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# INTEGRO-DIFFERENTIAL SWEEPING PROCESS APPROACH TO FRICTIONLESS CONTACT AND INTEGRO-DIFFERENTIAL COMPLEMENTARITY PROBLEMS

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Dedicated to the memory of Professor Jack Warga on the occasion of his 100th birthday

**Abstract.** The paper is concerned with integro-differential complementarity systems and frictionless contact problems. Well-posedness results of such problems are established by using an integro-differential sweeping process.

**Keywords.** Contact problem; Differential complementarity systems; Moreau's sweeping process; Prox-regular sets; Volterra integro-differential equation.

**2020 Mathematics Subject Classification.** 47J20, 49J40.

## 1. Introduction

The present paper continues our studies in [4, 5]. Our main aim here is to show the well-posedness of some important differential complementarity problems and frictionless contact problems by utilizing our recent well-posedness theorem in [4] on integro-differential sweeping process.

The sweeping process was introduced and largely treated by J. J. Moreau in a series of papers, in particular in [12, 13]. It was shown in [12] that such processes play a fundamental role in mechanics, especially in elasto-plasticity, quasi-statics, dynamics. The mathematical model of the sweeping process (see [12, 13]) corresponds to a point which is swept by a moving closed convex set $C(t)$ in a Hilbert space $H$ according to the differential inclusion

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) & a.e.\ t \in [T_0, T] \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{1.1}$$

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where $T_0, T \in \mathbb{R}$ with $0 \le T_0 < T$ and $N_{C(t)}(\cdot)$ denotes the normal cone of $C(t)$ (in the standard sense). The analysis of systems with external forces (see, e.g., [7, 20] and [6] for more details) led to consider and analyze the following perturbed variant

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f(t,x(t)) & a.e.\ t \in [T_0,T] \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{1.2}$$

where $f : [T_0,T] \times H \to H$ is a Carathéodory mapping, i.e., $f(t,\cdot)$ is continuous and $f(\cdot,x)$ is Bochner measurable for $[T_0,T]$ endowed with the Borel $\sigma$-field $\mathscr{B}([T_0,T])$. By Bochner measurable mapping we mean here any limit of uniformly convergent sequence of simple mappings from $[T_0,T]$ into $H$ with $[T_0,T]$ endowed with its Borel $\sigma$-field.

In [4, 5] we studied a general *integro-differential sweeping process of Volterra type* in the form $(P_{f_1,f_2})$,

$$\begin{cases} -\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t,x(t)) + \displaystyle\int_{T_0}^{t} f_2(t,s,x(s))ds & \text{a.e. } t \in [T_0,T] \\ x(T_0) = x_0 \in C(T_0), \end{cases} \tag{1.3}$$

where $C(t)$ is a prox-regular moving set of the Hilbert space $H$ (see Section 2 for the definition of such sets) and $N_{C(t)}(\cdot)$ is its normal cone. Many differential complementarity problems have been studied through sweeping processes in the form (1.2) by, e.g., Adly [1] and Adly et al [2, 3]. Sofonea et al [9, 11, 16, 17, 18] largely employed the sweeping process (1.2) to analyze several types of contact mechanical problems. In this work, we show how our analysis in [4] of integro-differential sweeping process of type (1.3) furnishes an efficient approach to study integro-differential complementarity problems and frictionless contact mechanical problems.

## 2. Well-posedness of integro-differential sweeping processes

Throughout the paper, $H$ denotes a Hilbert space endowed with its inner product $\langle \cdot,\cdot \rangle$ and associated norm $\|\cdot\|$. We use standard notation with $B_H(x,\delta)$ (resp. $B_H[x,\delta]$) as the open (resp. closed) ball around $x \in H$ with radius $\delta > 0$. When there is no risk of ambiguity, we will remove the subscript $H$. Let $S$ be a subset of $H$. Considering the distance function and the projection multimapping

$$d_S(x) = d(x,S) := \inf_{y \in S} \|x-y\| \quad \text{and} \quad \Pi_S(x) := \left\{ w \in S \,\middle|\, \|x-w\| = d_S(x) \right\}, \quad \text{for all } x \in H,$$

the basic *proximal normal cone* $N_S(x)$ to $S$ at $x$ can be defined by

$$N_S(x) := \left\{ v \in \mathbb{R}^n \middle|\, \exists \alpha > 0 \ \text{ such that } \ x \in \Pi_S(x + \alpha v) \right\} \ \text{ if } \ x \in S \tag{2.1}$$

and by $N_S(x) := \emptyset$ otherwise, see e.g., [15, 19] for equivalent descriptions and further references. Notice that the cone $N_S(x)$ is convex but it may not be closed.

Based on (2.1), we describe now the concept of prox-regularity of sets, it plays a fundamental role in variational analysis and the theory of sweeping processes. The set $S$ in $H$ is said to be *r-prox-regular* for an extended real $r \in ]0,\infty]$, if it is closed and if for all $x \in \operatorname{bd} S$ (the boundary of $S$) and $v \in N_S(x)$ with $\|v\| = 1$ one has

$$B_H[x + \eta v, \eta] \cap S = \{x\} \quad \text{for every real } \eta \in ]0,r[,$$

which is equivalent to

$$\langle v, y-x\rangle \le \frac{|v|}{2r}|y-x|^2 \ \text{ for all } \ y\in S,\ x\in \operatorname{bd} S \ \text{ and } \ v\in N_S(x). \tag{2.2}$$

We refer the reader to [14, 15, 19] and the references therein for more details and several references. It is also worth mentioning that any closed convex subset in $H$ is $r$-prox-regular with $r=\infty$.

Next, we present our important result in [4] ensuring the existence and uniqueness issues for the generalized integro-differential sweeping process (1.3). The result will be used several times in the sequel.
Our study in [4] of problem (1.3) is based on a new Gronwall-like differential inequality and on a new semi-discretization method for perturbed sweeping processes. Recently, in another paper [5] we also proved the solvability of another version of the same problem by setting up an appropriate catching-up algorithm (full discretization).

