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A DYNAMIC PROGRAMMING APPROACH TO OPTIMAL POLLUTION CONTROL UNDER UNCERTAIN IRREVERSIBILITY: THE POISSON CASE

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Dedicated to the memory of the 100th birthday of Professor Jack Warga

Abstract. We solve a bimodal optimal control problem with a non-concavity and uncertainty through a Poisson process underlying the transition from a mode to another. We use a dynamic programming approach and are able to uncover the global optimal dynamics (including optimal non-monotonic paths) under a few linear-quadratic assumptions, which do not get rid of the non-concavity of the problem. This is in contrast to the related literature on pollution control under irreversibility which usually explores local dynamics along monotonic solution paths to first-order Pontryagin conditions.

Keywords. Irreversible pollution; Multi-stage optimal control; Poisson process, HJB equations.

2020 Mathematics Subject Classification. 49L20, 90C39.

1. The Problem and Related Literature

Pollution irreversibility is being a crucial aspect of the current debate on sustainable development, in particular in relation with global warming. Pollution is irreversible when its impact on Nature and Humanity can no longer be reverted. There are strong reasons to believe that irreversible change due to global warming is under way as documented by Boucekkine et al. [1]. The problem is however under scrutiny since the 70s in hard sciences (for example, see Holling [5], for an example in ecology). It has become the object of deep investigation since the mid-90s in mathematical economics and operations research (see for example, Tsur and Zemel [9], and, in particular, Tahvonen and Withagen [8]).

A key complication in the mathematical treatment of optimization problems involving irreversible pollution is the induced non-concavity of the problem. For example, in Tahvonen and

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Withagen [8], pollution may turn irreversible because above a certain threshold level of pollution (the state variable), Nature self-cleaning capacity drops suddenly to zero, which makes the problem non-concave. As a result, beside multiple stationary states and potential complex dynamics, establishing the optimality of solution paths derived from first-order conditions may not be easy. Nonetheless, the related literature uses the standard hamiltonian-based Pontryagin method to tackle the optimization problems involved, at the cost of burdensome posterior elaborations in the best case (see in particular, the seminal work of Tahvonen and Withagen [8]. Together with the multi-stage (or multi-modal) intrinsic nature of the optimal control problems under study (as the associated state equations will feature two modes: reversible vs irreversible pollution), this makes the analysis highly tricky. Quite often, the analysis provided falls however short to identify the global optimal dynamics and to provide with deep non-local analysis.

In this paper, we propose a dynamic programming (DP) approach, which in our view fits better the structure of irreversible pollution control problems. While it will not of course eliminate the complexity of the problem, we will show that it does allow to provide with the full picture for (optimal) global dynamics. To this end, we consider the basic deterministic model studied in Tahvonen and Withagen [8]. We extend it by introducing uncertainty in the following way: we assume that the move from the reversible to irreversible pollution mode occurs through a Poisson process with constant arrival rate. This is probably the simplest stochastic extension of the basic model, it allows us to show in a way the flexibility of the DP approach followed.

There are a few papers studying irreversibility in stochastic environmental problems. The most known is due to Tsur and Zemel [9] who study the optimal pace of underground water extraction in the context where there exists a threshold of water reserves under which further extraction is no longer feasible. The threshold is unknown and it's assumed that it follows a random process with given distribution. Another interesting work taking this avenue has been proposed by Le Kama et al. [7], it is closer to our frame though both the mathematical and economic modelling are different. Here the state variable under pressure is the environmental quality: the irreversibility threshold for this variable is assumed to be reached at an uncertain time with given distribution.

As argued above, both papers do not study optimal global dynamics. In the case of Le Kama et al. [7], the results are derived for the steady state equilibria and their respective neighborhood. Moreover, both papers specialise in nondecreasing state variable paths. While this restriction makes economic sense, it also hides part of the complexity of the problem. By applying the DP approach to a generic irreversibility problem with Poisson arrival rates for the irreversible mode, we are able to produce the big picture of the optimal dynamics under a few linear-quadratic specifications (which do not remove the non-concave nature of our optimization problem). With respect to the literature quoted above, our main contributions are twofold: (1) we present the complete possible dynamics under different modes, which are essentially attracted or repelled by the two potential long-run steady states of the two modes, showing the potential emergence of optimal non-monotonic dynamics, and (2) we investigate when the irreversible regime, under Poisson process, can be triggered or not.

The rest of the paper is organized as following. Section 2 describes the model and Section 3 provides solutions to the optimal control problems in two difference modes– reversible and irreversible environmental regimes via dynamic programming. Section 4 presents possible outcomes from the solutions and draws the main contributions.

2. THE MODEL

Following Tahvonen and Withagen [8], we investigate a situation where the decision maker faces irreversible pollution accumulation. For simplicity, the pollution emission, $y(t)$, is used to measure the output level. The objective of the decision maker is to maximize social welfare:

$$\max_y W = \int_0^{+\infty} (U(y) - D(z)) e^{-rt} dt, \tag{2.1}$$

where r is time preference, $z(t)$ is accumulated pollution, $U(y)$ is the utility from enjoying final output generated with pollution $y(t)$, and $D(z)$ is damaging function from aggregate pollution stock z. Pollution stock $z(t)$ may decay at rate $\delta(z)$. However, the decay rate may spontaneously and irreversibly drop to zero. In other words, the pollution accumulation is given by the following:

$$\dot{z} = y - \delta(z), \;\; z(0) = z_0 \;\text{ given}, \tag{2.2}$$

where $\delta(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is the decay function which can abruptly drops to zero. After the drop, no decay is possible. Hence, there are two modes, with and without decay, denoted by $m = 1$ and 0, respectively. The jump from mode 1 to 0 when the state variable takes the value z occurs with rate

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \Pr\{m(t + \Delta t) = 0 | m(t) = 1\} = q(z). \tag{2.3}$$

In other words, the probability of the mode change during the interval $(t, t + \Delta t]$, given that state z and the mode at t is 1, is proportional to Δt, that is, the arrival of the irreversible regime follows with intensity $q(z) \geq 0$. Obviously, when $q(z) = 0$, no regime change happens.

As a result, the planner's optimal control problem are divided in periods I and II, corresponding to modes 1 and 0, respectively, as follows.
Period I.

$$\dot{z} = y - \delta(z), \;\; z(0) = z_0.$$

Period II.

$$\dot{z} = y, \;\; z(T) = z\left(T^-\right),$$

where T is the time of mode switching.

