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CONVEX KERNELS AND TANGENT CONE CHAIN RULES

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Dedicated to Francis Clarke on the occasion of his 75th birthday

Abstract. In this paper we present formulas for the contingent and adjacent cones to the graph of a composition of set-valued mappings, under conditions involving these cones' convex kernels. Special cases of the formulas include chain rules for epiderivatives of compositions of nonsmooth functions.

Keywords. Adjacent cone; Contingent cone; Convex kernel; Chain rule; Epiderivative.

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1. Introduction

Let X be a real normed space. We say that a set $C \subset X$ is a cone if whenever $x \in C$ and $\lambda > 0$, we have $\lambda x \in C$.

One of the building blocks of nonsmooth and variational analysis is the geometric concept of tangent cone. Tangent cones give local conical approximations to sets. Two approximations of this type that are the most effective—both in terms of having nice analytic properties and providing relatively "close" approximations—are the contingent cone and adjacent cone.

Definition 1.1. Let $S \subset X$ and $x \in S$.

(a) The contingent cone to S at x is defined by

$$T(S,x) := \left\{ z \in X \;\middle|\; \exists \{(t_j, z^j)\} \to (0^+, z) \text{ such that } x + t_j z^j \in S \quad \forall j \right\}.$$

(b) The adjacent cone to S at x is defined by

$$A(S,x) := \left\{ z \in X \;\middle|\; \forall \{t_j\} \to 0^+, \exists \{z^j\} \to z \text{ such that } x + t_j z^j \in S \quad \forall j \right\}.$$

For all $S \subset X$ and $x \in S$, the contingent cone and adjacent cone are closed cones containing 0. From the definitions, we can see that the inclusion $A(S,x) \subset T(S,x)$ is always true. For more

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on the properties and applications of these cones, see [1, 4, 5, 7, 8, 13] and their references. In particular, there are a number of equivalent ways to define these cones [1, §4.1], including

$$T(S,x) = \{z \in X \,\Big|\, \forall \varepsilon > 0, \forall \lambda > 0, \exists t \in (0,\lambda), \exists z' \in N_\varepsilon(z) \text{ such that } x + tz' \in S\Big\};$$

$$A(S,x) = \{z \in X \,\Big|\, \forall \varepsilon > 0, \exists \lambda > 0 \text{ such that } \forall t \in (0,\lambda), \exists z' \in N_\varepsilon(z) \text{ with } x + tz' \in S\Big\};$$

where $N_\varepsilon(x) := \{y \in X \,|\, \|y - x\| < \varepsilon\}$.

This paper investigates the calculus of the contingent and adjacent cones, continuing a study begun in [6]. As in [6], we will establish calculus rules for T and A under hypotheses involving their convex kernels. For $S \subset X$ and $x \in S$, the convex kernels of T and A are defined, respectively, by

$$T^\infty(S,x) := \{z \in X \,|\, y + z \in T(S,x) \quad \forall y \in T(S,x)\}$$

and

$$A^\infty(S,x) := \{z \in X \,|\, y + z \in A(S,x) \quad \forall y \in A(S,x)\}.$$

Since $T(S,x)$ and $A(S,x)$ are closed cones, $T^\infty(S,x)$ and $A^\infty(S,x)$ are always closed convex cones ([1], §4.5.1). Note also that since $0 \in T(S,x)$ and $0 \in A(S,x)$, we have $T^\infty(S,x) \subset T(S,x)$ and $A^\infty(S,x) \subset A(S,x)$.

The usefulness of these convex kernels is enhanced by the fact that direct characterizations have been established for them. Penot gave a sequential characterization of A^∞ in [7] and applied it in studying tangent cone calculus. More recently, Neuhaus [6] has found a sequential characterization of T^∞.

Proposition 1.2. *([7], Proposition 4.6; [6], Theorem 2.1) Let $S \subset X$, $x \in S$. Then*

$$A^\infty(S,x) = \{y \,|\, \forall x^j \to_S x, \forall t_j \to 0^+ \text{ with } \lim_{j\to\infty} (x^j - x)/t_j \in A(S,x),$$
$$\exists y^j \to y \text{ with } x^j + t_j y^j \in S \;\forall j\}$$

and

$$T^\infty(S,x) = \{y \,|\, \forall x^j \to_S x, \forall t_j \to 0^+ \text{ with } (x^j - x)/t_j \text{ convergent},$$
$$\exists t_j' \to 0^+, \exists y^j \to y \text{ with } x + t_j'(y^j + (x^j - x)/t_j) \in S \;\forall j\},$$

where $x^j \to_S x$ means that $x^j \to x$ with each $x^j \in S$.

This paper is organized as follows. In §2 we derive chain rules for the contingent and adjacent cones to the graphs of compositions of set-valued mappings. We apply these chain rules in §3 and §4 to establish chain rules for the contingent and adjacent epiderivatives of compositions of the form $g \circ h$, where in §3 h is Hadamard directionally differentiable, as in [16]; and in §4, g is isotone on a set containing the range of h, as in [14, 15].