Consider for a multimapping $C:[0,T]\rightrightarrows H$ and mappings $f_1$, $f_2$ the following assumptions:

($\mathscr{H}_1$) For each $t\in I:=[T_0,T]$, $C(t)$ is a nonempty closed subset of $H$ which is $r$-prox-regular for some constant $r\in ]0,+\infty]$, and has an absolutely continuous variation in the sense that there is some absolutely continuous function $\upsilon:[T_0,T]\longrightarrow\mathbb{R}$ such that

$$C(t)\subset C(s)+|\upsilon(t)-\upsilon(s)|B_H[0,1], \quad \forall\, t,s\in[T_0,T]. \tag{2.3}$$

($\mathscr{H}_2$) The mapping $f_1:[T_0,T]\times H\longrightarrow H$ is Bochner measurable in time (i.e., $f(\cdot,x)$ is Bochner measurable for each $x\in H$), and such that

($\mathscr{H}_{2,1}$) there exists a non-negative function $\beta_1(\cdot)\in L^1([T_0,T],\mathbb{R})$ such that

$$\|f_1(t,x)\|\le\beta_1(t)(1+\|x\|), \text{ for all } t\in[T_0,T] \text{ and for any } x\in\bigcup_{t\in[T_0,T]}C(t).$$

($\mathscr{H}_{2,2}$) for each real $\eta>0$ there exists a non-negative function $L_1^{\eta}(\cdot)\in L^1([T_0,T],\mathbb{R})$ such that for any $t\in[T_0,T]$ and for any $(x,y)\in B_H[0,\eta]\times B_H[0,\eta]$,

$$\|f_1(t,x)-f_1(t,y)\|\le L_1^{\eta}(t)\|x-y\|.$$

($\mathscr{H}_3$) The mapping $f_2:[T_0,T]^2\times H\longrightarrow H$ is Bochner measurable in $(s,t)\in[T_0,T]^2$ (i.e., $f_2(\cdot,\cdot,x)$ is Bochner measurable on $[T_0,T]^2$ for each $x\in H$) and such that

($\mathscr{H}_{3,1}$) there exists a non-negative function $\beta_2(\cdot,\cdot)\in L^1(Q_\Delta,\mathbb{R})$ such that

$$\|f_2(t,s,x)\|\le\beta_2(t,s)(1+\|x\|), \text{ for all } (t,s)\in Q_\Delta \text{ and for any } x\in\bigcup_{t\in[T_0,T]}C(t).$$

($\mathscr{H}_{3,2}$) for each real $\eta>0$ there exists a non-negative function $L_2^{\eta}(\cdot)\in L^1([T_0,T],\mathbb{R})$ such that for all $(t,s)\in Q_\Delta$ and for any $(x,y)\in B[0,\eta]\times B[0,\eta]$,

$$\|f_2(t,s,x)-f_2(t,s,y)\|\le L_2^{\eta}(t)\|x-y\|.$$

Above $L^1([T_0,T],\mathbb{R})$ (resp. $L^1(Q_\Delta,\mathbb{R})$) stands for the space of Lebesgue integrable functions on $[T_0,T]$ (resp. $Q_\Delta$), where

$$Q_\Delta:=\{(t,s)\in[T_0,T]\times[T_0,T]: s\le t\}.$$

We are now in a position to recall our theorem in [4] on the *well-posedness of integro-differential sweeping processes*. For the knowledge of the reader, the result is recalled in its full statement. We refer to [4] for the proof.

**Theorem 2.1** ([4]). *Let $H$ be a real Hilbert space and assume that $(\mathscr{H}_1)$, $(\mathscr{H}_2)$ and $(\mathscr{H}_3)$ are satisfied. Then for any initial point $x_0 \in H$ with $x_0 \in C(T_0)$, there exists a unique absolutely continuous solution $x : [T_0, T] \longrightarrow H$ of the differential inclusion $(P_{f_1,f_2})$. This solution satisfies:*

*(1) For a.e. $t \in [T_0, T]$*

$$\|\dot{x}(t) + f_1(t,x(t)) + \int_{T_0}^{t} f_2(t,s,x(s))\,ds\| \le |\dot{\upsilon}(t)| + \|f_1(t,x(t))\| + \int_{T_0}^{t} \|f_2(t,s,x(s))\|\,ds.$$

*(2) If $\displaystyle\int_{T_0}^{T}\left[\beta_1(\tau) + \int_{T_0}^{\tau} \beta_2(\tau,s)\,ds\right]d\tau < \frac{1}{4}$, one has*

$$\|f_1(t,x(t))\| \le (1+M)\beta_1(t), \text{ for all } t \in [T_0,T],$$

$$\|f_2(t,s,x(s))\| \le (1+M)\beta_2(t,s), \text{ for all } (t,s) \in Q_\Delta,$$

*and for almost all $t \in [T_0, T]$*

$$\left\|\dot{x}(t) + f_1(t,x(t)) + \int_{T_0}^{t} f_2(t,s,x(s))\,ds\right\| \le (1+M)\left(\beta_1(t) + \int_{T_0}^{t} \beta_2(t,s)\,ds\right) + |\dot{\upsilon}(t)|,$$

*where $\displaystyle M := 2\left(\|x_0\| + \int_{T_0}^{T} |\dot{\upsilon}(\tau)|\,d\tau + \frac{1}{2}\right)$.*

*(3) Assume the following strengthened form of assumption $(\mathscr{H}_{3,1})$ on the function $f_2$ holds: $(\mathscr{H}'_{3,1})$ : there exist non-negative functions $\alpha(\cdot) \in L^1([T_0,T],\mathbb{R})$ and $g(\cdot) \in L^1(Q_\Delta,\mathbb{R})$ such that*

$$\|f_2(t,s,x)\| \le g(t,s) + \alpha(t)\|x\|, \text{ for any } (t,s) \in Q_\Delta \text{ and any } x \in \bigcup_{t\in[T_0,T]} C(t).$$

*Then one has*

$$\|x(t)\| \le \widetilde{M},$$

$$\|f_1(t,x(t))\| \le (1+\widetilde{M})\beta_1(t), \text{ for all } t \in [T_0,T],$$

$$\|f_2(t,s,x(s))\| \le g(t,s) + \alpha(t)\widetilde{M}, \text{ for a.e. } (t,s) \in Q_\Delta,$$

*and for almost all $t \in [T_0, T]$*

$$\|\dot{x}(t) + f_1(t,x(t)) + \int_{T_0}^{t} f_2(t,s,x(s))\,ds\| \le |\dot{\upsilon}(t)| + (1+\widetilde{M})\beta_1(t) + \int_{T_0}^{t} g(t,s)\,ds + T\alpha(t)\widetilde{M},$$

*where*

$$\widetilde{M} := \|x_0\| \exp\left(\int_{T_0}^{T} (b(\tau)+1)\,d\tau\right)$$
$$+\exp\left(\int_{T_0}^{T} (b(\tau)+1)\,d\tau\right)\int_{T_0}^{T}\left(|\dot{\upsilon}(s)|+2\beta_1(s)+2\int_{T_0}^{T} g(s,\tau)\,d\tau\right)ds,$$

$$\text{and} \quad b(t) := 2\max\{\beta_1(t),\alpha(t)\} \quad \text{for all} \quad t\in[T_0,T].$$

## 3. Nonlinear integro-differential complementarity systems

In the present section, we show how Theorem 2.1 allows us to derive the existence and uniqueness of solutions for nonlinear integro-differential complementarity systems (NIDCS), our results complements those in [3].