2.1. A special case.

In the most part of the paper we focus on the following linear-quadratic functional forms to obtain closed form solutions and equilibria,

$$U(y) = ay - \frac{y^2}{2}, \qquad D(z) = -\frac{c}{2} z^2, \qquad q(z) = \lambda \tag{2.4}$$

and

$$\delta(z) = \max\{\alpha - \beta z, 0\} \tag{2.5}$$

where a, c, λ, α, and β are positive constants.

3. THE OPTIMAL CHOICES

The proposed model is a piecewise-deterministic process which has been studied for decades. General framework have been layed out and existence of solutions, in particular, viscosity solutions, have been provided in the literature. (See, e.g., Davis (1984) [2], Vermes (1985) [10], Lenhart and Yamada (1992) [6], Farid (1997) [4], and Dockner *et al.* (2000) [3], among others.) An excellent exposition can be found in Dockner *et al.* (2000) [3, Chapter 8].

3.1. **Hamiltonians.** We formulate the HJB equation for the value functions $V_m(z)$ in mode $m(=0,1)$ as follows. After mode change so that there is no decay, $V_0(z)$ for any $z>0$ is the optimal welfare,

$$V_0(z)=\sup_{y(\cdot)\in\mathscr{U}_0}\int_0^\infty [U(y(t))-D(w(t))]e^{-rt}dt,$$

where $\mathscr{U}_m$ is the set of feasible controls in Mode m for $m=0,1$, and $w(t)$ satisfies

$$\dot{w}=y(t), \qquad w(0)=z.$$

On the other hand, before mode change, the value function $V_1(z)$ is

$$V_1(z)=\sup_{y(\cdot)\in\mathscr{U}_1}\mathbb{E}\int_0^T [U(y(t))-D(w(t))]e^{-rt}dt+e^{-rT}V_0(w(T)),$$

where $w(t)$ satisfies

$$\dot{w}=y(t)-\delta(w(t)), \qquad w(0)=z.$$

Define functions f_m by

$$f_m(z,y)=y-m\delta(z) \qquad \text{for } m=0,1, \tag{3.1}$$

and let the Hamiltonians $H_m(z,p)$ in mode m be

$$H_m(z,p)=U(y_m^*)-D(z)+pf_m(z,y_m^*),$$

where y^* is the maximizer

$$y_m^*=\arg\max_{y\in U_m}\{U(y)-D(z)+pf_m(z,y)\}, \tag{3.2}$$

where U_m is the set of feasible controls at state value z and Mode $m\in\{0,1\}$. By the standard dynamic programming, we find the HJB equation for V_0 in Mode 0 has the form

$$rV_0(z)=\max_{y\in U_0}\left\{U(y)-D(z)+V_0'(z)y\right\}\equiv H_0\left(z,V_0'(z)\right). \tag{3.3}$$

To derive the HJB equation for V_1, we use Theorem 8.1 in [3] to obtain

$$rV_1(z)=\max_{y\in U_1}\left\{U(y)-D(z)+V_1'(z)[y-\delta(z)]+q(z)[V_0(z)-V_1(z)]\right\},$$

where $q(z)$ is given by (2.3). The above equation can be written as

$$(r+q(z))V_1(z)=H_1\left(z,V_1'(z)\right)+q(z)V_0(z), \tag{3.4}$$

where

$$H_1\left(z,V_1'(z)\right)=\max_{y\in U_1}\left\{U(y)-D(z)+V_1'(z)[y-\delta(z)]\right\}.$$

In the special case (2.4) and (2.5), the Hamiltonians take the form

$$\begin{aligned} H_0(z,p) &= ay_0^*-\frac{1}{2}(y_0^*)^2-\frac{c}{2}z^2+py_0^*, \\ H_1(z,p) &= ay_1^*-\frac{1}{2}(y_1^*)^2-\frac{c}{2}z^2+p[y_1^*-\delta(z)]. \end{aligned}$$

The feasible controls U_m are the nonnegative values of y_m. Hence,

$$y_m^*=\max\{a+p,0\} \qquad \text{for } m=0,1.$$

In this case HJB equations (3.3) and (3.4) become

$$2rV_0(z) = \begin{cases} \left(a+V_0'(z)\right)^2 - cz^2 & \text{if } V_0'(z) > -a, \\ -cz^2 & \text{if } V_0'(z) \le -a, \end{cases} \tag{3.5}$$

and

$$2(r+\lambda)V_1(z) = \begin{cases} \begin{array}{c} (a+V_1'(z))^2 - 2\delta(z)V_1'(z) \\ -cz^2 + 2\lambda V_0(z) \end{array} & \text{if } V_1'(z) > -a, \\ -2\delta(z)V_1'(z) - cz^2 + 2\lambda V_0(z) & \text{if } V_1'(z) \le -a, \end{cases} \tag{3.6}$$

respectively.

3.2. **Value function in Mode 0.** The value function V_0 in mode 0 satisfies the first equation in (3.5). One of the solutions is piecewise quadratic in the form

$$V_0(z) = \begin{cases} \frac{A}{2}z^2 + Bz + C & \text{for } z < \bar{z}_0, \\ -\frac{cz^2}{2r} & \text{for } z \ge \bar{z}_0, \end{cases}$$

where

$$\bar{z}_0 = \frac{ra}{c}, \tag{3.7}$$

and the coefficients A, B, and C satisfies

$$rA = A^2 - c, \quad rB = A(B+a), \quad 2rC = (B+a)^2.$$

The quadratic equation for A has two roots,

$$y_1 = \frac{r - \sqrt{r^2+4c}}{2}, \qquad y_2 = \frac{r+\sqrt{r^2+4c}}{2}. \tag{3.8}$$

