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ROBUST SUFFICIENT CONDITIONS FOR THE OBSERVABILITY OF A LINEAR TIME-INVARIANT SINGULARLY PERTURBED SYSTEM WITH DELAY

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Dedicated to the memory of Professor Rafail Gabasov

Abstract. In this paper, a singularly perturbed linear time-invariant system with a small parameter multilpying a part of the derivatives in the system and with a constant delay in the state (SPLTISD) is considered. On the basis of decomposition by means of linear non-degenerate change of variables the independent of the parameter sufficient rank type conditions of spectral observability of SPLTISD, which are valid for all sufficiently small values of the parameter, are proved.

Keywords. Delay; Decomposition; Robust sufficient conditions; Singularly perturbed system; Small parameter; Spectral observability.

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1. INTRODUCTION

Observability is a structural property of dynamic systems and is one of the most important conditions for their performance. To control the system, it is necessary to have information about its current state at each moment of time. In particular, system states play an important role in feedback control, so it is important that the states are observable. However, some state variables in the mathematical model of a dynamic system are abstract variables, have no analogue in a real system, and therefore cannot be directly measured. Measured and observed in the system are physical output variables, through which all components of the state vector must be uniquely expressed (i.e. restored).

Delay systems arise when modeling various processes in engineering, economics, technology, biology, ecology, social sphere, etc. Due to the complication of the concept of state for systems with delay (the phase space of a system with delay is infinite-dimensional, the state of such systems is characterized not by a set of a finite number of values, but by a set of functions), the concept of observability of systems with delay turns out to be more diverse. In this regard,

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there are, for example, relative observability, pointwise, complete, spectral observability etc. (see [3, 7, 8] and references therein).

Systems of differential equations with a small parameter at a part of the highest derivatives belong to the class of singularly perturbed systems (SPS) and arise in the study and modeling of objects that simultaneously perform slow and fast motions (multi-time scale systems). SPS arise in hydrodynamics, electric power engineering, radio engineering, flight dynamics, economics, in the theory of automatic control, etc. For instance, the motion of systems of rigid bodies is a complex composition of fast and slow motions. In the case of gyroscopic systems, there are fast - nutational and slow - precessional oscillations. In the case of electromechanical systems, the role of fast variables is played by variables describing electrical oscillations, and slow variables are variables that describe the mechanical part of the energy system of two different types of power plants.

In most studies known to the author, the observability of SPS is studied for systems without delay. Most of the results known to the author on the observability of SPS with delay (SPSD) refer to relative observability ([2, 5]). In [6], complete observability conditions were obtained for linear time-invariant singularly perturbed dynamical systems with pure delay, in [2] observability problems in the Euclidean space (relative observability), as well as in the state space for systems with a small delay, are considered. At the same time, as the author notes, the cases of systems with delay and with a small delay differ from each other and require different approaches to study.

In the present paper, we consider the conditions of spectral observability for singularly perturbed time-invariant dynamical systems with nonsmall delay in slow state variables.

2. Problem statement

Consider the singularly perturbed linear time invariant system with delay (SPLTISD)

$$SPLTISD: \begin{array}{l} \dot{x}(t)=A_{10}x(t)+A_{11}x(t-h)+A_2y(t), x\in\mathbb{R}^{n_1}, y\in\mathbb{R}^{n_2}, \\ \mu\dot{y}(t)=A_{30}x(t)+A_{31}x(t-h)+A_4y(t),\, t\in T=[0,t_1], \\ x(0)=x_0,\ y_0(0)=y_0,\, x(\theta)=\phi(\theta), \theta\in[-h,0), \end{array} \tag{2.1}$$

with output

$$v(t)=C_1x(t)+C_2y(t), t\in T, v\in\mathbb{R}^m, m\le n_1+n_2. \tag{2.2}$$

Here A_{ij}, $i=1,3$, $j=0,1, A_k, k=2,4, C_j, j=1,2$, are real matrices of appropriate sizes; $0<h$ is constant delay; $x_0\in\mathbb{R}^{n_1}$, $y_0\in\mathbb{R}^{n_2}$, $\varphi(\theta)$ are unknown initial vectors and continuous n_1- vector-function; μ is a small parameter, $\mu\in(0,\mu^0]$, $\mu^0\ll 1$.