We next list some terminology and notation that will be used throughout the paper. For real normed spaces X and Y and a relation or set-valued mapping $G : X \rightrightarrows Y$, the graph of G is the set

$$\operatorname{gph} G := \{(x,y) \in X \times Y \,|\, y \in G(x)\},$$

and the range of G is defined by

$$\operatorname{ran} G := \{y \in Y \,|\, y \in G(x) \text{ for some } x \in X\}.$$

For a tangent cone R and $y \in G(x)$, we will use the shorthand notation

$$R(G;x,y) := R(\operatorname{gph} G,(x,y)).$$

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be the set of extended real numbers. For a function $f : X \to \bar{\mathbb{R}}$, define

$$\operatorname{dom} f := \{x \in X \,|\, f(x) < +\infty\}.$$

We will say that f is *proper* if $\operatorname{dom} f$ is nonempty and f never takes on the value $-\infty$. The epigraph of f is the set

$$\operatorname{epi} f := \{(x,r) \in X \times \mathbb{R} \,|\, f(x) \le r\}.$$

We define the relation $Ef : X \rightrightarrows \mathbb{R}$ by

$$Ef(x) := \{r \,|\, f(x) \le r\}.$$

In other words, EF is the relation whose graph is the epigraph of f. Similarly, for $f = (f_1,\dots,f_n)$, we define $Ef : X \rightrightarrows \mathbb{R}^n$ by

$$Ef(x) := \{r = (r_1,\dots,r_n) \in \mathbb{R}^n \,|\, f_i(x) \le r_i, i = 1,\dots,n\}.$$

2. TANGENT CONE CHAIN RULES

In this section we present chain rules for the contingent and adjacent cones to the graphs of compositions of set-valued mappings. The proofs will be based on a well-established technique [2, 6, 7, 10, 11, 13, 15] utilizing "interior cones" of tangent cones.

Definition 2.1. Let X be a real normed space. Let $S \subset X$ and $x \in S$. Define the tangent cones
(a) $IT(S,x) := \{z \in X \,|\, \exists t_j \to 0^+$ such that $\forall z^j \to z, x + t_j z^j \in S$ for large enough j $\}$.
(b) $IA(S,x) := \{z \in X \,|\, \forall t_j \to 0^+, \forall z^j \to z, x + t_j z^j \in S$ for large enough j $\}$.
(c)

$$IT^\infty(S,x) := \{y \in X \,|\, \forall x^j \to_S x, \forall t_j \to 0^+ \text{ with } (x^j - x)/t_j \text{ convergent}, \exists t_j{}' \to 0^+ \\ \text{such that } \forall y^j \to y, x + t_j{}'(y^j + (x^j - x)/t_j) \in S \text{ for large enough j}\}.$$

(d)

$$IA^\infty(S,x) := \{y \in X \,|\, \forall x^j \to_S x, \forall t_j \to 0^+ \text{ with } \lim_{j\to\infty}(x^j - x)/t_j \in A(S,x), \\ \forall y^j \to y,\, x^j + t_j y^j \in S \text{ for large enough j }\}.$$

For all $S \subset X$ and $x \in S$, $IT(S,x)$, $IA(S,x)$, $IT^\infty(S,x)$, and $IA^\infty(S,x)$ are all (possibly empty) open cones, with $IT^\infty(S,x)$ and $IA^\infty(S,x)$ convex (see [6]). The following inclusions can be established readily from the definitions of these tangent cones.

Proposition 2.2. *([6], Proposition 2.5). Let $S \subset X$ and $x \in S$. Then*
(a) $T(S,x) + IT^\infty(S,x) \subset IT(S,x)$
(b) $T^\infty(S,x) + IT^\infty(S,x) \subset IT^\infty(S,x)$
(c) $A(S,x) + IA^\infty(S,x) \subset IA(S,x)$
(d) $A^\infty(S,x) + IA^\infty(S,x) \subset IA^\infty(S,x)$

We next present some chain rules that follow quickly from the definitions of T, A, IT, and IA. In proving these results, it can be helpful to keep in mind that in Definition 1.1, the existence of a sequence $z^j \to z$ with $x + t_j z^j \in S$ for all j is equivalent to the existence of $z^j \to z$ such that $x + t_j z^j \in S$ for all j large enough. Indeed, suppose that $x + t_j z^j \in S$ for all $j > N$. Then for the sequence $\{y^j\}$ defined by $y^j = 0$ for $j \leq N$ and $y^j = z^j$ for $j > N$, we have $y^j \to z$ and $x + t_j y^j \in S$ for all j.

Proposition 2.3. *Let* $H : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$, $(x,y) \in \operatorname{gph} H$, $(y,z) \in \operatorname{gph} G$. *Then*
(a) $IA(G;y,z) \circ T(H;x,y) \subset T(G \circ H;x,z)$
(b) $A(G;y,z) \circ IT(H;x,y) \subset T(G \circ H;x,z)$
(c) $IT(G;y,z) \circ A(H;x,y) \subset T(G \circ H;x,z)$
(d) $T(G;y,z) \circ IA(H;x,y) \subset T(G \circ H;x,z)$
(e) $IA(G;y,z) \circ A(H;x,y) \subset A(G \circ H;x,z)$
(f) $A(G;y,z) \circ IA(H;x,y) \subset A(G \circ H;x,z)$

Proof. To prove (a), let $(d,r) \in IA(G;y,z) \circ T(H;x,y)$. Then there exists $s \in Y$ such that $(d,s) \in T(H;x,y)$ and $(s,r) \in IA(G;y,z)$. Since $(d,s) \in T(H;x,y)$, there exist sequences $t_j \to 0^+$, $d^j \to d$, $s^j \to s$ with

$$(x,y) + t_j(d^j, s^j) \in \operatorname{gph} H.$$

Now for any $r^j \to r$,

$$(y,z) + t_j(s^j, r^j) \in \operatorname{gph} G$$

for j large enough. In other words, $(x + t_j d^j, y + t_j s^j) \in \operatorname{gph} H$ for all j, and for j large enough, $(y + t_j s^j, z + t_j r^j) \in \operatorname{gph} G$. So

$$(x,z) + t_j(d^j, r^j) \in \operatorname{gph}(G \circ H)$$

for all j large enough, and we have $(d,r) \in T(G \circ H;x,z)$. The proof of (d) is similar.

