\bar{x} \in M$. Assume that X is a finite dimensional linear normed space, $C \subset X$ is convex, S is a closed convex cone in Z, $int\, S \neq \emptyset$, $G : X \to W$ and $H : X \to Z$ areGâteaux differentiable at $\bar{x}$, F_s is p-times Gâteaux differentiable at $\bar{x}$ for some s, $1 \leq s \leq r$, Z and W are linear normed spaces Suppose also that the following two conditions hold:*

a) $(F_s)_G^{(i)}(\bar{x})(v)^i \geq 0$, for all $v \in T_{\bar{x}}C \cap \mathbb{S}$, $i = 1, 2, \ldots, n-1$,

b) $(F_s)_G^{(p)}(\bar{x})(v)^p > 0$, for all $v \in T_{\bar{x}}C \cap \mathbb{S} \cap \{u \in X:\ H'_G(\bar{x})(u) \in -S_{H(\bar{x})},\ G'_G(\bar{x})(u) = 0\}$. Here $S_{H(\bar{x})} = cl(cone(S + H(\bar{x})))$, and $\mathbb{S}$ is the unit sphere in X.

Then $\bar{x}$ is a strict local Pareto minimum of order p of F on M.

Note that $S_{H(\bar{x})} \neq \emptyset$ if $int\, S \neq \emptyset$.

Theorem 3.9. *(Theorem 4.1, [14]) Consider the problem (MOP) with the feasible set $M = S \cap D_G$. Let $F = (F_1, \ldots, F_r) : U \to \mathbb{R}^r$ be defined on an open subset U of the finite dimensional normed space X. Suppose that $G : U \to \mathbb{R}^k$ and $F_s : U \to \mathbb{R}$ for some $s \in \{1, \ldots, r\}$ are p-times Gâteaux differentiable at $\bar{x} \in M = S \cap D_G$, where $D_G = \{x \in U;\ G(x) = 0\}$, $p \geq 2$, and S is an arbitrary subset of X, $S \subseteq U \subseteq X$. Suppose that there exists some $\lambda \in \mathbb{R}^k$ such that*

i) $[(F_s)_G^{(j)}(\bar{x}) - \lambda G_G^{(j)}(\bar{x})][y]^j \geq 0$, for all $y \in X$, $1 \leq j \leq p-1$, and

ii) $[(F_s)_G^{(p)}(\bar{x}) - \lambda G_G^{(p)}(\bar{x})][y]^p > 0$, for all $y \in \Gamma(\bar{x}, M)$, $y \neq 0$.

Then $\bar{x}$ is a strict local Pareto minimum of order p of F on M.

Theorem 3.10. *(Theorem 4.2, [14]) Consider the problem (MOP) with the feasible set $M = S \cap D_G$. Suppose that*

a) $F = (F_1, \ldots, F_r) : U \to \mathbb{R}^r$ is defined on an open subset U of the finite dimensional normed space X, $F_s : U \to \mathbb{R}$ for some $s \in \{1, \ldots, r\}$ is p-times Gâteaux differentiable at $\bar{x} \in M$, where $D_G = \{x \in U;\ G(x) = 0\}$, $p \geq 2$, S is an arbitrary subset of X, $S \subseteq U \subset X$.

b) There is a positive integer m such that G is m-times Gâteaux differentiable at $\bar{x}$, $G_G^{(j)}(\bar{x}) = 0$, $0 \leq j \leq m-1$, and $G_G^{(m)}(\bar{x})$ is not identically zero.

c) $(F_s)_G^{(j)}(\bar{x})[y]^j \geq 0$, $\forall y \in X$, $1 \leq j \leq p-1$, and $(F_s)_G^{(p)}(\bar{x})[y]^p > 0$, $\forall y \neq 0$ with $G_G^{(m)}(\bar{x})[y]^m = 0$ and $y \in \Gamma(\bar{x}, S)$.