Denote $n=n_1+n_2$. Itroduce vector $z=(x',y')'\in\mathrm{R}^n$ ($'$ is the transpose symbol).

Let $p\triangleq\frac{d}{dt}$ be differentiation operator, e^{-ph} be delay operator: $e^{-ph}z(t)\triangleq z(t-h)$, $\mathsf{A}_\mathsf{i}\left(e^{-ph}\right)\triangleq A_{i0}+A_{i1}e^{-ph}$, $i=1,3$, be matrix-valued operators. Introduce for a given $\mu>0$ the following matrix-valued operators, that depend on the parameter:

$$\mathsf{A}\left(\mu,e^{-ph}\right)=\begin{pmatrix} \mathsf{A}_1\left(e^{-ph}\right) & A_2 \\ \frac{\mathsf{A}_3\left(e^{-ph}\right)}{\mu} & \frac{A_4}{\mu}\end{pmatrix}, \quad C=\begin{pmatrix} C_1 & C_2\end{pmatrix}, \quad \mu>0.$$

Using the above notations we can rewrite SPLTISD (2.1) in the equivalent operator form:

$$\begin{array}{l} pz(t) = \mathsf{A}\left(\mu, e^{-ph}\right) z(t),\ z \in \mathrm{R}^n,\ t \in T, \\ v(t) = Cz(t),\ v \in \mathrm{R}^m,\ t \in T, \\ z(0) = z_0,\ Hz(\theta) = \phi(\theta), \theta \in [-h, 0),\ H = [I_{n_1},\ 0_{n_1 \times n_2}]. \end{array} \tag{2.3}$$

For a given $\mu \in (0, \mu^0]$ by

$$\sigma(\mu) = \left\{ \lambda \in \mathbb{C} : \det\left(\lambda I_n - \mathsf{A}\left(\mu, e^{-\lambda h}\right)\right) = 0 \right\} \tag{2.4}$$

denote the spectrum (set of the eigenvalues) of the SPLTISD (2.1).

Let us denote by $\Sigma_\lambda(\mu)$ the finite-dimensional system that is the projection of SPLTISD (2.1) on the generalized proper subspace, associated with its eigenvalue $\lambda \in \sigma(\mu)$.

Definition 2.1. For a given $\mu \in (0, \mu^0]$ SPLTISD (2.1) is spectrally observable if any finite-dimensional system $\Sigma_\lambda(\mu)$, associated with the eigenvalues $\lambda \in \sigma(\mu)$ is observable.

Definition 2.2. If there exists a value $\mu^* > 0$ that SPLTISD (2.1) is spectrally observable for any $\mu \in (0, \mu^*]$, we say that it is spectrally observable robustly with respect to $\mu \in (0, \mu^*]$.

Define the matrix function:

$$\mathsf{N}(\mu, \lambda, e^{-\lambda h}) \stackrel{def}{=} \begin{bmatrix} \lambda I_n - \mathsf{A}\left(\mu, e^{-\lambda h}\right) \\ C \end{bmatrix}.$$

The following criterion of SPLTISD spectral observability for a given $\mu \in (0, \mu^0]$ follows from the [1] and [8].

Proposition 2.3. *For a given $\mu \in (0, \mu^0]$ the SPLTISD (2.1) is spectrally observable if and only if*

$$rank\, \mathsf{N}\left(\mu, \lambda, e^{-\lambda h}\right) = n_1 + n_2, \quad \forall \lambda \in \sigma(\mu).$$

In the analysis of singularly perturbed systems it is important to establish their properties not only for several separate values of the small parameter but for all sufficiently small ones.

Objective of the paper is to obtain parametric rank-type sufficient conditions for spectral observability of the SPLTISD in terms of spectral observability of two independent of small parameter subsystems of lower dimensions than the investigated SPLTISD, that are approximation of the original SPLTISD (2.1). The conditions do not depend on the parameter and robust with respect to μ for all its sufficiently small values.