We use the negative one, y_1. As the result,

$$A = y_1, \quad B = \frac{y_1 a}{r - y_1}, \quad C = \frac{(B+a)^2}{2r}.$$

Hence,

$$2rV_0(z) = \left(y_1 z + \frac{y_1 a}{r-y_1} + a\right)^2 - cz^2.$$

Since

$$\begin{aligned} \frac{a}{r-y_1} + \frac{a}{y_1} &= \frac{ar}{(r-y_1)y_1} = \frac{4ar}{\left(r+\sqrt{r^2+4c}\right)\left(r-\sqrt{r^2+4c}\right)} \\ &= -\frac{ar}{c} = -\bar{z}_0, \end{aligned}$$

it follows that

$$2rV_0(z) = (z-\bar{z}_0)^2 y_1^2 - cz^2$$

for $z < \bar{z}_0$. As a result,

$$V_0(z) = \begin{cases} \frac{1}{2r}\left[(z-\bar{z}_0)^2 y_1^2 - cz^2\right] & \text{for } z < \bar{z}_0, \\ -\frac{c}{2r}z^2 & \text{for } z \ge \bar{z}_0. \end{cases} \tag{3.9}$$

3.3. **Value function in Mode 1.** For $z \geq \bar{z}$ where $\bar{z}$ is the minimum zero of $\delta(z)$, there is no difference whether mode changes or not. So $V_1(z) = V_0(z)$ for such z. In particular,

$$V_1(\bar{z}) = V_0(\bar{z}) \tag{3.10}$$

which serves as the transversality condition. Hence, one only needs solve $V_1(z)$ for $z < \bar{z}$ from the HJB equation (3.6). The HJB equation is linear if $V_1'(z) \leq -a$ and nonlinear if $V_1'(z) > -a$. In either case there exist linear or quadratic solutions to the HJB equation. (These solutions are indeed derived below near the end of this section.) However, only in special cases these quadratic solutions satisfy (3.10). In the case the quadratic solutions fail to satisfy (3.10), the solution must be solved from the boundary value problem (3.6) and (3.10). As to be seen below, such a solution is hyperbolically shaped.

To solve the boundary value problem for $V_1(z)$, we first solve the associated equation for $f_1(z)$ defined by (3.1) with $m = 1$, which is the pollution accumulation rate in Mode 1. By definition,

$$f_0(z) = V_0'(z) + a, \qquad f_1(z) = V_1'(z) + a - \delta(z).$$

To derive a differential equation for f_1, we differentiate the both sides of (3.6) with respect to z to obtain

$$f_1(z) f_1'(z) = (r+\lambda) f_1(z) + (r-\beta+\lambda)\delta(z) + (\beta - r) a + cz - \lambda f_0(z) \tag{3.11}$$

if $f_1(z) > -\delta(z)$ and

$$-\delta(z) f_1'(z) = (r-\beta+\lambda) f_1(z) + (r-2\beta+\lambda)\delta(z) + (\beta - r) a + cz - \lambda f_0(z) \tag{3.12}$$

if $f_1(z) \leq -\delta(z)$.

The boundary value $f_1(\bar{z})$ is determined by the assumption

$$V_1(\bar{z}) = V_0(\bar{z}), \qquad \delta(\bar{z}) = 0, \tag{3.13}$$

as follows. If $f_0(\bar{z}) > 0$, by (3.5)

$$2rV_0(\bar{z}) + c\bar{z}^2 = (f_0(\bar{z}))^2 > 0.$$

By (3.6) and (3.13) it follows that

$$2(r+\lambda)V_1(\bar{z}) - 2\lambda V_0(\bar{z}) + c\bar{z}^2 = 2rV_0(\bar{z}) + c\bar{z}^2 > 0.$$

In view of (3.6), the above inequality implies that $f_1(\bar{z}) > 0$ and

$$\begin{aligned}(f_1(\bar{z}))^2 &= 2(r+\lambda)V_1(\bar{z}) - 2\lambda V_0(\bar{z}) + c\bar{z}^2 \\ &= 2rV_0(\bar{z}) + c\bar{z}^2 = (f_0(\bar{z}))^2.\end{aligned}$$

Hence, $f_1(\bar{z}) = f_0(\bar{z})$. If $f_0(\bar{z}) \leq 0$, by (3.5),

$$V_0(\bar{z}) = -\frac{c\bar{z}^2}{2r}.$$

By (3.6) and (3.13) again it follows that $f_1(\bar{z}) \leq 0$. From (3.9) it can be seen that $f_0(\bar{z}) \leq 0$ if and only if $\bar{z} \geq \bar{z}_0$ and in this case

$$f_0(\bar{z}) = a - \frac{c\bar{z}}{r}. \tag{3.14}$$

We use an asymptotic expansion to find $f_1(\bar{z})$ in this case. Suppose

$$V_1(z) = \frac{A_1}{2}(z-\bar{z})^2 + B_1(z-\bar{z}) + V_0(\bar{z}) + o\left((z-\bar{z})^2\right)$$

and assume that $V_1(z)$ is differentiable at $\bar{z}$. Substituting the right-hand side into (3.6), and using

$$\delta(z) = -\beta(z-\bar{z}), \qquad V_0(z) = -\frac{cz^2}{2r},$$

we abtain

$$\begin{aligned}&(r+\lambda)\left[A_1(z-\bar{z})^2 + 2B_1(z-\bar{z}) + 2V_0(\bar{z})\right] \\ =\ & 2\beta(z-\bar{z})\left[A_1(z-\bar{z})+B_1\right] - c\left(1+\frac{\lambda}{r}\right)z^2 + o\left((z-\bar{z})^2\right).\end{aligned}$$

By comparing coefficients of powers of $z-\bar{z}$, we find

$$\begin{aligned}(r+\lambda)A_1 &= 2A_1\beta - c\left(1+\frac{\lambda}{r}\right), \\ 2(r+\lambda)B_1 &= 2B_1\beta - 2c\left(1+\frac{\lambda}{r}\right)\bar{z}.\end{aligned}$$

This leads to

$$f_1(\bar{z}) = a + B_1 = a - \frac{c(r+\lambda)\bar{z}}{r(r+\lambda-\beta)}. \tag{3.15}$$

It is clear that $f_1(\bar{z}) < f_0(\bar{z}) < 0$ if $\bar{z} > \bar{z}_0$ and $\beta < r+\lambda$.