To prove (b), let $(d,r) \in A(G;y,z) \circ IT(H;x,y)$. Then there exists $s \in Y$ such that $(d,s) \in IT(H;x,y)$ and $(s,r) \in A(G;y,z)$. Since $(d,s) \in IT(H;x,y)$, there exists a sequence $t_j \to 0^+$ such that for all $d^j \to d$, $w^j \to s$, we have

$$(x,y) + t_j(d^j, w^j) \in \operatorname{gph} H$$

for large enough j. For this sequence $\{t_j\}$, there exist $s^j \to s$ and $r^j \to r$ with

$$(y,z) + t_j(s^j, r^j) \in \operatorname{gph} G.$$

So for any $d^j \to d$, both $(x + t_j d^j, y + t_j s^j) \in \operatorname{gph} H$ and $(y + t_j s^j, z + t_j r^j) \in \operatorname{gph} G$ for j large enough. Therefore

$$(x,z) + t_j(d^j, r^j) \in \operatorname{gph}(G \circ H)$$

for all j large enough, and we have $(d,r) \in T(G \circ H;x,z)$. The proof of (c) is similar.

To prove (e), let $(d,r) \in IA(G;y,z) \circ A(H;x,y)$. Then there exists $s \in Y$ such that $(d,s) \in A(H;x,y)$ and $(s,r) \in IA(G;y,z)$. Let $t_j \to 0^+$. Since $(d,s) \in A(H;x,y)$, there exist sequences $d^j \to d$, $s^j \to s$ with

$$(x,y) + t_j(d^j, s^j) \in \operatorname{gph} H.$$

Now for any $r^j \to r$,

$$(y,z) + t_j(s^j, r^j) \in \operatorname{gph} G$$

for j large enough. In other words, $(x + t_j d^j, y + t_j s^j) \in \operatorname{gph} H$ for all j, and for j large enough, $(y + t_j s^j, z + t_j r^j) \in \operatorname{gph} G$. So

$$(x,z) + t_j(d^j, r^j) \in \operatorname{gph}(G \circ H)$$

for all j large enough, and we have $(d,r) \in A(G \circ H;x,z)$. The proof of (f) is similar. □

Proposition 2.3 raises a question: When can we replace IA by A and IT by T in Proposition 2.3? We can use Proposition 2.2 to give one answer to this question.

Theorem 2.4. *Let $H : X \rightrightarrows Y$, $G : Y \rightrightarrows Z$, $(x,y) \in \operatorname{gph} H$, $(y,z) \in \operatorname{gph} G$.*
(a) If

$$IA^{\infty}(G;y,z) \circ T^{\infty}(H;x,y) \neq \emptyset \tag{2.1}$$

or

$$A^{\infty}(G;y,z) \circ IT^{\infty}(H;x,y) \neq \emptyset, \tag{2.2}$$

then

$$A(G;y,z) \circ T(H;x,y) \subset T(G \circ H;x,z). \tag{2.3}$$

(b) If

$$IT^{\infty}(G;y,z) \circ A^{\infty}(H;x,y) \neq \emptyset \tag{2.4}$$

or

$$T^{\infty}(G;y,z) \circ IA^{\infty}(H;x,y) \neq \emptyset, \tag{2.5}$$

then

$$T(G;y,z) \circ A(H;x,y) \subset T(G \circ H;x,z). \tag{2.6}$$

(c) If

$$IA^{\infty}(G;y,z) \circ A^{\infty}(H;x,y) \neq \emptyset \tag{2.7}$$

or

$$A^{\infty}(G;y,z) \circ IA^{\infty}(H;x,y) \neq \emptyset, \tag{2.8}$$

then

$$A(G;y,z) \circ A(H;x,y) \subset A(G \circ H;x,z). \tag{2.9}$$

Proof. To prove (a), let $(d,r) \in A(G;y,z) \circ T(H;x,y)$ and suppose (2.1) holds. Let $(v,w) \in IA^{\infty}(G;y,z) \circ T^{\infty}(H;x,y)$. Then there exists $s \in Y$ with $(d,s) \in T(H;x,y)$, $(s,r) \in A(G;y,z)$, and there exists $u \in Y$ with $(v,u) \in T^{\infty}(H;x,y)$, $(u,w) \in IA^{\infty}(G;y,z)$, For all $t > 0$,

$$(d,s) + t(v,u) \in T(H;x,y)$$

by the definition of T^{∞}; and

$$(s,r) + t(u,w) \in IA(G;y,z)$$

by Proposition 2.2(c). In other words,

$$(d + tv, s + tu) \in T(H;x,y)$$

and

$$(s + tu, r + tw) \in IA(G;x,y),$$

so that

$$(d + tv, r + tw) \in IA(G;y,z) \circ T(H;x,y).$$

By Proposition 2.3(a), it follows that

$$(d + tv, r + tw) \in T(G \circ H;x,z) \quad \forall t > 0,$$

and since $T(G \circ H;x,z)$ is closed, we have $(d,r) \in T(G \circ H;x,z)$. Therefore (2.3) holds. The proof of (2.3) under assumption (2.2) is analogous, using Proposition 2.2(b) and Proposition 2.3(b). The proofs of parts (b) and (c) are also analogous to this one. $\square$

To put the hypotheses of Theorem 2.4 in context, it is helpful to compare the definitions of IA^∞ and IT^∞ with that of the cone of hypertangent directions [2,9,10,11], defined for $S \subset X$ and $x \in S$ by

$$IC(S,x) = \{y \,|\, \forall x^j \to_S x, \forall t_j \to 0^+, \forall y^j \to y, x^j + t_j y^j \in S \text{ for large enough j}\}.$$

Clearly $IC(S,x) \subset IA^\infty(S,x)$. Taking $t_j{}' = t_j$, we also can see that $IC(S,x) \subset IT^\infty(S,x)$. As a result, $IA^\infty(S,x)$ and $IT^\infty(S,x)$ are nonempty whenever $IC(S,x)$ is nonempty. Sets for which $IC(S,x)$ is nonempty are said to be *epi-Lipschitzian at x* ([9], Theorem 3; [11], pp. 20-21).