3. Slow and fast subsystems of SPLTISD

Let $\det A_4 \neq 0$. With $n_1 + n_2-$dimensional system (2.1) are associated two independent of μ subsystems: the slow and the fast ones. The slow subsystem – *degenerate system* (DS) – has the form

$$\begin{array}{rcl} \dot{x}_s(t) &=& A_{s0} x_s(t) + A_{s1} x_s(t-h),\ x_s \in \mathrm{R}^{n_1},\ t \geq 0, \\ v_s(t) &=& C_{s0} x_s(t) + C_{s1} x_s(t-h),\ v_s \in \mathrm{R}^m,\ t \in T, \\ x_s(0) &=& x_0, x_s(\theta) = \phi(\theta), \theta \in [-h, 0), \end{array} \tag{3.1}$$

where $A_{sj} \stackrel{\Delta}{=} A_{1j} - A_2 A_4^{-1} A_{3j},\ j = 0, 1,\ C_{s0} \stackrel{\Delta}{=} C_1 - C_2 A_4^{-1} A_{30}, C_{s1} \stackrel{\Delta}{=} -C_2 A_4^{-1} A_{31}$.

The degenerate system (3.1) is a linear time-invariant n_1-dimensional system with delay in the equation of state and in the output. It is obtained from (2.1) by setting there formally $\mu = 0$, expressing $y_s(t) = A_4^{-1}\left[\mathsf{A}_3\left(e^{-ph}\right)x_s(t)\right]$ from the second equation and substituting it into the first one.

The spectrum of the DS (3.1):

$$\sigma_s = \left\{\lambda \in \mathrm{C} : \det\left[\lambda I_{n_1} - \mathsf{A}_s\left(e^{-\lambda h}\right)\right] = 0\right\} \tag{3.2}$$

is a finite or countable set of complex numbers.

The boundary layer system (BLS) has the form:

$$\begin{aligned} \frac{dy_f(\tau)}{d\tau} &= A_4 y_f(\tau),\, y_f \in \mathrm{R}^{n_2},\, \tau = \frac{t}{\mu} \in T_\mu \triangleq \left[0, \frac{t_1}{\mu}\right], \\ v_f(\tau) &= C_2 y_f(\tau), \\ y_f(0) &= y_0 - A_4^{-1}\left[\mathsf{A}_3\left(e^{-\lambda h}\right)\phi(0)\right], \end{aligned} \tag{3.3}$$

where $y_f(\tau) = y(\mu\tau) - y_s(\mu\tau)$.

The boundary layer system (3.3) is a linear time-invariant n_2-dimensional system without delay and is derived from second equation of (2.1) for the fast state variable y in the following way: (i) the terms containing the slow state variable x are removed from second equation of (2.1); (ii) the transformation of variables $t = \mu\tau$, $y(\mu\tau) = y_f(\tau)$, is done in the resulting equation, where τ, y_f are new independent variables (the stretched time) and state.

The spectrum of the BLS (3.3) is a finite set of complex numbers:

$$\sigma_f = \{\lambda \in \mathrm{C} : \det[\lambda I_{n_2} - A_4] = 0\}. \tag{3.4}$$

Note that subsystems (3.1) and (3.3) have smaller dimensions than the original SPLTISD (2.1) and do not depend on a small parameter μ.

Let us denote by $\Sigma_{s\lambda}$, $\Sigma_{f\lambda}$ the finite-dimensional systems that is the projection of DS (3.1) and BLS (3.3) on the generalized proper subspace, associated with its eigenvalue $\lambda \in \sigma_s$ and $\lambda \in \sigma_f$, respectively.

Definition 3.1. The DS (3.1) is said to be *spectrally observable* if any finite-dimensional system $\Sigma_{s\lambda}$, associated with the eigenvalues $\lambda \in \sigma_s$, is observable.

Definition 3.2. The BLS (3.3) is said to be *spectrally observable* if any finite-dimensional system $\Sigma_{f\lambda}$, associated with the eigenvalues $\lambda \in \sigma_f$, is observable.

So, the main objective of the article is to obtain conditions of spectral observability of the SPLTISD (2.1) (Definition 2.1) in terms of spectral observability of its subsystems (3.1), (3.3) (Definitions 3.1, 3.2), robust with respect to μ (Definition 2.2) .