We now solve the boundary value problem associated with the equations (3.11) and (3.12). We first consider (3.11). Since $f_0(z)$ is linear, the equation can be written in the form

$$f_1(z)f_1'(z) = (r+\lambda)f_1(z) + B_2 z + C_2$$

where

$$\begin{aligned}B_2 &= \beta(\beta-r-\lambda) + c - \lambda y_1, \\ C_2 &= (r-\beta+\lambda)\alpha + (\beta-r)a + \lambda y_1\bar{z}_0.\end{aligned} \tag{3.16}$$

Using a change of variable $x = z + C_2/B_2$, the equation becomes

$$W_1(x)W_1'(x) = (r+\lambda)W_1(x) + B_2 x.$$

where $W_1(x) = f_1(z)$. The general solution in implicit form is

$$|W_1 - xY_1|^{p_1}|W_1 - xY_2|^{p_2} = K$$

where C is a constant,

$$\begin{aligned}Y_1 &= \frac{1}{2}\left[r+\lambda-\sqrt{(r+\lambda)^2+4B_2}\right], \\ Y_2 &= \frac{1}{2}\left[r+\lambda+\sqrt{(r+\lambda)^2+4B_2}\right],\end{aligned} \tag{3.17}$$

and

$$p_1 = \frac{Y_1}{Y_2-Y_1}, \qquad p_2 = \frac{-Y_2}{Y_2-Y_1}.$$

This leads to

$$|f_1(z) - (z-\bar{z}_1)Y_1|^{p_1}|f_1(z) - (z-\bar{z}_1)Y_2|^{p_2} = K,$$

where

$$\bar{z}_1 = -\frac{C_2}{B_2} = -\frac{(r-\beta+\lambda)\alpha + (\beta-r)a + \lambda y_1\bar{z}_0}{\beta(\beta-r-\lambda)+c-\lambda y_1}. \tag{3.18}$$

Obviously if there were no mode change, $\bar{z}_1$ would be the "potential" long-run steady state in the reversible environmental regime. Within the current framework there is either uncertain

Poisson process or pollution accumulation across the threshold $\bar{z}$, nevertheless, $\bar{z}_1$ plays an important role in determining the trajectory of the dynamics. The detail results will be presented in the following Theorem 4.1 and 4.4.

Note that Eq. (3.12) for $f_1(z) \le -\delta(z)$ is linear, which can be readily solved. With $f_1(z)$ solved, one can find V_1 from the second equation of (3.6) as

$$V_1(z) = \frac{1}{2(r+\lambda)}\left[f_1(z)^2 - \delta(z)^2 + 2a\delta(z) - cz^2 + 2\lambda V_0(z)\right]. \tag{3.19}$$

In the special case where $K=0$, there are two value functions $V_{1,i}(z)$ with

$$f_{1,i}(z) = (z-\bar{z}_1)Y_i \qquad \text{for } i=1,2.$$

These value functions may not match the value functions $V_0(z)$ in Mode 0 at $\bar{z}$. In other words, the linear-quadratic autonomous system may not generate linear state strategy in Mode 1, even though in the Mode 0, the strategy is linear in state variable. In the general case, $K>0$, and so, the solution is hyperbolic shaped. More precisely, due to the transversality condition at $\bar{z}$, $V_0(\bar{z}) = V_1(\bar{z})$, linear strategy, thus linear-quadratic value functions, may not hold in both regimes at the same time.

We comment that the value function $V_1(z)$ constructed above is continuous but not smooth at $\bar{z}$ in the case $\bar{z} > \bar{z}_0$. This can be seen from (3.15) which leads to

$$V_1'(\bar{z}) = f_1(\bar{z}) - a \neq f_0(\bar{z}) - a = V_0'(\bar{z}).$$

In the case where $\bar{z} < \bar{z}_0$, as to be seen in the next section, the physically reasonable solutions can be non-smooth at certain point (such as $\bar{z}_1^*$ in Theorem 4.1). Nevertheless, In all such cases $f_1(z)$ is bounded, which implies that $V_1(z)$ is Lipschitz continuous. In such cases the verification theorem, e.g. [3, Theorem 8.1], is still valid. (See the comments after Theorem 8.1 in [3].)

4. Possible Outcomes

As shown above, in Period II, the only possible steady state is $\bar{z}_0$ given by (3.7). The outcome depends on whether the threshold value $\bar{z}$ is below or above $\bar{z}_0$. In each case we construct a physically meaningful value function $V_1(z)$ and derive behavior of the stock of pollution $z(t)$ based on the value function.

By "physically meaningful value function" we mean a value function that is defined for all z and is non-increasing in z. As can be seen from the HJB equation (3.5), the only physically meaningful value function in Mode 0, $V_0(z)$, is the one given by (3.9). The associated pollution accumulation rate, $f_0(z)$, is affine. On the other hand, due to the transversality condition, often a physically meaningful value function in Mode 1, $V_1(z)$, needs to be constructed with discontinuous rate, $f_1(z)$. Such a value function is still Lipschitz continuous, and the point of discontinuity of $f_1(z)$ is a (usually unstable) steady state of the system dynamics.

4.1. **Case 1:** $\bar{z} < \bar{z}_0$.

In this case $f_0(\bar{z}) > 0$. As a result, the stock of pollution, $z(t)$, increases and converges to $\bar{z}_0$ in Period II. We give possible behavior of $z(t)$ in Period I in the following theorem.

Theorem 4.1. *Suppose $\bar{z} < \bar{z}_0$. The following results are true.*

(1) *Suppose either*

$$\bar{z}_1 > \bar{z}, \qquad Y_1 > 0, \tag{4.1}$$

or

$$\bar{z}_1 < \bar{z}, \qquad Y_1 > 0, \qquad (\bar{z} - \bar{z}_1) Y_2 < (\bar{z} - \bar{z}_0) y_1 \tag{4.2}$$

or

$$\bar{z}_1 < \bar{z}, \qquad Y_1 < 0. \tag{4.3}$$

Furthermore, suppose $|Y_1|$ is sufficiently small. Then there exist $\bar{z}_1^* < \bar{z}$ (possibly negative) such that for any $z_0 < \bar{z}_1^*$, $z(t)$ is decreasing in Period I, and for any z_0 that satisfies $\bar{z}_1^* < z_0 < \bar{z}$, $z(t)$ is increasing in Period I. In the case where $\bar{z}_1^* < 0$, $z(t)$ is increasing in Period I for any $z_0 < \bar{z}$.