3. CHAIN RULES FOR CONTINGENT AND ADJACENT EPIDERIVATIVES

In this section and the next one, we consider special cases of Theorem 2.4 for relations whose graphs are epigraphs or graphs of functions. We begin by reviewing the concepts of directional derivative that are defined via tangent cones of epigraphs.

Let R denote some concept of tangent cone. (Examples defined in this paper include $R = T, A, IT, IA, T^\infty, A^\infty, IT^\infty, IA^\infty$.) Suppose that $f : X \to \bar{\mathbb{R}}$ is finite at x. We define the R-epiderivative of f at x in the direction $y \in X$ by

$$f^R(x;y) = \inf\{r \,|\, (y,r) \in R(\operatorname{epi} f, (x, f(x)))\}. \tag{3.1}$$

In cases where R is a closed cone (e.g., $R = T, A, T^\infty, A^\infty$), then (3.1) implies that

$$\operatorname{epi} f^R(x;\cdot) = R(\operatorname{epi} f, (x, f(x))). \tag{3.2}$$

More explicit formulas for a number of epiderivatives can readily be obtained ([1], Chapter 6; [13], Theorem 5.4). In particular,

$$f^T(x;y) = \sup_{\varepsilon>0} \sup_{\lambda>0} \inf_{0<t<\lambda} \inf_{\|v-y\|<\varepsilon} (f(x+tv) - f(x))/t;$$
$$f^A(x;y) = \sup_{\varepsilon>0} \inf_{\lambda>0} \sup_{0<t<\lambda} \inf_{\|v-y\|<\varepsilon} (f(x+tv) - f(x))/t;$$
$$f^{IA}(x;y) = \inf_{\varepsilon>0} \inf_{\lambda>0} \sup_{0<t<\lambda} \sup_{\|v-y\|<\varepsilon} (f(x+tv) - f(x))/t;$$
$$f^{IT}(x;y) = \inf_{\varepsilon>0} \sup_{\lambda>0} \inf_{0<t<\lambda} \sup_{\|v-y\|<\varepsilon} (f(x+tv) - f(x))/t.$$

In the case where f is Lipschitzian near x, then these epiderivatives reduce to the familiar lower and upper Dini directional derivatives ([13], Theorem 5.8); that is,

$$f^T(x;y) = f^{IT}(x;y) = \sup_{\lambda>0} \inf_{0<t<\lambda} (f(x+ty) - f(x))/t$$

and

$$f^A(x;y) = f^{IA}(x;y) = \inf_{\lambda>0} \sup_{0<t<\lambda} (f(x+ty) - f(x))/t.$$

With the help of Theorem 2.4, we can derive calculus rules for A- and T-epiderivatives that are valid for large classes of non-Lipschitzian functions. In this section we examine compositions of the form $g \circ h$, where h is Hadamard directionally differentiable.

Definition 3.1. The function $f : X \to Y$ is said to Hadamard directionally differentiable at $\bar{x} \in X$ in the direction $x \in X$ if the Hadamard directional derivative

$$f'(\bar{x};x) = \lim_{t\downarrow 0, v\to x} (f(\bar{x}+tv) - f(\bar{x}))/t$$

exists as an element of Y.

Much information about Hadamard directional differentiability (also known as semidifferentiability) can be found in [3, 12]. From the definitions of the contingent and adjacent cones, one can quickly deduce the following fact:

Proposition 3.2. *(see e.g. [16]) Let $f : X \to Y$ be Hadamard directionally differentiable at $x \in X$ in the direction $y \in X$. Then for the graph of f, denoted by*

$$\operatorname{gph} f := \{(x,r) \in \mathbb{R}^m \times \mathbb{R} \,|\, f(x) = r\},$$

we have

$$\{L \,|\, (y,L) \in T(\operatorname{gph} f, (x, f(x)))\} = \{L \,|\, (y,L) \in A(\operatorname{gph} f, (x, f(x)))\} = \{f'(x;y)\}. \tag{3.3}$$

We next establish an inclusion involving compositions $G \circ H$, where H is the relation whose graph is $\operatorname{gph} h$ for a function h that is Hadamard directionally differentiable at x in some direction.

Proposition 3.3. *Let $h : X \to Y$ be Hadamard directionally differentiable at $x \in X$ in the direction $d \in X$. Let $G : Y \rightrightarrows Z$ with $z \in G(h(x))$, and let H be the relation whose graph is* $\operatorname{gph} h$.
(a) If $(d,r) \in A(G \circ H; x, z)$, then $(d,r) \in A(G; h(x), z) \circ A(H; x, h(x))$.
(b) If $(d,r) \in T(G \circ H; x, z)$, then $(d,r) \in T(G; h(x), z) \circ T(H; x, h(x))$.

Proof. To prove (a), let $(d,r) \in A(G \circ H; x, z)$, and let $t_j \to 0^+$. There exist sequences $d^j \to d$, $r^j \to r$ with

$$(x,z) + t_j(d^j, r^j) \in \operatorname{gph}(G \circ H).$$

Then for each j, there exists s^j such that

$$(x + t_j d^j, s^j) \in \operatorname{gph} H$$

and

$$(s^j, z + t_j r^j) \in \operatorname{gph} G.$$

By the definition of H, $s^j = h(x + t_j d^j)$.

Now define $y^j = (s^j - h(x))/t_j$. Since h is Hadamard directionally differentiable at x in the direction d, $y^j \to y := h'(x;d)$. It follows that

$$(x, h(x)) + t_j(d^j, y^j) \in \operatorname{gph} H$$

and

$$(h(x), z) + t_j(y^j, r^j) \in \operatorname{gph} G,$$

which means that $(d,y) \in A(H; x, h(x))$ and $(y,r) \in A(G; h(x), z)$. Therefore

$$(d,r) \in A(G; h(x), z) \circ A(H; x, h(x)).$$

The proof of (b) is analogous to this one. □

One consequence of Proposition 3.3 is an inequality for compositions of functions $g \circ h$, where h is Hadamard directionally differentiable at x in some direction.

