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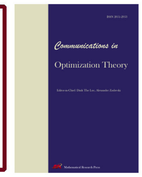

A MODEL FOR FLOW IN DEFORMABLE POROUS MEDIA

VIVETTE GIRAULT[1,*], MARÍA GONZÁLEZ TABOADA[2], FRÉDÉRIC HECHT[3], KUMBAKONAM R. RAJAGOPAL[4]

[1]Laboratoire Jacques-Louis Lions (LJLL), Sorbonne-Université, CNRS,
Université de Paris, F-75005 Paris, France
[2]Department of Mathematics and CITIC, Universidade da Coruña, Campus de Elviña s/n, 15071, A Coruña, Spain
[3]Laboratoire Jacques-Louis Lions (LJLL), Sorbonne-Université, CNRS,
Université de Paris, F-75005 Paris, France
[4]Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843-3123, USA

Dedicated to the memory of our dear friend and colleague Roland Glowinski

Abstract. Within the context of mixture theory, we consider a linearized model for the flow of a fluid though a deformable porous elastic solid, where with respect to the fluid, the solid is undergoing small but not negligible velocity. Albeit linear, the corresponding system of equations is fairly complex, because it is coupled by the solid and fluid velocities. The theoretical analysis (existence and uniqueness of solutions) of the system as well as the numerical analysis (uniform stability of the discrete solution) of its discretized version are done by superposition: splitting the solid's displacement into a part that depends only on the data and a part that depends on the velocity of the fluid. A simple time lagging decoupling algorithm is studied and a sharper iterative algorithm proposed at the end. Numerical experiments confirm the performance of the numerical scheme and the validity of the model.
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1. Introduction

The success achieved by the equations developed by Darcy [10], to study the flow of fluids through porous solids, has unfortunately led to its use where it is not appropriate. Darcy's equations are valid for a very specific class of flows through porous media. It holds when the solid through which the fluid is flowing is rigid, the fluid under consideration being an incompressible Navier-Stokes fluid. It also presumes that the flows are slow so that inertial

*Corresponding author.

E-mail address: vivette.girault@sorbonne-universite.fr (V. Girault), maria.gonzalez.taboada@udc.es (M. González Taboada), frederic.hecht@sorbonne-universite.fr (F. Hecht), krajagopal@tamu.edu (K.R. Rajagopal).

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effects may be neglected. These equations do not apply if the flows are not slow or if the solid under question undergoes deformation. Another serious drawback of Darcy's equation is that it is silent with respect to the states of stress and strain in the solid. In important applications like hydraulic fracturing, one needs to know the stresses that develop in the solid. Modifications or generalizations of Darcy's equation due to Brinkman [7, 8] and Forchheimer [12] only address the flow of the fluid and have nothing to say about the deformation of the solid or the stresses and strains that develop in the solid. An approach that will allow us to determine kinematical quantities as well as the stresses in both the fluid and the porous solid is the theory of mixtures, an approach that was given a formal mathematical structure by Truesdell [30, 31].

The basic assumption of mixture theory is that each point in the mixture is occupied, in a homogenized sense, by a particle belonging to each constituent of the mixture. Then, balance laws are postulated for each constituent with the possibility of conversion of mass and interactions such as momentum transfer between the constituents (see Bowen [5], Atkin and Craine [2], Samohyl [25], Rajagopal and Tao [24]). Darcy's equation and Fick's equation (see Fick [11]) are very special cases that stem as approximations from the theory of mixtures. Based on the assumptions that are made concerning the constituents, the kinematics associated with the constituents, and the constraints imposed, one obtains the governing equations for the problem under consideration. For instance, one may suppose the solid is rigid, a linearized elastic solid, a nonlinear elastic solid, viscoelastic solid, etc., and that the fluid is a Navier-Stokes fluid, a non-Newtonian fluid, etc., and furthermore, we can make additional assumptions concerning the kinematics that the fluid is flowing slowly, the solid is undergoing small deformations, etc. We could also enforce constraints that the fluid is incompressible, the volume of the constituents is additive (see Mills [20]), and the like. In general, we could have reacting mixtures, in which case we will have to also provide the equations governing the reactions, constitutive relations for the mass production of the constituents, the momentum supply to the constituents, etc., but in our study, we do not consider the interconversion of mass of the constituents. We will have interactions between the constituents that contributes to the momentum and energy of each constituent. Some of the interactions that come into play in the balance of linear momentum are the drag force (due to differences in velocity of the constituents), virtual mass force (due to differences in acceleration of the constituents), Magnus effect (due to differences in spin), Basset forces (due to differences in the histories of the motion), and numerous other interaction terms due to differences in densities, density gradients, lift forces, etc. (see Johnson et al. [18] for a detailed discussion of the interaction forces that arise in mixture theory). Once specific assumptions have been made concerning all the above issues, we can derive the appropriate governing equations for each of the constituents from the balance laws for the constituents.

A serious difficulty in the use of mixture theory wherein we use balance equations for each of the constituents arises from the assumption that the constituents of the mixture co-occupy every point belonging to the mixture, albeit in a homogenized sense. Invariably we only know the boundary and initial conditions associated with the mixture as a whole, we do not know how this condition is to be apportioned to each constituent. There are several approaches suggested to generate these boundary conditions (see Rajagopal and Tao [24], and the recent study appealing to thermodynamics by Soucek et al. [28]). The Flory-Huggins equation (see Treloar [29]), obtained using thermodynamic arguments concerning saturated swollen state of polymers has been used at the boundary of a porous solid through which a fluid is flowing, to study the diffusion of fluids through polymeric solids (see Shi et al. [26]). Given the variety of boundary conditions being used, Prasad and Rajagopal [22] employed several of them to study the flow of fluids through polymeric solids undergoing large deformations and found that for global quantities such as the flow rate, the results were nearly the same for all of them. We discuss the issue of boundary and initial conditions relevant to our problem in Sections 2.1 and 2.3. The same difficulties are associated with the specification of initial conditions.

In reference [23], Rajagopal developed a hierarchy of equations for the flow of fluids through porous elastic solids, depending on the nature of the porous solid and the fluid flowing through it, kinematical considerations for both the constituents, and the type of interaction mechanisms that come into play. In this study, we assume that the partial stress in each constituent is completely determined by only properties and kinematics associated with that constituent, and that one of the constituents is a Navier-Stokes fluid and the other an isotropic and homogeneous linearized elastic solid. Furthermore, we assume that the only interaction is drag that depends on the relative velocities of the fluid and solid. Thus, if the velocity of the solid is ignored, there would be no coupling between the equations governing the fluid and the solid[1], and the problem governing the fluid motion would reduce to Darcy's or Brinkman's equation and having determined the velocity for the fluid, the governing equation for the solid can be solved. There are however practical situations involving unsteady motions of flow through porous media wherein the solid is undergoing small but not negligible velocity, and in such problems, one obtains the following equations governing the deformation of the solid and fluid, respectively:

$$(\lambda^s+\mu^s)\nabla(\nabla\cdot\mathbf{u}^s)+\mu^s\Delta\mathbf{u}^s+\rho^s\mathbf{b}_e+\alpha(\mathbf{v}^f-\mathbf{v}^s)=\mathbf{0}, \tag{1.1}$$

$$-\nabla p^f+\mu^f\Delta\mathbf{v}^f+\rho^f\mathbf{b}_e-\alpha(\mathbf{v}^f-\mathbf{v}^s)=\rho^f\frac{\partial\mathbf{v}^f}{\partial t}, \tag{1.2}$$

with the incompressibility condition

$$\nabla\cdot\mathbf{v}^f=0. \tag{1.3}$$

In the above equation, $\mathbf{u}^s$ is the displacement of the solid, and $\mathbf{v}^s$ and $\mathbf{v}^f$ are the velocity of the solid and fluid, respectively, p^f is the fluid pressure, λ^s and μ^s are the Lamé parameters, ρ^s and ρ^f denote the solid and fluid density, respectively, μ^f is the dynamic viscosity of the fluid, $\mathbf{b}_e$ is the external volume force, and α is the interaction parameter between the fluid and solid. Also, while we have assumed that the velocity of the solid cannot be neglected, its acceleration is negligible and hence ignored. The above equations are all written almost everywhere in a region of space and time that will be specified below.

This article is divided into six sections, including the introduction. The theoretical analysis of the problem is carried out in Sections 2 and 3. The numerical discretization and analysis is done in Section 4. Section 5 is devoted to the analysis of a simple decoupling algorithm, and a more elaborated algorithm is proposed at the end. Numerical simulations are described in Section 6.

[1]If we allow the kinematical quantities of both the constituents to play a role in determining the stresses in both the constituents, even when the velocity of the solid is ignored, the governing equations would be coupled (see Shi et al. [26]).

1.1. **Notation.** In this work, we shall use the following notation for a bounded connected open set $\Omega \subset \mathbb{R}^d$, $d = 2,3$. The scalar product of $L^2(\Omega)$ is denoted by $(\cdot,\cdot)_\Omega$

$$\forall f,g \in L^2(\Omega), \quad (f,g)_\Omega = \int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x},$$

and the index Ω is omitted when the domain of integration is clear from the context. For any non-negative integer m, the classical Sobolev space $H^m(\Omega)$ is defined by (cf. [1] or [21]),

$$H^m(\Omega) = \{v \in L^2(\Omega)\,;\, \partial^k v \in L^2(\Omega)\, \forall\, |k| \le m\},$$

where k is a multi index, e.g. when $d = 3$, $k = (k_1,k_2,k_3)$, $k_i \ge 0$, $|k| = k_1+k_2+k_3$,

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}},$$

equipped with the following seminorm and norm for which it is a Hilbert space:

$$|v|_{H^m(\Omega)} = \left[\sum_{|k|=m} \int_\Omega |\partial^k v(\mathbf{x})|^2\, d\mathbf{x}\right]^{\frac{1}{2}}, \quad \|v\|_{H^m(\Omega)} = \left[\sum_{0\le|k|\le m} |v|^2_{H^k(\Omega)}\right]^{\frac{1}{2}}.$$

This definition is extended to any real number $s = m + s'$ for an integer $m \ge 0$ and $0 < s' < 1$ by defining in dimension d the fractional semi-norm and norm, see [19] and [15],

$$|v|_{H^s(\Omega)} = \left(\sum_{|k|=m} \int_\Omega \int_\Omega \frac{|\partial^k v(\mathbf{x}) - \partial^k v(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{d+2s'}}\, d\mathbf{x}\, d\mathbf{y}\right)^{\frac{1}{2}}, \ \|v\|_{H^s(\Omega)} = \left(\|v\|^2_{H^m(\Omega)} + |v|^2_{H^s(\Omega)}\right)^{\frac{1}{2}}.$$

These fractional order spaces are often used for traces. The following trace property holds in a domain Ω with a Lipschitz continuous boundary $\partial\Omega$: If v belongs to $H^s(\Omega)$ for some $s \in]\frac{1}{2},1]$, then its trace on $\partial\Omega$ belongs to $H^{s-\frac{1}{2}}(\partial\Omega)$ and there exists a constant C_s such that

$$\forall v \in H^s(\Omega)\,,\, \|v\|_{H^{s-\frac{1}{2}}(\partial\Omega)} \le C_s \|v\|_{H^s(\Omega)}. \tag{1.4}$$

In particular, $H^{\frac{1}{2}}(\partial\Omega)$ is the trace space of $H^1(\Omega)$, with norm

$$|v|_{H^{\frac{1}{2}}(\partial\Omega)} = \Big(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^d} d\mathbf{x}\, d\mathbf{y}\Big)^{\frac{1}{2}},$$

(here d refers to the dimension of Ω) and $H^{-\frac{1}{2}}(\partial\Omega)$ is the dual space of $H^{\frac{1}{2}}(\partial\Omega)$. Finally, if Γ is a subset of $\partial\Omega$ with positive measure, $|\Gamma| > 0$, we say that a function g in $H^{\frac{1}{2}}(\Gamma)$ belongs to $H^{\frac{1}{2}}_{00}(\Gamma)$ if its extension by zero to $\partial\Omega$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$. It is a proper subspace of $H^{\frac{1}{2}}(\Gamma)$, and is normed by

$$\|v\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = \Big(|v|^2_{H^{\frac{1}{2}}(\Gamma)} + \int_\Gamma |v(\mathbf{x})|^2\, \frac{d\mathbf{x}}{\mathrm{d}(\mathbf{x},\Gamma)}\Big)^{\frac{1}{2}}, \tag{1.5}$$

where $\mathrm{d}(\mathbf{x},\Gamma)$ denotes the distance from $\mathbf{x}$ to the boundary of Γ. We also recall Poincaré's inequality valid for all functions $\mathbf{v}$ in $H^1(\Omega)^d$ that vanish on a portion Γ of $\partial\Omega$ with positive measure:

$$\|v\|_{L^2(\Omega)} \le \mathscr{P} |v|_{H^1(\Omega)}, \tag{1.6}$$

where $\mathscr{P}$ is a constant depending only on Ω and Γ. We also recall a trace inequality on the complement of Γ on $\partial\Omega$, say $\tilde{\Gamma}$, assuming that $\tilde{\Gamma}$ has also positive measure:

$$\|v\|_{H^{\frac{1}{2}}_{00}(\tilde{\Gamma})} \leq C_N |v|_{H^1(\Omega)}. \tag{1.7}$$

To understand this last inequality, recall that if v in $H^1(\Omega)$ vanishes on Γ, then its trace on $\tilde{\Gamma}$ belongs to $H^{\frac{1}{2}}_{00}(\tilde{\Gamma})$, owing that its extension by zero to $\partial\Omega$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$.

The space $H(\mathrm{div},\Omega)$ is the Hilbert space

$$H(\mathrm{div},\Omega) = \{\mathbf{v} \in L^2(\Omega)^d\,;\, \nabla\cdot\mathbf{v} \in L^2(\Omega)\}, \tag{1.8}$$

equipped with the graph norm. Let $\mathbf{n}$ denote the exterior unit normal vector to $\partial\Omega$. The normal trace $\mathbf{v}\cdot\mathbf{n}$ of a function $\mathbf{v}$ of $H(\mathrm{div},\Omega)$ on $\partial\Omega$ belongs to $H^{-\frac{1}{2}}(\partial\Omega)$, the dual space of $H^{\frac{1}{2}}(\partial\Omega)$, see for instance [13]. Recall also the spaces

$$W = \{\mathbf{v} \in H^1(\Omega)^d\,;\, \nabla\cdot\mathbf{v} = 0\}, \quad V = W \cap H^1_0(\Omega)^d, \tag{1.9}$$

and

$$L^2_0(\Omega) = \{f \in L^2(\Omega)\,;\, \int_\Omega f(\mathbf{x})\,d\mathbf{x} = 0\}. \tag{1.10}$$

As usual, for handling time-dependent problems, it is convenient to consider measurable functions defined on a time interval $]a,b[$ with values in a functional space, say X (cf. [19]). More precisely, let $\|\cdot\|_X$ denote the norm of X; then for any number r, $1 \leq r \leq \infty$, we define

$$L^r(a,b;X) = \{f \text{ measurable in }]a,b[\,;\, \int_a^b \|f(t)\|_X^r dt < \infty\},$$

equipped with the norm

$$\|f\|_{L^r(a,b;X)} = \left(\int_a^b \|f(t)\|_X^r\, dt\right)^{\frac{1}{r}},$$

with the usual modification if $r = \infty$. It is a Banach space if X is a Banach space, and for $r = 2$, it is a Hilbert space if X is a Hilbert space. In particular,

$$L^2(a,b;L^2(\Omega)) = L^2(\Omega \times]a,b[).$$

Derivatives with respect to time are denoted by ∂_t and we define for instance

$$H^1(a,b;X) = \{f \in L^2(a,b;X)\,;\, \partial_t f \in L^2(a,b;X)\}.$$

In the sequel, if there is no ambiguity, the differential notation in integrals will be omitted.

2. Settings and splitting strategy

To write rigorously the system (1.1)–(1.3), we first complement it with boundary and initial conditions, but rather than setting it right away into a coupled variational formulation, to facilitate its theoretical analysis, it will be useful to split it into simpler subproblems.

2.1. **Boundary and initial conditions.** Let Ω be a bounded, connected, Lipschitz domain in $\mathbb{R}^d$, with boundary $\partial\Omega$ and unit exterior normal $\mathbf{n}$, let $T > 0$ be the final time, $Q = \Omega\times]0,T[$ and let

$$\partial\Omega = \Gamma_D^s \cup \Gamma_N^s,$$

where $\Gamma_D^s \cap \Gamma_N^s = \emptyset$, Γ_D^s and Γ_N^s are both Lipschitz-continuous, and $|\Gamma_D^s| > 0$. The system (1.1)–(1.3) must be complemented with boundary and initial conditions. Here we consider mixed boundary conditions for the displacement and Dirichlet boundary conditions for the flow. More precisely, $\mathbf{u}^s$ has a non homogeneous Dirichlet boundary condition on Γ_D^s a.e. on]0,T[,

$$\mathbf{u}^s|_{\Gamma_D^s} = \mathbf{u}_D^s, \tag{2.1}$$

and a non homogeneous natural boundary condition on Γ_N^s a.e. on]0,T[,

$$\Big(\big(\mu^s \nabla \mathbf{u}^s + (\lambda^s + \mu^s)(\nabla \cdot \mathbf{u}^s)\mathbf{I}\big)\mathbf{n}\Big)|_{\Gamma_N^s} = \mathbf{g}_N^s. \tag{2.2}$$

The flow velocity $\mathbf{v}^f$ has a non homogeneous Dirichlet boundary condition on $\partial\Omega$ a.e. on]0,T[,

$$\mathbf{v}^f|_{\partial\Omega} = \mathbf{v}_D^f, \tag{2.3}$$

satisfying the necessary compatibility condition

$$\int_{\partial\Omega} \mathbf{v}_D^f \cdot \mathbf{n} = 0. \tag{2.4}$$

Regarding initial conditions, we assume that $\mathbf{v}^f(0)$ is given, and because of the presence of $\mathbf{v}^s$ in (1.1), we assume that $\mathbf{u}^s(0)$ is also given,

$$\mathbf{u}^s(0) = \text{ given with } \mathbf{u}^s(0)|_{\Gamma_D^s} = \mathbf{u}_D^s(0), \tag{2.5}$$

$$\mathbf{v}^f(0) = \text{ given in } W \text{ with } \mathbf{v}^f(0)|_{\partial\Omega} = \mathbf{v}_D^f(0), \tag{2.6}$$

i.e., satisfying $\nabla \cdot \mathbf{v}^f(0) = 0$ with the necessary condition (2.4), i.e.

$$\int_{\partial\Omega} \mathbf{v}_D^f(0) \cdot \mathbf{n} = 0.$$

2.2. **Spaces.** Let

$$H^1_{0,\Gamma_D^s}(\Omega) = \{v \in H^1(\Omega)\,;\, v|_{\Gamma_D^s} = 0\}. \tag{2.7}$$

The assumptions in space on the data are fairly straightforward, with the possible exception of the natural boundary condition (2.2). The above observation that the trace on Γ_N^s of functions of $H^1_{0,\Gamma_D^s}(\Omega)$ belongs to $H^{\frac{1}{2}}_{00}(\Gamma_N^s)$ suggests to take each component of $\mathbf{g}_N^s(t)$ in the dual space $H^{\frac{1}{2}}_{00}(\Gamma_N^s)'$ of $H^{\frac{1}{2}}_{00}(\Gamma_N^s)$. Then the complete assumptions on the data of the displacement equation are

$$\mathbf{b}_e \in L^2(Q)^d,\ \mathbf{u}_D^s \in H^1(0,T;H^{\frac{1}{2}}(\Gamma_D^s)^d),\ \mathbf{g}_N^s \in H^1(0,T;\big(H^{\frac{1}{2}}_{00}(\Gamma_N^s)'\big)^d), \mathbf{u}^s(0) \in H^1(\Omega)^d. \tag{2.8}$$

Owing to the simpler boundary conditions on the velocity, the assumptions on the data for the flow equation are simpler

$$\mathbf{b}_e \in L^2(Q)^d,\ \mathbf{v}_D^f \in H^1(0,T;H^{\frac{1}{2}}(\partial\Omega)^d),\ \mathbf{v}^f(0) \in W, \tag{2.9}$$

with the restriction (2.4) satisfied at all time t in $[0,T]$.

For a given reasonably smooth $\mathbf{v}^s$, the system (1.2), (1.3), (2.3), (2.6) is a classical time dependent Stokes-like system with a non homogeneous Dirichlet boundary condition. Therefore, it is easily set into an equivalent variational form with solution $\mathbf{v}^f$ in $L^\infty(0,T;L^2(\Omega)^d)\cap L^2(0,T;H^1(\Omega)^d)$. The displacement system (1.1), which is a standard elasticity-like equation given a reasonably smooth $\mathbf{v}^f$, has also an equivalent variational form with solution $\mathbf{u}^s$ in $L^\infty(0,T;L^2(\Omega)^d)\cap L^2(0,T;H^1(\Omega)^d)$, but this is a little less straightforward because of its mixed boundary conditions, as discussed in the next subsection.

2.3. **Meaning of the natural boundary conditions, case $d=3$.** When the problem does not depend on time, the natural boundary conditions are defined by setting the problem in divergence form in Ω and using the properties of $H(\mathrm{div},\Omega)$. The idea, which is similar when the problem depends on time, is to write (1.1) in divergence form in the space-time cylinder $Q_t=\Omega\times]0,t[$; with the notation, $Q=Q_T$. Take the case $d=3$ and consider the first line of the left-hand side of (1.1), set $X=(\mathbf{x},t)\in\mathbb{R}^4$ and define the following three vectors, each with four components,

$$\begin{aligned}
U_1&=(-(\lambda^s+\mu^s)(\nabla\cdot\mathbf{u}^s)-\mu^s\partial_{x_1}u_1^s,-\mu^s\partial_{x_2}u_1^s,-\mu^s\partial_{x_3}u_1^s,\alpha u_1^s),\\
U_2&=(-\mu^s\partial_{x_1}u_2^s,-(\lambda^s+\mu^s)(\nabla\cdot\mathbf{u}^s)-\mu^s\partial_{x_2}u_2^s,-\mu^s\partial_{x_3}u_2^s,\alpha u_2^s),\\
U_3&=(-\mu^s\partial_{x_1}u_3^s,-\mu^s\partial_{x_2}u_3^s,-(\lambda^s+\mu^s)(\nabla\cdot\mathbf{u}^s)-\mu^s\partial_{x_3}u_3^s,\alpha u_3^s).
\end{aligned}$$

It is easy to check that, with this notation, the line i of (1.1) reads

$$\nabla_X\cdot U_i=\rho^s b_{e,i}+\alpha v_i^f,\quad i=1,2,3,$$

where the divergence is taken with respect to $X=(\mathbf{x},t)$ and the right-hand side belongs to $L^2(Q)$, assuming that v_i^f belongs to $L^2(Q)$. Thus, the reasonable space for each U_i is $L^2(Q)^4$ with $\nabla\cdot U_i$ in $L^2(Q)$, i.e. U_i belongs to $H(\mathrm{div},Q)$. This gives a meaning to the normal trace $U_i\cdot\mathbf{n}$ in $H^{-\frac{1}{2}}(\partial Q)$, where $\mathbf{n}$ denotes the exterior normal vector to Q. As $\mathbf{x}$ and t are orthogonal, the exterior normal $\mathbf{n}$ to Ω at each time t coincides with the exterior normal vector to Q. Hence this gives meaning to $\big(\mu^s\nabla\mathbf{u}^s+(\lambda^s+\mu^s)(\nabla\cdot\mathbf{u}^s)\mathbf{I}\big)\mathbf{n}$ as a vector valued distribution on $\Gamma_N^s\times]0,T[$, without prescribing from the onset that $\partial_t\mathbf{u}^s$ is in $L^2(\Omega)^3$, although this regularity will be derived further on.