Introduce the following matrix-valued operators:

$$\begin{aligned} &\mathsf{A}_s\left(e^{-ph}\right) \triangleq \mathsf{A}_1\left(e^{-ph}\right) - A_2 A_4^{-1}\mathsf{A}_3\left(e^{-ph}\right), \\ &\mathsf{C}_s\left(e^{-ph}\right) = C_{s0} + C_{s1}e^{-ph}, \end{aligned} \tag{3.5}$$

and define the matrix-valued functions

$$\mathsf{N}_s(\lambda, e^{-\lambda h}) \triangleq \begin{bmatrix} \lambda I_{n_1} - \mathsf{A}_s\left(e^{-\lambda h}\right) \\ \mathsf{C}_s\left(e^{-\lambda h}\right) \end{bmatrix}, N_f(\lambda) \triangleq \begin{bmatrix} \lambda I_{n_2} - A_4 \\ C_2 \end{bmatrix}, \lambda \in \mathbb{C}.$$

Applying the conditions of spectral observability from [8] to DS (3.1) and BLS (3.3), we obtain that the following propositions are valid.

Lemma 3.3. *The DS (3.1) is spectrally observable if and only if the following condition is satisfied*

$$rank\,\mathsf{N}_s\left(\lambda, e^{-\lambda h}\right) = n_1 \quad \forall\lambda \in \sigma_s. \tag{3.6}$$

Lemma 3.4. *The BLS (3.3) is spectrally observable if and only if the following condition is satisfied*

$$rank\,N_f(\lambda) = n_2 \quad \forall\lambda \in \sigma_f. \tag{3.7}$$

Along with conditions (3.6), (3.7) we formulate some more applicable conditions for the spectral observability of the subsystems, which simplify the procedure for checking this property. To do this, we define the matrix-valued function

$$\mathsf{P}_\mathsf{s}(z) \triangleq \left[C_s', C_s'\mathsf{A}_\mathsf{s}'(z), \ldots, C_s'\mathsf{A}_\mathsf{s}'^{\,n_1-1}(z)\right], \, z \in \mathbb{C}, \tag{3.8}$$

and the matrix

$$\mathsf{P}_\mathsf{f} \triangleq \left[C_2', C_2'\mathsf{A}_4', \ldots, C_2'\mathsf{A}_4'^{\,n_2-1}\right]. \tag{3.9}$$

Similar to Theorems 4, 5 [11] it can be proved

Lemma 3.5. *Let*

$$rank\ \mathsf{P}_s\left(e^{-\lambda h}\right) = n_1 \ \textit{for some}\ \lambda \in \mathbb{C}. \tag{3.10}$$

Then for this λ

$$rank\,\mathsf{N}_s\left(\lambda, e^{-\lambda h}\right) = n_1. \tag{3.11}$$

Lemma 3.6. *The following equality*

$$rank\,\mathsf{N}_f(\lambda) = n_2\,\forall\lambda \in \mathbb{C} \tag{3.12}$$

holds if and only if

$$rank\ \mathsf{P}_f = n_2. \tag{3.13}$$

Similar to Theorems 6, 7 [11] along with conditions of Theorems 3.3, 3.4 we formulate the conditions of subsystems spectral observability, which does not require the computation of all eigenvalues from (3.2), (3.4). To do this lets define the following sets of complex numbers:

$$\Lambda_s = \left\{\lambda \in \mathbb{C} : rank\ \mathsf{P}_\mathsf{s}\left(e^{-\lambda h}\right) < n_1\right\}, \, Z_s = \left\{z \in \mathbb{C} : rank\ \mathsf{P}_\mathsf{s}(z) < n_1\right\}.$$

Define also the set

$$\Omega_s \triangleq \sigma_s \cap \Lambda_s.$$

Lemma 3.7. *Let*

1) there exist $z \in \mathbb{C}$, $z \neq 0$, *that* $rank\ \mathsf{P}_s(z) = n_1$;
2) $rank\,\mathsf{N}_s(\lambda, z) = n_1\ \forall\lambda \in \Omega_s$, $z \in Z_s$.

Then the DS (3.1) is spectrally observable.

4. Decomposition of SPLTISD and its spectrum

Let apply to SPLTISD (2.1) the change of variables [10], [11]:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \mathsf{T}\left(\mu, e^{-ph}\right)\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix}, \quad \xi(t) \in R^{n_1},\ \eta(t) \in R^{n_2},\ t \in T, \tag{4.1}$$

$$\mathsf{T}\left(\mu, e^{-ph}\right) = \begin{pmatrix} I_{n_1} & \mu\,\mathsf{H}\left(\mu, e^{-ph}\right) \\ -\mathsf{L}\left(\mu, e^{-ph}\right) & I_{n_2} - \mu\,\mathsf{L}\left(\mu, e^{-ph}\right)\mathsf{H}\left(\mu, e^{-ph}\right) \end{pmatrix}, det\,\mathsf{T} \equiv 1. \tag{4.2}$$