Then $\bar{x}$ is a strict local Pareto minimum of order p of F on M.

Remark 3.11. If the set S is convex, then in the above theorem (Theorem 4.2, [14]), the contingent cone $\Gamma(\bar{x}, S)$ can be replaced by the tangent cone $T_{\bar{x}}S$ as these cones coincide.

Example 3.12. Let us consider the function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ subject to $G(x_1, x_2) = x_1^5 - x_2^4 + x_2^2 - x_1^3 x_2 = 0$, $G : \mathbb{R}^2 \to \mathbb{R}$, where $f_1(x_1, x_2) = x_1^2 - x_2^2$ and

$$f_2(x_1, x_2) = \begin{cases} -x_2, \ if\ x_1 = x_2^2 \\ 0, \ if\ otherwise. \end{cases}$$

The functions f_1 and G are polynomials, so they are (Fréchet) differentiable at $(0,0)$ of any order and therefore Gâteaux differentiable of any order. The function f_2 is Gâteaux differentiable at $(0,0)$ with $(f_2)'_G(0,0) = 0$, but f_2 is not (Fréchet) differentiable at $(0,0)$. Indeed,

$\lim\limits_{\|h\|\to 0} \dfrac{|f_2((0,0)+h)-f_2(0,0)|}{\|h\|} = 1 \neq 0$, for $h(t) = (t^2,t)$ with $t \to 0^+$, so the (Fréchet) derivative of f_2 does not exist at $(0,0)$.

First we show that the origin is an isolated local minimum of order two of f_1 on D_G.

We notice that $\bar{x} = (0,0)$ belongs to the constraint set D_G. Obviously, $\bar{x}$ verifies the well-known first-order necessary optimality conditions $f_1'(\bar{x})(v) \geq 0$, for all $v \in T_{\bar{x}}D_G$, as $f_1'(0,0) = 0$. Also it can be seen that $\bar{x}$ satisfies our second-order necessary optimality conditions given in [9, Theorem 2.1]: $f_1''(\bar{x})(v)(v) + f_1'(\bar{x})(w) = f_1''(\bar{x})(v)(v) = 2v_1^2 - 2v_2^2 \geq 0$, for all $v = (v_1,v_2) \in S_w \subset T_{\bar{x}}D_G \subseteq \Gamma(\bar{x},D_G) \subseteq \{v \in \mathbb{R}^2; G''(0,0)[v]^2 = 0\} = \{v \in \mathbb{R}^2; v_2 = 0\}$ (see [9] for the definition of the set S_w).

The assumptions of Theorem 3.4 are satisfied with $f_1 = F$, $S = \mathbb{R}^2$, $p = 2$, $m = 2$. Indeed, $f_1'(\bar{x})(y) = 0 \geq 0$, for all $y \in \mathbb{R}^2$, and $f_1''(\bar{x})(y)(y) = 2y_1^2 - 2y_2^2 > 0$, for all $y \neq 0$ with $G''(\bar{x})[y]^2 = 0$, as $y_2 = 0$, and so $y_1 \neq 0$. Therefore, by Theorem 3.4 (Theorem 3.3, [14]) the origin is an isolated local minimum of order two of f_1 subject to $G(x_1,x_2) = 0$.

The hypotheses of Theorem 3.10 (Theorem 4.2, [14]) are verified with $f_s = f_1$, $S = \mathbb{R}^2$, $p = 2$, $m = 2$, and therefore, the origin is a strict local Pareto minimum of order two of f subject to $G(x_1,x_2) = 0$.

Theorem 4.2, [18] is applicable with $f_s = f_1$, $h = G$, $g(x) = 0$, for all $x \in \mathbb{R}^2$, $C = \mathbb{R}^2$, but the hypotheses of Luu's result are not satisfied as $f_1''(0,0)[y]^2 = 2y_1^2 - 2y_2^2 \not> 0$, for all y with $\|y\| = 1$. Thus, the origin can not be recognized as a strict local Pareto minimum of order two of f on D_G by means of [18, Theorem 4.2].