2.4. **Splitting the displacement equation.** As the problem is linear, it is convenient to split (1.1) into a system embodying all its data, but without $\mathbf{v}^f$, and a system with only $\mathbf{v}^f$ and all zero data. To be specific, $\mathbf{u}^s$ is split as follows:

$$\mathbf{u}^s=\check{\mathbf{u}}^s+\hat{\mathbf{u}}^s(\mathbf{v}^f),$$

where $\check{\mathbf{u}}^s$ solves

$$\begin{aligned}
\alpha\partial_t\check{\mathbf{u}}^s-\mu^s\Delta\check{\mathbf{u}}^s-(\lambda^s+\mu^s)\nabla(\nabla\cdot\check{\mathbf{u}}^s)&=\rho^s\mathbf{b}_e,\ \text{a.e. in } Q,\\
\check{\mathbf{u}}^s&=\mathbf{u}_D^s,\ \text{a.e. on } \Gamma_D^s\times]0,T[,\\
\big(\mu^s\nabla\check{\mathbf{u}}^s+(\lambda^s+\mu^s)(\nabla\cdot\check{\mathbf{u}}^s)\mathbf{I}\big)\mathbf{n}&=\mathbf{g}_N^s,\ \text{a.e. on } \Gamma_N^s\times]0,T[,\\
\check{\mathbf{u}}^s(0)&=\mathbf{u}^s(0),\ \text{a.e. in } \Omega,
\end{aligned}\tag{2.10}$$

and $\hat{\mathbf{u}}^s(\mathbf{v}^f)$ solves

$$\begin{aligned}\alpha\partial_t\hat{\mathbf{u}}^s(\mathbf{v}^f)-\mu^s\Delta\hat{\mathbf{u}}^s(\mathbf{v}^f)-(\lambda^s+\mu^s)\nabla(\nabla\cdot\hat{\mathbf{u}}^s(\mathbf{v}^f))=\alpha\mathbf{v}^f,\ \text{a.e. in } Q,\\ \hat{\mathbf{u}}^s(\mathbf{v}^f)=\mathbf{0},\ \text{a.e. on } \Gamma_D^s\times]0,T[,\\ \big(\mu^s\nabla\hat{\mathbf{u}}^s(\mathbf{v}^f)+(\lambda^s+\mu^s)(\nabla\cdot\hat{\mathbf{u}}^s(\mathbf{v}^f))\mathbf{I}\big)\mathbf{n}=\mathbf{0},\ \text{a.e. on } \Gamma_N^s\times]0,T[,\\ \hat{\mathbf{u}}^s(\mathbf{v}^f)(0)=\mathbf{0},\ \text{a.e. in } \Omega.\end{aligned}\tag{2.11}$$

We shall see below that the system (2.10) has a unique solution $\check{\mathbf{u}}^s\in L^\infty(0,T;L^2(\Omega)^d)\cap L^2(0,T;H^1(\Omega)^d)$, $\partial_t\check{\mathbf{u}}^s\in L^2(Q)^d$, and the system (2.11) has a unique solution $\hat{\mathbf{u}}^s(\mathbf{v}^f)\in L^\infty(0,T;L^2(\Omega)^d)\cap L^2(0,T;H^1(\Omega)^d)$, $\partial_t\hat{\mathbf{u}}^s(\mathbf{v}^f)\in L^2(Q)^d$ for any $\mathbf{v}^f\in L^2(Q)^d$.

2.5. **Splitting the flow equation.** Now, the unknown in the left-hand side of (1.2) can be split as

$$\alpha\partial_t\mathbf{u}^s=\alpha\partial_t\check{\mathbf{u}}^s+\alpha\partial_t\hat{\mathbf{u}}^s(\mathbf{v}^f).$$

Since $\check{\mathbf{u}}^s$ depends only on the data, by proceeding as above, we split $\mathbf{v}^f$ into

$$\mathbf{v}^f=\check{\mathbf{v}}^f+\mathbf{v}_0^f,$$

where $\check{\mathbf{v}}^f$ solves the non homogeneous Stokes system depending only on the data

$$\begin{aligned}\rho^f\partial_t\check{\mathbf{v}}^f-\mu^f\Delta\check{\mathbf{v}}^f+\nabla\check{p}^f+\alpha\check{\mathbf{v}}^f=\rho^f\mathbf{b}_e+\alpha\partial_t\check{\mathbf{u}}^s,\ \text{a.e. in } Q,\\ \nabla\cdot\check{\mathbf{v}}^f=0,\ \text{a.e. in } Q,\\ \check{\mathbf{v}}^f=\mathbf{v}_D^f,\ \text{a.e. on } \partial\Omega\times]0,T[,\\ \check{\mathbf{v}}^f(0)=\mathbf{v}^f(0),\ \text{a.e. in } \Omega,\end{aligned}\tag{2.12}$$

and $\mathbf{v}_0^f$ solves the homogeneous implicit problem,

$$\begin{aligned}\rho^f\partial_t\mathbf{v}_0^f-\mu^f\Delta\mathbf{v}_0^f+\nabla p_0^f+\alpha\mathbf{v}_0^f=\alpha\partial_t\hat{\mathbf{u}}^s(\check{\mathbf{v}}^f)+\alpha\partial_t\hat{\mathbf{u}}^s(\mathbf{v}_0^f),\ \text{a.e. in } Q,\\ \nabla\cdot\mathbf{v}_0^f=0,\ \text{a.e. in } Q,\\ \mathbf{v}_0^f=\mathbf{0},\ \text{a.e. on } \partial\Omega\times]0,T[,\\ \mathbf{v}_0^f(0)=\mathbf{0},\ \text{a.e. in } \Omega.\end{aligned}\tag{2.13}$$

We shall see below that, on the one hand, the system (2.12) has a unique solution $\check{\mathbf{v}}^f$ in the space $L^\infty(0,T;L^2(\Omega)^d)\cap L^2(0,T;H^1(\Omega)^d)$; hence according to the results announced in Section 2.4, $\partial_t\hat{\mathbf{u}}^s(\check{\mathbf{v}}^f)$ belongs to $L^2(Q)^d$. On the other hand, the system (2.13) is implicit and a direct proof of existence of its solution is not so clear. However, for any given $\mathbf{v}_0^f\in L^2(Q)^d$ in the right-hand side of the first equation of (2.13), we shall see that the system (2.13) has a unique solution $\mathbf{v}_0^f\in L^\infty(0,T;L^2(\Omega)^d)\cap L^2(0,T;H^1(\Omega)^d)$ and $\partial_t\hat{\mathbf{u}}^s(\mathbf{v}_0^f)\in L^2(Q)^d$. Therefore (2.13) has a regularizing effect and thus Schauder's fixed-point theorem will be used to prove existence of its solution.

3. EXISTENCE

First, problems (2.10)–(2.13) are set into equivalent variational form; equivalence following easily from the material of Section 2. Next, the displacement and flow involve the solution of non homogeneous problems, namely (2.10) and (2.12), that depend only on the data; each is easily solved. Moreover, the displacement of the solid can be expressed in terms of the velocity of the fluid, and this problem, namely (2.11), is also readily solved. Finally, there remains (2.13), an implicit equation for the velocity, that will be solved by a fixed-point argument.

3.1. **The solution of Problem** (2.10).

To begin with, it is easy to check that Problem (2.10) cannot have more than one solution in $L^\infty(0,T;L^2(\Omega)^d)\cap L^2(0,T;H^1(\Omega)^d)$.

To establish existence, as usual, the non homogenous essential Dirichlet boundary condition is lifted so that the problem analyzed is homogeneous. Let $\widetilde{\mathbf{u}}^s_D \in H^1(0,T;H^1(\Omega)^d)$ be a lifting of $\mathbf{u}^s_D$ for all $t\in[0,T]$:

$$\widetilde{\mathbf{u}}^s_D|_{\Gamma^s_D} = \mathbf{u}^s_D, \quad \|\widetilde{\mathbf{u}}^s_D\|_{H^1(0,T;H^1(\Omega)^d)} \le C\|\mathbf{u}^s_D\|_{H^1(0,T;H^{\frac{1}{2}}(\Gamma^s_D)^d)}. \tag{3.1}$$

Let us set

$$\bar{\mathbf{u}}^0 = \check{\mathbf{u}}^s - \widetilde{\mathbf{u}}^s_D. \tag{3.2}$$

Then $\bar{\mathbf{u}}^0$ solves the problem: Find $\bar{\mathbf{u}}^0 \in H^1(0,T;L^2(\Omega)^d)\cap L^2(0,T;H^1_{0,\Gamma^s_D}(\Omega)^d)$ such that for all $\mathbf{w}\in H^1_{0,\Gamma^s_D}(\Omega)^d$ and a.e. in $]0,T[$,

$$\begin{aligned}\alpha(\partial_t\bar{\mathbf{u}}^0,\mathbf{w})+&\mu^s(\nabla\bar{\mathbf{u}}^0,\nabla\mathbf{w})+(\lambda^s+\mu^s)(\nabla\cdot\bar{\mathbf{u}}^0,\nabla\cdot\mathbf{w}) = \rho^s(\mathbf{b}_e,\mathbf{w})+\langle\mathbf{g}^s_N,\mathbf{w}\rangle_{\Gamma^s_N}\\ &-\alpha(\partial_t\widetilde{\mathbf{u}}^s_D,\mathbf{w})-\mu^s(\nabla\widetilde{\mathbf{u}}^s_D,\nabla\mathbf{w})-(\lambda^s+\mu^s)(\nabla\cdot\widetilde{\mathbf{u}}^s_D,\nabla\cdot\mathbf{w}),\end{aligned} \tag{3.3}$$

with the initial condition in Ω

$$\bar{\mathbf{u}}^0(0) = \mathbf{u}^s(0) - \widetilde{\mathbf{u}}^s_D(0). \tag{3.4}$$

It is easily established that $\bar{\mathbf{u}}^0$ satisfies a variational formulation equivalent to (2.10) in the spaces chosen above. Regarding uniqueness of $\bar{\mathbf{u}}^0$, since problem (3.3)–(3.4) is linear, an immediate calculation shows that when its right-hand sides are zero, then its only solution in $L^\infty(0,T;L^2(\Omega)^d)\cap L^2(0,T;H^1_{0,\Gamma^s_D}(\Omega)^d)$ is necessarily zero.

Existence of the solution of Problem (3.3)–(3.4) follows readily by Galerkin's construction. Let $\{\mathbf{w}_i\}_{i\ge1}$ be a basis of $H^1_{0,\Gamma^s_D}(\Omega)^d$, let W_m be the space spanned by $\mathbf{w}_i$, $1\le i\le m$, and consider the function

$$\bar{\mathbf{u}}^0_m(\mathbf{x},t) = \sum_{k=1}^m g_k(t)\mathbf{w}_k(\mathbf{x}),$$

where $g_i\in H^1(0,T)$, $1\le i\le m$, solves

$$\begin{aligned}\alpha(\partial_t\bar{\mathbf{u}}^0_m,\mathbf{w}_j)+&\mu^s(\nabla\bar{\mathbf{u}}^0_m,\nabla\mathbf{w}_j)+(\lambda^s+\mu^s)(\nabla\cdot\bar{\mathbf{u}}^0_m,\nabla\cdot\mathbf{w}_j) = \rho^s(\mathbf{b}_e,\mathbf{w}_j)+\langle\mathbf{g}^s_N,\mathbf{w}_j\rangle_{\Gamma^s_N}\\ &-\alpha(\partial_t\widetilde{\mathbf{u}}^s_D,\mathbf{w}_j)-\mu^s(\nabla\widetilde{\mathbf{u}}^s_D,\nabla\mathbf{w}_j)-(\lambda^s+\mu^s)(\nabla\cdot\widetilde{\mathbf{u}}^s_D,\nabla\cdot\mathbf{w}_j),\ 1\le j\le m,\end{aligned} \tag{3.5}$$

with initial condition

$$(\nabla\bar{\mathbf{u}}^0_m(0),\nabla\mathbf{w}_j) = (\nabla\mathbf{u}^s(0),\nabla\mathbf{w}_j) - (\nabla\widetilde{\mathbf{u}}^s_D(0),\nabla\mathbf{w}_j),\ 1\le j\le m. \tag{3.6}$$

Problem (3.5) is a square system of m linear ODEs of order one in time with initial condition (3.6). The matrix of the time derivative is invertible with constant coefficients and its other coefficients belong to $L^2(0,T)$ since both $(\mathbf{b}_e, \mathbf{w}_j)$ and $(\partial_t \widetilde{\mathbf{u}}_D^s, \mathbf{w}_j)$ belong to $L^2(0,T)$ and the other terms are smoother. Therefore, it has a unique solution in $H^1(0,T)$ bounded as follows:

Proposition 3.1. *Under the assumptions* (2.5) *and* (2.8)*, the solution* $\bar{\mathbf{u}}_m^0$ *of* (3.5)–(3.6) *satisfies the estimates, with constants* C_i *independent of* m*, for all* $0 < t \le T$,

$$\begin{aligned} \alpha \|\bar{\mathbf{u}}_m^0(t)\|^2_{L^2(\Omega)} + \mu^s \|\nabla \bar{\mathbf{u}}_m^0\|^2_{L^2(Q_t)} + (\lambda^s + \mu^s)\|\nabla \cdot \bar{\mathbf{u}}_m^0\|^2_{L^2(Q_t)} \le \\ \le \alpha \|\bar{\mathbf{u}}_m^0(0)\|^2_{L^2(\Omega)} + C_1 + C_2 \|\mathbf{b}_e\|^2_{L^2(Q_t)}, \\ \|\nabla \bar{\mathbf{u}}_m^0(0)\|_{L^2(\Omega)} \le \|\nabla \mathbf{u}^s(0)\|_{L^2(\Omega)} + \|\nabla \widetilde{\mathbf{u}}_D^s(0)\|_{L^2(\Omega)} \le C_3. \end{aligned} \tag{3.7}$$

Proof. By testing (3.5) with $\bar{\mathbf{u}}_m^0$, we derive

$$\begin{aligned} \frac{\alpha}{2}\frac{d}{dt}\|\bar{\mathbf{u}}_m^0\|^2_{L^2(\Omega)} + \mu^s \|\nabla \bar{\mathbf{u}}_m^0\|^2_{L^2(\Omega)} + (\lambda^s + \mu^s)\|\nabla \cdot \bar{\mathbf{u}}_m^0\|^2_{L^2(\Omega)} \le \rho^s \|\mathbf{b}_e\|_{L^2(\Omega)} \|\bar{\mathbf{u}}_m^0\|_{L^2(\Omega)} \\ + \|\mathbf{g}_N^s\|_{H_{00}^{\frac12}(\Gamma_N^s)'} \|\bar{\mathbf{u}}_m^0\|_{H_{00}^{\frac12}(\Gamma_N^s)} + \alpha \|\partial_t \widetilde{\mathbf{u}}_D^s\|_{L^2(\Omega)} \|\bar{\mathbf{u}}_m^0\|_{L^2(\Omega)} \\ + \mu^s \|\nabla \widetilde{\mathbf{u}}_D^s\|_{L^2(\Omega)} \|\nabla \bar{\mathbf{u}}_m^0\|_{L^2(\Omega)} + (\lambda^s + \mu^s)\|\nabla \cdot \widetilde{\mathbf{u}}_D^s\|_{L^2(\Omega)} \|\nabla \cdot \bar{\mathbf{u}}_m^0\|_{L^2(\Omega)}. \end{aligned}$$

Here we can use Poincaré's inequality (1.6) and the trace inequality (1.7) since $\bar{\mathbf{u}}_m^0$ vanishes on Γ_D^s. Thus, we bound $\|\bar{\mathbf{u}}_m^0\|_{L^2(\Omega)}$ and $\|\bar{\mathbf{u}}_m^0\|_{H_{00}^{\frac12}(\Gamma_N^s)}$ and apply suitably Young's inequality. Then, by integrating in time from 0 to t, we immediately derive the first part of (3.7) with a constant C_2 that depends only on ρ^s and μ^s, and a constant C_1 that depends only on λ^s, μ^s, $\|\partial_t \widetilde{\mathbf{u}}_D^s\|_{L^2(Q_t)}$, $\|\mathbf{g}_N^s\|_{L^2(0,t;(H_{00}^{\frac12}(\Gamma_N^s))')}$, $\|\nabla \cdot \widetilde{\mathbf{u}}_D^s\|_{L^2(Q_t)}$, and $\|\nabla \widetilde{\mathbf{u}}_D^s\|_{L^2(Q_t)}$. Finally, the second part of (3.7) follows from (3.6). □

Note that the first part of (3.7) does not require the second part; an L^2 bound of the initial data is sufficient. The H^1 bound will be used in deriving the additional estimate below.

Lemma 3.2. *Under the assumptions* (2.5) *and* (2.8)*, the solution* $\bar{\mathbf{u}}_m^0$ *of* (3.5)–(3.6) *satisfies the following estimate, with constants* C_1 *and* C_2 *independent of* m*, for all* $0 < t \le T$:

$$\alpha \|\partial_t \bar{\mathbf{u}}_m^0\|^2_{L^2(Q_t)} + \frac{\mu^s}{2}\|\nabla \bar{\mathbf{u}}_m^0(t)\|^2_{L^2(\Omega)} + (\lambda^s + \mu^s)\frac12 \|\nabla \cdot \bar{\mathbf{u}}_m^0(t)\|^2_{L^2(\Omega)} \le C_1 + C_2 \|\mathbf{b}_e\|^2_{L^2(Q_t)}. \tag{3.8}$$

Proof. Here, (3.5) is tested with $\partial_t \bar{\mathbf{u}}_m^0$, which is allowable since the coefficients g_k belong to $H^1(0,T)$. This leads to

$$\begin{aligned} \alpha \|\partial_t \bar{\mathbf{u}}_m^0\|^2_{L^2(\Omega)} + \frac{\mu^s}{2}\frac{d}{dt}\|\nabla \bar{\mathbf{u}}_m^0\|^2_{L^2(\Omega)} + (\lambda^s + \mu^s)\frac12 \frac{d}{dt}\|\nabla \cdot \bar{\mathbf{u}}_m^0\|^2_{L^2(\Omega)} \\ \le \rho^s \|\mathbf{b}_e\|_{L^2(\Omega)} \|\partial_t \bar{\mathbf{u}}_m^0\|_{L^2(\Omega)} + \alpha \|\partial_t \widetilde{\mathbf{u}}_D^s\|_{L^2(\Omega)} \|\partial_t \bar{\mathbf{u}}_m^0\|_{L^2(\Omega)} \\ + \langle \mathbf{g}_N^s, \partial_t \bar{\mathbf{u}}_m^0 \rangle_{\Gamma_N^s} - (\lambda^s + \mu^s)(\nabla \cdot \widetilde{\mathbf{u}}_D^s, \nabla \cdot \partial_t \bar{\mathbf{u}}_m^0) - \mu^s (\nabla \widetilde{\mathbf{u}}_D^s, \nabla \partial_t \bar{\mathbf{u}}_m^0). \end{aligned}$$

This inequality is integrated in time from 0 to t, but its last line must be integrated by parts because the left-hand side cannot control $\partial_t \nabla \bar{\mathbf{u}}_m^0$. For instance,

$$\begin{aligned}\int_0^t \langle \mathbf{g}_N^s, \partial_t \bar{\mathbf{u}}_m^0 \rangle_{\Gamma_N^s} &= -\int_0^t \langle \partial_t \mathbf{g}_N^s, \bar{\mathbf{u}}_m^0 \rangle_{\Gamma_N^s} + \langle \mathbf{g}_N^s(t), \bar{\mathbf{u}}_m^0(t) \rangle_{\Gamma_N^s} - \langle \mathbf{g}_N^s(0), \bar{\mathbf{u}}_m^0(0) \rangle_{\Gamma_N^s} \\ &\le C_N \Big(\|\partial_t \mathbf{g}_N^s\|_{L^2(0,t;(H_{00}^{\frac{1}{2}}(\Gamma_N^s))')} \|\nabla \bar{\mathbf{u}}_m^0\|_{L^2(Q_t)} + \|\mathbf{g}_N^s(t)\|_{H_{00}^{\frac{1}{2}}(\Gamma_N^s)'} \|\nabla \bar{\mathbf{u}}_m^0(t)\|_{L^2(\Omega)} \\ &\quad + \|\mathbf{g}_N^s(0)\|_{H_{00}^{\frac{1}{2}}(\Gamma_N^s)'} \|\nabla \bar{\mathbf{u}}_m^0(0)\|_{L^2(\Omega)} \Big).\end{aligned}$$

The remaining two terms are treated similarly. Then, (3.8) follows by suitably applying Young's inequality and using the bounds of Proposition 3.1. The constant C_2 depends only on ρ^s, μ^s, and α, and the constant C_1 depends only on α, μ^s, λ^s, $\|\mathbf{u}_D^s\|_{H^1(0,t;H^{\frac{1}{2}}(\Gamma_D^s)^d)}$, $\|\mathbf{g}_N^s\|_{H^1(0,t;(H_{00}^{\frac{1}{2}}(\Gamma_N^s)')^d)}$, and $|\mathbf{u}^s(0)|_{H^1(\Omega)}$. ☐

3.1.1. *Passing to the limit in* (3.5)–(3.6). Proposition 3.1 and Lemma 3.2 yield the uniform estimates,

$$\forall m \ge 1, \quad \|\bar{\mathbf{u}}_m^0\|_{H^1(0,T;L^2(\Omega)^d)} \le C, \quad \|\bar{\mathbf{u}}_m^0\|_{L^\infty(0,T;H^1(\Omega)^d)} \le C.$$

Therefore, there exists a function $\bar{\mathbf{z}} \in H^1(0,T;L^2(\Omega)^d) \cap L^\infty(0,T;H^1_{0,\Gamma_D^s}(\Omega)^d)$ such that, up to a subsequence,

$$\bar{\mathbf{u}}_m^0 \to \bar{\mathbf{z}}, \text{ weakly in } H^1(0,T;L^2(\Omega)^d) \quad \text{and weakly* in } L^\infty(0,T;H^1_{0,\Gamma_D^s}(\Omega)^d). \tag{3.9}$$

Considering that the set of functions of the form

$$\sum_{i=1}^m f_i(t)\mathbf{w}_i(\mathbf{x}),$$

with $f_i \in H^1(0,T)$, is dense in $H^1(0,T;H^1_{0,\Gamma_D^s}(\Omega)^d)$, we can pass to the limit in (3.5) and obtain that $\bar{\mathbf{z}}$ satisfies for all $\varphi \in H^1(0,T;H^1_{0,\Gamma_D^s}(\Omega)^d)$,

$$\begin{aligned}\alpha \int_0^T (\partial_t \bar{\mathbf{z}}, \varphi) + \mu^s \int_0^T (\nabla \bar{\mathbf{z}}, \nabla \varphi) + (\lambda^s + \mu^s) \int_0^T (\nabla \cdot \bar{\mathbf{z}}, \nabla \cdot \varphi) \\ = \rho^s \int_0^T (\mathbf{b}_e, \varphi) + \int_0^T \langle \mathbf{g}_N^s, \varphi \rangle_{\Gamma_N^s} - \alpha \int_0^T (\partial_t \widetilde{\mathbf{u}}_D^s, \varphi) \\ -\mu^s \int_0^T (\nabla \widetilde{\mathbf{u}}_D^s, \nabla \varphi) - (\lambda^s + \mu^s) \int_0^T (\nabla \cdot \widetilde{\mathbf{u}}_D^s, \nabla \cdot \varphi).\end{aligned} \tag{3.10}$$

This yields (3.3) in $L^2(0,T)$. To recover the initial condition, observe that, owing to equation (3.6) and to the uniform bound in the second part of (3.7), $\bar{\mathbf{u}}_m^0(0)$ converges, up to a subsequence, to $\mathbf{u}^s(0) - \widetilde{\mathbf{u}}_D^s(0)$ weakly in $H^1_{0,\Gamma_D^s}(\Omega)^d$. Thus we deduce from (3.5) after an integration by parts and passing to the limit, that for all $\varphi \in H^1(0,T;H^1_{0,\Gamma_D^s}(\Omega)^d)$ with $\varphi(T) = \mathbf{0}$,

$$\begin{aligned}-\alpha \int_0^T (\bar{\mathbf{z}}, \partial_t \varphi) - \alpha(\mathbf{u}^s(0) - \widetilde{\mathbf{u}}_D^s(0), \varphi(0)) + \mu^s \int_0^T (\nabla \bar{\mathbf{z}}, \nabla \varphi) + (\lambda^s + \mu^s) \int_0^T (\nabla \cdot \bar{\mathbf{z}}, \nabla \cdot \varphi) \\ = \rho^s \int_0^T (\mathbf{b}_e, \varphi) + \int_0^T \langle \mathbf{g}_N^s, \varphi \rangle_{\Gamma_N^s} - \alpha \int_0^T (\partial_t \widetilde{\mathbf{u}}_D^s, \varphi) \\ -\mu^s \int_0^T (\nabla \widetilde{\mathbf{u}}_D^s, \nabla \varphi) - (\lambda^s + \mu^s) \int_0^T (\nabla \cdot \widetilde{\mathbf{u}}_D^s, \nabla \cdot \varphi).\end{aligned}$$

Then another integration by parts shows that $\bar{\mathbf{z}}(0) = \mathbf{u}^s(0) - \tilde{\mathbf{u}}^s_D(0)$. Hence Problem (3.3)–(3.4) has a solution and since this solution is unique, the full sequence $\bar{\mathbf{u}}^0_m$ converges to $\bar{\mathbf{u}}^0$. This is summarized in the next theorem.