Here $\mathsf{H}\left(\mu, e^{-ph}\right)$ and $\mathsf{L}\left(\mu, e^{-ph}\right)$ are matrix-valued operators depending on the parameter μ and satisfying the following matrix-valued functional equations:

$$\begin{aligned} &\mathsf{A}_3\left(e^{-ph}\right) - A_4\mathsf{L}\left(\mu, e^{-\lambda h}\right) + \mu\,\mathsf{L}\left(\mu, e^{-\lambda h}\right)\mathsf{A}_1\left(e^{-ph}\right) - \mu\,\mathsf{L}\left(\mu, e^{-\lambda h}\right)A_2\mathsf{L}\left(\mu, e^{-\lambda h}\right) = 0, \\ &\mu\left(\mathsf{A}_1\left(e^{-ph}\right) - A_2\mathsf{L}\left(\mu, e^{-\lambda h}\right)\right)\mathsf{H}\left(\mu, e^{-\lambda h}\right) - \\ &-\mathsf{H}\left(\mu, e^{-\lambda h}\right)\left(A_4 + \mu\,\mathsf{L}\left(\mu, e^{-\lambda h}\right)A_2\right) + A_2 = 0. \end{aligned} \tag{4.3}$$

Under assumption $\det A_4 \neq 0$ there exists a $\mu^* > 0$ such that for all $\mu \in [0, \mu^*]$ there is a continuously depending on μ solution $\mathsf{L}\left(\mu, e^{-ph}\right)$, $\mathsf{H}\left(\mu, e^{-ph}\right)$ of the equations (4.3), that could be represented in asymptotic series form:

$$\begin{aligned} \mathsf{L}\left(\mu, e^{-ph}\right) &= \sum_{i=0}^{k} \mu^i \mathsf{L}^i\left(e^{-ph}\right) + O\left(\mu^{k+1}\right), \\ \mathsf{H}\left(\mu, e^{-ph}\right) &= \sum_{i=0}^{k} \mu^i \mathsf{H}^i\left(e^{-ph}\right) + O\left(\mu^{k}\right), \end{aligned} \tag{4.4}$$

where terms of the asymptotic series (4.4) can be found according to the following iterative scheme:

$$\begin{aligned} &\mathsf{L}^{k+1}\left(e^{-ph}\right) = A_4^{-1}\left(\mathsf{L}^k\left(e^{-ph}\right)\mathsf{A}_1\left(e^{-ph}\right) - \sum_{j=0}^{k} \mathsf{L}^{k-j}\left(e^{-ph}\right)A_2\mathsf{L}^j\left(e^{-ph}\right)\right), \\ &\mathsf{H}^{k+1}\left(e^{-ph}\right) \\ &= A_4^{-1}\left(\mathsf{A}_1\left(e^{-ph}\right)\mathsf{H}^k\left(e^{-ph}\right) - A_2\sum_{i=0}^{k}\mathsf{L}^i\left(e^{-ph}\right)\mathsf{H}^{k-i}\left(e^{-ph}\right) - \sum_{i=0}^{k}\mathsf{H}^i\left(e^{-ph}\right)\mathsf{L}^{k-i}\left(e^{-ph}\right)A_2\right), \\ &\mathsf{L}^0\left(e^{-ph}\right) = A_4^{-1}\mathsf{A}_3\left(e^{-ph}\right),\ \mathsf{H}^0 = A_2A_4^{-1}. \end{aligned} \tag{4.5}$$

By $O(\mu)$ denote any vector function $f(t,\mu)$, $t \in [t_1, t_2]$, with the following property: there exist positive constants μ^* and c such that the Euclidian norm $|f(t,\mu)|$ satisfies the inequality $|f(t,\mu)| \leq c\mu$ for all $\mu \in (0, \mu^*]$ and all $t \in [t_1, t_2]$.