4. Flow-Invariance via Higher-Order Tangent Cones

In this section we provide conditions for a closed subset D of a Banach space X to be flow-invariant with respect to the n-th order autonomous differential equation

$$u^{(n)}(t) = F(u(t)), t \geq 0, \tag{4.1}$$

where $n \geq 3$, and $F : U \to X$ is a locally Lipschitz mapping.

The invariant sets for the first-order differential equations were studied by H. Brézis [3], M.G. Crandall [15], R.H. Jr. Martin [20], N.H. Pavel and F. Iacob [23] and many other authors.

In [25], N.H. Pavel and C. Ursescu treated the problem of flow-invariance of a set with respect to the second-order differential equation $u''(t) = F(u(t))$, $t \geq 0$, using the theory of tangent cones.

In [6, 7], we characterized the sets $D = D_G = \{x \in X; G(x) = 0\}$, that are flow-invariant with respect to the n-th order autonomous differential equation (4.1) when $n \geq 3$, and thus extended Theorem 2.6, [25].

Theorem 4.1. *(Definition 1.9, [21]) The nonempty set $D \subset U$ is said to be (right-hand) flow-invariant with respect to the n-th order differential equation (4.1) if the solution $u : [0,T) \to X$ to the Cauchy problem (CP) determined by (4.1) and the initial conditions*

$$u(0) = x, u'(0) = v_1, \ldots, u^{(n-1)}(0) = v_{n-1}, \tag{4.2}$$

with $x \in D$, $v_1 \in T_xD, \ldots,$ $v_{n-1} \in T_x^{n-1}D$, $F(x) \in T_x^nD$ having correspondent vectors $v_1, \ldots,$ v_{n-1}, satisfies

$$u(t) \in D, \forall t \geq 0, t \in \mathrm{dom}\, u. \tag{4.3}$$

The constraints imposed to $(x, v_1, \dots, v_{n-1})$ are necessary conditions to have the invariance property (4.3).

In [21], Pavel and Motreanu introduced the following set

$$\begin{aligned} M_D^{(n)} = \{ & (x, v_1, \dots, v_{n-1}) \in D \times X^{n-1} : v_i \in T_x^i D, i = 1, \dots, n-1, \\ & F(x) \in T_x^n D \text{ with associated vectors } v_1, \dots, v_{n-1}\}, n \geq 2. \end{aligned} \tag{4.4}$$

Note that $M_D^{(2)}$ coincides with the set M_D previously introduced by Pavel and Ursescu in [25] $M_D = \{(x, v) \in D \times X^{n-1} : v \in T_x D, F(x) \in T_x^2 D \text{ with associated vector } v\}$.

The choice in (4.2) for the initial conditions was expresses in [21] by means of (4.4) as follows

$$(u(0), u'(0), \dots, u^{(n-1)}(0)) = (x, v_1, \dots, v_{n-1}) \in M_D^{(n)}.$$

This was justified by the following result that extended Theorem 2.2 i) of Pavel and Ursescu, [25] established for second-order differential equations.

Proposition 4.2. *(Proposition 1.8, [21]) Consider the Cauchy problem (CP) determined by (4.1) and (4.2). If $u : [0, T) \to X$ is a solution of (CP), which satisfies the invariance property (4.3), then one has*

$$(u(t), u'(t), \dots, u^{(n-1)}(t)) \in M_D^{(n)}, \forall t \in [0, T). \tag{4.5}$$

N.H. Pavel and C. Ursescu ([21, 25]) reduced the problem of invariant sets for (4.1) to a similar problem for a first-order differential equation, fact that allowed them to utilize a theorem proved by M. Nagumo [22] and, independently, by H. Brézis (Theorem 1, [3]), in order to obtain the following characterization of flow-invariant sets $D \subset U$ with respect to the n-th order differential equation (4.1).