(2) *Suppose $\bar{z}_1 > \bar{z}$ and $Y_1 < 0$. Then $z(t)$ is increasing in Period I for any $z_0 < \bar{z}$.*

(3) *Suppose $\bar{z}_1 < \bar{z}$ and either*

$$Y_1 > 0, \qquad (\bar{z} - \bar{z}_1) Y_1 \le (\bar{z} - \bar{z}_0) y_1 \le (\bar{z} - \bar{z}_1) Y_2 \tag{4.4}$$

or

$$Y_1 < 0, \qquad (\bar{z} - \bar{z}_1) Y_2 = (\bar{z} - \bar{z}_0) y_1. \tag{4.5}$$

Then for any $z_0 < \bar{z}_1$, $z(t)$ is decreasing and for any z_0 such that $\bar{z}_1 < z_0 < \bar{z}$, $z(t)$ is increasing in Period I.

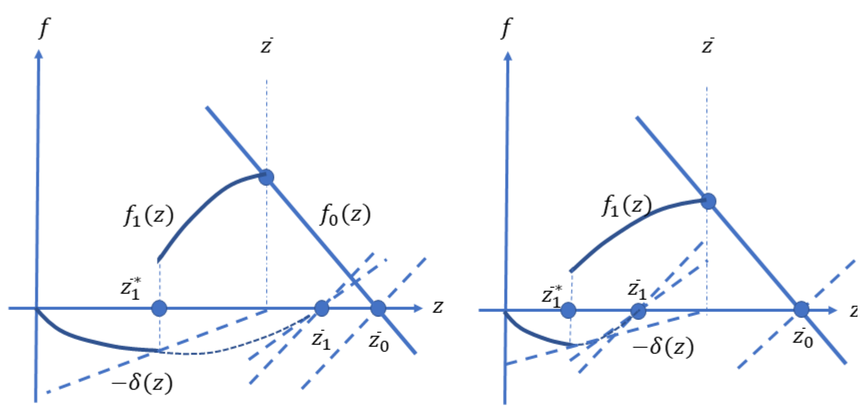

FIGURE 1. Type 1 with (4.1) holds (left) and with (4.2) holds (right).

Proof. Since in Mode 0, there is no decay of pollution, it follows that $V_1(z) = V_0(z)$ for $z \ge \bar{z}$. In addition, as shown in Subsection 3.3,

$$f_1(\bar{z}) = f_0(\bar{z}) > 0. \tag{4.6}$$

By continuity, $f_1(z) > 0$ for some $z < \bar{z}$. Note that Eq. (3.11) is equivalent to the differential equations

$$\begin{aligned} dx/d\tau &= (r+\lambda)x + (r - \beta + \lambda)\delta(z) + (\beta - r)a + cz - \lambda f_0(z), \\ dz/d\tau &= x \end{aligned} \tag{4.7}$$

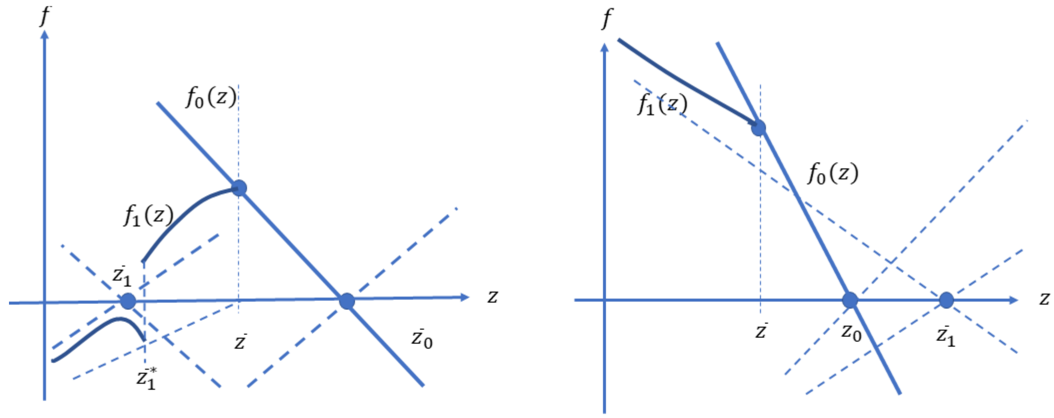

FIGURE 2. Type 1 with (4.3) holds and $|Y_1|$ sufficiently small (left) and Type 2 with $Y_1 < 0$ and $\bar{z}_1 > \bar{z}$ (right).

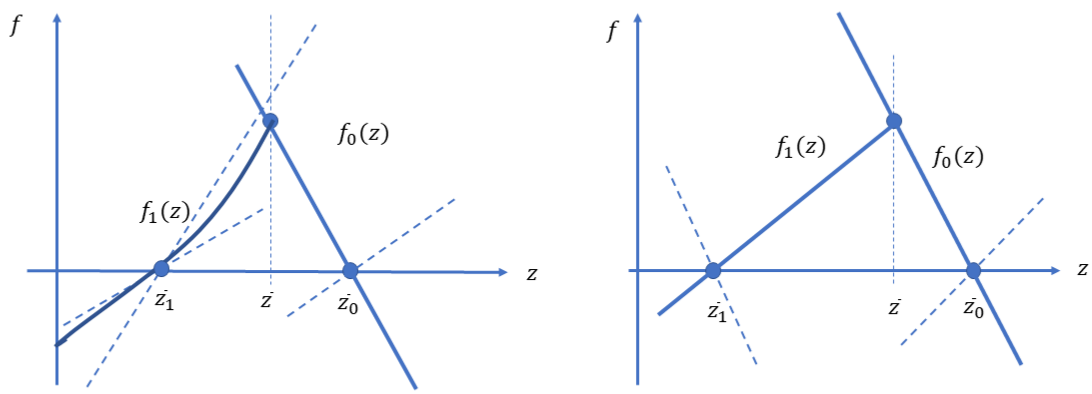

FIGURE 3. Type 3 with (4.4) holds (left) and with (4.5) holds (right).

and $(0,\bar{z}_1)$ is an equilibrium of this autonomous dynamical system. (Here, τ has nothing to do with the time variable, t.)Its Jacobian matrix takes the form

$$J = \begin{pmatrix} r+\lambda & B_2 \\ 1 & 0 \end{pmatrix}, \tag{4.8}$$

where B_2 is defined in (3.16). Matrix J has eigenvalues Y_1 and Y_2 given by (3.17). Every trajectory $(x(t), z(t))$ associates a value function with $f_1(z)$ that satisfies $f_1(z(t)) = x(t)$. Since value functions are defined for all z, only trajectories whose range include all $z \geq 0$ are acceptable. These include stable and unstable manifolds.