Let $T > T_0$ be real numbers, $I = [T_0,T]$, $n,m \in \mathbb{N}$, $f_1 : I\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$, $f_2 : I^2\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : I\times\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be given mappings. For $u_1,u_2 \in \mathbb{R}^m$ we will write $0 \le u_1 \perp u_2 \le 0$ to mean that $u_1 \in \mathbb{R}^m_+$, $u_2 \in -\mathbb{R}^m_+$ and $\langle u_1,u_2\rangle = 0$, where $\langle\cdot,\cdot\rangle$ is the canonic scalar product in $\mathbb{R}^m$. Assuming that $g(t,\cdot)$ is differentiable for each $t\in I$ and denoting $\nabla_2 g(t,y)$ the gradient of $g(t,\cdot)$ at $y$, the NIDCS (associated with $f_1$, $f_2$ and $g$) can be described as

$$\text{(NIDCS):}\begin{cases} -\dot{x}(t) = f_1(t,x(t)) + \displaystyle\int_{T_0}^{t} f_2(t,s,x(s))\,ds + \nabla_2 g(t,x(t))^T z(t) \\ 0 \le z(t) \perp g(t,x) \le 0, \end{cases}$$

where $z : I \longrightarrow \mathbb{R}^m$ is unknown mapping. The term $\nabla_2 g(t,x(t))^T z(t)$ can be seen as the generalized reactions due to the constraints in mechanics.
For a mapping $z : [T_0,T] \to \mathbb{R}^m$ we note that

$$z(t)\in\mathbb{R}^m_+ \quad \text{and} \quad \langle z(t),g(t,x)\rangle = 0 \iff z(t)\in N_{\mathbb{R}^m_+}(g(t,x)).$$

So, proceeding as, for example, in [3, Section 9.2] with $9.2-9.3$ therein, (NIDCS) is equivalent to

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f_1(t,x(t)) + \int_{T_0}^{t} f_2(t,s,x(s))\,ds, \tag{3.1}$$

where

$$C(t) := \{x\in\mathbb{R}^n : g_1(t,x)\le 0,\, g_2(t,x)\le 0,\dots,g_m(t,x)\le 0\}, \tag{3.2}$$

and where we set $g(t,\cdot) = (g_1(t,\cdot),g_2(t,\cdot),\dots,g_m(t,\cdot))$ for each $t\in I$.

**Theorem 3.1.** [3] *Let $C(t)$ be defined as in (3.2) and assume that, there exists an extended real $\rho \in ]0,+\infty]$ such that*

*(1) for all $t\in I$, for all $k\in\{1,\dots,m\}$, $g_k(t,\cdot)$ is continuously differentiable on $U_\rho(C(t)) := \{y\in\mathbb{R}^n : d_{C(t)}(y) < \rho\}$.*

*(2) there exists a real $\gamma > 0$ such that, for all $t \in I$, for all $k \in \{1,\dots,m\}$, and for all $x,y \in U_\rho(C(t))$*

$$\langle \nabla_2 g_k(t,x) - \nabla_2 g_k(t,y), x-y\rangle \geq -\gamma \|x-y\|^2,$$

*that is, $\nabla_2 g_k(t,\cdot)$ is $\gamma$-hypomonotone on $U_\rho(C(t))$.*

*(3) there is a real $\delta > 0$ such that for all $(t,x) \in I \times \mathbb{R}^n$ with $x \in \mathrm{bd}\,(C(t))$ (the boundary of $C(t)$), there exists $\bar{\upsilon} \in B[0,1]$ satisfying, for all $k \in \{1,\dots,m\}$*

$$\langle \nabla_2 g(t,x), \bar{\upsilon}\rangle \leq -\delta. \tag{3.3}$$

*Then for every $t \in I$, the set $C(t)$ is r-prox-regular with $r = \min\left\{\rho, \dfrac{\delta}{\gamma}\right\}$.*

The nonlinear differential complementarity system (NDCS) (i.e., (NIDCS) with $f_2 \equiv 0$ ) was studied in [3], where the authors transform the (NDCS) involving inequality constraints $C(t)$ to a perturbed sweeping process in the form (1.2). We extend this approach by employing the above transformation of (NIDCS) into an integro-differential sweeping process of the form (1.3). Also, in contrast to [3], we do not assume that the moving set $C(t)$ described by a finite number of inequalities is absolutely continuous with respect to the Hausdorff distance. Rather, inpired by [2] we provide sufficient verifiable conditions ensuring this type of regularity needed on $C(\cdot)$.

**Proposition 3.2.** *Let $C(t)$ be defined as in (3.2). Assume that there exist an absolutely continuous function $w$, a real $\delta > 0$ and a vector $y \in \mathbb{R}^n$ with $\|y\| = 1$ such that for any $i = 1,\dots,m$ and any $s,t \in I$*

$$g_i(t,x) \leq g_i(s,x) + |w(t) - w(s)|, \quad \text{for all} \quad x \in U_r(C(s)), \tag{3.4}$$

$$\langle \nabla_2 g_i(t,x), y\rangle \leq -\delta, \quad \text{for all} \quad t \in I,\ x \in U_r(C(t)), \tag{3.5}$$

*where $r$ denotes the prox-regularity constant of all sets $C(t)$. Then $C(\cdot)$ is absolutely continuous on $I$ in the sense of (2.3) with $\upsilon(\cdot) := \delta^{-1} w(\cdot)$.*

*Proof.* Let $\delta, y$ and $w(\cdot)$ be as given in the statement. Let $s,t \in I$, let $x \in C(s)$ and choose a subdivision $T_0 < T_1 < \dots < T_p = T$ such that $\int_{T_{k-1}}^{T_k} |\dot{\upsilon}(\tau)|\,d\tau < r$ for every $k = 1,\dots,p$. Fix any $k = 1,\dots,p$ and $s,t \in [T_{k-1}, T_k]$. Take any $i = 1,\dots,m$ and note that

$$\begin{aligned}
g_i(t, x + |\upsilon(t) - \upsilon(s)|y) &= (g_i(t, x + |\upsilon(t) - \upsilon(s)|y) - g_i(s, x + |\upsilon(t) - \upsilon(s)|y)) \\
&\quad + g_i(s, x + |\upsilon(t) - \upsilon(s)|y) \\
&\leq |w(t) - w(s)| + g_i(s, x + |\upsilon(t) - \upsilon(s)|y) \\
&= |w(t) - w(s)| + g_i(s,x) \\
&\quad + \int_0^1 \langle \nabla_2 g_i(s, x + \theta y|\upsilon(t) - \upsilon(s)|), y|\upsilon(t) - \upsilon(s)|\rangle d\theta.
\end{aligned}$$

According to (3.5) and to the inclusion $x \in C(s)$ it ensues that

$$g_i(t, x + |\upsilon(t) - \upsilon(s)|y) \leq |w(t) - w(s)| - \delta|\upsilon(t) - \upsilon(s)| \leq 0.$$