Theorem 4.3. *(Theorem 1.10, [21]) Assume that $M_D^{(n)}$ is a nonempty closed subset of $U \times X^{n-1}$, for a closed subset D of U, $n \geq 2$. Then $D \subset U$ is a flow-invariant set with respect with the n-th order differential equation (4.1) if and only if $(v_1, \dots, v_{n-1}, F(x)) \in X^n$ is a tangent vector to $M_D^{(n)}$ for all $(x, v_1, \dots, v_{n-1}) \in M_D^{(n)}$, i.e.,*

$$\lim_{t \downarrow 0} t^{-1} d((x, v_1, \dots, v_{n-1}) + t(v_1, \dots, v_{n-1}, F(x)); M_D^{(n)}) = 0.$$

The above necessary and sufficient conditions for a set to be flow-invariant with respect to the n^{th}-order differential equation (4.1) were inspired by Theorem 2.4, [25] given for the case $n = 2$.

Pavel and Ursescu gave a description of the sets $D_G = \{u \in U : G(u) = 0\}$ that are flow-invariant with respect to the second order differential equation $u''(t) = F(u(t)), t \geq 0$.

Theorem 4.4. *(Theorem 2.6, [25]) Assume that $G : U \to \mathbb{R}^s$, $s \geq 1$ is two times Fréchet differentiable and its first Fréchet derivative $G'(x) : X \to \mathbb{R}^s$ is onto for each $x \in D_G$. Then $M_{D_G}^{(2)}$ is given by*

$$M_{D_G}^{(2)} = \{(x, v_1) \in U \times X : G(x) = 0, G'(x)(v_1) = 0, G''(x)(v_1)(v_1) + G'(x)(F(x)) = 0\}. \tag{4.6}$$

Suppose further that G is three times Fréchet differentiable on U, the function $h : U \to \mathbb{R}^s$ given by

$$h(x) = G'(x)(F(x)), \forall x \in U,$$

is Fréchet differentiable, $M_{D_G}^{(2)}$ is nonempty and the mapping
$(G'(x)(\cdot), G''(x)(v_1)(\cdot)) : X \to \mathbb{R}^s \times \mathbb{R}^s$ *is onto for every* $(x, v_1) \in M_{D_G}^{(2)}$.

Then D_G is flow-invariant with respect to the differential equation $u''(t) = F(u(t)), t \geq 0$ if and only if

$$G'''(x)(v_1)(v_1)(v_1) + 2G''(x)(v_1)(F(x)) + h'(x)(v_1) = 0,\ \forall (x, v_1) \in M_{D_G}^{(2)}. \tag{4.7}$$

Note that if $G : U \to \mathbb{R}^s$, then $D_G = \{u \in U : G(x) = 0\}$ is closed in U.

Remark 4.5. (Remark 5.1, [25]) First recall that a function $F : U \to X$ is regarded as a field of force on U, in the sense that to each vector position $x \in U$ it is associated the vector force $F(x) \in X$.

The notion

$$D_G\ is\ a\ flow-invariant\ set\ for\ the\ equation\ u''(t) = F(u(t)),\ t \geq 0, \tag{4.8}$$

can be restated in terms of Flight Mechanics as follows:

A mass particle projected from a point $x \in D_G$ with a velocity $v_1 \in X$ such that $(x, v_1) \in M_{D_G}^{(2)}$ (given by (4.6)), describes (under the action of the force field F) an orbit which lies in D_G. Under the hypotheses of Theorem 4.4 upon G, this happens if and only if (4.7) holds.

We extended Theorem 2.6, [25] to third-order flow-invariance problems.