Theorem 3.3. *Under the assumptions* (2.5) *and* (2.8)*, Problem* (2.10) *has a unique solution* $\check{\mathbf{u}}^s \in H^1(0,T;L^2(\Omega)^d) \cap L^\infty(0,T;H^1(\Omega)^d)$.

3.2. **The solution of Problem** (2.11)**.** Problem (2.11) is a simplified version of Problem (2.10) with $\alpha \mathbf{v}^f$ instead of $\rho^s \mathbf{b}_e$, simpler owing that its boundary and initial conditions are all homogeneous. Hence, it has a unique solution as long as $\mathbf{v}^f$ belongs to $L^2(Q)^d$, and we have the following corollary:

Corollary 3.4. *In addition to the assumptions* (2.5) *and* (2.8)*, suppose that* $\mathbf{v}^f \in L^2(Q)^d$*. Then Problem* (2.11) *has a unique solution* $\hat{\mathbf{u}}^s(\mathbf{v}^f) \in H^1(0,T;L^2(\Omega)^d) \cap L^\infty(0,T;H^1_{0,\Gamma^s_D}(\Omega)^d)$*. It depends linearly on* $\mathbf{v}^f$ *and satisfies the bounds for all* $t \in]0,T[$

$$\alpha \|\hat{\mathbf{u}}^s(\mathbf{v}^f)(t)\|^2_{L^2(\Omega)} + \mu^s \|\nabla \hat{\mathbf{u}}^s(\mathbf{v}^f)\|^2_{L^2(Q_t)} + 2(\lambda^s + \mu^s)\|\nabla \cdot \hat{\mathbf{u}}^s(\mathbf{v}^f)\|^2_{L^2(Q_t)} \le \frac{\alpha^2 \mathscr{P}^2}{\mu^s} \|\mathbf{v}^f\|^2_{L^2(Q_t)},$$
$$\alpha \|\partial_t \hat{\mathbf{u}}^s(\mathbf{v}^f)\|^2_{L^2(Q_t)} + \mu^s \|\nabla \hat{\mathbf{u}}^s(\mathbf{v}^f)(t)\|^2_{L^2(\Omega)} + (\lambda^s + \mu^s)\|\nabla \cdot \hat{\mathbf{u}}^s(\mathbf{v}^f)(t)\|^2_{L^2(\Omega)} \le \alpha \|\mathbf{v}^f\|^2_{L^2(Q_t)}. \tag{3.11}$$

3.3. **The solution of Problem** (2.12)**.** Problem (2.12) is treated much like Problem (2.10) but requires somewhat more care on account of the divergence condition. As far as uniqueness is concerned, it cannot have more than one velocity solution in $L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;W)$ and this velocity determines a unique pressure, provided it has mean value zero in Ω.

For existence, let $\tilde{\mathbf{v}}^f_D \in H^1(0,T;H^1(\Omega)^d)$ be a divergence-free lifting of $\mathbf{v}^f_D$ for all $t \in [0,T]$:

$$\tilde{\mathbf{v}}^f_D|_{\partial\Omega} = \mathbf{v}^f_D, \quad \nabla \cdot \tilde{\mathbf{v}}^f_D = 0, \quad \|\tilde{\mathbf{v}}^f_D\|_{H^1(0,T;H^1(\Omega)^d)} \le C \|\mathbf{v}^f_D\|_{H^1(0,T;H^{\frac{1}{2}}(\partial\Omega)^d)}; \tag{3.12}$$

the divergence zero condition is possible owing to the compatibility condition (2.4). Proceeding as in Section 3.1, we set

$$\bar{\mathbf{v}}^f = \check{\mathbf{v}}^f - \tilde{\mathbf{v}}^f_D;$$

note that a.e. in $]0,T[$, $\bar{\mathbf{v}}^f$ belongs to V, defined by (1.9). Then $\bar{\mathbf{v}}^f$ solves the following variational formulation: Find $\bar{\mathbf{v}}^f \in H^1(0,T;L^2(\Omega)^d) \cap L^2(0,T;V)$ such that for all $\mathbf{w} \in H^1_0(\Omega)^d$ and a.e. in $]0,T[$,

$$\begin{aligned} \rho^f(\partial_t \bar{\mathbf{v}}^f, \mathbf{w}) + \mu^f(\nabla \bar{\mathbf{v}}^f, \nabla \mathbf{w}) - (\bar{p}^f, \nabla \cdot \mathbf{w}) + \alpha(\bar{\mathbf{v}}^f, \mathbf{w}) = \rho^f(\mathbf{b}_e, \mathbf{w}) + \alpha(\partial_t \check{\mathbf{u}}^s, \mathbf{w}) \\ - \rho^f(\partial_t \tilde{\mathbf{v}}^f_D, \mathbf{w}) - \mu^f(\nabla \tilde{\mathbf{v}}^f_D, \nabla \mathbf{w}) - \alpha(\tilde{\mathbf{v}}^f_D, \mathbf{w}), \end{aligned} \tag{3.13}$$

with the initial condition in Ω

$$\bar{\mathbf{v}}^f(0) = \mathbf{v}^f(0) - \tilde{\mathbf{v}}^f_D(0), \tag{3.14}$$

where $\check{\mathbf{u}}^s$ solves (2.10), see Theorem 3.3. Regarding uniqueness, since problem (3.13)–(3.14) is linear, we readily deduce that when its right-hand sides are zero, then its only velocity solution in $L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;V)$ is necessarily zero, and the pressure is uniquely determined by the velocity, provided it has mean value zero in space.

Existence of the solution of Problem (3.13)–(3.14), with the zero divergence condition, can also be deduced by Galerkin's construction. As the space V is separable, let $\{\mathbf{w}_i\}_{i\geq 1}$ be a basis of V, and let V_m be the space spanned by $\mathbf{w}_i$, $1 \leq i \leq m$. Consider the function

$$\bar{\mathbf{v}}_m^f(\mathbf{x},t) = \sum_{k=1}^{m} g_k(t)\mathbf{w}_k(\mathbf{x}),$$

and the discrete problem: Find $g_i \in H^1(0,T)$, $1 \leq i \leq m$, solution of

$$\begin{aligned}\rho^f(\partial_t \bar{\mathbf{v}}_m^f, \mathbf{w}_j) + \mu^f(\nabla \bar{\mathbf{v}}_m^f, \nabla \mathbf{w}_j) + \alpha(\bar{\mathbf{v}}_m^f, \mathbf{w}_j) = \rho^f(\mathbf{b}_e, \mathbf{w}_j) + \alpha(\partial_t \check{\mathbf{u}}^s, \mathbf{w}_j)\\ - \rho^f(\partial_t \tilde{\mathbf{v}}_D^f, \mathbf{w}_j) - \mu^f(\nabla \tilde{\mathbf{v}}_D^f, \nabla \mathbf{w}_j) - \alpha(\tilde{\mathbf{v}}_D^f, \mathbf{w}_j),\ 1 \leq j \leq m,\end{aligned} \tag{3.15}$$

with initial condition

$$(\nabla \bar{\mathbf{v}}_m^f(0), \nabla \mathbf{w}_j) = (\nabla \mathbf{v}^f(0), \nabla \mathbf{w}_j) - (\nabla \tilde{\mathbf{v}}_D^f(0), \nabla \mathbf{w}_j),\ 1 \leq j \leq m. \tag{3.16}$$

Problem (3.15)–(3.16) is a square system of m linear ODEs with constant coefficients on the left-hand side and coefficients in the right-hand side in $L^2(0,T)$. It has a unique solution in $H^1(0,T)$, bounded as follows:

Proposition 3.5. *In addition to the assumptions of Proposition 3.1, suppose that* (2.6) *and* (2.9) *hold. Then the solution $\bar{\mathbf{v}}_m^f$ of* (3.15)–(3.16) *satisfies the estimate for $0 < t \leq T$,*

$$\begin{aligned}\rho^f \|\bar{\mathbf{v}}_m^f(t)\|^2_{L^2(\Omega)} + \mu^f \|\nabla \bar{\mathbf{v}}_m^f\|^2_{L^2(Q_t)} \leq \rho^f \|\bar{\mathbf{v}}_m^f(0)\|^2_{L^2(\Omega)} + \mu^f \|\nabla \tilde{\mathbf{v}}_D^f\|^2_{L^2(Q_t)}\\ + \frac{2}{\alpha}\Big((\rho^f)^2 \big(\|\mathbf{b}_e\|^2_{L^2(Q_t)} + \|\partial_t \tilde{\mathbf{v}}_D^f\|^2_{L^2(Q_t)}\big) + \alpha^2 \big(\|\partial_t \check{\mathbf{u}}^s\|^2_{L^2(Q_t)} + \|\tilde{\mathbf{v}}_D^f\|^2_{L^2(Q_t)}\big)\Big),\end{aligned} \tag{3.17}$$

and

$$\|\nabla \bar{\mathbf{v}}_m^f(0)\|_{L^2(\Omega)} \leq \|\nabla \mathbf{v}^f(0)\|_{L^2(\Omega)} + \|\nabla \tilde{\mathbf{v}}_D^f(0)\|_{L^2(\Omega)}. \tag{3.18}$$

Proof. By testing (3.15) with $\bar{\mathbf{v}}_m^f$, we obtain a.e. in $]0,T[$

$$\begin{aligned}\frac{1}{2}\rho^f \frac{d}{dt}\|\bar{\mathbf{v}}_m^f\|^2_{L^2(\Omega)} + \mu^f \|\nabla \bar{\mathbf{v}}_m^f\|^2_{L^2(\Omega)} + \alpha \|\bar{\mathbf{v}}_m^f\|^2_{L^2(\Omega)} \leq \mu^f \|\nabla \tilde{\mathbf{v}}_D^f\|_{L^2(\Omega)} \|\nabla \bar{\mathbf{v}}_m^f\|_{L^2(\Omega)}\\ + \Big(\rho^f \big(\|\mathbf{b}_e\|_{L^2(\Omega)} + \|\partial_t \tilde{\mathbf{v}}_D^f\|_{L^2(\Omega)}\big) + \alpha \|\partial_t \check{\mathbf{u}}^s\|_{L^2(\Omega)} + \alpha \|\tilde{\mathbf{v}}_D^f\|_{L^2(\Omega)}\Big) \|\bar{\mathbf{v}}_m^f\|_{L^2(\Omega)}.\end{aligned}$$

Then Young's inequality gives a.e. in $]0,T[$

$$\begin{aligned}\frac{\rho^f}{2}\frac{d}{dt}\|\bar{\mathbf{v}}_m^f\|^2_{L^2(\Omega)} + \frac{\mu^f}{2}\|\nabla \bar{\mathbf{v}}_m^f\|^2_{L^2(\Omega)} \leq \frac{\mu^f}{2}\|\nabla \tilde{\mathbf{v}}_D^f\|^2_{L^2(\Omega)}\\ + \frac{1}{4\alpha}\Big(\rho^f \big(\|\mathbf{b}_e\|_{L^2(\Omega)} + \|\partial_t \tilde{\mathbf{v}}_D^f\|_{L^2(\Omega)}\big) + \alpha \big(\|\partial_t \check{\mathbf{u}}^s\|_{L^2(\Omega)} + \|\tilde{\mathbf{v}}_D^f\|_{L^2(\Omega)}\big)\Big)^2,\end{aligned} \tag{3.19}$$

and (3.17) follows by integrating over $]0,t[$ and (3.18) follows by using (3.16). □

Again, (3.17) does not require the gradient of the initial data. The H^1 bound will be used in estimating the time derivative of $\bar{\mathbf{v}}_m^f$. To simplify, the constants will not be specified.

Lemma 3.6. *Under the assumptions of Proposition 3.5, the solution $\bar{\mathbf{v}}_m^f$ of* (3.15)–(3.16) *satisfies the following estimate, with constants C_1 and C_2 independent of m, for all $0 < t \leq T$:*

$$\rho^f \|\partial_t \bar{\mathbf{v}}_m^f\|^2_{L^2(Q_t)} + \frac{\mu^f}{2}\|\nabla \bar{\mathbf{v}}_m^f(t)\|^2_{L^2(\Omega)} + \alpha \|\bar{\mathbf{v}}_m^f(t)\|^2_{L^2(\Omega)} \leq C_1 + C_2 \|\mathbf{b}_e\|^2_{L^2(Q_t)}. \tag{3.20}$$

Proof. For the time derivative, (3.15) is tested with $\partial_t \bar{\mathbf{v}}_m^f$:

$$\rho^f \|\partial_t \bar{\mathbf{v}}_m^f\|^2_{L^2(\Omega)} + \frac{\mu^f}{2}\frac{d}{dt}\|\nabla \bar{\mathbf{v}}_m^f\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\frac{d}{dt}\|\bar{\mathbf{v}}_m^f\|^2_{L^2(\Omega)} = \rho^f(\mathbf{b}_e, \partial_t \bar{\mathbf{v}}_m^f) + \alpha(\partial_t \check{\mathbf{u}}^s, \partial_t \bar{\mathbf{v}}_m^f)$$
$$- \rho^f(\partial_t \widetilde{\mathbf{v}}_D^f, \partial_t \bar{\mathbf{v}}_m^f) - \mu^f(\nabla \widetilde{\mathbf{v}}_D^f, \partial_t \nabla \bar{\mathbf{v}}_m^f) - \alpha(\widetilde{\mathbf{v}}_D^f, \partial_t \bar{\mathbf{v}}_m^f).$$

This is integrated in time over $]0,t[$, but again the factor of μ^f on the right-hand side must be integrated by parts because the left-hand side cannot control $\partial_t \nabla \bar{\mathbf{v}}_m^f$. This gives

$$\rho^f \|\partial_t \bar{\mathbf{v}}_m^f\|^2_{L^2(Q_t)} + \frac{\mu^f}{2}\|\nabla \bar{\mathbf{v}}_m^f(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\bar{\mathbf{v}}_m^f(t)\|^2_{L^2(\Omega)}$$
$$\leq \Big(\rho^f\|\mathbf{b}_e\|_{L^2(Q_t)} + \alpha\|\partial_t \check{\mathbf{u}}^s\|_{L^2(Q_t)} + \rho^f\|\partial_t \widetilde{\mathbf{v}}_D^f\|_{L^2(Q_t)} + \alpha\|\widetilde{\mathbf{v}}_D^f\|_{L^2(Q_t)}\Big)\|\partial_t \bar{\mathbf{v}}_m^f\|_{L^2(Q_t)}$$
$$+ \frac{\mu^f}{2}\|\nabla \bar{\mathbf{v}}_m^f(0)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\bar{\mathbf{v}}_m^f(0)\|^2_{L^2(\Omega)}$$
$$+ \mu^f \int_0^t (\partial_t \nabla \widetilde{\mathbf{v}}_D^f, \nabla \bar{\mathbf{v}}_m^f) - \mu^f(\nabla \widetilde{\mathbf{v}}_D^f(t), \nabla \bar{\mathbf{v}}_m^f(t)) + \mu^f(\nabla \widetilde{\mathbf{v}}_D^f(0), \nabla \bar{\mathbf{v}}_m^f(0)).$$

With Young's inequality and (3.17), we easily recover (3.20) with constants C_1 and C_2, independent of m. The constant C_2 depends only on α, ρ^f, ρ^s, μ^f, and μ^s. □

3.3.1. *Passing to the limit in* (3.15). The uniform estimates (3.17) and (3.20) imply that there exists a function $\bar{\mathbf{z}} \in H^1(0,T;L^2(\Omega)^d) \cap L^\infty(0,T;V)$ such that, up to a subsequence,

$$\bar{\mathbf{v}}_m^f \to \bar{\mathbf{z}}, \text{ weakly in } H^1(0,T;L^2(\Omega)^d) \quad \text{and weakly* in } L^\infty(0,T;V). \tag{3.21}$$

Since the set of functions of the form

$$\sum_{i=1}^m f_i(t)\mathbf{w}_i(\mathbf{x}),$$

with $f_i \in H^1(0,T)$, is dense in $H^1(0,T;V)$, we can pass to the limit in (3.15) and obtain that $\bar{\mathbf{z}}$ satisfies for all $\varphi \in H^1(0,T;V)$,

$$\begin{aligned} &\rho^f \int_0^T (\partial_t \bar{\mathbf{z}}, \varphi) + \mu^f \int_0^T (\nabla \bar{\mathbf{z}}, \nabla \varphi) + \alpha \int_0^T (\bar{\mathbf{z}}, \varphi) = \rho^f \int_0^T (\mathbf{b}_e, \varphi) \\ &+ \alpha \int_0^T (\partial_t \check{\mathbf{u}}^s, \varphi) - \rho^f \int_0^T (\partial_t \widetilde{\mathbf{v}}_D^f, \varphi) - \mu^f \int_0^T (\nabla \widetilde{\mathbf{v}}_D^f, \nabla \varphi) - \alpha \int_0^T (\widetilde{\mathbf{v}}_D^f, \varphi). \end{aligned} \tag{3.22}$$

As all terms above are well-defined when $\varphi \in L^2(0,T;V)$, (3.22) implies that in $L^2(0,T)$, for all $\varphi \in V$

$$\langle \rho^f \partial_t \bar{\mathbf{z}} - \mu^f \Delta \bar{\mathbf{z}} + \alpha \bar{\mathbf{z}} - \rho^f \mathbf{b}_e - \alpha \partial_t \check{\mathbf{u}}^s + \rho^f \partial_t \widetilde{\mathbf{v}}_D^f - \mu^f \Delta \widetilde{\mathbf{v}}_D^f + \alpha \widetilde{\mathbf{v}}_D^f, \varphi \rangle = 0,$$

where the duality is taken between V' and V. But, as the first argument of the duality belongs to $H^{-1}(\Omega)^d$ for almost every time, De Rham's Lemma implies that there exists a function p, a.e. in time, unique in $L^2_0(\Omega)$ such that

$$\rho^f \partial_t \bar{\mathbf{z}} - \mu^f \Delta \bar{\mathbf{z}} + \alpha \bar{\mathbf{z}} - \rho^f \mathbf{b}_e - \alpha \partial_t \check{\mathbf{u}}^s + \rho^f \partial_t \widetilde{\mathbf{v}}_D^f - \mu^f \Delta \widetilde{\mathbf{v}}_D^f + \alpha \widetilde{\mathbf{v}}_D^f = \nabla p, \tag{3.23}$$

and, see for instance [13] ,

$$\|p\|_{L^2(\Omega)} \leq \frac{1}{\beta}\|\rho^f \partial_t \bar{\mathbf{z}} - \mu^f \Delta \bar{\mathbf{z}} + \alpha \bar{\mathbf{z}} - \rho^f \mathbf{b}_e - \alpha \partial_t \check{\mathbf{u}}^s + \rho^f \partial_t \widetilde{\mathbf{v}}_D^f - \mu^f \Delta \widetilde{\mathbf{v}}_D^f + \alpha \widetilde{\mathbf{v}}_D^f\|_{H^{-1}(\Omega)}.$$

The constant $\beta > 0$ depends only on the domain. The above inequality holds a.e. in time and since its right-hand side belongs to $L^2(0,T)$, we have

$$\|p\|_{L^2(Q)} \leq \frac{1}{\beta}\|\rho^f \partial_t \bar{\mathbf{z}} - \mu^f \Delta \bar{\mathbf{z}} + \alpha \bar{\mathbf{z}} - \rho^f \mathbf{b}_e - \alpha \partial_t \check{\mathbf{u}}^s + \rho^f \partial_t \tilde{\mathbf{v}}_D^f - \mu^f \Delta \tilde{\mathbf{v}}_D^f + \alpha \tilde{\mathbf{v}}_D^f\|_{L^2(0,T;H^{-1}(\Omega)^d)}. \tag{3.24}$$

Thus $\bar{\mathbf{z}}$ and p satisfy (3.13). To recover the initial data, we infer from (3.16) that $\bar{\mathbf{v}}_m^f(0)$ converges weakly to some function ζ in V and ζ satisfies

$$\forall \varphi \in V, \quad (\nabla \zeta, \nabla \varphi) = (\nabla(\mathbf{v}^f(0) - \tilde{\mathbf{v}}_D^f(0)), \nabla \varphi).$$

As both ζ and $\mathbf{v}^f(0) - \tilde{\mathbf{v}}_D^f(0)$ belong to V, this implies that

$$\zeta = \mathbf{v}^f(0) - \tilde{\mathbf{v}}_D^f(0).$$

From here, by proceeding as above, we deduce that $\bar{\mathbf{z}}(0) = \zeta$. Hence, we have the analogue of Theorem 3.3.

Theorem 3.7. *Under the assumptions of Lemma 3.6, Problem (2.12) has a unique solution* $\check{\mathbf{v}}^f \in H^1(0,T;L^2(\Omega)^d) \cap L^\infty(0,T;W)$ *and* $\check{p}^f$ *in* $L^2(0,T;L_0^2(\Omega))$.