This being true for every $i = 1,\dots,m$, it follows that $x + |\upsilon(t) - \upsilon(s)|y$ belongs to $C(t)$, otherwise stated, $x \in C(t) + |\upsilon(t) - \upsilon(s)|(-y)$. It results that $C(s) \subset C(t) + |\upsilon(t) - \upsilon(s)|B[0,1]$. Since the variables s and t play symmetric roles, the multimapping $C(\cdot)$ has an absolutely continuous variation on $[T_{k-1}, T_k]$ in the sense of (2.3). From this we clearly derive that $C(\cdot)$ has an absolutely continuous variation on $I$. □

**Example 3.3.** Let $m = 1$, $n = 2$, $T_0 = 0$, $T = 1$, $g(t,x) = t^{\frac{1}{3}} - x_1 - x_2^2$, and define

$$C(t) = \{x \in \mathbb{R}^2 : g(t,x) \leq 0\}.$$

Clearly, $C(t)$ is r-prox-regular, since $g(t,\cdot)$ satisfies all assumptions of Theorem 3.1 for all $t \in I$. Now we check (3.4) and (3.5). Let $x \in \mathbb{R}^2$, $t,s \in I$. Fix any $\delta \in (0,1]$ and put $y = (1,0)$. Then for $w(t) := t^{1/3}$ we have

$$g(t,x) - g(s,x) = t^{\frac{1}{3}} - s^{\frac{1}{3}} \leq |t^{\frac{1}{3}} - s^{\frac{1}{3}}| = |w(t) - w(s)|,$$
$$\langle \nabla_2 g(t,x), y \rangle = -1 \leq -\delta.$$

We see that $w(t) = t^{1/3}$ is not Lipschitz on $I$ but it is absolutely continuous there. Then $C(\cdot)$ has an absolutely continuous variation $\upsilon$ on $I$ in the sense of (2.3), with $\upsilon(t) = t^{1/3}/\delta$.

**Theorem 3.4.** *Assume that, with $f_1$, $f_2$ and $g$ as avove, the assumptions in Theorem 3.1 and Proposition 3.2 hold and that conditions $(\mathscr{H}_2)$, $(\mathscr{H}_3)$ are satisfied. Then, for every initial data $x_0$ with $g(0,x_0) \leq 0$, problem (NIDCS) has one and only one solution $x(\cdot)$.*

*Proof.* Like, for example, in [3, Section 9.2], the result follows from the above equivalence between the problem (NIDCS) and the integro-differential sweeping process (3.1) since all assumptions of Theorem 2.1 are satisfied according to Theorem 3.1 and Proposition 3.2. □

## 4. AN INTEGRO-DIFFERENTIAL SWEEPING PROCESS APPROACH TO A FRICTIONLESS CONTACT PROBLEM

This section is concerned with another application of Theorem 2.1. Here we consider an important frictionless contact problem. We need first to introduce some preliminaries and notation which will be employed in the description of the contact problem.
Let $d \in \{1,2,3\}$ and let $\mathbb{S}^d$ denote the space of second-order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order $d$. As usual for mechanical contact problems, generic vectors and tensors in $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted by boldface characters, and index notation will be utilized for their components, so $\zeta \in \mathbb{R}^d$ and $\alpha \in \mathbb{S}^d$ can be written as $\zeta = (\zeta_i)$ and $\alpha = (\alpha_{ij})$. The zero elements of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ will be denoted $0_{\mathbb{R}^d}$ and $0_{\mathbb{S}^d}$ respectively. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are canonically defined by

$$\zeta \cdot \xi = \sum_i \zeta_i \cdot \xi_i, \quad \|\zeta\| = (\zeta \cdot \zeta)^{\frac{1}{2}} \quad \text{for all } \zeta = (\zeta_i),\, \xi = (\xi_i) \in \mathbb{R}^d,$$

$$\alpha \cdot \beta = \sum_{i,j} \alpha_{ij} \cdot \beta_{ij}, \quad \|\alpha\| = (\alpha \cdot \alpha)^{\frac{1}{2}} \quad \text{for all } \alpha = (\alpha_{ij}),\, \beta = (\beta_{ij}) \in \mathbb{S}^d,$$

where the indices $i,j$ in the above sums run from 1 to $d$. Here it is convenient to denote $\zeta \cdot \xi$ the inner product instead of $\langle \zeta, \xi \rangle$.
We consider a viscoelastic body which occupies a domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\Gamma$. We denote by $\overline{\Omega} = \Omega \cup \Gamma$ the closure of $\Omega$ in $\mathbb{R}^d$. The boundary $\Gamma$ is decomposed into three parts $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ and $\overline{\Gamma}_3$ with $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ being relatively open and mutually disjoint and, moreover, the (area/surface) measure $\text{meas}(\Gamma_1)$ relative to $\Gamma$ is positive, i.e., $\text{meas}(\Gamma_1) > 0$.

As usual, $H^1(\Omega)$ is the Sobolev space of real-valued functions in $L^2(\Omega)$ with first order distributional derivatives in $L^2(\Omega)$ as well. Denoting $H^1(\Omega)^d$ the space of mappings $v:\Omega\to\mathbb{R}^d$ with $v_i\in H^1(\Omega)$, $i=1,\cdots,d$, we will use the spaces

$$V=\{v\in H^1(\Omega)^d\ :\ v=0\ \text{ on}\,\Gamma_1\},$$

$$Q=\{\theta=(\theta_{ij})\ :\ \theta_{ij}=\theta_{ji}\in L^2(\Omega)\}.$$

The spaces $Q$ and $V$ are endowed with the canonical inner products given by

$$(\theta,\tau)_Q=\int_\Omega \theta\cdot\tau\,dx,\quad (u,v)_{\mathscr{E}}=\int_\Omega \varepsilon(u)\cdot\varepsilon(v)\,dx=(\varepsilon(u),\varepsilon(v))_Q.$$

Here $\varepsilon$ represents the deformation operator, that is

$$\varepsilon(u)=(\varepsilon_{ij}(u)),\quad \varepsilon_{ij}(u)=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\quad i,j=1,\dots,d,$$

and the index "$\mathscr{E}$" is utilized to emphasize that the inner product $(u,v)_{\mathscr{E}}$ is constructed by means of the function $\varepsilon(\cdot)$. Put $\|\tau\|=(\tau,\tau)_Q^{1/2}$ and $\|v\|_{\mathscr{E}}=(v,v)_{\mathscr{E}}^{1/2}$. The space $Q$ endowed with the inner product $(\cdot,\cdot)_Q$ and the associated norm $\|\cdot\|_Q$ is clearly a Hilbert space. Regarding $V$, by the assumption meas$(\Gamma_1)>0$ Korn's inequality (see, e.g., [10, Lemma 6.2, p 115]) tells us that for some constant $\kappa>0$ we have $\kappa\|v\|_{H^1(\Omega)^d}\le\|v\|_{\mathscr{E}}$ for all $v\in V$, and from this and the definition of $\|\cdot\|_{\mathscr{E}}$ we see that $\|\cdot\|_{\mathscr{E}}$ is a norm on $V$ which is equivalent to $\|\cdot\|_{H^1(\Omega)^d}$ on $V$. Therefore, the space $V$ endowed with the inner product $(\cdot,\cdot)_{\mathscr{E}}$ and the associated norm $\|\cdot\|_{\mathscr{E}}$ is also a Hilbert space.