Theorem 4.6. *(Theorem 3, [7], Theorem 4.3, [6]) Assume that $G : U \to \mathbb{R}^s$, $s \geq 1$ is three times Fréchet differentiable and its first Fréchet derivative $G'(x) : X \to \mathbb{R}^s$ is onto for each $x \in D_G$. Then $M_{D_G}^{(3)}$ is given by*

$$\begin{aligned} M_{D_G}^{(3)} = \{(x, v_1, v_2) \in U \times X \times X : G(x) = 0, G'(x)(v_1) = 0,\\ G''(x)(v_1)(v_1) + G'(x)(v_2) = 0,\\ G'''(x)(v_1)(v_1)(v_1) + 3G''(x)(v_1)(v_2) + G'(x)(F(x)) = 0\}. \end{aligned} \tag{4.9}$$

Suppose further that G is four times Fréchet differentiable on U, the function $h : U \to \mathbb{R}^s$ given by

$$h(x) = G'(x)(F(x)), \forall x \in U, \tag{4.10}$$

is Fréchet differentiable, $M_{D_G}^{(3)}$ is nonempty and the mapping
$(G'(x)(\cdot), G''(x)(v_1)(\cdot)) : X \to \mathbb{R}^s \times \mathbb{R}^s$ *is onto for every* $(x, v_1, v_2) \in M_{D_G}^{(3)}$.

Then D_G is flow-invariant with respect to the differential equation $u'''(t) = F(u(t)), t \geq 0$ if and only if

$$\begin{aligned} G^{(4)}(x)(v_1)(v_1)(v_1)(v_1) + 6G'''(x)(v_1)(v_1)(v_2) +\\ + 3G''(x)(v_2)(v_2) + 3G''(x)(v_1)(F(x)) + h'(x)(v_1) = 0. \end{aligned} \tag{4.11}$$

Our result below [7] represents a generalization of the previous theorem for the case of equation (4.1), that is, for the case of higher-order flow-invariance problems.

Theorem 4.7. *(Theorem 4, [7], Theorem 4.4, [6]) Assume that $G : U \to \mathbb{R}^s$ is n times Fréchet differentiable and its first Fréchet derivative $G'(x) : X \to \mathbb{R}^s$ is onto for each $x \in D_G$. Then $M_{D_G}^{(n)}$ is given by*

$$M_{D_G}^{(n)} = \Big\{(x, v_1, ..., v_{n-1}) \in U \times X^{n-1} : G(x) = 0,$$

$$S_j^G(x,v_1,...,v_j)=0, 1\le j\le n-1,$$

$$G'(x)(F(x))+\sum_{k=2}^{n}\frac{n!}{k!}\Big[\sum_{\substack{i_1+...+i_k=n\\ i_1,...,i_k\in\{1,...,n-1\}}}\frac{1}{i_1!\cdots i_k!}G^{(k)}(x)(v_{i_1})\dots(v_{i_k})\Big]=0\Big\}. \tag{4.12}$$

Suppose further that G is $n+1$ times Fréchet differentiable on U, the function $h:U\to\mathbb{R}^s$ given by

$$h(x)=G'(x)(F(x)),\forall x\in U,$$

is Fréchet differentiable, $M_{D_G}^{(n)}$ is nonempty and the mapping $(G'(x)(\cdot),G''(x)(v_1)(\cdot)):X\to\mathbb{R}^s\times\mathbb{R}^s$ is onto for any $(x,v_1,\dots v_{n-1})\in M_{D_G}^{(n)}$.

Then D_G is flow-invariant with respect to the differential equation $u^{(n)}(t)=F(u(t)),t\ge 0$ if and only if

$$h'(x)(v_1)+\sum_{k=3}^{n}\frac{n!}{k!}\Big\{\sum_{\substack{i_1+...+i_k=n\\ i_1,...,i_k\in\{1,...,n-2\}}}\frac{1}{i_1!\cdots i_k!}\Big[G^{(k+1)}(x)(v_1)(v_{i_1})\dots(v_{i_k})+$$

$$+G^{(k)}(x)(v_{i_1+1})(v_{i_2})\dots(v_{i_k})+\dots+G^{(k)}(x)(v_{i_1})\dots(v_{i_{k-1}})(v_{i_k+1})\Big]\Big\}+$$

$$+nG'''(x)(v_1)(v_1)(v_{n-1})+nG''(x)(v_2)(v_{n-1})+nG''(x)(v_1)(F(x))=0. \tag{4.13}$$