It can be seen that $f_1(\bar{z}_1) = 0$ if and only if the trajectory is either a stable or unstable manifold. Furthermore, the eigenvectors corresponding the eigenvalue Y_i are parallel to the vector $\langle 1, Y_i \rangle$ for $i = 1, 2$. In particular, at least one unstable manifold emanating from the equilibrium with a positive angle to the z-axis. By optimality, $f_1(z)$ is the least provided that $f_1(z)$ is defined for $z \in [0, \bar{z}]$. On the other hand, without assuming that the value function V_1 to be differentiate, we can have f_1 discontinuous at certain points.

Part 1. Suppose (4.1) holds. Then the trajectory (z,x) that passes through $(\bar{z}, f_0(\bar{z}))$ intersects the z-axis at some $\hat{z} < \bar{z}$ and approaches the equilibrium $(\bar{z}_1, 0)$ along Y_1 as $\tau \to -\infty$. Hence, there is a point $\bar{z}_1^*$, between $\hat{z}$ and $\bar{z}$, such that $f_1(\bar{z}_1^*) = \delta(\bar{z}_1^*)$. Either $\bar{z}_1^* \leq 0$ or $\bar{z}_1^* > 0$. In the former case, $f_1(z) > \delta(z) > 0$ on the interval $(0, \bar{z})$. Hence, $z(t)$ is increasing in Period I for any $z_0 \in (0, \bar{z})$. In the latter case, for Y_1 sufficiently small, there is a trajectory passing through the point $(\bar{z}_1^*, -\delta(\bar{z}_1^*))$ such that $x < 0$ for $z \in (0, \bar{z}_1^*)$. (See the left graph in Fig. 1.) We define $f_1(z)$ by this trajectory on $(0, \bar{z}_1^*)$. This solution is defined for all $z \in (0, \bar{z})$.

In the case where (4.2) holds, the construction is similar. The last inequality in (4.2) implies that the trajectory that passes through $(\bar{z}, f_0(\bar{z}))$ does not converge to $(\bar{z}_1, 0)$ as $\tau \to -\infty$, but instead it intersects the z-axis somewhere to the left of $\bar{z}$. Hence, again there is $\bar{z}_1^*$ at which $f_1(\bar{z}_1^*) = \delta(\bar{z}_1^*)$. We then let the trajectory that passes through $(\bar{z}_1^*, -\delta(\bar{z}_1^*))$ to define $f_1(z)$ for $z < \bar{z}_1^*$ if $\bar{z}_1^* > 0$. (See the right graph in Fig. 1).

If (4.3) holds, then since $|Y_1|$ is sufficiently small, there is a point $\bar{z}_1^* < \bar{z}$ such that

$$f_1(\bar{z}_1^*) = \delta(\bar{z}_1^*) > |Y_1|(\bar{z}_1^* - \bar{z}_1).$$

As a result trajectory that passes through the point $(\bar{z}_1^*, -\delta(\bar{z}_1^*))$ will be below the z-axis for $z < \bar{z}_1^*$. We let $f_1(z)$ be defined by this trajectory on $(0, \bar{z}_1^*)$. (See the left graph in Fig. 2.) This solution is defined for all $z \in [0, \bar{z}]$.

Since in each case $f_1(z) < 0$ if $z < \bar{z}_1^*$ and $f_1(z) > 0$ if $\bar{z}_1^* < z < \bar{z}$, $z(t)$ is decreasing if $z_0 < \bar{z}_1^*$ and it is increasing if $\bar{z}_1^* < z_0 < \bar{z}$. This proves Part 1 of the theorem.

Part 2. Suppose $\bar{z}_1 > \bar{z}$ and $Y_1 < 0$. Since $f_1(\bar{z}) > 0$, the trajectory that passes through $(\bar{z}, f_1(\bar{z}))$ cannot intersect the z-axis to the left of $\bar{z}$. Therefore, $f_1(z) > 0$ for $0 \leq z \leq \bar{z}$. (See the right graph in Fig. 2.) It is clear that $\bar{z}_0$ is an attractor in Period II. This proves Part 2.

Part 3. Suppose (4.4) holds. Then the trajectory that passes through $(\bar{z}, f_0(\bar{z}))$ converges to $(\bar{z}_1, 0)$ as $\tau \to -\infty$, along the eigenvector $\langle 1, Y_1 \rangle$. For $z < \bar{z}_1$ we let $f_1(z)$ be defined by the unstable manifold in the direction $\langle 1, Y_1 \rangle$. Hence, $f_1(z) < 0$ if $0 < z < \bar{z}_1$ and $f_1(z) > 0$ if $\bar{z}_1 < z < \bar{z}$. (See the left graph of Fig. 3.) Suppose (4.5) holds. Then the unstable manifold from the equilibrium $(\bar{z}_1, 0)$ defines a solution $f_1(z)$ that is negative for $0 < z < \bar{z}_1$ and is positive for $\bar{z}_1 < z < \bar{z}$. (See the right graph of Fig. 3.) In both cases the conclusion in Part 3 follows.

This completes the proof.

The above theorem shows that in general in Period I, there is $\bar{z}_1^* < \bar{z}$ such that the stock of pollution, $z(t)$, is decreasing toward zero if $0 \leq z_0 < \bar{z}_1^*$ and it is increasing toward $\bar{z}$ if $\bar{z}_1^* < z_0 < \bar{z}$. Hence, the threshold, $\bar{z}$ is reached in finite time in Period I if $z_0 > \bar{z}_1^*$. In addition, since $f_1(z)$ is discontinuous at $\bar{z}_1^*$ in type 1, the value function $V_1(z)$ has a corner point. In contrast, $f_1(z)$ is continuous at the repeller $\bar{z}_1$, and hence there is no corner point on the value function. We show graphs of the value functions for types in Figure 4 for comparison.

To close this case, recall mentioned above that generally under multistage optimal control (and differential game) problems with endogenous stage changes, even with linear-quadratic autonomous framework, there is no guarantee that linear-state optimal control, thus linear-quadratic value functions, are possible in mode 1 because of the transversality condition between the two modes. Nonetheless, Part 3 of Theorem 4.1 shows that under the condition (4.5) there exists one group of linear-state dependent optimal choices, and thus linear-quadratic value functions, in both mode 0 and 1. The method provided above could be applied to other studies of multistage (or multi-mode) optimal control and differential game. We conclude the results in the following.