For a vector $v\in V$, its normal and tangential components are $v_\nu=v\cdot\nu$ and $v_\tau=v-v_\nu\nu$, respectively, where $\nu$ denotes the outward unit normal vector to the boundary $\Gamma$. The normal and tangential components of the stress tensor $\sigma$ on the boundary $\Gamma$ are denoted by $\sigma_\nu=(\sigma\nu)\cdot\nu$ and $\sigma_\tau=\sigma\nu-\sigma_\nu\nu$, respectively. In addition, we recall that the Sobolev trace theorem yields

$$\|v\|_{L^2(\Gamma_3)^d}\le c\|v\|_{\mathscr{E}}\quad\text{for all }\ v\in V, \tag{4.1}$$

$c$ being a positive constant which depends on $\Omega$, $\Gamma_1$ and $\Gamma_3$.

Next, we recall that the following Green's formula holds:

$$\int_\Omega \sigma\cdot\varepsilon(v)\,dx+\int_\Omega \operatorname{Div}\sigma\cdot v\,dx=\int_\Gamma \sigma\nu\cdot v\,da\quad\text{for all }\ v\in H^1(\Omega)^d, \tag{4.2}$$

where Div denotes the divergence operator given by $\operatorname{Div}\sigma=(\sum_j\frac{\partial\sigma_{ij}}{\partial x_j})$, that is, the sum $\sum_j\frac{\partial\sigma_{ij}}{\partial x_j}$ is the $i$-th component of $\operatorname{Div}\sigma$.

Let $Q_\infty$ be the space of fourth order tensor fields given by

$$Q_\infty=\{e=(e_{ijkh})\ :\ e_{ijkh}=e_{jikh}=e_{khij}\in L^\infty(\Omega),\ 1\le i,j,k,h\le d\}.$$

It is easy to see that $Q_\infty$ is a real Banach space with the norm

$$\|e\|_{Q_\infty}=\max_{1\le i,j,k,h\le d}\|e_{ijkh}\|_{L^\infty(\Omega)},$$

and, moreover,

$$\|e\tau\|_Q\le d\|e\|_{Q_\infty}\|\tau\|_Q\quad\text{for all } e\in Q_\infty,\ \tau\in Q, \tag{4.3}$$

where $e\tau$ is the tensor function in $Q$ given by its $i,j$ components as $e\tau = (\sum_{k,h} e_{ijkh}\tau_{kh})$. More on actions of tensors on vectors and matrices can be found, e.g., in [8].
Classically, for $u\colon \Omega\times[0,T]\to\mathbb{R}^d$ and $\sigma\colon \Omega\times[0,T]\to\mathbb{S}^d$ it can be convenient (as below) to denote by $u(t)$ and $\sigma(t)$ the mappings $u(\cdot,t)$ and $\sigma(\cdot,t)$.

We can now present the frictionless contact problem which we are interested in. It is a quasistatic problem which models the contact between a deformable body and an obstacle, the so-called foundation. The material is assumed to have a viscoelastic behavior which is modeled by a constitutive law with long-term memory, thus, at each moment of time, the stress tensor depends not only on the present strain tensor, but also on its whole history. The contact is frictionless and is modeled by the well-known Signorini conditions. We refer to [9, 11, 16, 18] for the modeling details of this kind of problem. For our purpose of motivation, the main concern is to derive a formulation of the problem, expressed in terms of integro-differential sweeping process, and to prove its unique solvability under mild regularity hypotheses.

The formulation of the problem is as follows.

**Problem 1.** Find $u\colon \Omega\times[0,T]\to\mathbb{R}^d$ and $\sigma\colon \Omega\times[0,T]\to\mathbb{S}^d$ with $u_i(\cdot,t)$ and $\sigma_{ij}(\cdot,t)$ in $H^1(\Omega)$ such that for a.e. $t\in]0,T[$

$$\sigma(t) = \mathscr{A}\varepsilon(\dot u(t)) + \mathscr{B}(t,\varepsilon(u(t))) + \int_0^t \mathscr{R}(t-s)\varepsilon(u(s))\,ds \quad \text{in} \quad \Omega, \tag{4.4}$$

$$\operatorname{Div}\sigma(t) + f_0(t) = 0 \quad \text{in} \quad \Omega, \tag{4.5}$$

$$\sigma(t)\nu = f_N(t) \quad \text{on} \quad \Gamma_2, \tag{4.6}$$

$$u_\nu(t)\le 0,\ \ \sigma_\nu(t)\le 0,\ \ \sigma_\nu(t)u_\nu(t)=0,\ \ \sigma_\tau(t)=0 \quad \text{on} \quad \Gamma_3, \tag{4.7}$$

and

$$u(t)=0 \quad \text{on} \quad \Gamma_1\times]0,T[, \tag{4.8}$$

$$u(0)=u_0 \quad \text{in} \quad \Omega. \tag{4.9}$$

Here, $\mathscr{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d$, $\mathscr{R}:[0,T]\to\mathbf{Q}_\infty$, $\mathscr{B}:[0,T]\times Q\to Q$ are prescribed mappings, and $\mathscr{B}$ is defined in the form $\mathscr{B}(t,\theta)(x)=\mathscr{B}_0(x,t,\theta(x))$ for all $x\in\Omega$, where $\mathscr{B}_0:\Omega\times]0,T[\times\mathbb{S}^d\to\mathbb{S}^d$.

Let us give a short description of the conditions in Problem 1. Equation (4.4) represents the viscoelastic constitutive law with long memory in which $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{R}$ denote the viscosity, elasticity and relaxation operators, respectively. Equations of type (4.4) are related to the Kelvin-Voigt law, so when $\mathscr{R}$ vanishes, (4.4) reduces to the well known Kelvin-Voigt constitutive law extensively studied in the literature, see Shillor, Sofonea and Telega [16, Chapter 8], and the references therein. Equation (4.5) is the equilibrium equation, while conditions (4.8) and (4.6) are the displacement and traction boundary conditions, respectively. Conditions (4.7) represent the frictionless Signorini contact conditions in which $u_\nu$ denotes the normal displacement, $\sigma_\nu$ represents the normal stress, and $\sigma_\tau$ is the tangential stress on the potential contact surface. Finally, (4.9) represents the initial condition in which $u_0$ is the initial displacement field.