Here, $S_j^G(x,v_1,...,v_j)$, $j\ge 1$ denotes the expression

$$S_j^G(x,v_1,...,v_j)=\sum_{k=1}^{j}\frac{j!}{k!}\Big[\sum_{\substack{i_1+...+i_k=j\\ i_1,...,i_k\in\{1,...,j\}}}\frac{1}{i_1!\cdots i_k!}G^{(k)}(x)(v_{i_1})\dots(v_{i_k})\Big].$$

We can derive necessary and sufficient conditions for the flow-invariance of the sphere $S(r)$ of radius $r>0$ centered at origin in an Hilbert space H by applying Theorems 4.6 and 4.7 with $G(x)=\frac{1}{2}(\|x\|^2-r^2),\forall x\in H$ as $S(r)=D_G$.

Corollary 4.8. *Let H be a real Hilbert space of inner product $<,>$ and norm $\|\cdot\|$.*

Then, in the case of the sphere $S(r)=\{x\in H,\|x\|=r\}$, $r>0$, the sets given by (4.9) and (4.12) become respectively

$$M_{S(r)}^{(3)}=\big\{(x,y_1,y_2)\in U\times H\times H,\|x\|=r,<x,y_1>=0,$$

$$\|y_1\|^2+<x,y_2>=0,<x,F(x)>+3<y_1,y_2>=0\big\},\ \textit{and} \tag{4.14}$$

$$M_{S(r)}^{(n)}=\big\{(x,y_1,\dots,y_{n-1})\in U\times H^{n-1},\|x\|=r,<x,y_1>=0,$$

$$\|y_1\|^2+<x,y_2>=0,<x,y_j>+\frac{1}{2}\sum_{k=1}^{j-1}\binom{j}{k}<y_k,y_{j-k}>=0, 3\le j\le n-1,$$

$$<x,F(x)>+\frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}<y_k,y_{n-k}>=0\big\}, n>3. \tag{4.15}$$

Corollary 4.9. *Let $U \subset H$ be an open subset of the Hilbert space H, with $S(r) \subset U$. Assume that $F : U \to H$ is locally Lipschitz and the mapping $h(x) =< x, F(x) >$ is Fréchet differentiable on U.*

Then $S(r)$ is a flow-invariant set with respect to the equation $u'''(t) = F(u(t)), t \geq 0$, if and only if

$$3\|y_2\|^2 + 3 < y_1, F(x) > +h'(x)(y_1) = 0, \forall (x, y_1, y_2) \in M^{(3)}_{S(r)}, \tag{4.16}$$

and $S(r)$ is a flow-invariant set for (4.1) if and only if

$$n < y_2, y_{n-1} > +n < y_1, F(x) > +h'(x)(y_1) = 0, \forall (x, y_1, \ldots, y_{n-1}) \in M^{(n)}_{S(r)}. \tag{4.17}$$

Remark 4.10. The result obtained when the order of the differential equation is equal to three has a geometrical interpretation. Suppose that at the initial moment a mass particle is at a point x in a given set D_G (for example, on a sphere of center the origin and radius r in the three dimensional space in the case $G(x) = \frac{1}{2}(\|x\|^2 - r^2)$, for all $x \in \mathbb{R}^3$), it has a certain velocity v_1 and a certain acceleration v_2, and its trajectory $u(t)$ satisfies the third-order autonomous differential equation under discussion, then the trajectory of the particle under the action of the force field F remains in that set as long as it exists if and only if the necessary and sufficient condition for the flow-invariance of the set with respect to that third-order differential equation are verified (under the hypotheses of Theorem 4.7 upon G, and $n = 3$).

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