Corollary 3.4. *Let $h : X \to Y$ be Hadamard directionally differentiable at $\bar{x} \in X$ in the direction $x \in X$. Let $g : Y \to \bar{\mathbb{R}}$ be finite at $h(\bar{x})$. Then*

$$(g \circ h)^A(\bar{x};x) \geq g^A(h(\bar{x});h'(\bar{x};x)). \tag{3.4}$$

and

$$(g \circ h)^T(\bar{x};x) \geq g^T(h(\bar{x});h'(\bar{x};x)). \tag{3.5}$$

Proof. Let H be the relation whose graph is $\operatorname{gph} h$, and let $G : Y \rightrightarrows \mathbb{R}$ be the relation Eg whose graph is $\operatorname{epi} g$. Suppose that $(g \circ h)^A(\bar{x};x) \leq r$. Then

$$(x,r) \in A(\operatorname{epi}(g \circ h), (\bar{x}, (g \circ h)(\bar{x}))) = A(G \circ H; \bar{x}, (g \circ h)(\bar{x})).$$

By Proposition 3.3, there exists y such that

$$(x,y) \in A(\operatorname{gph} h, (\bar{x}, h(\bar{x})))$$

and

$$(y,r) \in A(\operatorname{epi} g, (h(\bar{x}), (g \circ h)(\bar{x}))).$$

Since $y = h'(\bar{x};x)$ by Proposition 3.2, we have $g^A(h(\bar{x});h'(\bar{x};x)) \leq r$, and (3.4) holds. The proof of (3.5) is analogous to that of (3.4). □

We can use Theorem 2.4 to find conditions which guarantee equality in inequalities (3.4) and (3.5).

Theorem 3.5. *Let $h : X \to Y$ be Hadamard directionally differentiable at $\bar{x} \in X$ in the direction $x \in X$. Let $g : Y \to \bar{\mathbb{R}}$ be finite at $h(\bar{x})$.*

(a) If $\operatorname{dom} g^{IT^\infty}(h(\bar{x});\cdot) \neq \emptyset$, *then*

$$(g \circ h)^T(\bar{x};x) = g^T(h(\bar{x});h'(\bar{x};x)). \tag{3.6}$$

(b) If $\operatorname{dom} g^{IA^\infty}(h(\bar{x});\cdot) \neq \emptyset$, *then*

$$(g \circ h)^A(\bar{x};x) = g^A(h(\bar{x});h'(\bar{x};x)). \tag{3.7}$$

Proof. Define $H : X \rightrightarrows Y$ and $G : Y \rightrightarrows \mathbb{R}$ such that $\operatorname{gph} H = \operatorname{gph} h$ and $G = Eg$. To prove (a), suppose that $\operatorname{dom} g^{IT^\infty}(h(\bar{x});\cdot) \neq \emptyset$. It suffices to show that

$$(g \circ h)^T(\bar{x};x) \leq g^T(h(\bar{x});h'(\bar{x};x)). \tag{3.8}$$

Suppose $g^T(h(\bar{x});h'(\bar{x};x) \leq r$. Then

$$(\bar{x}, h'(\bar{x};x) \in A(H;\bar{x},h(\bar{x}))$$

and

$$(h'(\bar{x};x),r) \in T(G;h(\bar{x}),g(h(\bar{x}))),$$

so $(x,r) \in T(G;h(\bar{x}),g(h(\bar{x}))) \circ A(H;\bar{x},h(\bar{x}))$. Since $\operatorname{dom} g^{IT^\infty}(h(\bar{x});\cdot) \neq \emptyset$, either $g^{IT^\infty}(h(\bar{x});0) = 0$ or $g^{IT^\infty}(h(\bar{x});0) = -\infty$. In either case, there exists $s \in \mathbb{R}$ with $(0,s) \in IT^\infty(G;h(\bar{x}),g(h(\bar{x})))$. Since $(0,0) \in A^\infty(H;\bar{x},h(\bar{x})))$, it follows that

$$(0,s) \in IT^\infty(G;h(\bar{x}),g(h(\bar{x}))) \circ A^\infty(H;\bar{x},h(\bar{x}))).$$

By Theorem 2.4(b), we have $(x,r) \in T(G \circ H;\bar{x},g(h(\bar{x})))$. Therefore $(g \circ h)^T(\bar{x};x) \leq r$, and (3.8) holds. The proof of (b) is analogous to this one. □

Remark 3.6.

(a) Functions $f : Y \to \bar{\mathbb{R}}$ for which

$$\operatorname{dom} f^{IC}(y;\cdot) \neq \emptyset$$

are said to directionally Lipschitzian at y [2,10,11]. This class of functions includes locally Lipschitzian functions and other large classes of nonsmooth functions. Since the cones IT^{∞} and IA^{∞} are at least as large as IC, the hypotheses of Theorem 3.5 are satisfied when g is directionally Lipschitzian.

(b) Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \\ 2^{-n-1} & \text{if } 2^{-n-1} < |x| \leq 2^{-n},\ n = 0, \pm 1, \pm 2, \ldots \end{cases}.$$

Then

$$IC(\operatorname{epi} f, (0,0)) = \emptyset,$$

so f is not directionally Lipschitzian at 0. On the other hand,

$$IA^{\infty}(\operatorname{epi} f, (0,0)) = \{(x,y) \,|\, y > |x|\}$$

and

$$IT^{\infty}(\operatorname{epi} f, (0,0)) = \{(x,y) \,|\, y > |x|/2\},$$

so

$$\operatorname{dom} f^{IT^{\infty}}(0;\cdot) = \operatorname{dom} f^{IA^{\infty}}(0;\cdot) = \mathbb{R}$$

for this example.