3.4. **The solution of Problem (2.13).** As Problem (2.13) is an implicit system, it can be easily solved by using the following form of Schauder's Fixed Point Theorem, see for instance [6]:

Theorem 3.8. *Let H be a Banach space and E a non empty closed convex set in H. Let F be a continuous mapping from E into E such that $F(E)$ is contained in a compact subset $\mathscr{K}$ of E. Then F has at least one fixed point in $\mathscr{K}$.*

Regarding compactness, we shall use the following form of the Aubin–Lions–Simon Theorem, see [3, 27, 6]:

Theorem 3.9. *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces with continuous embeddings, such that the embedding of B_0 into B_1 is compact. Let p, r be two numbers such that $1 \leq p, r \leq \infty$ and let $T > 0$. Then the space*

$$\big\{v \in L^p(0,T;B_0)\,;\, \frac{dv}{dt} \in L^r(0,T;B_2)\big\},$$

is compactly embedded into $L^p(0,T;B_1)$.

According to Corollary 3.4, the fact that $\hat{\mathbf{u}}^s(\mathbf{v}_0^f)$ is well-defined for $\mathbf{v}_0^f$ in $L^2(Q)^d$ suggests to take $H = L^2(Q)^d$. To define the mapping F, let ψ be given in $L^2(Q)^d$ and consider the homogeneous problem

$$\begin{aligned} \rho^f \partial_t \mathbf{v}_0^f - \mu^f \Delta \mathbf{v}_0^f + \nabla p_0^f + \alpha \mathbf{v}_0^f &= \alpha \partial_t \hat{\mathbf{u}}^s(\check{\mathbf{v}}^f) + \alpha \partial_t \hat{\mathbf{u}}^s(\psi), \text{ a.e. in } Q, \\ \nabla \cdot \mathbf{v}_0^f &= 0, \text{ a.e. in } Q, \\ \mathbf{v}_0^f &= \mathbf{0}, \text{ a.e. on } \partial\Omega \times]0,T[, \\ \mathbf{v}_0^f(0) &= \mathbf{0}, \text{ a.e. in } \Omega. \end{aligned} \tag{3.25}$$

This is a time dependent Stokes system, similar to Problem (2.12), but simpler because it is homogeneous. According to Theorem 3.7, it has a unique solution $\mathbf{v}_0^f$ in $H^1(0,T;L^2(\Omega)^d) \cap$

$L^\infty(0,T;V)$ and p_0^f in $L^2(0,T;L^2_0(\Omega))$. Furthermore, $\mathbf{v}_0^f$ satisfies the following estimate a.e. in $]0,T[$:

$$\frac{1}{2}\rho^f\|\mathbf{v}_0^f(t)\|^2_{L^2(\Omega)} + \mu^f\|\nabla\mathbf{v}_0^f\|^2_{L^2(Q_t)} \leq \frac{\alpha}{2}\Big(\|\partial_t\hat{\mathbf{u}}^s(\psi)\|^2_{L^2(Q_t)} + \|\partial_t\hat{\mathbf{u}}^s(\check{\mathbf{v}}^f)\|^2_{L^2(Q_t)}\Big) \leq \frac{\alpha}{2}\Big(\|\psi\|^2_{L^2(Q_t)} + \|\check{\mathbf{v}}^f\|^2_{L^2(Q_t)}\Big). \tag{3.26}$$

The first inequality is standard and the second inequality follows from (3.11). Similarly, $\partial_t\mathbf{v}_0^f$ is bounded as follows a.e. in $]0,T[$:

$$\begin{aligned}
&\frac{1}{2}\rho^f\|\partial_t\mathbf{v}_0^f\|^2_{L^2(Q_t)} + \frac{\mu^f}{2}\|\nabla\mathbf{v}_0^f(t)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\mathbf{v}_0^f(t)\|^2_{L^2(\Omega)}\\
&\leq \frac{\alpha^2}{\rho^f}\Big(\|\partial_t\hat{\mathbf{u}}^s(\psi)\|^2_{L^2(Q_t)} + \|\partial_t\hat{\mathbf{u}}^s(\check{\mathbf{v}}^f)\|^2_{L^2(Q_t)}\Big)\\
&\leq \frac{\alpha^2}{\rho^f}\Big(\|\psi\|^2_{L^2(Q_t)} + \|\check{\mathbf{v}}^f\|^2_{L^2(Q_t)}\Big).
\end{aligned} \tag{3.27}$$

These bounds lead to the main results of this section.

Theorem 3.10. *Under the assumptions of Lemma 3.6, for $d=2,3$, Problem (2.13) has a unique solution $\mathbf{v}_0^f \in H^1(0,T;L^2(\Omega)^d)\cap L^\infty(0,T;V)$ and $p_0^f \in L^2(0,T;L^2_0(\Omega))$.*

Proof. Let F be the mapping $\psi \in L^2(Q)^d \mapsto \mathbf{v}_0^f \in L^2(Q)^d$, where $\mathbf{v}_0^f$ is the unique solution of (3.25). This mapping is affine, Lipschitz-continuous, and according to (3.26) and (3.27), the range space of F is contained in $H^1(0,T;L^2(\Omega)^d)\cap L^\infty(0,T;V)$. Here we can take $E = H = L^2(Q)^d$ and $F(E)\subset H^1(0,T;L^2(\Omega)^d)\cap L^\infty(0,T;V)$ which is compactly embedded into $L^2(Q)^d$, owing to Theorem 3.9. Therefore Theorem 3.8 guarantees that F has at least one fixed point in $H^1(0,T;L^2(\Omega)^d)\cap L^\infty(0,T;V)$. This yields existence of a velocity solution $\mathbf{v}_0^f$ of (2.13); uniqueness is easily established. By using the above argument, this velocity determines a unique pressure p_0^f in $L^2(0,T;L^2_0(\Omega))$. □

Corollary 3.11. *Under the assumptions (2.5), (2.8), (2.6), and (2.9), for $d=2,3$, the coupled problem (1.1)–(1.3) with the above initial and boundary conditions has a unique solution $\mathbf{u}^s$ and $\mathbf{v}^f$ both in $H^1(0,T;L^2(\Omega)^d)\cap L^\infty(0,T;H^1(\Omega)^d)$ and p^f in $L^2(0,T;L^2_0(\Omega))$.*

4. DISCRETIZATION OF PROBLEM (1.1)–(1.3)

To simplify, we discretize the problem where the Dirichlet boundary conditions are all homogeneous. In addition, for discretizing in time, it is simpler, but not fundamental, to suppose that $\mathbf{b}_e$ belongs to $\mathscr{C}^0([0,T];L^2(\Omega)^3)$, see Remark 4.1 below.

Let us consider a uniform refinement of the interval $[0,T]$ by N subintervals, $[t_{n-1},t_n]$, with step size $\Delta t = t_n - t_{n-1}$ for $n=1,\ldots,N$. We discretize the time derivatives by means of the backward Euler method, that is, we approximate

$$\frac{\partial\mathbf{v}^f}{\partial t}(t_n) \simeq \frac{\mathbf{v}^{f,n}-\mathbf{v}^{f,n-1}}{\Delta t}, \qquad \frac{\partial\mathbf{u}^s}{\partial t}(t_n) \simeq \frac{\mathbf{u}^{s,n}-\mathbf{u}^{s,n-1}}{\Delta t},$$

where we denote $\mathbf{v}^{f,n}$ the approximate value of $\mathbf{v}^f(t_n)$, for $n=1,\ldots,N$, and $\mathbf{v}^{f,0}=\mathbf{v}^f(0)$; similarly, $\mathbf{u}^{s,n}$ denotes the approximate value of $\mathbf{u}^s(t_n)$, for $n=1,\ldots,N$, and $\mathbf{u}^{s,0}=\mathbf{u}^s(0)$. Finally, we denote by $p^{f,n}$ the approximate value of $p^f(t_n)$, for $n=1,\ldots,N$.

Recall that $d = 2,3$. As usual, we denote by $\{\mathcal{T}_h\}_h$ a regular family of simplicial meshes of $\overline{\Omega}$, in the sense of Ciarlet [9]. For the discretization of the fluid and pressure, we introduce a pair of finite element spaces (V_h, Q_h) with $V_h \subset H^1_0(\Omega)^d$ and $Q_h \subset L^2_0(\Omega)$, and we define

$$V_h^0 = \{\mathbf{w}_h \in V_h\,;\, \forall q_h \in Q_h, \quad \int_\Omega q_h \nabla\cdot \mathbf{w}_h = 0\}.$$

For the discretization of the solid part, we consider a finite element space U_h such that $U_h \subset H^1_{0,\Gamma^s_D}(\Omega)^d$. Additional properties of these spaces will be prescribed when needed. An example is the classical Hood–Taylor finite element pair of order two for the fluid, see [17, 14, 4],

$$V_h = \{\mathbf{w}_h \in [\mathscr{C}^0(\bar{\Omega})]^d\,;\, \forall K \in \mathcal{T}_h, \quad \mathbf{w}_h|_K \in [\mathscr{P}_2(K)]^d\} \cap H^1_0(\Omega)^d\,,$$
$$Q_h = \{q_h \in \mathscr{C}^0(\bar{\Omega})\,;\, \forall K \in \mathcal{T}_h, \quad q_h|_K \in \mathscr{P}_1(K)\} \cap L^2_0(\Omega)\,,$$

and the classical finite element space of the same order for the solid,

$$U_h = \{\mathbf{z}_h \in [\mathscr{C}^0(\bar{\Omega})]^d\,;\, \forall K \in \mathcal{T}_h, \quad \mathbf{z}_h|_K \in [\mathscr{P}_2(K)]^d\} \cap H^1_{0,\Gamma^s_D}(\Omega)^d\,.$$

The given initial velocity is approximated by a discretization operator Π_h mapping $H^1_0(\Omega)^d$ into V_h that preserves weakly the divergence, i.e., for all $\mathbf{v} \in H^1_0(\Omega)^d$,

$$\forall q_h \in Q_h, \int_\Omega q_h \nabla\cdot \Pi_h(\mathbf{v}) = \int_\Omega q_h \nabla\cdot \mathbf{v}. \tag{4.1}$$

Then, we set

$$\mathbf{v}_h^{f,0} = \Pi_h(\mathbf{v}^f(0)) \in V_h^0. \tag{4.2}$$

The given initial displacement is approximated by a discretization operator P_h mapping $H^1_{0,\Gamma^s_D}(\Omega)^d$ into U_h,

$$\mathbf{u}_h^{s,0} = P_h(\mathbf{u}^s(0)). \tag{4.3}$$

Then, we introduce the following fully discrete scheme: For each $n = 1,\dots,N$, given $\mathbf{v}_h^{f,n-1}$ and $\mathbf{u}_h^{s,n-1}$, find $(\mathbf{v}_h^{f,n}, p_h^{f,n}, \mathbf{u}_h^{s,n}) \in V_h \times Q_h \times U_h$ such that,

$$\begin{aligned}
\rho^f \frac{1}{\Delta t}\int_\Omega (\mathbf{v}_h^{f,n} - \mathbf{v}_h^{f,n-1})\cdot \mathbf{w}_h + \mu^f \int_\Omega \nabla \mathbf{v}_h^{f,n} : \nabla \mathbf{w}_h - \int_\Omega p_h^{f,n} \nabla\cdot \mathbf{w}_h - \rho^f \int_\Omega \mathbf{b}_e(t_n)\cdot \mathbf{w}_h \\
+\alpha \int_\Omega \mathbf{v}_h^{f,n}\cdot \mathbf{w}_h - \alpha \frac{1}{\Delta t}\int_\Omega (\mathbf{u}_h^{s,n} - \mathbf{u}_h^{s,n-1})\cdot \mathbf{w}_h = 0,\\
\int_\Omega q_h \nabla\cdot \mathbf{v}_h^{f,n} = 0,\\
(\lambda^s + \mu^s)\int_\Omega (\nabla\cdot \mathbf{u}_h^{s,n})(\nabla\cdot \mathbf{z}_h) + \mu^s \int_\Omega \nabla \mathbf{u}_h^{s,n} : \nabla \mathbf{z}_h - \rho^s \int_\Omega \mathbf{b}_e(t_n)\cdot \mathbf{z}_h - \langle \mathbf{g}_N^{s,n}, \mathbf{z}_h\rangle_{\Gamma^s_N}\\
-\alpha \int_\Omega \mathbf{v}_h^{f,n}\cdot \mathbf{z}_h + \alpha \frac{1}{\Delta t}\int_\Omega (\mathbf{u}_h^{s,n} - \mathbf{u}_h^{s,n-1})\cdot \mathbf{z}_h = 0,
\end{aligned} \tag{4.4}$$

for all $(\mathbf{w}_h, q_h, \mathbf{z}_h) \in V_h \times Q_h \times U_h$, where we denoted $\mathbf{g}_N^{s,n} := \mathbf{g}_N^s(t_n)$, for $n = 1,\dots,N$.

Remark 4.1. This scheme uses the pointwise values $\mathbf{b}_e(t_n)$, whence the assumption that $\mathbf{b}_e$ is continuous in time. This is just a matter of convenience and we could have replaced the pointwise values by integral mean values in time, i.e., used $\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} \mathbf{b}_e$ instead of $\mathbf{b}_e(t_n)$, in which case, it would have sufficed that $\mathbf{b}_e$ be in $L^2(Q)^d$.

4.1. **Existence of the discrete solution of (4.4).** To begin with, let us prove that the scheme (4.4) has exactly one solution for any choice of discrete spaces, as long as the pair (V_h, Q_h) satisfies an inf-sup condition, for a constant $\beta^\star > 0$,

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{\int_\Omega q_h \nabla \cdot \mathbf{v}_h}{|\mathbf{v}_h|_{H^1(\Omega)} \|q_h\|_{L^2(\Omega)}} \geq \beta^\star. \tag{4.5}$$

This condition guarantees the existence of an approximation operator Π_h satisfying (4.1), see [13]. Since $\mathbf{u}_h^{s,0}$ and $\mathbf{v}_h^{f,0}$ are determined by the initial data, it suffices to consider an arbitrary step n of the scheme. As this is a square system of linear equations, we must prove that if all data are zero, then the solution is necessarily zero, i.e., if

$$\mathbf{v}_h^{f,n-1} = \mathbf{0}, \quad \mathbf{b}_e(t_n) = \mathbf{0}, \quad \mathbf{u}_h^{s,n-1} = \mathbf{0}, \quad \mathbf{g}_N^s(t_n) = \mathbf{0},$$

then

$$\mathbf{v}_h^{f,n} = \mathbf{0}, \quad \mathbf{u}_h^{s,n} = \mathbf{0}, \quad p_h^{f,n} = 0.$$

This homogeneous system reads

$$\begin{aligned} \frac{\rho^f}{\Delta t}\big(\mathbf{v}_h^{f,n}, \mathbf{w}_h\big) + \mu^f\big(\nabla \mathbf{v}_h^{f,n}, \nabla \mathbf{w}_h\big) - \big(p_h^{f,n}, \nabla \cdot \mathbf{w}_h\big) + \alpha\big(\mathbf{v}_h^{f,n}, \mathbf{w}_h\big) &= \frac{\alpha}{\Delta t}\big(\mathbf{u}_h^{s,n}, \mathbf{w}_h\big) \\ \big(q_h, \nabla \cdot \mathbf{v}_h^{f,n}\big) &= 0 \\ \frac{\alpha}{\Delta t}\big(\mathbf{u}_h^{s,n}, \mathbf{z}_h\big) + (\lambda^s + \mu^s)\big(\nabla \cdot \mathbf{u}_h^{s,n}, \nabla \cdot \mathbf{z}_h\big) + \mu^s\big(\nabla \mathbf{u}_h^{s,n}, \nabla \mathbf{z}_h\big) &= \alpha\big(\mathbf{v}_h^{f,n}, \mathbf{z}_h\big). \end{aligned} \tag{4.6}$$

Proposition 4.2. *If the inf-sup condition (4.5) holds, the system (4.6) has only the zero solution.*

Proof. To simplify, here the norm of $L^2(\Omega)$ is denoted without subscript. Let $\mathbf{v}_h^{f,n}$, $p_h^{f,n}$, and $\mathbf{u}_h^{s,n}$ solve (4.6) and test it with $\mathbf{w}_h = \mathbf{v}_h^{f,n}$, $q_h = p_h^{f,n}$, $\mathbf{z}_h = \mathbf{u}_h^{s,n}$. The first and second equation of (4.6) and the third equation of (4.6) imply, respectively

$$\begin{aligned} \frac{\rho^f}{\Delta t}\|\mathbf{v}_h^{f,n}\|^2 + \mu^f\|\nabla \mathbf{v}_h^{f,n}\|^2 + \alpha\|\mathbf{v}_h^{f,n}\|^2 &= \frac{\alpha}{\Delta t}\big(\mathbf{u}_h^{s,n}, \mathbf{v}_h^{f,n}\big), \\ \frac{\alpha}{\Delta t}\|\mathbf{u}_h^{s,n}\|^2 + (\lambda^s + \mu^s)\|\nabla \cdot \mathbf{u}_h^{s,n}\|^2 + \mu^s\|\nabla \mathbf{u}_h^{s,n}\|^2 &= \alpha\big(\mathbf{v}_h^{f,n}, \mathbf{u}_h^{s,n}\big). \end{aligned} \tag{4.7}$$

By combining these two, we obtain

$$\frac{\rho^f}{\Delta t}\|\mathbf{v}_h^{f,n}\|^2 + \mu^f\|\nabla \mathbf{v}_h^{f,n}\|^2 + \alpha\|\mathbf{v}_h^{f,n}\|^2 = \frac{\alpha}{\Delta t^2}\|\mathbf{u}_h^{s,n}\|^2 + \frac{\lambda^s + \mu^s}{\Delta t}\|\nabla \cdot \mathbf{u}_h^{s,n}\|^2 + \frac{\mu^s}{\Delta t}\|\nabla \mathbf{u}_h^{s,n}\|^2. \tag{4.8}$$

But (4.7) implies

$$\frac{\alpha}{\Delta t}\|\mathbf{u}_h^{s,n}\|^2 + (\lambda^s + \mu^s)\|\nabla \cdot \mathbf{u}_h^{s,n}\|^2 + \mu^s\|\nabla \mathbf{u}_h^{s,n}\|^2 \leq \frac{\alpha}{2}\Big(\Delta t\|\mathbf{v}_h^{f,n}\|^2 + \frac{1}{\Delta t}\|\mathbf{u}_h^{s,n}\|^2\Big). \tag{4.9}$$

On the one hand, this yields

$$\frac{\alpha}{2\Delta t}\|\mathbf{u}_h^{s,n}\|^2 + (\lambda^s + \mu^s)\|\nabla \cdot \mathbf{u}_h^{s,n}\|^2 + \mu^s\|\nabla \mathbf{u}_h^{s,n}\|^2 \leq \frac{\alpha}{2}\Delta t\|\mathbf{v}_h^{f,n}\|^2,$$

thus

$$\frac{\alpha}{2\Delta t}\|\mathbf{u}_h^{s,n}\|^2 \leq \frac{\alpha}{2}\Delta t\|\mathbf{v}_h^{f,n}\|^2.$$

On the other hand, by dividing (4.9) by Δt, and substituting this last inequality, we have

$$\frac{\alpha}{\Delta t^2}\|\mathbf{u}_h^{s,n}\|^2 + \frac{\lambda^s + \mu^s}{\Delta t}\|\nabla \cdot \mathbf{u}_h^{s,n}\|^2 + \frac{\mu^s}{\Delta t}\|\nabla \mathbf{u}_h^{s,n}\|^2 \leq \alpha\|\mathbf{v}_h^{f,n}\|^2.$$

Thus the right-hand side of (4.8) is bounded by $\alpha\|\mathbf{v}_h^{f,n}\|^2$, i.e.

$$\frac{\rho^f}{\Delta t}\|\mathbf{v}_h^{f,n}\|^2+\mu^f\|\nabla\mathbf{v}_h^{f,n}\|^2+\alpha\|\mathbf{v}_h^{f,n}\|^2\le\alpha\|\mathbf{v}_h^{f,n}\|^2.$$

Of course, we infer from this that $\mathbf{v}_h^{f,n}=\mathbf{0}$. In turn, this yields $\mathbf{u}_h^{s,n}=\mathbf{0}$. Which in turn implies that $\left(p_h^{f,n},\nabla\cdot\mathbf{w}_h\right)=0$. Owing to the inf-sup condition, this gives $p_h^{f,n}=0$, see [13]. □

4.2. **Stability of the fully discrete scheme (4.4).** Throughout this section, we denote by capital boldface letters the whole time sequence; for example, $\mathbf{V}_h^f=(\mathbf{v}_h^{f,n})_{n\ge 0}$. In order to obtain a priori estimates for the discrete solution, we follow the ideas used in proving existence of the continuous problem and split the discrete displacement into

$$\mathbf{u}_h^{s,n}=\check{\mathbf{u}}_h^{s,n}+\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_h^f),$$

where $\check{\mathbf{u}}_h^{s,n}\in U_h$ is the solution, for all $\mathbf{z}_h\in U_h$, to

$$\begin{aligned}(\lambda^s+\mu^s)\int_\Omega(\nabla\cdot\check{\mathbf{u}}_h^{s,n})(\nabla\cdot\mathbf{z}_h) \;&+\; \mu^s\int_\Omega\nabla\check{\mathbf{u}}_h^{s,n}:\nabla\mathbf{z}_h+\frac{\alpha}{\Delta t}\int_\Omega(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\cdot\mathbf{z}_h\\ &=\; \rho^s\int_\Omega\mathbf{b}_e(t_n)\cdot\mathbf{z}_h+\langle\mathbf{g}_N^{s,n},\mathbf{z}_h\rangle_{\Gamma_N^s},\end{aligned}\tag{4.10}$$

with $\check{\mathbf{u}}_h^{s,0}=P_h(\mathbf{u}^s(0))$ (see (4.3)), and for any sequence $\mathbf{W}_h=(\mathbf{w}_h^n)_n$, $\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\in U_h$ is the solution, for all $\mathbf{z}_h\in U_h$, of:

$$\begin{aligned}&(\lambda^s+\mu^s)\int_\Omega(\nabla\cdot\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h))(\nabla\cdot\mathbf{z}_h)+\mu^s\int_\Omega\nabla\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h):\nabla\mathbf{z}_h\\ &+\frac{\alpha}{\Delta t}\int_\Omega(\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h))\cdot\mathbf{z}_h=\alpha\int_\Omega\mathbf{w}_h^n\cdot\mathbf{z}_h,\end{aligned}\tag{4.11}$$

with $\hat{\mathbf{u}}_h^{s,0}(\mathbf{W}_h)=\mathbf{0}$. Then, we decompose

$$\mathbf{v}_h^{f,n}=\check{\mathbf{v}}_h^{f,n}+\mathbf{v}_{0,h}^{f,n},$$

where $\check{\mathbf{v}}_h^{f,n}\in V_h^0$ satisfies for all $\mathbf{w}_h\in V_h^0$:

$$\begin{aligned}\frac{\rho^f}{\Delta t}\int_\Omega(\check{\mathbf{v}}_h^{f,n}-\check{\mathbf{v}}_h^{f,n-1})\cdot\mathbf{w}_h \;&+\; \mu^f\int_\Omega\nabla\check{\mathbf{v}}_h^{f,n}:\nabla\mathbf{w}_h+\alpha\int_\Omega\check{\mathbf{v}}_h^{f,n}\cdot\mathbf{w}_h\\ &=\; \rho^f\int_\Omega\mathbf{b}_e(t_n)\cdot\mathbf{w}_h+\frac{\alpha}{\Delta t}\int_\Omega(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\cdot\mathbf{w}_h,\end{aligned}\tag{4.12}$$

with $\check{\mathbf{v}}_h^{f,0}:=\Pi_h(\mathbf{v}^f(0))$ (see (4.2)), and $\mathbf{v}_{0,h}^{f,n}\in V_h^0$ solves for all $\mathbf{w}_h\in V_h^0$:

$$\begin{aligned}\frac{\rho^f}{\Delta t}\int_\Omega(\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1})\cdot\mathbf{w}_h \;&+\; \mu^f\int_\Omega\nabla\mathbf{v}_{0,h}^{f,n}:\nabla\mathbf{w}_h+\int_\Omega\alpha\mathbf{v}_{0,h}^{f,n}\cdot\mathbf{w}_h\\ &=\; \frac{\alpha}{\Delta t}\int_\Omega(\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f))\cdot\mathbf{w}_h\\ &+\; \frac{\alpha}{\Delta t}\int_\Omega(\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f))\cdot\mathbf{w}_h,\end{aligned}\tag{4.13}$$

where $\mathbf{v}_{0,h}^{f,0}=\mathbf{0}$, and, owing to the linearity of the problem, we can write $\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_h^f)=\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)+\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f)$.