For $[0, \mu^*]$ by using the SPLTISD (2.1) matrix parameters and the matrix-valued functions $\mathsf{L}\left(\mu, e^{-ph}\right)$, $\mathsf{H}\left(\mu, e^{-ph}\right)$, we introduce the following continuously depending on μ matrix-valued functions

$$\begin{aligned} &\mathsf{A}_\xi\left(\mu, e^{-ph}\right) \triangleq \mathsf{A}_1\left(e^{-ph}\right) - A_2\mathsf{L}\left(\mu, e^{-ph}\right) \overset{(4.5),(3.5)}{=} \mathsf{A}_s\left(e^{-\lambda h}\right) + O(\mu), \\ &\mathsf{A}_\eta\left(\mu, e^{-ph}\right) \triangleq A_4 + \mu\,\mathsf{L}\left(\mu, e^{-ph}\right)A_2 \overset{(4.5)}{=} A_4 + O(\mu), \\ &\mathsf{C}_\xi\left(\mu, e^{-\lambda h}\right) \triangleq C_1 - C_2\mathsf{L}\left(\mu, e^{-ph}\right) \overset{(4.5),(3.5)}{=} \mathsf{C}_s\left(e^{-\lambda h}\right) + O(\mu), \\ &\mathsf{C}_\eta\left(\mu, e^{-ph}\right) \triangleq C_2 + \mu C_1\mathsf{H}\left(\mu, e^{-ph}\right) - \mu C_2\mathsf{L}\left(\mu, e^{-ph}\right)\mathsf{H}\left(\mu, e^{-ph}\right) \overset{(4.5)}{=} C_2 + O(\mu). \end{aligned} \tag{4.6}$$

As a result of the application to the system (2.1) of the transformation (4.1), taking into account (4.2), (4.6) the SPLTISD (2.1) goes into equivalent system with separated motions:

$$\begin{aligned}&\dot{\xi}(t) = \mathsf{A}_{\xi}\left(\mu, e^{-ph}\right)\xi(t),\, \xi \in \mathrm{R}^{n_1},\\ &\mu\dot{\eta}(t) = \mathsf{A}_{\eta}\left(\mu, e^{-ph}\right)\eta(t),\, \eta \in \mathrm{R}^{n_2},\\ &v(t) = \mathsf{C}_{\xi}\left(\mu, e^{-ph}\right)\xi(t) + \mathsf{C}_{\eta}\left(\mu, e^{-ph}\right)\eta(t),\, t \in T.\end{aligned} \tag{4.7}$$

Taking into account (4.6), we represent the system (4.7) as

$$\begin{aligned}&\dot{\xi}(t) = \left(\mathsf{A}_s\left(e^{-ph}\right) + O(\mu)\right)\xi(t),\\ &\mu\dot{\eta}(t) = \left(A_4 + O(\mu)\right)\eta(t),\, t \in T,\\ &v(t) = \left(\mathsf{C}_s\left(e^{-ph}\right) + O(\mu)\right)\xi(t) + \left(C_2 + O(\mu)\right)\eta(t),\, t \in T,\end{aligned} \tag{4.8}$$

whence it follows that decoupled system (4.7) is $O(\mu)$-close to the DS (3.1) and the BLS (3.3).

Proposition 4.1. [11] *For sufficiently small $\mu \in \left(0, \mu^0\right]$ the spectrum $\sigma(\mu)$ (2.4) of the SPLTISD (2.1) is separated into two disjoint parts:*

$$\sigma(\mu) = \sigma_x(\mu) \cup \sigma_y(\mu), \quad \sigma_x(\mu) \cap \sigma_y(\mu) = \emptyset.$$

The "slow" part

$$\sigma_x(\mu) = \left\{\lambda \in C : \det\left[\lambda I_{n_1} - \mathsf{A}_{\xi}\left(\mu, e^{-\lambda h}\right)\right] = 0\right\}$$

consists of an elements which for sufficient small μ are the functions $\lambda(\mu)$ that continuously depend on μ and tend to the elements of the DS (3.1) spectrum (3.2) as $\mu \to 0$:

$$\lim_{\mu \to 0} \lambda_i(\mu) = \lambda_{si} \in \sigma_s.$$

The fast part

$$\sigma_y(\mu) = \left\{\lambda \in C : \det\left[\lambda I_{n_2} - \mathsf{A}_{\eta}\left(\mu, e^{-\lambda h}\right)\right] = 0\right\}$$

consists of n_2 elements that tend to infinity, with the rate μ^{-1} and are of the form $\frac{\lambda_i(\mu)}{\mu}$, where