4. COMPOSITIONS WITH ISOTONE FUNCTIONS

Definition 4.1. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be elements of $\mathbb{R}^n$. We say $x \leq y$ if $x_i \leq y_i$ for each i. The function $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be isotone on $S \subset \mathbb{R}^n$ if $g(x) \leq g(y)$ whenever $x, y \in S$ and $x \leq y$.

In this section we present chain rules for compositions $g \circ h$, where $h = (h_1, \ldots, h_n)$, each $h_i : X \to \mathbb{R}$, and $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ is isotone on $\operatorname{ran}(Eh)$. In such compositions we adopt the convention that $(g \circ h)(x) = +\infty$ if at least one $h_i(x) = +\infty$. Compositions of this form have the following key property:

Lemma 4.2. *Let $h_i : X \to \mathbb{R}$, $i = 1, \ldots, n$, and suppose that $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ is isotone on* $\operatorname{ran}(Eh)$*, where $h = (h_1, \ldots, h_n)$. Then*

$$\operatorname{epi}(g \circ h) = \operatorname{gph}(Eg \circ Eh). \tag{4.1}$$

Proof. Let $(x,r) \in \operatorname{epi}(g \circ h)$. Then $g(h(x)) \leq r$, and so $h(x) \in Eh(x)$ with $r \in Eg(h(x))$, which means $(x,r) \in \operatorname{gph}(Eg \circ Eh)$. On the other hand, suppose $(x,r) \in \operatorname{gph}(Eg \circ Eh)$. Then there exists $y \in \mathbb{R}^n$ such that $y \in Eh(x)$ and $r \in Eg(y)$. Since g is isotone on $\operatorname{ran} Rh$,

$$g(h(x)) \leq g(y) \leq r,$$

and so $(x,r) \in \operatorname{epi}(g \circ h)$. Therefore (4.1) holds. □

It is also important to note that $T-$ and $A-$epiderivatives of isotone functions are themselves isotone.

Lemma 4.3. *Let $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ be isotone on $N_\varepsilon(x)$ for some $x \in \mathbb{R}^n$ and $\varepsilon > 0$. Then $g^T(x;\cdot)$ and $g^A(x;\cdot)$ are isotone on $\mathbb{R}^n$.*

Proof. Let $y,z \in \mathbb{R}^n$ with $y \le z$. To show that $g^A(x;\cdot)$ is isotone, suppose that $g^A(x;z) \le r$. Then by (3.2), $(z,r) \in A(\operatorname{epi} g,(x,g(x)))$. Let $t_j \to 0^+$. There exist $z^j \to z$, $r^j \to r$ such that $(g(x+t_jz^j)-g(x))/t_j \le r^j$. Now define $y^j := y+z^j-z$. Then $y^j \to y$, and $y^j \le z^j$ for each j. For j large enough, $x+t_jy^j$ and $x+t_jz^j$ are in $N_\varepsilon(x)$. So for j large enough,

$$(g(x+t_jy^j)-g(x))/t_j \le (g(x+t_jz^j)-g(x))/t_j \le r^j,$$

and $(y,r) \in A(\operatorname{epi} g,(x,g(x)))$. Therefore $g^A(x;y) \le r$, and we have $g^A(x;\cdot)$ isotone on $\mathbb{R}^n$. The proof of the isotonicity of $g^T(x;\cdot)$ is analogous to this one. □

Using the definitions of the $A-$ and $T-$epiderivatives, one can establish some chain rule inequalities for compositions with isotone functions. In Proposition 4.4 below, inequalities (4.2) and (4.4) are demonstrated in Proposition 6.1 of [17], and the proof of (4.3) is analogous to those of (4.2) and (4.4).

Proposition 4.4. *([17], Proposition 6.1) Let $h_i : X \to \bar{\mathbb{R}}$, $i = 1,\ldots,n$ be finite at $x \in X$, and define $h(x) = (h_1(x),\ldots,h_n(x))$. Let $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ be isotone on $\operatorname{ran}(Eh) \cup N_\varepsilon(h(x))$ for some $\varepsilon > 0$. Then if $y \in X$ is such that ${h_i}^T(x;y)$, $i = 1,\ldots,n$ are finite,*

$$(g\circ h)^T(x;y) \ge g^T(h(x);{h_1}^T(x;y),\ldots,{h_n}^T(x;y)). \tag{4.2}$$

and

$$(g\circ h)^A(x;y) \ge g^A(h(x);{h_1}^T(x;y),\ldots,{h_n}^T(x;y)). \tag{4.3}$$

Alternatively, if ${h_1}^A(x;y), {h_2}^T(x;y),\ldots,{h_n}^T(x;y)$ are finite, then

$$(g\circ h)^A(x;y) \ge g^T(h(x);{h_1}^A(x;y),{h_2}^T(x;y)\ldots,{h_n}^T(x;y)). \tag{4.4}$$

We can apply Theorem 2.4 to prove inequalities that go the opposite direction under appropriate conditions. We look first at the case where $n = 1$.