We consider the following usual hypotheses (see, e.g., [17]):

H($\mathscr{A}$): The viscosity tensor $\mathscr{A}=(a_{ijkh})\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d$ satisfies the natural standard properties of symmetry and ellipticity:

(a) $a_{ijkh} \in L^\infty(\Omega)$.
(b) $\mathscr{A}\sigma\cdot\tau = \sigma\cdot\mathscr{A}\tau$ for all $\sigma,\tau\in\mathbb{S}^d$, a.e. in $\Omega$.
(c) $\exists m_{\mathscr{A}} > 0:$ $\mathscr{A}\tau\cdot\tau \geq m_{\mathscr{A}}\|\tau\|^2_{\mathbb{S}^d}$ for all $\tau\in\mathbb{S}^d$, a.e. in $\Omega$.

We recall that the $i,j$ component of the tensor function $\mathscr{A}\tau$ is $\sum_{kh} a_{ijkh}\tau_{kh}$.

H($\mathscr{B}$): The mapping $\mathscr{B}_0\colon \Omega\times]0,T[\times\mathbb{S}^d\to\mathbb{S}^d$ is such that
(a) There is $L_{\mathscr{B}}\geq 0$ such that

$$\|\mathscr{B}_0(x,t,\alpha_1)-\mathscr{B}_0(x,t,\alpha_2)\|_{\mathbb{S}^d}\leq L_{\mathscr{B}}\|\alpha_1-\alpha_2\|_{\mathbb{S}^d} \quad \text{for all } \alpha_1,\alpha_2\in\mathbb{S}^d, \text{ a.e. } (x,t)\in\Omega\times]0,T[.$$

(b) $\mathscr{B}_0(\cdot,\cdot,\varepsilon)$ is Borel measurable on $\Omega\times]0,T[$ for all $\varepsilon\in\mathbb{S}^d$.
(c) $\mathscr{B}_0(\cdot,t,0_{\mathbb{S}^d})$ belongs to $Q$ for all $t\in]0,T[$.

H($\mathscr{R},f_0,f_N$): The prescribed relaxation tensor $\mathscr{R}$ and densities of body forces $f_0$ and surface tractions $f_N$ are such that
(a) $\mathscr{R}\in\mathscr{C}([0,T],Q_\infty)$.
(b) $f_0\in\mathscr{C}([0,T],L^2(\Omega)^d)$.
(c) $f_N\in\mathscr{C}([0,T],L^2(\Gamma_2)^d)$.

Let us turn to an analysis of any eventual solution of Problem 1 (if any).
To this end we assume in what follows that the viscosity and elasticity operators satisfy assumptions H($\mathscr{A}$) and H($\mathscr{B}$), respectively. The relaxation operator, the densities of body forces and the surface tractions satisfy the assumption H($\mathscr{R},f_0,f_N$).
We also introduce the set of admissible displacements fields, defined by

$$U=\{v\in V\ :\ v_\nu\leq 0 \quad \text{a.e. on } \Gamma_3\}, \tag{4.10}$$

and we note that $U$ is a closed convex subset of $V$ such that $0_V\in U$. And, finally, the initial displacement satisfies $u_0\in U$. For $u,v\in V$ let

$$(u,v)_V=(\mathscr{A}\varepsilon(u),\varepsilon(v))_Q,\quad \|u\|_V=(u,u)_V^{\frac{1}{2}}. \tag{4.11}$$

Using the assumption H($\mathscr{A}$) we obtain that $(\cdot,\cdot)_V$ is an inner product on $V$ and $\|\cdot\|_V$ and $\|\cdot\|_{\mathscr{E}}$ are equivalent norms on $V$. Therefore, $(V,\|\cdot\|_V)$ is a real Hilbert space.
Next, with the volume measure $dx$ and the area/surface measure $da$ on $\Gamma$, we notice that

$$v\mapsto\int_\Omega f_0(t)\cdot v\,dx+\int_{\Gamma_2} f_N(t)\cdot v\,da \ \ \forall v\in V, t\in[0,T],$$

is a continuous linear functional on the space $V$. Therefore, we may apply the Riesz representation theorem to define the element $f(t)\in V$ by the equality

$$(f(t),v)_V=\int_\Omega f_0(t)\cdot v\,dx+\int_{\Gamma_2} f_N(t)\cdot v\,da \ \ \forall v\in V, t\in[0,T]. \tag{4.12}$$

Let $(u,\sigma)$ be a pair of feasible functions, satisfying (4.4)-(4.9). Fix any $t$ in a suitable (full Lebesgue measure) subset of $]0,T[$ over which (4.4)-(4.7) hold. Let $v\in U$. Using the Green formula (4.2) and using (4.5) we have

$$\int_\Omega \sigma(t)\cdot(\varepsilon(v)-\varepsilon(u(t)))\,dx=\int_\Gamma \sigma(t)\nu\cdot(v-u(t))\,da+\int_\Omega f_0(t)\cdot(v-u(t))\,dx.$$

From the boundary conditions (4.8), (4.6) and the following decomposition formula $\sigma(t)\nu\cdot(v-u(t))=\sigma_\nu(t)(v_\nu-u_\nu(t))+\sigma_\tau(t)\cdot(v_\tau-u_\tau(t))$ on $\Gamma_3$, it ensues that

$$\begin{aligned}\int_\Omega \sigma(t)\cdot(\varepsilon(v)-\varepsilon(u(t)))\,dx &= \int_{\Gamma_2} f_N(t)\cdot(v-u(t))\,da+\int_{\Gamma_3}\sigma_\nu(t)(v_\nu-u_\nu(t))\,da\\ &\quad+\int_{\Gamma_3}\sigma_\tau(t)\cdot(v_\tau-u_\tau(t))\,da+\int_\Omega f_0(t)\cdot(v-u(t))\,dx.\end{aligned}\tag{4.13}$$

Using (4.10) and putting (4.7) into (4.13) we obtain

$$\int_\Omega \sigma(t)\cdot(\varepsilon(v)-\varepsilon(u(t)))\,dx \geq \int_{\Gamma_2} f_N(t)\cdot(v-u(t))\,da+\int_\Omega f_0(t)\cdot(v-u(t))\,dx.\tag{4.14}$$

The inequality (4.14), the constitutive law (4.4) and the initial conditions (4.9) yield that any solution of Probem 1 is a solution of the following Problem 2.