4.2.1. *Stability of* $\check{\mathbf{U}}_h^s$. The following proposition establishes the stability of (4.10).

Proposition 4.3. *Let the inf-sup condition (4.5) hold. Assume* $\mathbf{b}_e \in \mathscr{C}^0([0,T];L^2(\Omega)^d)$ *and each component of* $\mathbf{g}_N^s$ *belongs to* $H^1(0,T;H_{00}^{1/2}(\Gamma_N^s)')$*. Then, there exists a positive constant* C_1*, independent of* h *and* Δt*, such that, for* $1 \le m \le N$*,*

$$(\lambda^s+\mu^s)\sum_{n=1}^m \Delta t\|\nabla\cdot\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}^2 + \frac{\mu^s}{2}\sum_{n=1}^m \Delta t\|\nabla\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|\check{\mathbf{u}}_h^{s,m}\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\sum_{n=1}^m \|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)}^2 \le C_1\,. \tag{4.14}$$

If, in addition, $\check{\mathbf{u}}_h^{s,0} = P_h(\mathbf{u}^s(0))$ *is bounded in* $H^1(\Omega)^d$*, there exists a positive constant* C_2*, independent of* h *and* Δt*, such that, for* $1 \le m \le N$*,*

$$\begin{aligned}\frac{\lambda^s+\mu^s}{2}\Big(\|\nabla\cdot\check{\mathbf{u}}_h^{s,m}\|_{L^2(\Omega)}^2 + \sum_{n=1}^m\|\nabla\cdot(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\|_{L^2(\Omega)}^2\Big)\\ +\frac{\mu^s}{4}\Big(\|\nabla\check{\mathbf{u}}_h^{s,m}\|_{L^2(\Omega)}^2 + \sum_{n=1}^m\|\nabla(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\|_{L^2(\Omega)}^2\Big)\\ +\frac{\alpha}{2}\sum_{n=1}^m\frac{1}{\Delta t}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)}^2 \le C_2\,.\end{aligned} \tag{4.15}$$

Proof. We start by testing (4.10) with $\mathbf{z}_h = \check{\mathbf{u}}_h^{s,n}$,

$$\begin{aligned}(\lambda^s+\mu^s)\|\nabla\cdot\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}^2 + \mu^s\|\nabla\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}^2\\ +\frac{\alpha}{2\,\Delta t}\Big(\|\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}^2 - \|\check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)}^2 + \|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)}^2\Big)\\ = \rho^s\int_\Omega \mathbf{b}_e(t_n)\cdot\check{\mathbf{u}}_h^{s,n} + \langle\mathbf{g}_N^{s,n},\check{\mathbf{u}}_h^{s,n}\rangle_{\Gamma_N^s}\,.\end{aligned} \tag{4.16}$$

By using the Cauchy–Schwarz inequality, Poincaré inequality (1.6), the trace inequality (1.7), and Young's inequality, we can bound the right hand side of (4.16) as follows:

$$\begin{aligned}|\rho^s\int_\Omega \mathbf{b}_e(t_n)\cdot\check{\mathbf{u}}_h^{s,n} \;&+\; \langle\mathbf{g}_N^{s,n},\check{\mathbf{u}}_h^{s,n}\rangle_{\Gamma_N^s}| \le\\ &\le \rho^s\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)}\|\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)} + \|\mathbf{g}_N^{s,n}\|_{H_{00}^{1/2}(\Gamma_N^s)'}\|\check{\mathbf{u}}_h^{s,n}\|_{H_{00}^{1/2}(\Gamma_N^s)}\\ &\le \Big(\rho^s\mathscr{P}\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)} + C_N\|\mathbf{g}_N^{s,n}\|_{H_{00}^{1/2}(\Gamma_N^s)'}\Big)\|\nabla\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}\\ &\le \frac{1}{2}\mu^s\|\nabla\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}^2 + \frac{1}{2\mu^s}\Big(\rho^s\mathscr{P}\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)} + C_N\|\mathbf{g}_N^{s,n}\|_{H_{00}^{1/2}(\Gamma_N^s)'}\Big)^2.\end{aligned} \tag{4.17}$$

From (4.16) and (4.17),

$$\begin{aligned}(\lambda^s+\mu^s)\|\nabla\cdot\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}^2 + \frac{\mu^s}{2}\|\nabla\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}^2\\ +\frac{\alpha}{2\,\Delta t}\Big(\|\check{\mathbf{u}}_h^{s,n}\|_{L^2(\Omega)}^2 - \|\check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)}^2 + \|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)}^2\Big)\\ \le \frac{1}{2\mu^s}\Big(\rho^s\mathscr{P}\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)} + C_N\|\mathbf{g}_N^{s,n}\|_{H_{00}^{1/2}(\Gamma_N^s)'}\Big)^2.\end{aligned}$$

Summing over n from $n=1$ to m, and multiplying by Δt, we obtain:

$$\begin{aligned}(\lambda^s+\mu^s)\sum_{n=1}^{m}\Delta t\|\nabla\cdot\check{\mathbf{u}}_h^{s,n}\|^2_{L^2(\Omega)}+\frac{\mu^s}{2}\sum_{n=1}^{m}\Delta t\|\nabla\check{\mathbf{u}}_h^{s,n}\|^2_{L^2(\Omega)}\\+\frac{\alpha}{2}\|\check{\mathbf{u}}_h^{s,m}\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\sum_{n=1}^{m}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}\\\leq\frac{1}{2\mu^s}\sum_{n=1}^{m}\Delta t\Big(\rho^s\mathscr{P}\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)}+C_N\|\mathbf{g}_N^{s,n}\|_{H^{1/2}_{00}(\Gamma_N^s)'}\Big)^2+\frac{\alpha}{2}\|\check{\mathbf{u}}_h^{s,0}\|^2_{L^2(\Omega)},\end{aligned}\tag{4.18}$$

and the right-hand side is bounded independently of h and Δt provided $\mathbf{b}_e\in\mathscr{C}^0([0,T];L^2(\Omega)^d)$ and each component of $\mathbf{g}_N^s$ belongs to $H^1(0,T;H^{1/2}_{00}(\Gamma_N^s)')$, which proves (4.14). At this stage, continuity in time is sufficient for $\mathbf{g}_N^s$, but derivability will be needed in the second part of the proof.

Next, by testing (4.10) with $\mathbf{z}_h=\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}$, we obtain that:

$$\begin{aligned}\frac{\lambda^s+\mu^s}{2}(\|\nabla\cdot\check{\mathbf{u}}_h^{s,n}\|^2_{L^2(\Omega)}-\|\nabla\cdot\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}+\|\nabla\cdot(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\|^2_{L^2(\Omega)})\quad+\\+\frac{\mu^s}{2}(\|\nabla\check{\mathbf{u}}_h^{s,n}\|^2_{L^2(\Omega)}-\|\nabla\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}+\|\nabla(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\|^2_{L^2(\Omega)})\quad+\\+\frac{\alpha}{\Delta t}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}\quad=\\=\rho^s\int_\Omega\mathbf{b}_e(t_n)\cdot(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})+\langle\mathbf{g}_N^{s,n},\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\rangle_{\Gamma_N^s}.\end{aligned}\tag{4.19}$$

The first term on the right-hand side of (4.19) can be bounded by using the Cauchy-Schwarz inequality and Young's inequality:

$$\begin{aligned}\Big|\rho^s\int_\Omega\mathbf{b}_e(t_n)\cdot(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\Big|&\leq\rho^s\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)}\\&\leq\frac{\Delta t(\rho^s)^2}{2\alpha}\|\mathbf{b}_e(t_n)\|^2_{L^2(\Omega)}+\frac{\alpha}{2\Delta t}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}.\end{aligned}\tag{4.20}$$

In order to bound the second term on the right-hand side of (4.19), we need to sum by parts:

$$\sum_{n=1}^{m}\langle\mathbf{g}_N^{s,n},\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\rangle_{\Gamma_N^s}=\langle\mathbf{g}_N^{s,m},\check{\mathbf{u}}_h^{s,m}\rangle_{\Gamma_N^s}-\langle\mathbf{g}_N^{s,1},\check{\mathbf{u}}_h^{s,0}\rangle_{\Gamma_N^s}-\sum_{n=2}^{m}\langle\mathbf{g}_N^{s,n}-\mathbf{g}_N^{s,n-1},\check{\mathbf{u}}_h^{s,n-1}\rangle_{\Gamma_N^s},\tag{4.21}$$

where the sum on the right-hand side is empty when $m = 1$. Then, by using (1.6), (1.7), and Young's inequality, we obtain

$$
\begin{aligned}
& \textstyle\sum_{n=1}^{m} \langle \mathbf{g}_N^{s,n}, \check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1} \rangle_{\Gamma_N^s} \\
\le\ & \|\mathbf{g}_N^{s,m}\|_{H_{00}^{1/2}(\Gamma_N^s)'} \|\check{\mathbf{u}}_h^{s,m}\|_{H_{00}^{1/2}(\Gamma_N^s)} + \|\mathbf{g}_N^{s,1}\|_{H_{00}^{1/2}(\Gamma_N^s)'} \|\check{\mathbf{u}}_h^{s,0}\|_{H_{00}^{1/2}(\Gamma_N^s)} \\
+\ & \sum_{n=2}^{m} \|\mathbf{g}_N^{s,n} - \mathbf{g}_N^{s,n-1}\|_{H_{00}^{1/2}(\Gamma_N^s)'} \|\check{\mathbf{u}}_h^{s,n-1}\|_{H_{00}^{1/2}(\Gamma_N^s)} \\
\le\ & C_N \Big(\|\mathbf{g}_N^{s,m}\|_{H_{00}^{1/2}(\Gamma_N^s)'} \|\nabla \check{\mathbf{u}}_h^{s,m}\|_{L^2(\Omega)} + \|\mathbf{g}_N^{s,1}\|_{H_{00}^{1/2}(\Gamma_N^s)'} \|\nabla \check{\mathbf{u}}_h^{s,0}\|_{L^2(\Omega)} \\
+\ & \sum_{n=2}^{m} \|\mathbf{g}_N^{s,n} - \mathbf{g}_N^{s,n-1}\|_{H_{00}^{1/2}(\Gamma_N^s)'} \|\nabla \check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)} \Big) \\
\le\ & \frac{C_N^2}{\mu^s} \|\mathbf{g}_N^{s,m}\|^2_{H_{00}^{1/2}(\Gamma_N^s)'} + \frac{\mu^s}{4} \|\nabla \check{\mathbf{u}}_h^{s,m}\|^2_{L^2(\Omega)} + C_N \|\mathbf{g}_N^{s,1}\|_{H_{00}^{1/2}(\Gamma_N^s)'} \|\nabla \check{\mathbf{u}}_h^{s,0}\|_{L^2(\Omega)} \\
+\ & \frac{C_N^2}{2\mu^s} \sum_{n=2}^{m} \frac{1}{\Delta t} \|\mathbf{g}_N^{s,n} - \mathbf{g}_N^{s,n-1}\|^2_{H_{00}^{1/2}(\Gamma_N^s)'} + \frac{\mu^s}{2} \sum_{n=2}^{m} \Delta t \|\nabla \check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)},
\end{aligned}
\tag{4.22}
$$

where the last two sums are empty when $m = 1$. Thus, by summing over n from $n = 1$ to $n = m$ in (4.19) and inserting (4.22), we have:

$$
\begin{aligned}
& \frac{\lambda^s + \mu^s}{2} \Big(\|\nabla \cdot \check{\mathbf{u}}_h^{s,m}\|^2_{L^2(\Omega)} + \sum_{n=1}^{m} \|\nabla \cdot (\check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1})\|^2_{L^2(\Omega)} \Big) \\
& \quad + \frac{\mu^s}{4} \|\nabla \check{\mathbf{u}}_h^{s,m}\|^2_{L^2(\Omega)} + \frac{\mu^s}{2} \sum_{n=1}^{m} \|\nabla (\check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1})\|^2_{L^2(\Omega)} \\
& \quad + \frac{\alpha}{2} \sum_{n=1}^{m} \frac{1}{\Delta t} \|\check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)} \\
\le\ & \frac{\lambda^s + \mu^s}{2} \|\nabla \cdot \check{\mathbf{u}}_h^{s,0}\|^2_{L^2(\Omega)} + \mu^s \|\nabla \check{\mathbf{u}}_h^{s,0}\|^2_{L^2(\Omega)} + \frac{(\rho^s)^2}{2\alpha} \sum_{n=1}^{m} \Delta t \|\mathbf{b}_e(t_n)\|^2_{L^2(\Omega)} \\
& \quad + \frac{C_N^2}{\mu^s} \|\mathbf{g}_N^{s,m}\|^2_{H_{00}^{1/2}(\Gamma_N^s)'} + \frac{C_N^2}{2\mu^s} \|\mathbf{g}_N^{s,1}\|^2_{H_{00}^{1/2}(\Gamma_N^s)'} \\
& \quad + \frac{C_N^2}{2\mu^s} \sum_{n=2}^{m} \frac{1}{\Delta t} \|\mathbf{g}_N^{s,n} - \mathbf{g}_N^{s,n-1}\|^2_{H_{00}^{1/2}(\Gamma_N^s)'} + \frac{\mu^s}{2} \sum_{n=2}^{m} \Delta t \|\nabla \check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where the last two sums are empty when $m = 1$. Thus using (4.18) to bound the last term above, we deduce

$$
\begin{aligned}
&\frac{\lambda^s+\mu^s}{2}\Big(\|\nabla\cdot\check{\mathbf{u}}_h^{s,m}\|^2_{L^2(\Omega)}+\sum_{n=1}^{m}\|\nabla\cdot(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\|^2_{L^2(\Omega)}\Big)\\
&\quad+\frac{\mu^s}{4}\|\nabla\check{\mathbf{u}}_h^{s,m}\|^2_{L^2(\Omega)}+\frac{\mu^s}{2}\sum_{n=1}^{m}\|\nabla(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\|^2_{L^2(\Omega)}\\
&\quad+\frac{\alpha}{2}\sum_{n=1}^{m}\frac{1}{\Delta t}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}\\
&\le\frac{\lambda^s+\mu^s}{2}\|\nabla\cdot\check{\mathbf{u}}_h^{s,0}\|^2_{L^2(\Omega)}+\mu^s\|\nabla\check{\mathbf{u}}_h^{s,0}\|^2_{L^2(\Omega)}+\frac{(\rho^s)^2}{2\alpha}\sum_{n=1}^{m}\Delta t\|\mathbf{b}_e(t_n)\|^2_{L^2(\Omega)}\\
&+\frac{C_N^2}{\mu^s}\|\mathbf{g}_N^{s,m}\|^2_{H^{1/2}_{00}(\Gamma_N^s)'}+\frac{C_N^2}{2\mu^s}\|\mathbf{g}_N^{s,1}\|^2_{H^{1/2}_{00}(\Gamma_N^s)'}+\frac{C_N^2}{2\mu^s}\sum_{n=2}^{m}\frac{1}{\Delta t}\|\mathbf{g}_N^{s,n}-\mathbf{g}_N^{s,n-1}\|^2_{H^{1/2}_{00}(\Gamma_N^s)'}\\
&+\frac{1}{2\mu^s}\sum_{n=1}^{m}\Delta t\Big(\rho^s\mathscr{P}\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)}+C_N\|\mathbf{g}_N^{s,n}\|_{H^{1/2}_{00}(\Gamma_N^s)'}\Big)^2+\frac{\alpha}{2}\|\check{\mathbf{u}}_h^{s,0}\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{4.23}
$$

The right-hand side is bounded independently of h and Δt if $\check{\mathbf{u}}_h^{s,0} = P_h(\mathbf{u}^s(0))$ is bounded in $H^1(\Omega)^d$, $\mathbf{b}_e \in \mathscr{C}^0([0,T];L^2(\Omega)^d)$ and each component of $\partial_t \mathbf{g}_N^s$ is in $L^2(0,T;H^{1/2}_{00}(\Gamma_N^s)')$. $\square$

4.2.2. *Stability of* $\hat{\mathbf{U}}_h^s(\mathbf{W}_h)$. Here $\mathbf{W}_h$ is any sequence in V_h, in fact the arguments developed below apply to any discrete sequence for which $\hat{\mathbf{U}}_h^s(\mathbf{W}_h)$ is well defined.

Proposition 4.4. *The following two inequalities hold:*

$$
\begin{aligned}
&(\lambda^s+\mu^s)\sum_{n=1}^{m}\Delta t\|\nabla\cdot\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|^2_{L^2(\Omega)}+\frac{\mu^s}{2}\sum_{n=1}^{m}\Delta t\|\nabla\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|^2_{L^2(\Omega)}\\
&\quad+\frac{\alpha}{2}\|\hat{\mathbf{u}}_h^{s,m}(\mathbf{W}_h)\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\sum_{n=1}^{m}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|^2_{L^2(\Omega)}\\
&\quad\le\frac{(\alpha\mathscr{P})^2}{2\mu^s}\sum_{n=1}^{m}\Delta t\|\mathbf{w}_h^n\|^2_{L^2(\Omega)},
\end{aligned}
\tag{4.24}
$$

where $\mathscr{P}$ *is the constant of* (1.6),

$$
\begin{aligned}
&\frac{\lambda^s+\mu^s}{2}\big(\|\nabla\cdot\hat{\mathbf{u}}_h^{s,m}(\mathbf{W}_h)\|^2_{L^2(\Omega)}+\sum_{n=1}^{m}\|\nabla\cdot(\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h))\|^2_{L^2(\Omega)}\big)\;+\\
&\quad+\frac{\mu^s}{2}\big(\|\nabla\hat{\mathbf{u}}_h^{s,m}(\mathbf{W}_h)\|^2_{L^2(\Omega)}+\sum_{n=1}^{m}\|\nabla(\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h))\|^2_{L^2(\Omega)}\big)\;+\\
&\quad+\frac{\alpha}{2\Delta t}\sum_{n=1}^{m}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|^2_{L^2(\Omega)}\\
&\quad\le\frac{\alpha}{2}\sum_{n=1}^{m}\Delta t\|\mathbf{w}_h^n\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{4.25}
$$

Proof. By taking $\mathbf{z}_h = \hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)$ in (4.11), we obtain:

$$\begin{aligned}(\lambda^s+\mu^s)\|\nabla\cdot\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \mu^s\|\nabla\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 \\ +\frac{\alpha}{2\Delta t}\Big(\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 - \|\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2\Big) \\ \le \alpha\|\mathbf{w}_h^n\|_{L^2(\Omega)}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}.\end{aligned} \tag{4.26}$$

Poincaré's and Young's inequalities on the right-hand side of (4.26) yield,

$$\begin{aligned}\alpha\|\mathbf{w}_h^n\|_{L^2(\Omega)}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)} &\le \alpha\,\mathscr{P}\,\|\mathbf{w}_h^n\|_{L^2(\Omega)}\|\nabla\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)} \\ &\le \frac{\mu^s}{2}\|\nabla\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \frac{(\alpha\,\mathscr{P})^2}{2\mu^s}\|\mathbf{w}_h^n\|_{L^2(\Omega)}^2,\end{aligned} \tag{4.27}$$

and (4.26) becomes

$$\begin{aligned}(\lambda^s+\mu^s)\|\nabla\cdot\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \frac{\mu^s}{2}\|\nabla\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 \\ +\frac{\alpha}{2\Delta t}\Big(\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 - \|\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2\Big) \\ \le \frac{(\alpha\,\mathscr{P})^2}{2\mu^s}\|\mathbf{w}_h^n\|_{L^2(\Omega)}^2.\end{aligned}$$

Summing over n from $n=1$ to $n=m$, multiplying by Δt, and using that $\hat{\mathbf{u}}_h^{s,0}(\mathbf{W}_h)=\mathbf{0}$, we obtain (4.24).

Now, in order to bound the derivative, we take $\mathbf{z}_h = \hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h) - \hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)$ in (4.11). Then,

$$\begin{aligned}\frac{\lambda^s+\mu^s}{2}\Big(\|\nabla\cdot\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 - \|\nabla\cdot\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \|\nabla\cdot(\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h))\|_{L^2(\Omega)}^2\Big) \\ +\frac{\mu^s}{2}\Big(\|\nabla\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 - \|\nabla\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \|\nabla(\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h))\|_{L^2(\Omega)}^2\Big) \\ +\frac{\alpha}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 \\ \le \frac{\alpha}{2\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\Delta t\|\mathbf{w}_h^n\|_{L^2(\Omega)}^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\lambda^s+\mu^s}{2}\Big(\|\nabla\cdot\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 - \|\nabla\cdot\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \|\nabla\cdot(\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h))\|_{L^2(\Omega)}^2\Big) \\ +\frac{\mu^s}{2}\Big(\|\nabla\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 - \|\nabla\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 + \|\nabla(\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h))\|_{L^2(\Omega)}^2\Big) \\ +\frac{\alpha}{2\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|_{L^2(\Omega)}^2 \le \frac{\alpha}{2}\Delta t\|\mathbf{w}_h^n\|_{L^2(\Omega)}^2.\end{aligned} \tag{4.28}$$

Summing over n from $n=1$ to $n=m$, we arrive at (4.25). □

4.2.3. *Stability of* $\check{\mathbf{V}}_h^f$ *and* $\hat{\mathbf{U}}_h^s(\check{\mathbf{V}}_h^f)$.