Theorem 4.5. *Let $h : X \to \bar{\mathbb{R}}$ be finite at $\bar{x}$ and $g : \mathbb{R} \to \bar{\mathbb{R}}$ finite at $h(\bar{x})$ and isotone on $\operatorname{ran}(Eh) \cup N_\varepsilon(h(\bar{x}))$ for some $\varepsilon > 0$.*

(a) If $h^T(\bar{x};\cdot)$ is proper and

$$\operatorname{dom} g^{IA^\infty}(h(\bar{x});\cdot) \cap \{s \,|\, h^{T^\infty}(\bar{x};v) < s \text{ for some } v \in X\} \ne \emptyset \tag{4.5}$$

or

$$\operatorname{dom} g^{A^\infty}(h(\bar{x});\cdot) \cap \{s \,|\, h^{IT^\infty}(\bar{x};v) < s \text{ for some } v \in X\} \ne \emptyset, \tag{4.6}$$

then for all $y \in X$,

$$(g\circ h)^T(\bar{x};y) \le g^A(h(\bar{x});h^T(\bar{x};y)). \tag{4.7}$$

Equality holds in (4.7) if $h^T(\bar{x};y)$ is finite and $g^A(h(\bar{x});h^T(\bar{x};y)) = g^T(h(\bar{x});h^T(\bar{x};y))$.
(b) If $h^A(\bar{x};\cdot)$ is proper and

$$\operatorname{dom} g^{IT^\infty}(h(\bar{x});\cdot) \cap \{s \,|\, h^{A^\infty}(\bar{x};v) < s \text{ for some } v \in X\} \ne \emptyset \tag{4.8}$$

or

$$\operatorname{dom} g^{T^\infty}(h(\bar{x});\cdot) \cap \{s \,|\, h^{IA^\infty}(\bar{x};v) < s \text{ for some } v \in X\} \ne \emptyset, \tag{4.9}$$

then for all $y \in X$,

$$(g\circ h)^T(\bar{x};y) \le g^T(h(\bar{x});h^A(\bar{x};y)). \tag{4.10}$$

Equality holds in (4.10) if $h^T(\bar{x};y)$ *is finite and* $h^T(\bar{x};y) = h^A(\bar{x};y)$.
(c) If $h^A(\bar{x};\cdot)$ *is proper and*

$$\operatorname{dom} g^{IA^\infty}(h(\bar{x});\cdot) \cap \{s \,|\, h^{A^\infty}(\bar{x};v) < s \text{ for some } v \in X\} \neq \emptyset \tag{4.11}$$

or

$$\operatorname{dom} g^{A^\infty}(h(\bar{x});\cdot) \cap \{s \,|\, h^{IA^\infty}(\bar{x};v) < s \text{ for some } v \in X\} \neq \emptyset, \tag{4.12}$$

then for all $y \in X$,

$$(g \circ h)^A(\bar{x};y) \le g^A(h(\bar{x});h^A(\bar{x};y)). \tag{4.13}$$

Equality holds in (4.13) if $h^T(\bar{x};y)$ *is finite and* $h^T(\bar{x};y) = h^A(\bar{x};y)$.

Proof. In Theorem 2.4 let $H := Eh$, $G := Eg$, $x := \bar{x}$, $y := h(\bar{x})$, and $z := g(h(\bar{x}))$. To prove (a), let $y \in X$ and $r \in \mathbb{R}$, and suppose that $h^T(\bar{x};\cdot)$ is proper and

$$g^A(h(\bar{x});h^T(\bar{x};y)) \le r.$$

Then $h^T(\bar{x};y)$ is finite, $(y,h^T(\bar{x};y)) \in T(H;\bar{x},h(\bar{x}))$ and $(h^T(\bar{x};y),r) \in A(G;h(\bar{x}),g(h(\bar{x})))$, so

$$(y,r) \in A(G;h(\bar{x}),g(h(\bar{x}))) \circ T(H;\bar{x},h(\bar{x})). \tag{4.14}$$

If (4.5) holds, then there exist $v \in X$ and $s,d \in \mathbb{R}$ with $(v,s) \in T^\infty(H,\bar{x},h(\bar{x}))$ and $(s,d) \in IA^\infty(G;h(\bar{x}),g(h(\bar{x})))$, so that (2.1) is satisfied. By Theorem 2.4, (2.3) holds, so (4.14) implies that

$$(y,r) \in T(G \circ H;(\bar{x},g(h(\bar{x}))).$$

By Lemma 4.2, we then have $(y,r) \in T(\operatorname{epi}(g \circ h),(\bar{x},g(h(\bar{x})))$, which means that

$$(g \circ h)^T(\bar{x};y) \le r.$$

Therefore (4.7) holds. The condition for equality in (4.7) follows from Proposition 4.4. The proofs of the other parts of the Theorem are analogous to this one. □

For the $n > 1$ case, we can establish chain rules via the strategy outlined in [15]. In implementing that strategy, we will make use of facts about tangent cones of finite intersections of sets.

Lemma 4.6. *([15], Lemma 3.2) Let* $S_i \subset X$, $i = 1,\ldots,n$, *and* $x \in \cap_{i=1}^n S_i$. *If*

$$A^\infty(S_1,x) \cap \cap_{i=2}^n IA^\infty(S_i,x) \neq \emptyset, \tag{4.15}$$

then

$$\cap_{i=1}^n A(S_i,x) = A(\cap_{i=1}^n S_i,x) \tag{4.16}$$

and

$$\cap_{i=1}^n A^\infty(S_i,x) \subset A^\infty(\cap_{i=1}^n S_i,x). \tag{4.17}$$

Theorem 4.7. *Let* $h_i : X \to \bar{\mathbb{R}}$ *be finite at* $\bar{x}$ *and proper, and let* $h = (h_1,\ldots,h_n)$. *Let* $g : \mathbb{R}^n \to \mathbb{R}$ *be finite at* $h(\bar{x})$ *and isotone on* $\operatorname{ran}(Eh) \cup N_\varepsilon(h(\bar{x}))$ *for some* $\varepsilon > 0$. *Suppose that* ${h_i}^A(\bar{x};\cdot)$ *is proper for each i; that*

$$\operatorname{dom} {h_1}^{A^\infty}(\bar{x};\cdot) \cap \cap_{i=2}^n \operatorname{dom} {h_i}^{IA^\infty}(\bar{x};\cdot) \neq \emptyset; \tag{4.18}$$

and that for some $v \in X$ *and* $s_i \ge {h_i}^{A^\infty}(\bar{x};v)$, $i = 1,\ldots,n$,

$$(s_1,\ldots,s_n) \in \operatorname{dom} g^{IT^\infty}(h(\bar{x};\cdot).$$

Then for all $y \in X$,

$$(g \circ h)^T(\bar{x}; y) \leq g^T(h(\bar{x}); h_1^A(\bar{x}; y), \ldots, h_n^A(\bar{x}; y)). \tag{4.19}$$

Equality holds in (4.19) if each $h_i^A(\bar{x}; y)$ is finite and $h_i^A(\bar{x}; y) = h_i^T(\bar{x}; y)$, $i = 1, \ldots, n$.