**Problem 2.** Find the displacement field $u\colon[0,T]\to V$, such that

$$\begin{cases}\big(\mathscr{A}\varepsilon(\dot u(t)),\varepsilon(v)-\varepsilon(u(t))\big)_Q+\big(\mathscr{B}(t,\varepsilon(u(t))),\varepsilon(v)-\varepsilon(u(t))\big)_Q\\ +\left(\int_0^t \mathscr{R}(t-s)\varepsilon(u(s))\,ds,\varepsilon(v)-\varepsilon(u(t))\right)_Q\geq (f(t),v-u(t))_V\quad \forall v\in U,\\ u(0)=u_0,\ u(t)\in U\ \text{ for } t\in[0,T].\end{cases}\tag{4.15}$$

We will see below that Problem 2 has one and only one solution. Thus, consider the unique solution $u$ of Problem 2. Let $D:=D(\Omega;\mathbb{R}^d)=\mathscr{C}_0^\infty(\Omega;\mathbb{R}^d)$ denote the space of all mappings defined on $\Omega$ with values in $\mathbb{R}^d$ which are infinitely differentiable and have compact support in $\Omega$. Consider any $\varphi\in D$ and take $v:=u(t)+\varphi$. Clearly, $v\in U$ since $D(\Omega;\mathbb{R}^d)\subset U$. Then by the inequality in (4.15), by (4.4) and (4.12) we have in the sense of distribution that

$$\langle\sigma(t),\varepsilon(\varphi)\rangle_{D'\times D}\geq\langle f_0(t),\varphi\rangle_{D'\times D}\quad\text{for all}\quad \varphi\in D(\Omega;\mathbb{R}^d)\ \text{ a.e. } t\in[0,T].$$

We perform integrations by parts to obtain that

$$\langle-\operatorname{Div}\sigma(t),\varphi\rangle_{D'\times D}\geq\langle f_0(t),\varphi\rangle_{D'\times D}\quad\text{for all}\quad \varphi\in D(\Omega;\mathbb{R}^d).$$

Similarly, taking $v:=u(t)-\varphi$ and using the same arguments we also have that

$$\langle-\operatorname{Div}\sigma(t),\varphi\rangle_{D'\times D}\leq\langle f_0(t),\varphi\rangle_{D'\times D}\quad\text{for all}\quad \varphi\in D(\Omega;\mathbb{R}^d).$$

So, it follows that

$$\operatorname{Div}\sigma(t)+f_0(t)=0.$$

On the other hand, it is clear by definition of the spaces $V$ and $U$ that

$$u(t)=0\quad\text{on}\quad \Gamma_1\times[0,T],$$

and for a.e. $t\in]0,T[$

$$u_\nu(t)\leq 0,\quad\text{on}\quad\Gamma_3.$$

Notice also that $u(0)=u_0$ in $\Omega$.
Suppose in addition that the mapping $u$ is smooth, in the sense that $u(\cdot,t)\in\mathscr{C}^2(\Omega)$, and that $\Gamma_2$

and $\Gamma_3$ are $\mathscr{C}^\infty$-smooth for example. Then Theorem 6.3 in the book [10] of Kikuchi and Oden along with the comments subsequent to that theorem in that book, ensue that for a.e. $t \in ]0,T[$

$$\sigma(t)\nu = f_N(t) \quad \text{on} \quad \Gamma_2,$$

$$\sigma_\nu(t) \leq 0, \;\; \sigma_\nu(t)u_\nu(t) = 0, \;\; \sigma_\tau(t) = 0 \quad \text{on} \quad \Gamma_3.$$

So, under the above smoothness conditions, $u$ is solution of Problem 1.

To summarize, Problem 2 admits one and only one solution (as we will see below), and any solution of Problem 1 (if any) coincides with the solution of Problem 2. Furthermore, under the above regularity of $\Gamma_2$ and $\Gamma_3$, if the unique solution $u$ of Problem 2 possesses the regularity $u(\cdot,t) \in \mathscr{C}^2(\Omega)$, then it is a solution of Problem 1. The conclusion is that the unique solution $u$ of Problem 2 (furnished by the next theorem) is a **right weak solution** for the concerned Problem 1.

After the preceding analysis, we establish our existence and uniqueness result for Problem 2.

**Theorem 4.1.** *Under the above assumptions, for each $u_0 \in U$, Problem 2 has a unique absolutely continuous solution $u$.*

*Proof.* The proof consists of two parts in which we rewrite Problem 2 in an equivalent form of integro-differential sweeping process and apply the result of Theorem 2.1. To this end, denoting by $\mathscr{L}(V)$ the space of continuous linear operators from $V$ into itself, we apply the Riesz representation theorem to define the operators $B\colon [0,T] \times V \to V$ and $R\colon [0,T] \to \mathscr{L}(V)$ by

$$(B(t,v),w)_V = \big(\mathscr{B}(t,\varepsilon(v)),\varepsilon(w)\big)_Q \quad (R(t)v,w)_V = (\mathscr{R}(t)\varepsilon(v),\varepsilon(w))_Q \text{ for all } v,w \in V,\ t \in [0,T]. \tag{4.16}$$

Moreover, using (4.11) and inequality (4.15), we derive the following variational inequality for a.e. $t \in ]0,T[$

$$\begin{cases} \big(\dot u(t), v-u(t)\big)_V + \big(B(t,u(t)), v-u(t)\big)_V \\ + \left(\displaystyle\int_0^t R(t-s)u(s)\,ds, v-u(t)\right)_V \geq (f(t), v-u(t))_V \quad \text{for all } v \in U, \end{cases} \tag{4.17}$$

along with $u(0) = u_0$ and $u(t) \in U$. Then, the variational inequality (4.17) subject to the latter conditions is equivalent to the following integro-differential inclusion

$$\begin{cases} -\dot u(t) \in N_U(u(t)) + B(t,u(t)) - f(t) + \displaystyle\int_0^t R(t-s)u(s)\,ds \quad \text{a.e. } t \in [0,T] \\ u(0) = u_0 \in U. \end{cases} \tag{4.18}$$

Now, we prove the existence and uniqueness result for problem (4.18), by applying Theorem 2.1. In what follows, we will verify that the data of problem (4.18) satisfy hypotheses of Theorem 2.1 on the space $H = V$.