$$\lim_{\mu \to 0} \lambda_i(\mu) = \lambda_{fi} \in \sigma_f.$$

If in the spectrum σ_s (3.2) of the DS (3.1) there are no multiple values and in the spectrum σ_f (3.4) of BLS (3.3) there are no multiple values (it is allowed $\sigma_s \cap \sigma_f \neq \emptyset$), then the eigenvalues of the SPLTISD (2.1) are approximated as

$$\lambda_i(\mu) = \lambda_{si} + O(\mu),\ \lambda_{si} \in \sigma_s,\ \forall \lambda_i(\mu) \in \sigma_x,$$

$$\lambda_i(\mu) = \frac{\lambda_{fi} + O(\mu)}{\mu},\ \lambda_{fi} \in \sigma_f,\ \forall \lambda_i(\mu) \in \sigma_y.$$

5. Spectral observability of SPLTISD

Theorem 5.1. *If the DS (3.1) is spectrally observable and the BLS (3.3) is spectrally observable then $\exists \mu^* > 0$ that the SPLTISD (2.1), (2.2) is spectrally observable for all $\mu \in (0, \mu^*]$.*

Proof. By applying to (2.1), (2.2) the spectral observability condition [1], decoupling transformation [10], taking into account the invariance of the spectrum and preserving the matrix rank under nondegenerate transformations, it is determined that SPLTISD (2.1), (2.2) is spectrally observable at a given $\mu > 0$ if and only if the followiong condition is satisfied

$$rank\, \mathsf{N}_{\xi\,\eta}\left(\lambda, e^{-\lambda h}, \mu\right) \triangleq$$
$$= rank \begin{pmatrix} \lambda I_{n_1} - \mathsf{A}_\xi\left(\mu, e^{-\lambda h}\right) & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \mu\lambda I_{n_2} - \mathsf{A}_\eta\left(\mu, e^{-\lambda h}\right) \\ C_\xi\left(\mu, e^{-\lambda h}\right) & C_\eta\left(\mu, e^{-\lambda h}\right) \end{pmatrix} = n_1 + n_2\ \forall\lambda \in \sigma(\mu), \tag{5.1}$$

where due to (4.6)

$$\mathsf{N}_{\xi\,\eta}\left(\lambda, e^{-\lambda h}, \mu\right) =$$

$$= diag\{T^{-1}\left(\mu, e^{-ph}\right), I_m\} \cdot \mathsf{N}\left(\mu, e^{-\lambda h}\right) \cdot T\left(\mu, e^{-ph}\right) =$$
$$= \begin{pmatrix} \lambda I_{n_1} - \mathsf{A}_s\left(e^{-\lambda h}\right) + O(\mu) & 0_{n_1\times n_2} \\ 0_{n_2\times n_1} & \mu\lambda I_{n_2} - A_4 + O(\mu) \\ \mathsf{C}_s\left(e^{-\lambda h}\right) + O(\mu) & C_2 + O(\mu) \end{pmatrix}.$$

Let DS (3.1) is spectrally observable, i.e (3.6) be satisfied. Since $\forall\lambda_i(\mu) \in \sigma_x(\mu)$ it is true $\lambda_i(\mu) \underset{\mu\to 0}{\to} \lambda_{si} \in \sigma_s$, and taking into account the preservation of the matrix rank under small additive perturbations similar to Lemma 5 [11] it can be proved, that for sufficiently small $\mu > 0$

$$rank \begin{pmatrix} \lambda(\mu) I_{n_1} - \mathsf{A}_s\left(e^{-\lambda(\mu)h}\right) + O(\mu) \\ \mathsf{C}_s\left(e^{-\lambda h}\right) + O(\mu) \end{pmatrix} =$$
$$= rank \begin{pmatrix} \lambda(\mu) I_{n_1} - \mathsf{A}_\xi\left(\mu, e^{-\lambda(\mu)h}\right) \\ C_\xi\left(\mu, e^{-\lambda(\mu)h}\right) \end{pmatrix} = n_1 \quad \forall\lambda(\mu) \in \sigma_x(\mu). \tag{5.2}$$

Further, since for small $\mu > 0$ the sets $\sigma_x(\mu)$ and $\sigma_y(\mu)$ do not intersect, then, taking into account the preservation of the rank of the matrix under small additive perturbations, for any $\lambda = \lambda(\mu) \in \sigma_x(\mu) = \sigma(\mu) \setminus \sigma_y(\mu)$ and for sufficiently small $\mu > 0$

$$rank\left[\mu\lambda I_{n_2} - \mathsf{A}_\eta\left(\mu, e^{-\lambda h}\right)\right] = rank\left[\mu\lambda I_{n_2} - A_4 + O(\mu)\right] = n_2. \tag{5.3}$$

Combining (5.2), (5.3) we make sure that for all sufficiently small $\mu > 0$ $rank\, \mathsf{N}_{\xi\eta}(\lambda(\mu)) = n_1 + n_2$ for any $\lambda(\mu) \in \sigma_x(\mu)$.