Proposition 4.5. *Let the inf-sup condition (4.5) hold. Assume* $\mathbf{b}_e \in \mathscr{C}^0([0,T];L^2(\Omega)^d)$*, each component of* $\mathbf{g}_N^s$ *is in* $H^1(0,T;H_{00}^{1/2}(\Gamma_N^s)')$*,* $\check{\mathbf{v}}_h^{f,0}$ *is bounded in* $L^2(\Omega)^d$*, and* $\check{\mathbf{u}}_h^{s,0}$ *is bounded in* $H^1(\Omega)^d$*. Then, there exists a positive constant* C_3*, independent of* h *and* Δt*, such that*

$$\frac{\rho^f}{2}\Big(\|\check{\mathbf{v}}_h^{f,m}\|_{L^2(\Omega)}^2 + \sum_{n=1}^m \|\check{\mathbf{v}}_h^{f,n} - \check{\mathbf{v}}_h^{f,n-1}\|_{L^2(\Omega)}^2\Big) + \mu^f \sum_{n=1}^m \Delta t \|\nabla \check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}^2 \le C_3. \tag{4.29}$$

Proof. By testing (4.12) with $\mathbf{w}_h = \check{\mathbf{v}}_h^{f,n}$, we obtain:

$$\begin{aligned}\frac{\rho^f}{2\Delta t}\Big(\|\check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}^2 - \|\check{\mathbf{v}}_h^{f,n-1}\|_{L^2(\Omega)}^2 + \|\check{\mathbf{v}}_h^{f,n} - \check{\mathbf{v}}_h^{f,n-1}\|_{L^2(\Omega)}^2\Big) + \mu^f \|\nabla \check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}^2 + \alpha \|\check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}^2 \\ \le \rho^f \|\mathbf{b}_e(t_n)\|_{L^2(\Omega)} \|\check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)} + \frac{\alpha}{\Delta t}\|\check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)} \|\check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)} \\ \le \frac{(\rho^f)^2}{2\alpha}\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2(\Delta t)^2}\|\check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)}^2 + \alpha \|\check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}^2 .\end{aligned} \tag{4.30}$$

After summing over n from 1 to m in (4.30) and multiplying by Δt, this becomes:

$$\begin{aligned}\frac{\rho^f}{2}\Big(\|\check{\mathbf{v}}_h^{f,m}\|_{L^2(\Omega)}^2 + \sum_{n=1}^m \|\check{\mathbf{v}}_h^{f,n} - \check{\mathbf{v}}_h^{f,n-1}\|_{L^2(\Omega)}^2\Big) + \mu^f \sum_{n=1}^m \Delta t \|\nabla \check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}^2 \\ \le \frac{(\rho^f)^2}{2\alpha}\sum_{n=1}^m \Delta t \|\mathbf{b}_e(t_n)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\sum_{n=1}^m \frac{1}{\Delta t}\|\check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1}\|_{L^2(\Omega)}^2 + \frac{\rho^f}{2}\|\check{\mathbf{v}}_h^{f,0}\|_{L^2(\Omega)}^2.\end{aligned}$$

Then by virtue of (4.23), we infer

$$\begin{aligned}\frac{\rho^f}{2}\Big(\|\check{\mathbf{v}}_h^{f,m}\|_{L^2(\Omega)}^2 + \sum_{n=1}^m \|\check{\mathbf{v}}_h^{f,n} - \check{\mathbf{v}}_h^{f,n-1}\|_{L^2(\Omega)}^2\Big) + \mu^f \sum_{n=1}^m \Delta t \|\nabla \check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}^2 \\ \le \frac{1}{2\alpha}\big((\rho^f)^2 + (\rho^s)^2\big)\sum_{n=1}^m \Delta t \|\mathbf{b}_e(t_n)\|_{L^2(\Omega)}^2 + \frac{\rho^f}{2}\|\check{\mathbf{v}}_h^{f,0}\|_{L^2(\Omega)}^2 \\ + \frac{\lambda^s + \mu^s}{2}\|\nabla\cdot\check{\mathbf{u}}_h^{s,0}\|_{L^2(\Omega)}^2 + \mu^s \|\nabla \check{\mathbf{u}}_h^{s,0}\|_{L^2(\Omega)}^2 \\ + \frac{C_N^2}{\mu^s}\Big(\|\mathbf{g}_N^{s,m}\|_{H_{00}^{1/2}(\Gamma_N^s)'}^2 + \frac{1}{2}\|\mathbf{g}_N^{s,1}\|_{H_{00}^{1/2}(\Gamma_N^s)'}^2\Big) + \frac{C_N^2}{2\mu^s}\sum_{n=2}^m \frac{1}{\Delta t}\|\mathbf{g}_N^{s,n} - \mathbf{g}_N^{s,n-1}\|_{H_{00}^{1/2}(\Gamma_N^s)'}^2 \\ + \frac{1}{2\mu^s}\sum_{n=1}^m \Delta t \Big(\rho^s \mathscr{P}\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)} + C_N \|\mathbf{g}_N^{s,n}\|_{H_{00}^{1/2}(\Gamma_N^s)'}\Big)^2 + \frac{\alpha}{2}\|\check{\mathbf{u}}_h^{s,0}\|_{L^2(\Omega)}^2,\end{aligned} \tag{4.31}$$

and the right-hand side is bounded independently of h and Δt, provided $\mathbf{b}_e \in \mathscr{C}^0([0,T];L^2(\Omega)^d)$, each component of $\mathbf{g}_N^s$ is in $H^1(0,T;H_{00}^{1/2}(\Gamma_N^s)')$, $\check{\mathbf{v}}_h^{f,0}$ is bounded in $L^2(\Omega)^d$, and $\check{\mathbf{u}}_h^{s,0}$ is bounded in $H^1(\Omega)^d$. □

It follows from (4.29) that

$$\sum_{n=1}^m \Delta t \|\check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}^2 \le \frac{2C_3}{\rho^f} m\Delta t \le \frac{2C_3}{\rho^f} T, \tag{4.32}$$

a quantity bounded independently of h and Δt. Therefore, by substituting this bound into (4.24) and (4.25) with $\mathbf{W}_h = \check{\mathbf{V}}_h^f$, we immediately derive that

$$(\lambda^s+\mu^s)\sum_{n=1}^m \Delta t\|\nabla\cdot\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}^2 + \frac{\mu^s}{2}\sum_{n=1}^m \Delta t\|\nabla\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}^2$$
$$+\frac{\alpha}{2}\|\hat{\mathbf{u}}_h^{s,m}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\sum_{n=1}^m \|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}^2 \le \frac{(\alpha\mathscr{P})^2 C_3}{\mu^s\rho^f}m\Delta t\,, \tag{4.33}$$

$$\frac{\lambda^s+\mu^s}{2}\big(\|\nabla\cdot\hat{\mathbf{u}}_h^{s,m}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}^2 + \sum_{n=1}^m \|\nabla\cdot(\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f))\|_{L^2(\Omega)}^2\big)$$
$$+\frac{\mu^s}{2}\big(\|\nabla\hat{\mathbf{u}}_h^{s,m}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}^2 + \sum_{n=1}^m \|\nabla(\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f))\|_{L^2(\Omega)}^2\big) \tag{4.34}$$
$$+\frac{\alpha}{2\Delta t}\sum_{n=1}^m \|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}^2 \le \frac{\alpha C_3}{\rho^f}m\Delta t\,.$$

4.2.4. *Stability of* $\mathbf{V}_{0,h}^f$ *and of the velocity part of* (4.4). Regarding the velocity solution, it remains to derive a priori estimates for $\mathbf{V}_{0,h}^f$.

Proposition 4.6. *Under the assumptions of Proposition 4.5, there exists a positive constant* C_4*, independent of* h *and* Δt*, such that*

$$\frac{\rho^f}{2}\|\mathbf{v}_{0,h}^{f,m}\|_{L^2(\Omega)}^2 + \frac{\rho^f}{2}\sum_{n=1}^m \|\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1}\|_{L^2(\Omega)}^2 + \frac{\mu^f}{2}\sum_{n=1}^m \Delta t\|\nabla\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}^2 \le C_4\,. \tag{4.35}$$

Proof. By choosing $\mathbf{w}_h = \mathbf{v}_{0,h}^{f,n}$ in (4.13) and by applying the Cauchy-Schwarz, Poincaré and Young inequalities, we infer

$$\frac{\rho^f}{2\Delta t}\Big(\|\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}^2 - \|\mathbf{v}_{0,h}^{f,n-1}\|_{L^2(\Omega)}^2 + \|\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1}\|_{L^2(\Omega)}^2\Big) + \mu^f\|\nabla\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}^2 + \alpha\|\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}^2$$
$$\le \frac{\alpha}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}\|\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}$$
$$+\frac{\alpha}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)\|_{L^2(\Omega)}\|\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}$$
$$\le \frac{(\alpha\mathscr{P})^2}{2(\Delta t)^2\mu^f}\|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}^2 + \frac{\mu^f}{2}\|\nabla\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}^2$$
$$+\frac{\alpha}{2(\Delta t)^2}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}^2\,.$$

Then,

$$\frac{\rho^f}{2\Delta t}\Big(\|\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}^2 - \|\mathbf{v}_{0,h}^{f,n-1}\|_{L^2(\Omega)}^2 + \|\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1}\|_{L^2(\Omega)}^2\Big) + \frac{\mu^f}{2}\|\nabla\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}^2$$
$$\le \frac{(\alpha\mathscr{P})^2}{2(\Delta t)^2\mu^f}\|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2(\Delta t)^2}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)\|_{L^2(\Omega)}^2\,.$$

Upon summing from $n=1$ to $n=m$, and multiplying by Δt, this gives:

$$\frac{\rho^f}{2}\|\mathbf{v}_{0,h}^{f,m}\|^2_{L^2(\Omega)} + \frac{\rho^f}{2}\sum_{n=1}^m \|\mathbf{v}_{0,h}^{f,n} - \mathbf{v}_{0,h}^{f,n-1}\|^2_{L^2(\Omega)} + \frac{\mu^f}{2}\sum_{n=1}^m \Delta t\|\nabla \mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\sum_{n=1}^m \Delta t\|\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}$$
$$\leq \frac{(\alpha\mathscr{P})^2}{2\mu^f}\sum_{n=1}^m \frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f) - \hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\sum_{n=1}^m \frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f) - \hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)\|^2_{L^2(\Omega)}. \tag{4.36}$$

Finally, in view of (4.34), the first term on the right-hand side of (4.36) is bounded independently of h and Δt by

$$\frac{(\alpha\mathscr{P})^2}{\mu^f\rho^f}C_3 m\Delta t \leq \frac{(\alpha\mathscr{P})^2}{\mu^f\rho^f}C_3 T.$$

The second term on the right-hand side of (4.36) is bounded by

$$\frac{\alpha}{2}\sum_{n=1}^m \Delta t\|\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}$$

by virtue of (4.25) with $\mathbf{W}_h = \mathbf{V}_{0,h}^f$. Then, (4.35) follows from these two upper bounds. □

As a by-product of Proposition 4.6, we have, for $1 \leq m \leq N$,

$$\sum_{n=1}^m \Delta t\|\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)} \leq \frac{2}{\rho^f}C_4 m\Delta t \leq \frac{2}{\rho^f}C_4 T. \tag{4.37}$$

The next theorem summarizes the conclusions of Propositions 4.3, 4.4, 4.5, and 4.6.

Theorem 4.7. *Let the inf-sup condition (4.5) hold. For $d=2,3$, assume that $\mathbf{b}_e$ belongs to $\mathscr{C}^0([0,T];L^2(\Omega)^d)$, each component of $\mathbf{g}_N^s$ is in $H^1(0,T;H_{00}^{1/2}(\Gamma_N^s)')$, $\check{\mathbf{v}}_h^{f,0}$ is bounded in $L^2(\Omega)^d$, and $\check{\mathbf{u}}_h^{s,0}$ is bounded in $H^1(\Omega)^d$. Then, there exist positive constants, C_5, C_6 and C_7, independent of h and Δt, such that*

$$(\lambda^s+\mu^s)\sum_{n=1}^m \Delta t\|\nabla\cdot\mathbf{u}_h^{s,n}\|^2_{L^2(\Omega)} + \frac{\mu^s}{2}\sum_{n=1}^m \Delta t\|\nabla\mathbf{u}_h^{s,n}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|\mathbf{u}_h^{s,m}\|^2_{L^2(\Omega)}$$
$$+\frac{\alpha}{2}\sum_{n=1}^m \|\mathbf{u}_h^{s,n} - \mathbf{u}_h^{s,n-1}\|^2_{L^2(\Omega)} \leq C_5, \tag{4.38}$$

$$\frac{\lambda^s+\mu^s}{2}\Big(\|\nabla\cdot\mathbf{u}_h^{s,m}\|^2_{L^2(\Omega)} + \sum_{n=1}^m \|\nabla\cdot(\mathbf{u}_h^{s,n} - \mathbf{u}_h^{s,n-1})\|^2_{L^2(\Omega)}\Big) + \frac{\alpha}{2}\sum_{n=1}^m \frac{1}{\Delta t}\|\mathbf{u}_h^{s,n} - \mathbf{u}_h^{s,n-1}\|^2_{L^2(\Omega)}$$
$$+\frac{\mu^s}{4}\Big(\|\nabla\mathbf{u}_h^{s,m}\|^2_{L^2(\Omega)} + \sum_{n=1}^m \|\nabla(\mathbf{u}_h^{s,n} - \mathbf{u}_h^{s,n-1})\|^2_{L^2(\Omega)}\Big) \leq C_6, \tag{4.39}$$

$$\rho^f\|\mathbf{v}_h^{f,m}\|^2_{L^2(\Omega)} + \rho^f\sum_{n=1}^m \|\mathbf{v}_h^{f,n} - \mathbf{v}_h^{f,n-1}\|^2_{L^2(\Omega)} + \mu^f\sum_{n=1}^m \Delta t\|\mathbf{v}_h^{f,n}\|^2_{L^2(\Omega)} \leq C_7. \tag{4.40}$$

Proof. To derive (4.38), we use the decomposition $\mathbf{u}_h^{s,n} = \check{\mathbf{u}}_h^{s,n} + \hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f) + \hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f)$, the triangle inequality and bounds (4.14), (4.33) and (4.24) with $\mathbf{W}_h = \mathbf{V}_{0,h}^f$, together with (4.37). Inequality (4.39) follows from (4.15), (4.34), (4.25) with $\mathbf{W}_h = \mathbf{V}_{0,h}^f$, and (4.37). Finally, (4.40) is derived from the decomposition $\mathbf{v}_h^{f,n} = \check{\mathbf{v}}_h^{f,n} + \mathbf{v}_{0,h}^{f,n}$, after applying (4.29) and (4.35). □

4.2.5. *Stability of* p_h^f. As a rule, stability of the pressure stems from stability of the velocity's time derivative. Regarding $\check{\mathbf{v}}_h^f$, a straightforward argument shows that, by testing (4.12) with $\check{\mathbf{v}}_h^{f,n} - \check{\mathbf{v}}_h^{f,n-1}$, we obtain, for $1 \le m \le N$,

$$\frac{\rho^f}{2}\sum_{n=1}^{m}\frac{1}{\Delta t}\|\check{\mathbf{v}}_h^{f,n}-\check{\mathbf{v}}_h^{f,n-1}\|^2_{L^2(\Omega)}+\frac{\mu^f}{2}\big(\|\nabla\check{\mathbf{v}}_h^{f,m}\|^2_{L^2(\Omega)}+\sum_{n=1}^{m}\|\nabla(\check{\mathbf{v}}_h^{f,n}-\check{\mathbf{v}}_h^{f,n-1})\|^2_{L^2(\Omega)}\big)$$
$$\frac{\alpha}{2}\big(\|\check{\mathbf{v}}_h^{f,m}\|^2_{L^2(\Omega)}+\sum_{n=1}^{m}\|\check{\mathbf{v}}_h^{f,n}-\check{\mathbf{v}}_h^{f,n-1}\|^2_{L^2(\Omega)}\big)\le\frac{\mu^f}{2}\|\nabla\check{\mathbf{v}}_h^{f,0}\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\|\check{\mathbf{v}}_h^{f,0}\|^2_{L^2(\Omega)}$$
$$+\rho^f\sum_{n=1}^{m}\Delta t\|\mathbf{b}_e(t_n)\|^2_{L^2(\Omega)}+\frac{\alpha^2}{\rho^f}\sum_{n=1}^{m}\frac{1}{\Delta t}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)},\tag{4.41}$$

a quantity that is bounded independently of m, h, and Δt, provided that, in addition to the assumptions of Theorem 4.7, $\check{\mathbf{v}}_h^{f,0}$ is stable in $H^1(\Omega)^d$.

Likewise, by testing (4.13) with $\mathbf{v}_{0,h}^{f,n} - \mathbf{v}_{0,h}^{f,n-1}$, we derive, for $1 \le m \le N$,

$$\rho^f\sum_{n=1}^{m}\frac{1}{\Delta t}\|\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1}\|^2_{L^2(\Omega)}+\mu^f\big(\|\nabla\mathbf{v}_{0,h}^{f,m}\|^2_{L^2(\Omega)}+\sum_{n=1}^{m}\|\nabla(\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1})\|^2_{L^2(\Omega)}\big)$$
$$+\alpha\big(\|\mathbf{v}_{0,h}^{f,m}\|^2_{L^2(\Omega)}+\sum_{n=1}^{m}\|\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1}\|^2_{L^2(\Omega)}\big)$$
$$\le\frac{2\alpha^2}{\rho^f}\Big(\sum_{n=1}^{m}\frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\check{V}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{V}_h^f)\|^2_{L^2(\Omega)}+\sum_{n=1}^{m}\frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(V_{0,h}^f)-\hat{\mathbf{u}}_h^{s,n-1}(V_{0,h}^f)\|^2_{L^2(\Omega)}\Big),\tag{4.42}$$

again a quantity that is bounded independently of m, h, and Δt, see (4.34) and (4.37).

The stability of p_h^f follows readily from (4.41) and (4.42).

Theorem 4.8. *In addition to the assumptions of Theorem 4.7, suppose that* $\check{\mathbf{v}}_h^{f,0}$ *is bounded in* $H^1(\Omega)^d$ *and that the discrete inf-sup condition* (4.5) *holds with a constant* $\beta^\star > 0$ *independent of* h. *Then, there exists a positive constant* C_8*, independent of* h *and* Δt*, such that, for* $1 \le m \le N$,

$$\sum_{n=1}^{m}\Delta t\|p_h^{f,n}\|^2_{L^2(\Omega)}\le C_8.\tag{4.43}$$

Proof. A consequence of the inf-sup condition (4.5) is that for each $p_h^{f,n}$ in Q_h, there exists a function $\mathbf{w}_h^{f,n} \in V_h$ such that (see, for instance [13])

$$-(p_h^{f,n},\nabla\cdot\mathbf{w}_h^{f,n})=\|p_h^{f,n}\|^2_{L^2(\Omega)}\quad,\quad\|\nabla\mathbf{w}_h^{f,n}\|_{L^2(\Omega)}\le\frac{1}{\beta^*}\|p_h^{f,n}\|_{L^2(\Omega)}.\tag{4.44}$$

Then, by testing the first equation in (4.4) with $\mathbf{w}_h^{f,n}$, we obtain, in view of (4.44) and Poincaré's inequality,

$$\|p_h^{f,n}\|_{L^2(\Omega)}\le\frac{1}{\beta^*}\Big(\mu^f\|\nabla\mathbf{v}_h^{f,n}\|_{L^2(\Omega)}+\mathscr{P}\big(\alpha\|\mathbf{v}_h^{f,n}\|_{L^2(\Omega)}+\rho^f\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)}$$
$$+\frac{\alpha}{\Delta t}\|\mathbf{u}_h^{s,n}-\mathbf{u}_h^{s,n-1}\|_{L^2(\Omega)}+\frac{\rho^f}{\Delta t}\|\mathbf{v}_h^{f,n}-\mathbf{v}_h^{f,n-1}\|_{L^2(\Omega)}\big)\Big).$$

Then (4.43) follows by squaring, multiplying by Δt, summing over n from 1 to m, and applying the upper bounds of Theorems 4.7 and 4.8. □

5. SIMPLE DECOUPLING ALGORITHM

Solving monolithically Problem (4.4) involves a large number of unknowns, sometimes too large for the available memory. It also couples two systems with very different numerical properties. To reduce it to a sequence of smaller problems each one separately that is easier to solve, we propose a simple decoupling algorithm, without iterations. Note that the first equation of (4.4) is coupled to the third equation because it uses the unknown value of the discrete time derivative of $\mathbf{u}_h^{s,n}$. As the structure is expected to move more slowly than the fluid, the two equations could be decoupled by replacing this discrete time derivative by the discrete time derivative of $\mathbf{u}_h^{s,n-1}$. This raises the issue of approximating the initial time derivative. By reverting to the displacement equation (1.1), we see that a possible approximation is: Find $\mathbf{d}_h^{s,0} \in U_h$ solution of

$$\begin{aligned}\alpha \int_\Omega \mathbf{d}_h^{s,0}\cdot \mathbf{z}_h = -(\lambda^s+\mu^s)\int_\Omega(\nabla\cdot\mathbf{u}_h^{s,0})(\nabla\cdot\mathbf{z}_h) - \mu^s\int_\Omega \nabla\mathbf{u}_h^{s,0}:\nabla\mathbf{z}_h + \rho^s\int_\Omega \mathbf{b}_e(0)\cdot\mathbf{z}_h\\ +\alpha\int_\Omega \mathbf{v}_h^{f,0}\cdot\mathbf{z}_h + \langle \mathbf{g}_N^s(0),\mathbf{z}_h\rangle_{\Gamma_N^s},\end{aligned} \tag{5.1}$$

for all $\mathbf{z}_h \in U_h$. Clearly, this system is uniquely solvable. Then, starting from $\mathbf{v}_h^{f,0} \in V_h^0$, $\mathbf{u}_h^{s,0} \in U_h$, and $\mathbf{d}_h^{s,0} \in U_h$, the general step $n \geq 1$ of the algorithm is: find $(\mathbf{v}_h^{f,n}, p_h^{f,n}, \mathbf{u}_h^{s,n}, \mathbf{d}_h^{s,n})$ in $V_h \times Q_h \times U_h \times U_h$ such that

$$\begin{aligned}&\frac{\rho^f}{\Delta t}\int_\Omega(\mathbf{v}_h^{f,n}-\mathbf{v}_h^{f,n-1})\cdot\mathbf{w}_h + \mu^f\int_\Omega\nabla\mathbf{v}_h^{f,n}:\nabla\mathbf{w}_h - \int_\Omega p_h^{f,n}\nabla\cdot\mathbf{w}_h - \rho^f\int_\Omega \mathbf{b}_e(t_n)\cdot\mathbf{w}_h\\ &\qquad +\alpha\int_\Omega\left(\mathbf{v}_h^{f,n}-\mathbf{d}_h^{s,n-1}\right)\cdot\mathbf{w}_h = 0\\ &\int_\Omega q_h\nabla\cdot\mathbf{v}_h^{f,n} = 0\\ &(\lambda^s+\mu^s)\int_\Omega(\nabla\cdot\mathbf{u}_h^{s,n})(\nabla\cdot\mathbf{z}_h) + \mu^s\int_\Omega\nabla\mathbf{u}_h^{s,n}:\nabla\mathbf{z}_h - \rho^s\int_\Omega\mathbf{b}_e(t_n)\cdot\mathbf{z}_h - \langle\mathbf{g}_N^s,\mathbf{z}_h\rangle_{\Gamma_N^s}\\ &\qquad -\alpha\int_\Omega\mathbf{v}_h^{f,n}\cdot\mathbf{z}_h + \frac{\alpha}{\Delta t}\int_\Omega(\mathbf{u}_h^{s,n}-\mathbf{u}_h^{s,n-1})\cdot\mathbf{z}_h = 0,\\ &\int_\Omega \mathbf{d}_h^{s,n}\cdot\mathbf{z}_h = \frac{1}{\Delta t}\int_\Omega(\mathbf{u}_h^{s,n}-\mathbf{u}_h^{s,n-1})\cdot\mathbf{z}_h,\end{aligned} \tag{5.2}$$

for all $(\mathbf{w}_h, q_h, \mathbf{z}_h) \in V_h \times Q_h \times U_h$. The first two equations are a generalized Stokes system, the third equation is a linear elasticity system, and the last equation is a simple projection. As the equations are now decoupled, it is easy to check by inspection that each system generates a unique solution. Stability of this scheme is established in the next section, provided the starting value $\mathbf{d}_h^{s,0}$ is stable.