Proof. In Theorem 2.4 let $H := Eh$, $G := Eg$, $x := \bar{x}$, $y := h(\bar{x})$, and $z := g(h(\bar{x}))$. Set

$$\Omega := \{(y, r) \mid g^T(h(\bar{x}); h_1^A(\bar{x}; y), \ldots, h_n^A(\bar{x}; y) \leq r\}.$$

To prove (4.19), we want to show that $\Omega \subset \operatorname{epi}(g \circ h)^T(\bar{x}; \cdot)$, which is equivalent to showing $\Omega \subset T(G \circ H; \bar{x}, (g \circ h)(\bar{x}))$ since

$$\operatorname{epi}(g \circ h)^T(\bar{x}; \cdot) = T(Eg \circ Eh; \bar{x}, (g \circ h)(\bar{x})) = T(G \circ H; \bar{x}, (g \circ h)(\bar{x})).$$

Since $g^T(\bar{x}; \cdot)$ is isotone on $\mathbb{R}^n$ and each ${h_i}^A(\bar{x}; \cdot)$ is proper, we have

$$\Omega = \{(y, r) \mid \exists d \in \mathbb{R}^n \text{ with } g^T(h(\bar{x}); d) \leq r, (h_1^A(\bar{x}; y), \ldots, h_n^A(\bar{x}; y)) \leq d\}.$$

As in [15], for $i = 1, \ldots, n$ define

$$S_i := \{(x, y_1, \ldots, y_n) \mid h_i(x) \leq y_i\},$$

and let $S = \cap_{i=1}^n S_i$. Note that $S = \operatorname{gph} H$. From the definition in (3.1), one can readily verify that for $i = 1, \ldots, n$,

$$\{(y, d_1, \ldots, d_n) \mid {h_i}^A(\bar{x}; y) \leq d_i\} \subset A(S_i, (\bar{x}, h(\bar{x})));$$
$$\{(y, d_1, \ldots, d_n) \mid {h_i}^{A^\infty}(\bar{x}; y) \leq d_i\} \subset A^\infty(S_i, (\bar{x}, h(\bar{x})));$$

and

$$\{(y, d_1, \ldots, d_n) \mid {h_i}^{IA^\infty}(\bar{x}; y) < d_i\} \subset IA^\infty(S_i, (\bar{x}, h(\bar{x}))).$$

Assumption (4.18) then implies that

$$A^\infty(S_1, (\bar{x}, h(\bar{x}))) \cap \cap_{i=2}^n IA^\infty(S_i, (\bar{x}, h(\bar{x}))) \neq \emptyset,$$

so that by Lemma 4.6,

$$\{(y, d_1, \ldots, d_n) \mid {h_i}^{A^\infty}(\bar{x}; y) \leq d_i, i = 1, \ldots, n\} \subset \cap_{i=1}^n A^\infty(S_i, (\bar{x}, h(\bar{x}))) \subset A^\infty(H; \bar{x}, h(\bar{x})) \tag{4.20}$$

and

$$\{(y, d_1, \ldots, d_n) \mid {h_i}^A(\bar{x}; y) \leq d_i, i = 1, \ldots, n\} \subset \cap_{i=1}^n A(S_i, (\bar{x}, h(\bar{x}))) = A(H; \bar{x}, h(\bar{x})). \tag{4.21}$$

By (4.20), our assumptions also imply that

$$IT^\infty(G; h(\bar{x}), g(h(\bar{x}))) \circ A^\infty(H; \bar{x}, h(\bar{x})) \neq \emptyset,$$

so that we may apply Theorem 2.4(b). We then have

$$\begin{aligned}
\Omega &= \{(y, r) \mid \exists d \in \mathbb{R}^n \text{ with } g^T(h(\bar{x}); d) \leq r, (h_1^A(\bar{x}; y), \ldots, h_n^A(\bar{x}; y)) \leq d\} \\
&\subset \{(y, r) \mid \exists d \in \mathbb{R}^n \text{ with } g^T(h(\bar{x}); d) \leq r, (y, d) \in A(H; \bar{x}, h(\bar{x}))\} \text{ by (4.21)} \\
&= T(G; h(\bar{x}), g(h(\bar{x})) \circ A(H; \bar{x}, h(\bar{x})) \\
&\subset T(G \circ H, \bar{x}, g(h(\bar{x}))),
\end{aligned}$$

as desired. Equality in (4.19) under the given conditions follows from Proposition 4.4. □

Remark 4.8. Some of the possible choices for g in Theorem 4.7 are $g(y_1,\ldots,y_n) := \Sigma y_i$, $g(y_1,\ldots,y_n) := \max_{i\leq i\leq n} y_i$, and for $y \geq 0$, $g(y_1,\ldots,y_n) := \Pi y_i$.

5. CONCLUSION

The availability of intrinsic characterizations of the convex kernels of the contingent and adjacent cones facilitates the development of chain rules for the contigent and adjacent cones of compositions of set-valued mappings. Special cases include a variety of chain rules for contingent and adjacent epiderivatives of functions that are valid for large classes of functions.

The calculus of contingent and adjacent cones and epiderivatives can also be developed under metric subregularity assumptions, as in [4]. Those assumptions are neither weaker nor stronger, in general, than the ones used here involving convex kernels. Examples 2.10 and 2.11 in [6] give instances where one set of assumptions is satisfied and the other is not.

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