**(I).** Clearly, $C(\cdot) = U$ satisfies $(\mathscr{H}_1)$ since $U$ is a fixed nonempty closed convex subset of $V$.

**(II).** The function $f_1$ defined by $f_1(t,v) = B(t,v) - f(t)$ for all $t \in [0,T]$ and all $v \in V$ satisfies for some real constant $k > 0$ the hypothesis $(\mathscr{H}_2)$ with $\beta_1(t) = \max\left(k^2 L_{\mathscr{B}}, k\|\mathscr{B}(t,0_{\mathbb{S}^d})\|_Q +\right.$

$\|f(t)\|_V)$ and $L_1(t) = k^2 L_{\mathscr{B}}$ for all $t \in [0,T]$.
Indeed, by definition of the operator $B$ in (4.16) we see for all $v \in V$ and all $t \in [0,T]$ that

$$\begin{aligned}\|B(t,v)\|_V^2 &= \big(B(t,v),B(t,v)\big)_V = \big(\mathscr{B}(t,\varepsilon(v)),\varepsilon(B(t,v))\big)_Q \\ &\leq \|\mathscr{B}(t,\varepsilon(v))\|_Q \cdot \|\varepsilon(B(t,v))\|_Q = \|\mathscr{B}(t,\varepsilon(v))\|_Q \cdot \|B(t,v)\|_{\mathscr{E}},\end{aligned}$$

so recalling that the norms $\|\cdot\|_V$ and $\|\cdot\|_\varepsilon$ are equivalent we obtain some real constant $k > 0$ such that for all $v \in V$ and all $t \in [0,T]$

$$\|B(t,v)\|_V^2 = \big(B(t,v),B(t,v)\big)_V \leq k\|\mathscr{B}(t,\varepsilon(v))\|_Q \cdot \|B(t,v)\|_V.$$

On the other hand, using H($\mathscr{B}$) yields

$$\begin{aligned}\|B(t,v)\|_V &\leq k\|\mathscr{B}(t,\varepsilon(v))\|_Q \leq k(\|\mathscr{B}(t,\varepsilon(v)) - \mathscr{B}(t,0_{\mathbb{S}^d})\|_Q + \|\mathscr{B}(t,0_{\mathbb{S}^d})\|_Q) \\ &\leq k(L_{\mathscr{B}}\|\varepsilon(v)\|_Q + \|\mathscr{B}(t,0_{\mathbb{S}^d})\|_Q) = k(L_{\mathscr{B}}\|v\|_E + \|\mathscr{B}(t,0_{\mathbb{S}^d})\|_Q) \\ &\leq k(kL_{\mathscr{B}}\|v\|_V + \|\mathscr{B}(t,0_{\mathbb{S}^d}\|_Q).\end{aligned}$$

We conclude that

$$\begin{aligned}\|f_1(t,v)\|_V &\leq \|B(t,v)\|_V + \|f(t)\|_V \leq k(kL_{\mathscr{B}}\|v\|_V + \|\mathscr{B}(t,0_{\mathbb{S}^d}\|_Q) + \|f(t)\|_V \\ &\leq \max\,(k^2 L_{\mathscr{B}}, k\|\mathscr{B}(t,0_{\mathbb{S}^d})\|_Q + \|f(t)\|_V)(1+\|v\|_V).\end{aligned}$$

Similarly, given $v_1, v_2 \in V$ we have by the way that $B(t,v)$ has been defined

$$\begin{aligned}\|B(t,v_1) - B(t,v_2)\|_V &= \sup_{\|w\|_V \leq 1} (\mathscr{B}(t,\varepsilon(v_1)) - \mathscr{B}(t,\varepsilon(v_2)),\varepsilon(v))_Q \\ &\leq \sup_{\|w\|_V \leq 1} \|\mathscr{B}(t,\varepsilon(v_1)) - \mathscr{B}(t,\varepsilon(v_2))\|_Q \|\varepsilon(w)\|_Q \\ &\leq k\|\mathscr{B}(t,\varepsilon(v_1)) - \mathscr{B}(t,\varepsilon(v_2))\|_Q.\end{aligned}$$

From this and H($\mathscr{B}$) we obtain for all $v_1, v_2 \in V$

$$\begin{aligned}\|B(t,v_1) - B(t,v_2)\|_V &\leq kL_{\mathscr{B}}\|\varepsilon(v_1) - \varepsilon(v_2)\|_Q = kL_{\mathscr{B}}\|v_1 - v_2\|_{\mathscr{E}} \\ &\leq k^2 L_{\mathscr{B}}\|v_1 - v_2\|_V.\end{aligned}$$

**(III).** The function $f_2(t,s,v) = R(t-s)v$ for all $(t,s) \in Q_\Delta$ and $v \in V$ satisfies, for the above real constant $k > 0$, the hypothesis $(\mathscr{H}_3)$ with $\beta_2(t,s) = k^2 d\|\mathscr{R}(t-s)\|_{Q_\infty}$ and $L_2(t) = k^2 d \sup_{t\in[0,T]} \|\mathscr{R}(t)\|_{Q_\infty}$ for all $(t,s) \in Q_\Delta$.
Indeed, by definition of operator $R$ in (4.16) we have for all $v \in V$ and all $(t,s) \in Q_\Delta$ that

$$\begin{aligned}\|R(t-s)v\|_V^2 &= (R(t-s)v, R(t-s)v)_V = (\mathscr{R}(t-s)\varepsilon(v), \varepsilon(R(t-s)v))_Q \\ &\leq \|\mathscr{R}(t-s)\varepsilon(v)\|_Q \cdot \|\varepsilon(R(t-s)v)\|_Q = \|\mathscr{R}(t-s)\varepsilon(v)\|_Q \cdot \|R(t-s)v\|_{\mathscr{E}} \\ &\leq k\|\mathscr{R}(t-s)\varepsilon(v)\|_Q \cdot \|R(t-s)v\|_V.\end{aligned}$$

Next, using assumptions H($\mathscr{R}, f_0, f_N$)-($a$) and the inequality (4.3), we obtain

$$\begin{aligned}\|f_2(t,s,v)\|_V = \|R(t-s)v\|_V &\leq kd\|\mathscr{R}(t-s)\|_{Q_\infty} \cdot \|\varepsilon(v)\|_Q = kd\|\mathscr{R}(t-s)\|_{Q_\infty} \cdot \|v\|_{\mathscr{E}} \\ &\leq k^2 d\|\mathscr{R}(t-s)\|_{Q_\infty} \cdot \|v\|_V \\ &\leq k^2 d\|\mathscr{R}(t-s)\|_{Q_\infty} \cdot (1+\|v\|_V).\end{aligned}$$

Further, by (4.19) we have for any $v_1, v_2 \in V$ and $(t,s) \in Q_\Delta$ that

$$\begin{aligned}\|f_2(t,s,v_1) - f_2(t,s,v_2)\|_V &= \|R(t-s)v_1 - R(t-s)v_2\|_V = \|R(t-s)(v_1 - v_2)\|_V \\ &\leq k^2 d \|\mathscr{R}(t-s)\|_{Q_\infty} \cdot \|v_1 - v_2\|_V \\ &\leq k^2 d \sup_{t\in[0,T]} \|\mathscr{R}(t)\|_{Q_\infty} \cdot \|v_1 - v_2\|_V.\end{aligned}$$

We have verified that all hypotheses of Theorem 2.1 are satisfied. Hence, we deduce that problem 4.18 has a unique absolutely continuous solution $u$, so Problem 2 has a unique solution. The proof of the theorem is then complete. □

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