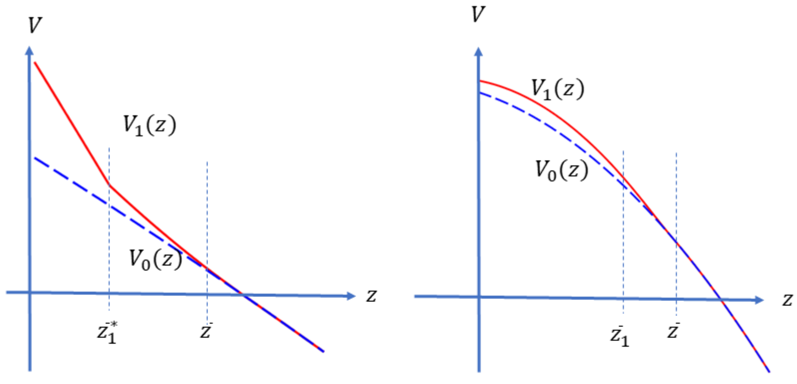

FIGURE 4. Value functions $V_0(z)$ and $V_1(z)$ for Type 1 (left) and Type 3 (right). A corner point is present in the graph of $V_1(z)$ for Type 1, but not for Type 3.

Proposition 4.2. *Suppose $\bar{z} < \bar{z}_0$ and $(\bar{z}-\bar{z}_0)\, y_1 = (\bar{z}-\bar{z}_1)\, Y_2$ hold. Then there exists linear-state dependent optimal pollution control in both mode $m = 0$ and $m = 1$ which is given by: for any $z \geq 0$,*

$$\begin{cases} y_0^*(z) = f_0(z) = (z-\bar{z}_0)y_1, \\ y_1^*(z) = f_1(z) + \delta(z) = (z-\bar{z}_1)Y_2 + \delta(z). \end{cases} \tag{4.9}$$

The corresponding value functions are

$$\begin{cases} V_0(z) = \dfrac{1}{2r}\left[(z-\bar{z}_0)^2 y_1^2 - cz^2\right], \\ V_1(z) = \dfrac{1}{2(r+\lambda)}\left[f_1^2(z) - \delta^2(z) + 2a\delta(z) - cz^2 + 2\lambda V_0(z)\right], \end{cases} \tag{4.10}$$

provided $V_1(z)$ is concave in term of z. Furthermore, the time to reach $\bar{z}$ from any z_0 on the interval $(\bar{z}_1, \bar{z})$ is given by

$$T = \int_{z_0}^{\bar{z}} \frac{dz}{(z-\bar{z}_1)Y_2} = \frac{1}{Y_2}\ln\left(\left|\frac{\bar{z}-\bar{z}_1}{z_0-\bar{z}_1}\right|\right). \tag{4.11}$$

Proof. From Part 3 of Theorem 4.1 we see that the linear function

$$f_1(z) = (z-\bar{z}_1)\, Y_2$$

satisfy (3.11). Hence (4.9) and (4.10) hold. In addition, (4.11) follows from the definition $\dot{z} = f_1(z)$ in Period I. This completes the proof.

Remark 4.3. Assumption $(\bar{z}-\bar{z}_0)\, y_1 = (\bar{z}-\bar{z}_1)\, Y_2$ is not indicating one special point, rather a manifold which satisfies this equality condition.

4.2. **Case 2: $\bar{z} > \bar{z}_0$.** In this case, $f_0(\bar{z})$ and $f_1(\bar{z})$ are given by (3.14) and (3.15) respectively, which are both negative. As a result, either $z(t)$ is decreasing in the entire Period I if $z_0 < \bar{z}$ and is near $\bar{z}$. Hence, $\bar{z}$ is never reached. We show that under certain conditions $z(t)$ is decreasing for any $z_0 < \bar{z}$.

Theorem 4.4. *Suppose that $\bar{z} > \bar{z}_0$ and that $\beta < r + \lambda$.*

(1) If $\bar{z}_1 < \bar{z}$ and either $Y_1 > 0$, or $Y_1 < 0$ and $|Y_1|$ is sufficiently small, then $z(t)$ is decreasing in Period I for any $z_0 \in (0, \bar{z})$.
(2) If $\bar{z}_1 > \bar{z}$ and either $Y_1 < 0$ or $Y_1 > 0$ and is sufficiently small, then $z(t)$ is decreasing in Period I for any $z_0 \in (0, \bar{z})$.

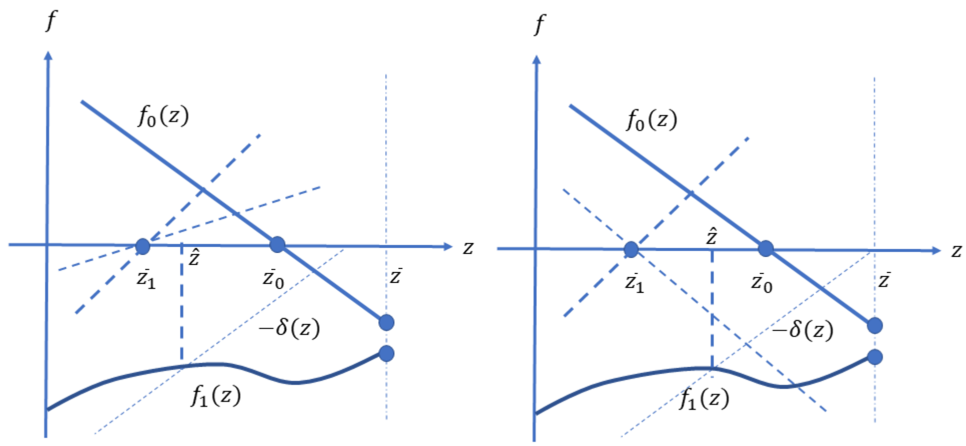

FIGURE 5. $\bar{z}_1 < \bar{z}$ with $Y_1 > 0$ (left) or $Y_1 < 0$ and $|Y_1|$ is sufficiently small (right).