Let BLS (3.3) is spectrally observable, i.e (3.7) be satisfied. Since $\forall\lambda(\mu) = \frac{1}{\mu}\lambda_i(\mu) \in \sigma_y(\mu)$ it is true $\lambda_i(\mu) \underset{\mu\to 0}{\to} \lambda_{fi} \in \sigma_f$, and taking into account the preservation of the rank of the matrix under small additive perturbations, the finiteness of the spectrum of the SP σ_f, for all sufficiently small $\mu > 0$ is true

$$rank \begin{pmatrix} \mu\lambda I_{n_2} - \mathsf{A}_\eta\left(\mu, e^{-\lambda h}\right) \\ C_\eta\left(\mu, e^{-\lambda h}\right) \end{pmatrix} =$$
$$= rank \begin{pmatrix} \mu\lambda(\mu) I_{n_2} - A_4 + O(\mu) \\ C_2 + O(\mu) \end{pmatrix} = n_2 \quad \forall \lambda(\mu) \in \sigma_y(\mu).$$

Further, since for small $\mu > 0$ the sets $\sigma_x(\mu)$ and $\sigma_y(\mu)$ do not intersect, then, taking into account the preservation of the rank of the matrix under small additive perturbations, for any $\lambda(\mu) \in \sigma_y(\mu) = \sigma(\mu) \setminus \sigma_x(\mu)$ and for sufficiently small $\mu > 0$

$$rank\left[\lambda(\mu) I_{n_1} - \mathsf{A}_\xi\left(\mu, e^{-\lambda(\mu)h}\right)\right] = rank\left[\lambda(\mu) I_{n_1} - \mathsf{A}_s\left(e^{-\lambda(\mu)h}\right) + O(\mu)\right] = n_2.$$

It follows from the last two conditions that for sufficiently small $\mu > 0$ (5.1) holds for any $\lambda(\mu) \in \sigma_y(\mu)$.

Also, since for $\lambda(\mu) \notin \sigma(\mu)$ $rank\, \mathsf{N}_{\xi\eta}(\lambda(\mu), e^{-\lambda(\mu)h}, \mu) = n_1 + n_2$, then condition (5.1) is true for any complex λ for all sufficiently small $\mu > 0$. This completes the proof. □

6. Conclusion

Rank-type sufficient conditions for the spectral observability of a singularly perturbed linear time-invariant system with delay in slow state variables by the decomposition of the system is proved. The conditions are expressed in terms of matrix parameters of systems of smaller dimensions than the original one, do not depend on a small parameter, and are valid for all sufficiently small values of it. Similar conditions for spectral controllability for SPLTISD (2.1) were proved in [10].

For the special case ($A_{11} = 0, A_{31} = 0$) SPLTISD (2.1) is a linear time-invariant singularly perturbed system without delay, the concept of spectral observability is equivalent to the concept of (complete) observability [4]. An approach to the analysis of the observability of such systems based on decomposition as a result of a nondegenerate change of variables of the Chang type was implemented in [4]. For $A_{11} = 0, A_{31} = 0$, the assertion of Theorem 5.1 coincides with the assertion from Theorem 6.2. [4, p.78].

Since the conditions of spectral observability coincide (see, for example, [3, 8]) or are part of the conditions of other types of observability ([9]) for systems with delay, the results of this work can be applied to formulate robust sufficient conditions of other types of observability for SPLTISD.

If conditions (3.6), (3.7) are considered only for $\lambda \in \sigma_s$, $\lambda \in \sigma_f$, with $Re\,\lambda > 0$, then they turn into detectability conditions of DS (3.1) and BLS (3.3). Therefore, the assertion of the Theorem 5.1 with obvious modifications is also valid for the detectability of SPLTISD (2.1).

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