5.1. **Stability of the discrete scheme (5.2).** Let us adapt to (5.2) the decomposition used for (4.4). We first observe that the third equation in (5.2) is formally equal to the third equation in (4.4). We then decompose again $\mathbf{u}_h^{s,n} = \check{\mathbf{u}}_h^{s,n} + \hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_h^f)$, where $\check{\mathbf{u}}_h^{s,n}$ is the solution to (4.10) with

$\check{\mathbf{u}}_h^{s,0} = P_h(\mathbf{u}^s(0))$ (see (4.3)), and $\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_h)$ is the solution to (4.11) with $\mathbf{W}_h = \mathbf{V}_h := (\mathbf{v}_h^{f,n})_{n\geq 0}$. Next, we split

$$\mathbf{d}_h^{s,n} = \check{\mathbf{d}}_h^{s,n} + \hat{\mathbf{d}}_h^{s,n}(\mathbf{V}_h),$$

where

$$\check{\mathbf{d}}_h^{s,n} = \frac{1}{\Delta t}\big(\check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1}\big), \quad \check{\mathbf{d}}_h^{s,0} = \mathbf{0},$$

$$\hat{\mathbf{d}}_h^{s,n}(\mathbf{V}_h) = \frac{1}{\Delta t}\big(\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_h) - \hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_h)\big), \quad \hat{\mathbf{d}}_h^{s,0}(\mathbf{V}_h) = \mathbf{d}_h^{s,0},$$

with $\mathbf{d}_h^{s,0}$ defined by (5.1). Finally, $\mathbf{v}_h^{f,n}$ is split into

$$\mathbf{v}_h^{f,n} = \check{\mathbf{v}}_h^{f,n} + \mathbf{v}_{0,h}^{f,n},$$

where for all $\mathbf{w}_h \in V_h^0$,

$$\begin{aligned}\frac{\rho^f}{\Delta t}\int_\Omega (\check{\mathbf{v}}_h^{f,n} - \check{\mathbf{v}}_h^{f,n-1})\cdot \mathbf{w}_h + \mu^f \int_\Omega \nabla \check{\mathbf{v}}_h^{f,n} : \nabla \mathbf{w}_h + \alpha \int_\Omega \check{\mathbf{v}}_h^{f,n}\cdot \mathbf{w}_h = \\ = \rho^f \int_\Omega \mathbf{b}_e(t_n)\cdot \mathbf{w}_h + \alpha \int_\Omega \check{\mathbf{d}}_h^{s,n-1}\cdot \mathbf{w}_h\,,\end{aligned} \tag{5.3}$$

with $\check{\mathbf{v}}_h^{f,0} := \Pi_h(\mathbf{v}^f(0))$ (see (4.2)), and for all $\mathbf{w}_h \in V_h^0$,

$$\begin{aligned}\frac{\rho^f}{\Delta t}\int_\Omega (\mathbf{v}_{0,h}^{f,n} - \mathbf{v}_{0,h}^{f,n-1})\cdot \mathbf{w}_h + \mu^f \int_\Omega \nabla \mathbf{v}_{0,h}^{f,n} : \nabla \mathbf{w}_h + \alpha \int_\Omega \mathbf{v}_{0,h}^{f,n}\cdot \mathbf{w}_h = \\ = \alpha \int_\Omega \big(\hat{\mathbf{d}}_h^{s,n-1}(\check{\mathbf{V}}_h^f) + \hat{\mathbf{d}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)\big)\cdot \mathbf{w}_h\,,\end{aligned} \tag{5.4}$$

where $\mathbf{v}_{0,h}^{f,0} = \mathbf{0}$.

The stability argument of (5.2) follows the same lines as that of (4.4), except that obviously (5.1) does not lead to a uniform bound for $\mathbf{d}_h^{s,0}$ in $L^2(\Omega)^d$. Instead, it gives rise to a bound in a discrete negative norm. More precisely, the dual norm defined by

$$\|\mathbf{d}_h^{s,0}\|_{U_h'} = \sup_{\mathbf{z}_h\in U_h} \frac{1}{\|\nabla \mathbf{z}_h\|_{L^2(\Omega)}}\Big|\int_\Omega \mathbf{d}_h^{s,0}\cdot \mathbf{z}_h\Big|, \tag{5.5}$$

leads to

$$\begin{aligned}\|\mathbf{d}_h^{s,0}\|_{U_h'} \leq \mathscr{K} := \frac{1}{\alpha}\Big[\sqrt{d}(\lambda^s + \mu^s)\|\nabla\cdot P_h(\mathbf{u}^s(0))\|_{L^2(\Omega)} + \mu^s\|\nabla(P_h(\mathbf{u}^s(0)))\|_{L^2(\Omega)} \\ + \mathscr{P}\big(\rho^s\|\mathbf{b}_e^s(0)\|_{L^2(\Omega)} + \alpha\|\Pi_h(\mathbf{v}^f(0))\|_{L^2(\Omega)}\big) + C_N\|\mathbf{g}_N^s(0)\|_{H_{00}^{\frac{1}{2}}(\Gamma_N)'}\Big],\end{aligned} \tag{5.6}$$

a quantity bounded independently of h. To see the influence of this bound on the subsequent velocity and displacement, consider $\mathbf{v}_h^{f,1}$. It solves for all $\mathbf{w}_h \in V_h^0$,

$$\frac{\rho^f}{\Delta t}\big(\mathbf{v}_h^{f,1} - \mathbf{v}_h^{f,0}, \mathbf{w}_h\big) + \mu^f\big(\nabla \mathbf{v}_h^{f,1}, \nabla \mathbf{w}_h\big) + \alpha\big(\mathbf{v}_h^{f,1}, \mathbf{w}_h\big) = \rho^f\big(\mathbf{b}_e(t_1), \mathbf{w}_h\big) + \alpha\big(\mathbf{d}_h^{s,0}, \mathbf{w}_h\big)\,.$$

In order to use (5.6) for estimating the last term above, we assume that

$$V_h \subset U_h. \tag{5.7}$$

Although this assumption holds in the example chosen at the beginning of Section 4, it does not necessarily hold in other situations and so may be restrictive, see Remark 5.2 below. Now, suppose (5.7) holds; the choice $\mathbf{w}_h = \mathbf{v}_h^{f,1}$ yields

$$\frac{\rho^f}{2\Delta t}\big(\|\mathbf{v}_h^{f,1}\|^2_{L^2(\Omega)}+\|\mathbf{v}_h^{f,1}-\mathbf{v}_h^{f,0}\|^2_{L^2(\Omega)}\big)+\mu^f\|\nabla\,\mathbf{v}_h^{f,1}\|^2_{L^2(\Omega)}+\alpha\|\mathbf{v}_h^{f,1}\|^2_{L^2(\Omega)}\le$$
$$\le\frac{\rho^f}{2\Delta t}\|\mathbf{v}_h^{f,0}\|^2_{L^2(\Omega)}+\rho^f\|\mathbf{b}_e(t_1)\|_{L^2(\Omega)}\|\mathbf{v}_h^{f,1}\|_{L^2(\Omega)}+\alpha\mathscr{K}\|\nabla\,\mathbf{v}_h^{f,1}\|_{L^2(\Omega)},$$

which implies that

$$\rho^f\big(\|\mathbf{v}_h^{f,1}\|^2_{L^2(\Omega)}+\|\mathbf{v}_h^{f,1}-\mathbf{v}_h^{f,0}\|^2_{L^2(\Omega)}\big)+\mu^f\Delta t\|\nabla\,\mathbf{v}_h^{f,1}\|^2_{L^2(\Omega)}+\alpha\Delta t\|\mathbf{v}_h^{f,1}\|^2_{L^2(\Omega)}\le$$
$$\le\rho^f\|\mathbf{v}_h^{f,0}\|^2_{L^2(\Omega)}+\frac{(\rho^f)^2}{\alpha}\Delta t\|\mathbf{b}_e(t_1)\|^2_{L^2(\Omega)}+\frac{\alpha^2}{\mu^f}\Delta t\mathscr{K}^2. \tag{5.8}$$

Of course, if we use the splitting $\mathbf{v}_h^{f,1}=\check{\mathbf{v}}_h^{f,1}+\mathbf{v}_{0,h}^{f,1}$, and take into account the initial conditions, we obtain more precise inequalities for each part

$$\rho^f\big(\|\check{\mathbf{v}}_h^{f,1}\|^2_{L^2(\Omega)}+\|\check{\mathbf{v}}_h^{f,1}-\check{\mathbf{v}}_h^{f,0}\|^2_{L^2(\Omega)}\big)+2\mu^f\Delta t\|\nabla\,\check{\mathbf{v}}_h^{f,1}\|^2_{L^2(\Omega)}+\alpha\Delta t\|\check{\mathbf{v}}_h^{f,1}\|^2_{L^2(\Omega)}\le$$
$$\le\rho^f\|\check{\mathbf{v}}_h^{f,0}\|^2_{L^2(\Omega)}+\frac{(\rho^f)^2}{\alpha}\Delta t\|\mathbf{b}_e(t_1)\|^2_{L^2(\Omega)}, \tag{5.9}$$

and

$$\rho^f\|\mathbf{v}_{0,h}^{f,1}\|^2_{L^2(\Omega)}+\Delta t\frac{\mu^f}{2}\|\nabla\,\mathbf{v}_{0,h}^{f,1}\|^2_{L^2(\Omega)}+\alpha\Delta t\|\mathbf{v}_{0,h}^{f,1}\|^2_{L^2(\Omega)}\le\frac{\alpha^2}{2\mu^f}\Delta t\mathscr{K}^2. \tag{5.10}$$

From here, the estimates of $\check{\mathbf{u}}_h^{s,n}$ and $\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)$ for $n\ge1$ are respectively the same as in (4.14), (4.15) and in (4.24), (4.25), to be specific,

$$(\lambda^s+\mu^s)\sum_{n=1}^m\Delta t\|\nabla\cdot\check{\mathbf{u}}_h^{s,n}\|^2_{L^2(\Omega)}+\frac{\mu^s}{2}\sum_{n=1}^m\Delta t\|\nabla\,\check{\mathbf{u}}_h^{s,n}\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\|\check{\mathbf{u}}_h^{s,m}\|^2_{L^2(\Omega)}$$
$$+\frac{\alpha}{2}\sum_{n=1}^m\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}\le C_1,$$

$$\frac{\lambda^s+\mu^s}{2}\Big(\|\nabla\cdot\check{\mathbf{u}}_h^{s,m}\|^2_{L^2(\Omega)}+\sum_{n=1}^m\|\nabla\cdot(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\|^2_{L^2(\Omega)}\Big)$$
$$+\frac{\mu^s}{4}\Big(\|\nabla\,\check{\mathbf{u}}_h^{s,m}\|^2_{L^2(\Omega)}+\sum_{n=1}^m\|\nabla(\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1})\|^2_{L^2(\Omega)}\Big)$$
$$+\frac{\alpha}{2}\sum_{n=1}^m\frac{1}{\Delta t}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}\le C_2,$$

$$(\lambda^s+\mu^s)\sum_{n=1}^m\Delta t\|\nabla\cdot\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|^2_{L^2(\Omega)}+\frac{\mu^s}{2}\sum_{n=1}^m\Delta t\|\nabla\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\|\hat{\mathbf{u}}_h^{s,m}(\mathbf{W}_h)\|^2_{L^2(\Omega)}$$
$$+\frac{\alpha}{2}\sum_{n=1}^m\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|^2_{L^2(\Omega)}\le\frac{(\alpha\mathscr{P})^2}{2\mu^s}\sum_{n=1}^m\Delta t\|\mathbf{w}_h^n\|^2_{L^2(\Omega)},$$

$$\frac{\lambda^s+\mu^s}{2}\big(\|\nabla\cdot\hat{\mathbf{u}}_h^{s,m}(\mathbf{W}_h)\|^2_{L^2(\Omega)}+\sum_{n=1}^m\|\nabla\cdot(\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h))\|^2_{L^2(\Omega)}\big)$$
$$+\frac{\mu^s}{2}\big(\|\nabla\hat{\mathbf{u}}_h^{s,m}(\mathbf{W}_h)\|^2_{L^2(\Omega)}+\sum_{n=1}^m\|\nabla(\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h))\|^2_{L^2(\Omega)}\big)$$
$$+\frac{\alpha}{2\Delta t}\sum_{n=1}^m\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{W}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{W}_h)\|^2_{L^2(\Omega)}\le\frac{\alpha}{2}\sum_{n=1}^m\Delta t\|\mathbf{w}_h^n\|^2_{L^2(\Omega)}.$$

We need to examine $\mathbf{v}_h^{f,n}$ for $n\ge 2$. First, regarding $\check{\mathbf{v}}_h^{f,n}$, we have

$$\frac{\rho^f}{2\Delta t}\big(\|\check{\mathbf{v}}_h^{f,n}\|^2_{L^2(\Omega)}+\|\check{\mathbf{v}}_h^{f,n}-\check{\mathbf{v}}_h^{f,n-1}\|^2_{L^2(\Omega)}\big)+\mu^f\|\nabla\check{\mathbf{v}}_h^{f,n}\|^2_{L^2(\Omega)}+\alpha\|\check{\mathbf{v}}_h^{f,n}\|^2_{L^2(\Omega)}\le$$
$$\le\frac{\rho^f}{2\Delta t}\|\check{\mathbf{v}}_h^{f,n-1}\|^2_{L^2(\Omega)}+\rho^f\|\mathbf{b}_e(t_n)\|_{L^2(\Omega)}\|\check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}+\alpha\|\check{\mathbf{d}}_h^{s,n-1}\|_{L^2(\Omega)}\|\check{\mathbf{v}}_h^{f,n}\|_{L^2(\Omega)}.$$

Thus

$$\frac{\rho^f}{2}\big(\|\check{\mathbf{v}}_h^{f,n}\|^2_{L^2(\Omega)}+\|\check{\mathbf{v}}_h^{f,n}-\check{\mathbf{v}}_h^{f,n-1}\|^2_{L^2(\Omega)}\big)+\mu^f\Delta t\|\nabla\check{\mathbf{v}}_h^{f,n}\|^2_{L^2(\Omega)}\le$$
$$\le\frac{\rho^f}{2}\|\check{\mathbf{v}}_h^{f,n-1}\|^2_{L^2(\Omega)}+\frac{(\rho^f)^2}{2\alpha}\Delta t\|\mathbf{b}_e(t_n)\|^2_{L^2(\Omega)}+\frac{\alpha}{2\Delta t}\|\check{\mathbf{u}}_h^{s,n-1}-\check{\mathbf{u}}_h^{s,n-2}\|^2_{L^2(\Omega)}.$$

By summing over n from $n=2$ to $n=m$, we obtain

$$\begin{aligned}&\frac{\rho^f}{2}\big(\|\check{\mathbf{v}}_h^{f,m}\|^2_{L^2(\Omega)}+\sum_{n=2}^m\|\check{\mathbf{v}}_h^{f,n}-\check{\mathbf{v}}_h^{f,n-1}\|^2_{L^2(\Omega)}\big)+\mu^f\sum_{n=2}^m\Delta t\|\nabla\check{\mathbf{v}}_h^{f,n}\|^2_{L^2(\Omega)}\le\\&\le\frac{\rho^f}{2}\|\check{\mathbf{v}}_h^{f,1}\|^2_{L^2(\Omega)}+\frac{(\rho^f)^2}{2\alpha}\sum_{n=2}^m\Delta t\|\mathbf{b}_e(t_n)\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\sum_{n=1}^{m-1}\frac{1}{\Delta t}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}.\end{aligned}\tag{5.11}$$

Then, we substitute (5.9) into the above right-hand side and obtain

$$\begin{aligned}&\frac{\rho^f}{2}\big(\|\check{\mathbf{v}}_h^{f,m}\|^2_{L^2(\Omega)}+\sum_{n=1}^m\|\check{\mathbf{v}}_h^{f,n}-\check{\mathbf{v}}_h^{f,n-1}\|^2_{L^2(\Omega)}\big)+\mu^f\sum_{n=1}^m\Delta t\|\nabla\check{\mathbf{v}}_h^{f,n}\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\Delta t\|\check{\mathbf{v}}_h^{f,1}\|^2_{L^2(\Omega)}\\&\le\frac{\rho^f}{2}\|\check{\mathbf{v}}_h^{f,0}\|^2_{L^2(\Omega)}+\frac{(\rho^f)^2}{2\alpha}\sum_{n=1}^m\Delta t\|\mathbf{b}_e(t_n)\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\sum_{n=1}^{m-1}\frac{1}{\Delta t}\|\check{\mathbf{u}}_h^{s,n}-\check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)}\le C_9,\end{aligned}\tag{5.12}$$

a quantity bounded above independently of h and Δt. Note that in view of (5.9), (5.12) is also valid when $m=1$. From (5.12), we deduce the analogue of (4.32),

$$\sum_{n=1}^m\Delta t\|\check{\mathbf{v}}_h^{f,n}\|^2_{L^2(\Omega)}\le\frac{2}{\rho^f}C_9m\Delta t\le\frac{2}{\rho^f}C_9T.\tag{5.13}$$

With (4.24) and (4.25), this implies

$$\begin{aligned}&(\lambda^s+\mu^s)\sum_{n=1}^m\Delta t\|\nabla\cdot\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h)\|^2_{L^2(\Omega)}+\frac{\mu^s}{2}\sum_{n=1}^m\Delta t\|\nabla\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h)\|^2_{L^2(\Omega)}+\frac{\alpha}{2}\|\hat{\mathbf{u}}_h^{s,m}(\check{\mathbf{V}}_h)\|^2_{L^2(\Omega)}\\&+\frac{\alpha}{2}\sum_{n=1}^m\|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h)\|^2_{L^2(\Omega)}\le\frac{(\alpha\mathscr{P})^2}{\mu^s\rho^f}C_9T,\end{aligned}\tag{5.14}$$

$$
\begin{aligned}
&\frac{\lambda^s+\mu^s}{2}\big(\|\nabla\cdot\hat{\mathbf{u}}_h^{s,m}(\check{\mathbf{V}}_h)\|^2_{L^2(\Omega)}+\sum_{n=1}^m\|\nabla\cdot(\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h))\|^2_{L^2(\Omega)}\big)\\
&+\frac{\mu^s}{2}\big(\|\nabla\hat{\mathbf{u}}_h^{s,m}(\check{\mathbf{V}}_h)\|^2_{L^2(\Omega)}+\sum_{n=1}^m\|\nabla(\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h))\|^2_{L^2(\Omega)}\big)\\
&+\frac{\alpha}{2\Delta t}\sum_{n=1}^m\|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h)\|^2_{L^2(\Omega)}\le\frac{\alpha}{\rho^f}C_9T.
\end{aligned}
\tag{5.15}
$$

It remains to study $\mathbf{v}_{0,h}^{f,n}$ for $n\ge 2$. We have

$$
\begin{aligned}
&\frac{\rho^f}{2\Delta t}\big(\|\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}+\|\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1}\|^2_{L^2(\Omega)}\big)+\mu^f\|\nabla\,\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}+\alpha\|\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}\le\\
&\quad\le\frac{\rho^f}{2\Delta t}\|\mathbf{v}_{0,h}^{f,n-1}\|^2_{L^2(\Omega)}+\frac{\alpha}{\Delta t}\big(\|\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-2}(\check{\mathbf{V}}_h^f)\|_{L^2(\Omega)}\\
&\quad+\|\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)-\hat{\mathbf{u}}_h^{s,n-2}(\mathbf{V}_{0,h}^f)\|_{L^2(\Omega)}\big)\|\mathbf{v}_{0,h}^{f,n}\|_{L^2(\Omega)}.
\end{aligned}
$$

By suitably using Young's inequality and summing over n from $n=2$ to $n=m$, we infer

$$
\begin{aligned}
&\frac{\rho^f}{2}\big(\|\mathbf{v}_{0,h}^{f,m}\|^2_{L^2(\Omega)}+\sum_{n=2}^m\|\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1}\|^2_{L^2(\Omega)}\big)+\frac{\mu^f}{2}\sum_{n=2}^m\Delta t\|\nabla\,\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}\\
&\quad+\frac{\alpha}{2}\sum_{n=2}^m\Delta t\|\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}\le\frac{\rho^f}{2}\|\mathbf{v}_{0,h}^{f,1}\|^2_{L^2(\Omega)}\\
&\quad+\frac{(\alpha\mathscr{P})^2}{2\mu^f}\sum_{n=1}^{m-1}\frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)\|^2_{L^2(\Omega)}\\
&\quad+\frac{\alpha}{2}\sum_{n=1}^{m-1}\frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{5.16}
$$

Let us substitute (5.10) into (5.16); this yields

$$
\begin{aligned}
&\frac{\rho^f}{2}\big(\|\mathbf{v}_{0,h}^{f,m}\|^2_{L^2(\Omega)}+\sum_{n=2}^m\|\mathbf{v}_{0,h}^{f,n}-\mathbf{v}_{0,h}^{f,n-1}\|^2_{L^2(\Omega)}\big)+\frac{\mu^f}{2}\big(\frac{\Delta t}{2}\|\nabla\,\mathbf{v}_{0,h}^{f,1}\|^2_{L^2(\Omega)}+\sum_{n=2}^m\Delta t\|\nabla\,\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}\big)\\
&+\frac{\alpha}{2}\sum_{n=1}^m\Delta t\|\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}\le\frac{(\alpha\mathscr{K})^2}{4\mu^f}\Delta t+\frac{(\alpha\mathscr{P})^2}{2\mu^f}\sum_{n=1}^{m-1}\frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)\|^2_{L^2(\Omega)}\\
&\qquad+\frac{\alpha}{2}\sum_{n=1}^{m-1}\frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)\|^2_{L^2(\Omega)},
\end{aligned}
\tag{5.17}
$$

an inequality that is also valid when $m=1$.