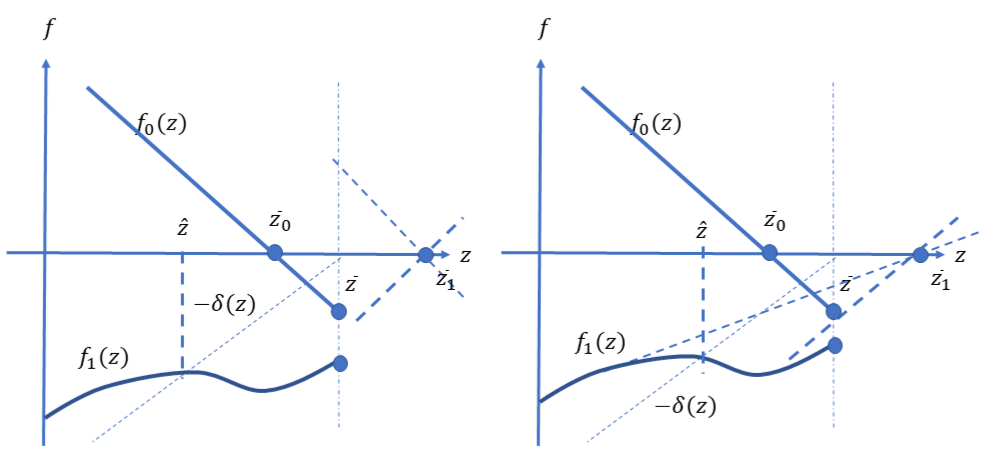

FIGURE 6. $\bar{z}_1 > \bar{z}$ with $Y_1 < 0$ (left) or $Y_1 > 0$ and is sufficiently small (right).

Proof. Since $\bar{z} > \bar{z}_0$ and $\beta < r + \lambda$, it is easy to see from (3.14) and (3.15) that

$$f_1(\bar{z}) < f_0(\bar{z}) < 0. \tag{4.12}$$

Furthermore, Eq. (3.11) is equivalent to the dynamical system, and steady state $\bar{z}_1$ in Period I is equivalent to the equilibrium $(0, \bar{z}_1)$. Since $\delta(\bar{z}) = 0$, it follows that $f_1(z)$ satisfies (3.12) for $z < \bar{z}$ and is near $\bar{z}$. The solution $f_1(z)$ may or may not stay below $-\delta(z)$. If it does, then

$f_1(z) < 0$ for all $z \in (0, \bar{z})$ and therefore $z(t)$ is decreasing for any $z_0 \in (0, \bar{z})$. If it does not, we use $\hat{z}$ to denote the solution of the equation $f_1(z) = -\delta(z)$.

We examine the trajectories that passes the point $(\hat{z}, f_1(\hat{z}))$ in various cases.

Suppose $\bar{z}_1 < \bar{z}$ and $Y_1 > 0$. Then $(\bar{z}_1, 0)$ is a repeller for the dynamical system. The trajectory passing through $(\hat{z}, f_1(\hat{z}))$ moves away from $(\bar{z}_1, 0)$ along the unstable manifold that is tangent to $\langle 1, Y_2 \rangle$. Hence, it does not intersect the z-axis. Therefore, $f_1(z) < 0$ for all $z \in (0, \bar{z})$. (See the left graph in Fig. 5.)

Suppose $\bar{z}_1 < \bar{z}$ and $Y_1 < 0$. Then $(\bar{z}_1, 0)$ is a saddle point. If $|Y_1|$ is small enough so that the stable manifold passes $\hat{z}$ above the point $(\hat{z}, f_1(\hat{z}))$, then the trajectory stays below the z-axis for all $z < \hat{z}$. Hence, again $f_1(z) < 0$ for all $z \in (0, \bar{z})$. (See the right graph in Fig. 5.)

Suppose $\bar{z}_1 > \bar{z}$ and $Y_1 < 0$. Then the trajectory that passes through $(\hat{z}, f_1(\hat{z}))$ moves away from $(\bar{z}_1, 0)$ and approaches the unstable manifold. Hence $f_1(z)$ for all $z < \hat{z}$. (See the left graph in Fig. 6.)

Suppose $\bar{z}_1 > \bar{z}$ and $Y_1 > 0$. Then $(\bar{z}_1, 0)$ is a repeller. If Y_1 is small enough such that the unstable manifold passing through $\hat{z}$ above the point $(\hat{z}, f_1(\hat{z}))$. As a result, the trajectory that passes through $(\hat{z}, f_1(\hat{z}))$ moves away from $(\bar{z}_1, 0)$ and stays between the two unstable manifolds. As a result, $f_1(z) < 0$ on $(0, \hat{z})$. (See the right graph in Fig. 6.)

Since $f_1(z) < 0$ implies that $z(t)$ is decreasing, the conclusion of the theorem follows.

This completes the proof.

REFERENCES

[1] R. Boucekkine, A. Pommeret, F. Prieur, Technological vs. ecological switch and the Environmental Kuznets Curve, American Journal of Agricultural Economics, 95 (2013) 252-260.

[2] M.H.A. Davis, Piecewise-deterministic Markov processes: a general class of non-diffusion stochastic models, Journal of the Royal Statistical Society (B), 46 (1984) 353-388.

[3] E. J. Dockner, S. Jϕrgensen, N. V. Long, G. Sorger, Differential Games in Economics and Management Science, Cambridge, Cambridge University Press, 2000.

[4] M. Farid, Optimal control, piecewise-deterministic processes and viscosity solutions, Ph.D. Thesis, University of London, 1997.

[5] C. Holling, Resilience and stability of ecological systems, Review of Ecology and Systematics 4 (1973) 1-23.

[6] S.M. Lenhar, N. Yamada, Viscosity solutions assciated with switching game for piecewise-deterministic processes, Stochastics and Stochastics Reports 38 (1992) 27-47.

[7] A. Le Kama, A. Pommeret, F. Prieur, Optimal emission policy under the risk of irreversible pollution, Journal of Public Economic Theory, 16 (2014) 959-980.

[8] O. Tahvonen, C. Withagen, Optimality of irreversible pollution accumulation, Journal of Economic Dynamics and Control, 20 (1996) 1775-1795.

[9] Y. Tsur, A. Zemel, Uncertainty and irreversibility in groundwater resource management, Journal of Environmental Economics and Management, 29 (1995) 149- 161.

[10] D. Vermes, Optimal control of piecewise-deterministic Markov processes, Stochastics, 14 (1985) 165- 208.