Now, by (4.25), we have on the one hand

$$
\sum_{n=1}^m\frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f)-\hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)\|^2_{L^2(\Omega)}\le\sum_{n=1}^m\Delta t\|\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)},
\tag{5.18}
$$

and on the other hand, in view of (5.13),

$$
\sum_{n=1}^m\frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\check{\mathbf{V}}_h^f)-\hat{\mathbf{u}}_h^{s,n-1}(\check{\mathbf{V}}_h^f)\|^2_{L^2(\Omega)}\le\sum_{n=1}^m\Delta t\|\check{\mathbf{v}}_h^{f,n}\|^2_{L^2(\Omega)}\le\frac{2}{\rho^f}C_9T.
\tag{5.19}
$$

When these two bounds are substituted into (5.17), we derive

$$\begin{aligned}\frac{\rho^f}{2}\big(\|\mathbf{v}_{0,h}^{f,m}\|^2_{L^2(\Omega)} + \sum_{n=1}^m \|\mathbf{v}_{0,h}^{f,n} - \mathbf{v}_{0,h}^{f,n-1}\|^2_{L^2(\Omega)}\big) + \frac{\mu^f}{2}\big(\frac{\Delta t}{2}\|\nabla\,\mathbf{v}_{0,h}^{f,1}\|^2_{L^2(\Omega)} \\ + \sum_{n=2}^m \Delta t\|\nabla\,\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)}\big) \le C_{10} := \frac{(\alpha\mathscr{K})^2}{4\mu^f}\Delta t + \frac{(\alpha\mathscr{P})^2}{\mu^f\rho^f}C_9 T,\end{aligned} \tag{5.20}$$

that is again bounded independently of h and Δt. Of course, this also implies

$$\sum_{n=1}^m \Delta t\|\mathbf{v}_{0,h}^{f,n}\|^2_{L^2(\Omega)} \le \frac{2}{\rho^f} m\Delta t C_{10} \le \frac{2}{\rho^f}C_{10}T, \tag{5.21}$$

that, in view of (5.18), implies

$$\sum_{n=1}^m \frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_{0,h}^f) - \hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_{0,h}^f)\|^2_{L^2(\Omega)} \le \frac{2}{\rho^f}C_{10}T. \tag{5.22}$$

A combination of (5.19) and (5.22) immediately yields

$$\sum_{n=1}^m \frac{1}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_h^f) - \hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_h^f)\|^2_{L^2(\Omega)} \le \frac{4}{\rho^f}\big(C_9 + C_{10}\big)T. \tag{5.23}$$

This induces an estimate for the auxiliary variable $\mathbf{d}_h^{s,n}$. Indeed, from the fourth equation in (5.2), taking $\mathbf{z}_h = \mathbf{d}_h^{s,n}$ and using the triangle and Cauchy-Schwarz inequalities,

$$\|\mathbf{d}_h^{s,n}\|^2_{L^2(\Omega)} \le \|\frac{\mathbf{u}_h^{s,n} - \mathbf{u}_h^{s,n-1}}{\Delta t}\|^2_{L^2(\Omega)} \le 2\|\frac{\check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1}}{\Delta t}\|^2_{L^2(\Omega)} + 2\|\frac{\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_h^f) - \hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_h^f)}{\Delta t}\|^2_{L^2(\Omega)}.$$

Summing over n from $n=1$ to $n=m$, and multiplying by Δt, we obtain, owing to (4.15) and (5.23):

$$\begin{aligned}\sum_{n=1}^m \Delta t\|\mathbf{d}_h^{s,n}\|^2_{L^2(\Omega)} &\le \sum_{n=1}^m \frac{2}{\Delta t}\|\check{\mathbf{u}}_h^{s,n} - \check{\mathbf{u}}_h^{s,n-1}\|^2_{L^2(\Omega)} + \sum_{n=1}^m \frac{2}{\Delta t}\|\hat{\mathbf{u}}_h^{s,n}(\mathbf{V}_h^f) - \hat{\mathbf{u}}_h^{s,n-1}(\mathbf{V}_h^f)\|^2_{L^2(\Omega)} \\ &\le \frac{4}{\alpha}C_2 + \frac{4}{\rho^f}\big(C_9 + C_{10}\big)T.\end{aligned} \tag{5.24}$$

Thus stability for the displacement and velocity is established under the previous hypotheses plus assumption (5.7).

Finally, we turn to the pressure. Again, a bound for the pressure relies on a bound for the velocity's time derivative. This bound is readily obtained by testing the first equation of (5.2) with $\mathbf{v}_h^{f,n} - \mathbf{v}_h^{f,n-1}$,

$$\begin{aligned}&\rho^f \sum_{n=1}^m \frac{1}{\Delta t}\|\mathbf{v}_h^{f,n} - \mathbf{v}_h^{f,n-1}\|^2_{L^2(\Omega)} + \mu^f\big(\|\nabla\,\mathbf{v}_h^{f,m}\|^2_{L^2(\Omega)} + \sum_{n=1}^m \|\nabla(\mathbf{v}_h^{f,n} - \mathbf{v}_h^{f,n-1})\|^2_{L^2(\Omega)}\big) \\ &+ \alpha\big(\|\mathbf{v}_h^{f,m}\|^2_{L^2(\Omega)} + \sum_{n=1}^m \|\mathbf{v}_h^{f,n} - \mathbf{v}_h^{f,n-1}\|^2_{L^2(\Omega)}\big) \le \mu^f\|\nabla\,\mathbf{v}_h^{f,0}\|^2_{L^2(\Omega)} \\ &+ \alpha\|\mathbf{v}_h^{f,0}\|^2_{L^2(\Omega)} + \frac{1}{\rho^f}\sum_{n=1}^m \Delta t\big(\rho^f\|\mathbf{b}_e^{t_n}\|_{L^2(\Omega)} + \alpha\|\mathbf{d}_h^{s,n-1}\|_{L^2(\Omega)}\big)^2 \le C_{11},\end{aligned}$$

with C_{11} bounded above independently of m, h, and Δt owing to (5.24) and (5.7). From here, stability of the pressure is a consequence of the inf-sup condition (4.5) with constant $\beta^* > 0$ independent of h, as in the proof of Theorem 4.8. Hence the scheme (5.2) is stable.

Theorem 5.1. *In addition to the hypotheses of Theorems 4.7 and 4.8, suppose that* (5.7) *holds. Then, the statement of these theorems extend to the solution of* (5.2) *and moreover the auxiliary variable* $\mathbf{d}_h^s$ *satisfies* (5.24).

Remark 5.2. The hypothesis (5.7) is only used once, namely for deriving a bound for the initial term $\mathbf{d}_h^{s,0}$ in a dual norm. The subsequent terms $\mathbf{d}_h^{s,n}$ for $n \geq 1$ are bounded in L^2 as in Section 4.2. But the algorithm (5.2) is fairly crude in the sense that it is a simple time lagging. It could be refined by iteration at each time step, in which case the initial error would not be so influential, and we could take

$$\mathbf{d}_h^{s,0} = \mathbf{0}.$$

Here is a possible iterative algorithm at each time step $n \geq 1$, starting from $\mathbf{v}_h^{f,0} \in V_h^0$, $\mathbf{u}_h^{s,0} \in U_h$, and $\mathbf{d}_h^{s,0} = \mathbf{0}$, where the superscript ℓ denotes the algorithmic step. Let

$$\mathbf{d}_h^{s,n,0} = \mathbf{d}_h^{s,n-1}.$$

Then for $\ell \geq 1$ until convergence is attained, find $(\mathbf{v}_h^{f,n,\ell}, p_h^{f,n,\ell}, \mathbf{u}_h^{s,n,\ell}, \mathbf{d}_h^{s,n,\ell})$ in $V_h \times Q_h \times U_h \times U_h$ such that

$$\begin{aligned}
&\frac{\rho^f}{\Delta t}\int_\Omega (\mathbf{v}_h^{f,n,\ell} - \mathbf{v}_h^{f,n-1}) \cdot \mathbf{w}_h + \mu^f \int_\Omega \nabla \mathbf{v}_h^{f,n,\ell} : \nabla \mathbf{w}_h - \int_\Omega p_h^{f,n,\ell} \nabla \cdot \mathbf{w}_h - \rho^f \int_\Omega \mathbf{b}_e(t_n) \cdot \mathbf{w}_h \\
&\qquad + \alpha \int_\Omega \left(\mathbf{v}_h^{f,n,\ell} - \mathbf{d}_h^{s,n,\ell-1}\right) \cdot \mathbf{w}_h = 0 \\
&\int_\Omega q_h \nabla \cdot \mathbf{v}_h^{f,n,\ell} = 0 \\
&(\lambda^s + \mu^s) \int_\Omega (\nabla \cdot \mathbf{u}_h^{s,n,\ell})(\nabla \cdot \mathbf{z}_h) + \mu^s \int_\Omega \nabla \mathbf{u}_h^{s,n,\ell} : \nabla \mathbf{z}_h - \rho^s \int_\Omega \mathbf{b}_e(t_n) \cdot \mathbf{z}_h - \langle \mathbf{g}_N^s, \mathbf{z}_h \rangle_{\Gamma_N^s} \\
&\qquad - \alpha \int_\Omega \mathbf{v}_h^{f,n,\ell} \cdot \mathbf{z}_h + \frac{\alpha}{\Delta t} \int_\Omega (\mathbf{u}_h^{s,n,\ell} - \mathbf{u}_h^{s,n-1}) \cdot \mathbf{z}_h = 0, \\
&\int_\Omega \mathbf{d}_h^{s,n,\ell} \cdot \mathbf{z}_h = \frac{1}{\Delta t} \int_\Omega (\mathbf{u}_h^{s,n,\ell} - \mathbf{u}_h^{s,n-1}) \cdot \mathbf{z}_h,
\end{aligned} \tag{5.25}$$

for all $(\mathbf{w}_h, q_h, \mathbf{z}_h) \in V_h \times Q_h \times U_h$.

6. Numerical experiments

In this section, we present some numerical results obtained with the monolithic algorithm (4.4) in the case $d = 2$. We consider the backward Euler method for the discretization in time, the classical Hood-Taylor element for the discretization of the flow equations and continuous piecewise quadratic polynomials for the discretization of the solid displacement. The numerical results were obtained with FreeFem++ [16]. We also implemented the decoupled algorithm (5.2). In that case, we obtained basically the same results although the CPU time is lower (see the remark at the end of this section).

Let Ω be the rectangle $]0,2[\times]0,1[$ (see Figure 1). The parameter values in the International System are as follows: $\lambda^s = 1.65 \times 10^9\,Pa$, $\mu^s = 2.475 \times 10^9\,Pa$, $\rho^s = 2430\,Kg/m^3$, $\rho^f = 867\,Kg/m^3$, $\mu^f = 0.016\,Pa \cdot s$, corresponding to shale for the solid and WTI for the fluid. We consider $\alpha = 0.9\,s^{-1}$ and a mesh with 4024 triangles. The final time is $T = 1$ and we take a time step $\Delta t = 0.1$.

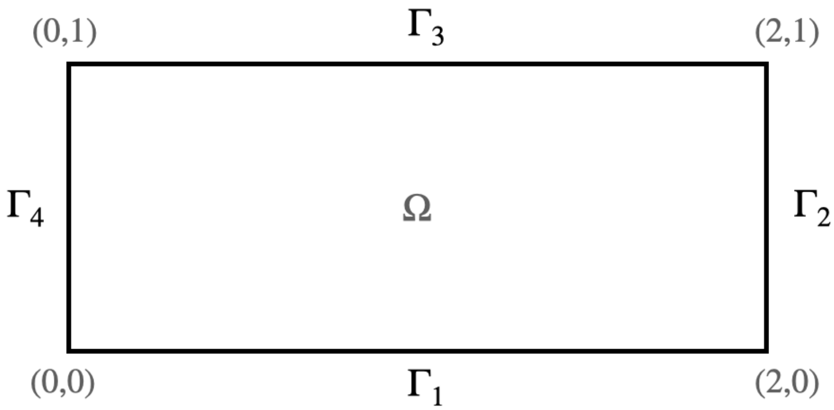

FIGURE 1. Geometry

6.1. **Test 1.** We assume homogeneous Dirichlet boundary conditions for the displacement on the left and right sides of the rectangle:

$$\mathbf{u}^s = \mathbf{0}, \quad \text{on } (\Gamma_2 \cup \Gamma_4) \times]0,T[,$$

a free normal traction boundary condition on the top:

$$\Big((\mu^s \nabla \mathbf{u}^s + (\lambda^s + \mu^s)(\nabla \cdot \mathbf{u}^s)\mathbf{I})\mathbf{n} \Big) = \mathbf{0}, \quad \text{on } \Gamma_3 \times]0,T[,$$

and a given normal traction boundary condition on the bottom:

$$\Big((\mu^s \nabla \mathbf{u}^s + (\lambda^s + \mu^s)(\nabla \cdot \mathbf{u}^s)\mathbf{I})\mathbf{n} \Big) = \mathbf{g}_N^s, \quad \text{on } \Gamma_1 \times]0,T[,$$

where

$$\mathbf{g}_N^s = \begin{pmatrix} 0 \\ -0.5\sin(t) \end{pmatrix}.$$

As external body force we consider gravity:

$$\mathbf{b}_e = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix}.$$

For the velocity, we choose no-slip boundary conditions on the top and bottom boundaries of the domain, and a do-nothing boundary condition on the left and right sides:

$$\mathbf{v}^f = \mathbf{0}, \quad \text{on } (\Gamma_1 \cup \Gamma_3) \times]0,T[,$$

$$(-p^f \mathbf{I} + \nabla \mathbf{v}^f)\mathbf{n} = \mathbf{0}, \quad \text{on } (\Gamma_2 \cup \Gamma_4) \times]0,T[.$$

Initial conditions are $\mathbf{u}^s(0) = \mathbf{0}$ and $\mathbf{v}^f(0) = \mathbf{0}$ in Ω.

In Figure 2 we show the displacements at time steps $t = 0.1$, $t = 0.5$ and $t = 1$. Solid velocities at the first time step are plotted in Figure 3. Finally, fluid velocities and pressure are plotted in Figures 4 and 5, respectively.

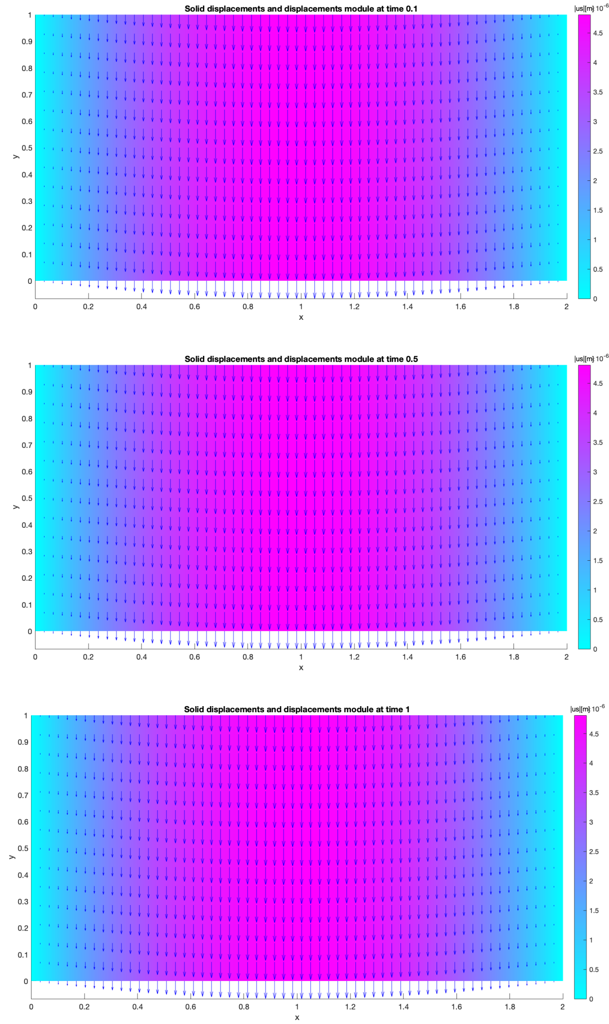

FIGURE 2. Test 1: Displacement at times $t = 0.1$, $t = 0.5$ and $t = 1$.

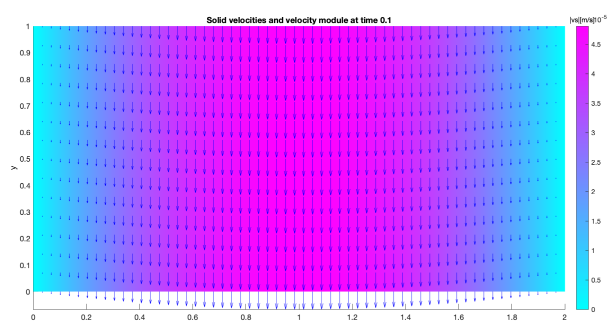

FIGURE 3. Test 1: Solid velocity at time $t = 0.1$

6.2. **Test 2:** In this test, the only difference are the boundary conditions for the flow. For the velocity, we choose no-slip boundary conditions on the left and right boundaries of the domain,

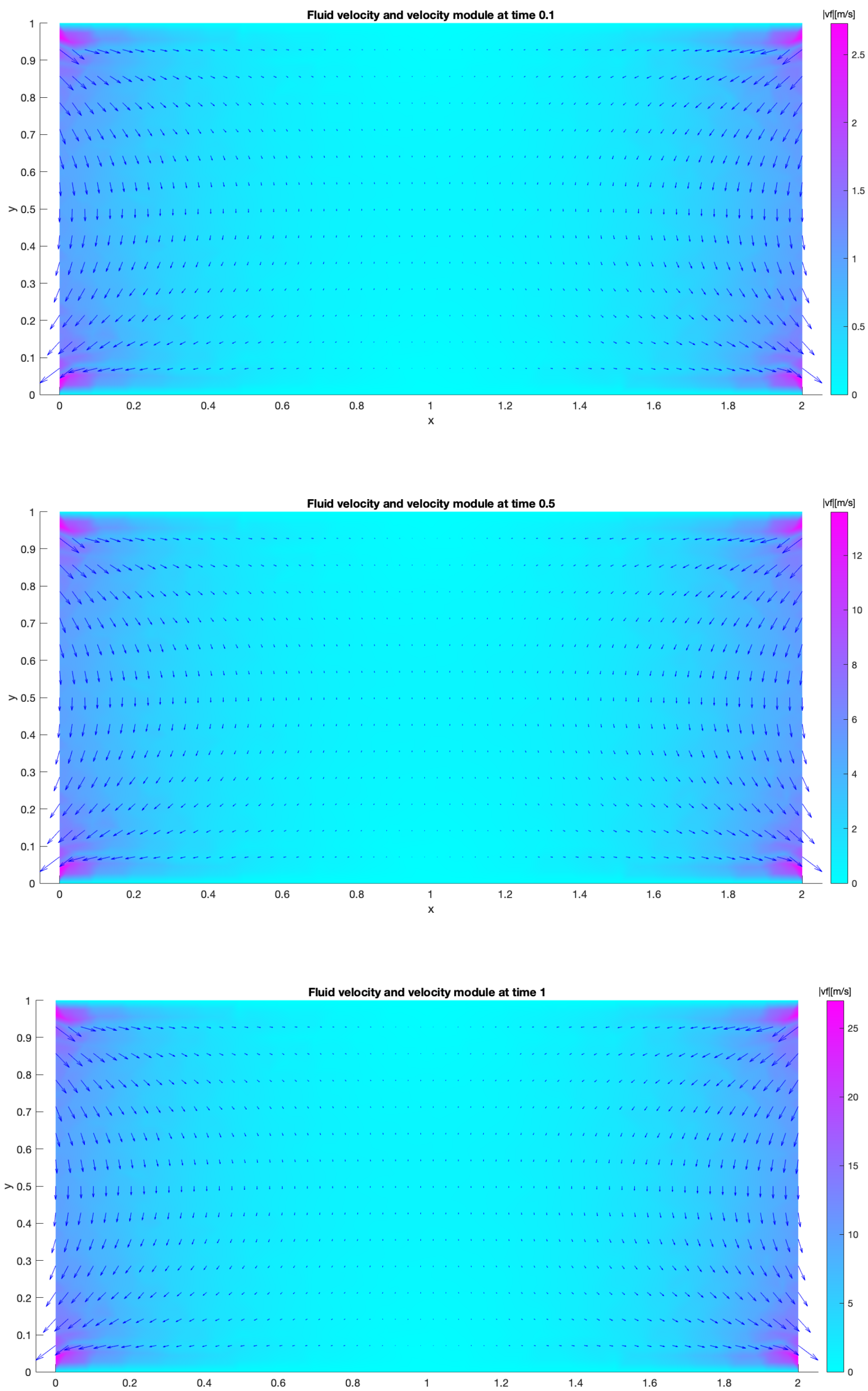

FIGURE 4. Test 1: Fluid velocity at times $t = 0.1$, $t = 0.5$ and $t = 1$.

and a do-nothing boundary condition on the top and bottom sides:

$$\mathbf{v}^f = \mathbf{0}, \quad \text{on } (\Gamma_2 \cup \Gamma_4) \times]0, T[,$$

$$(-p^f \mathbf{I} + \nabla \mathbf{v}^f)\mathbf{n} = \mathbf{0}, \quad \text{on } (\Gamma_1 \cup \Gamma_3) \times]0, T[.$$

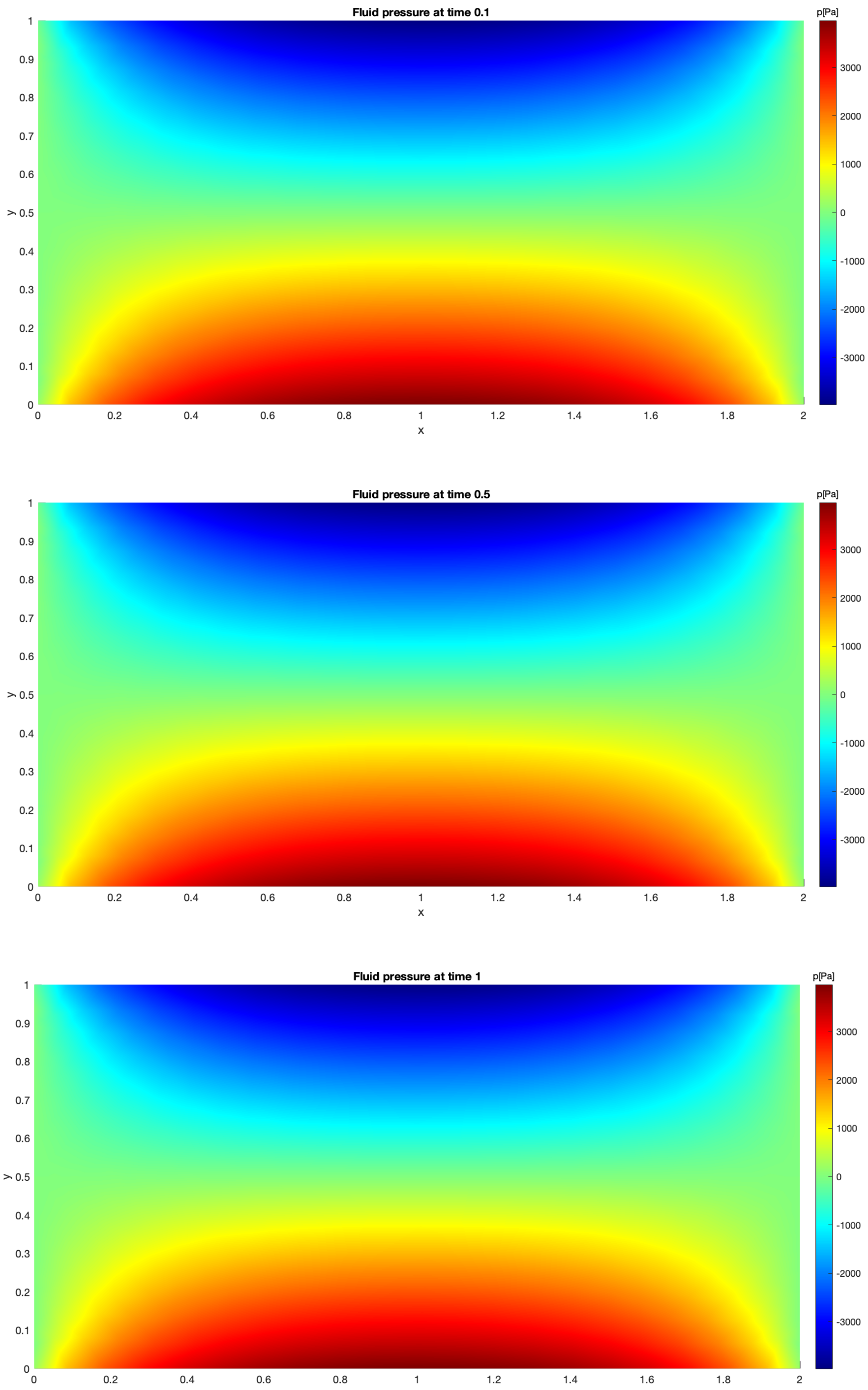

FIGURE 5. Test 1: Fluid pressure at times $t = 0.1$, $t = 0.5$ and $t = 1$.

In Figure 6 we show the displacements at time steps $t = 0.1$, $t = 0.5$, and $t = 1$. Solid velocities at the first time step are plotted in Figure 7. Finally, fluid velocities and pressure are plotted in Figures 8 and 9, respectively.

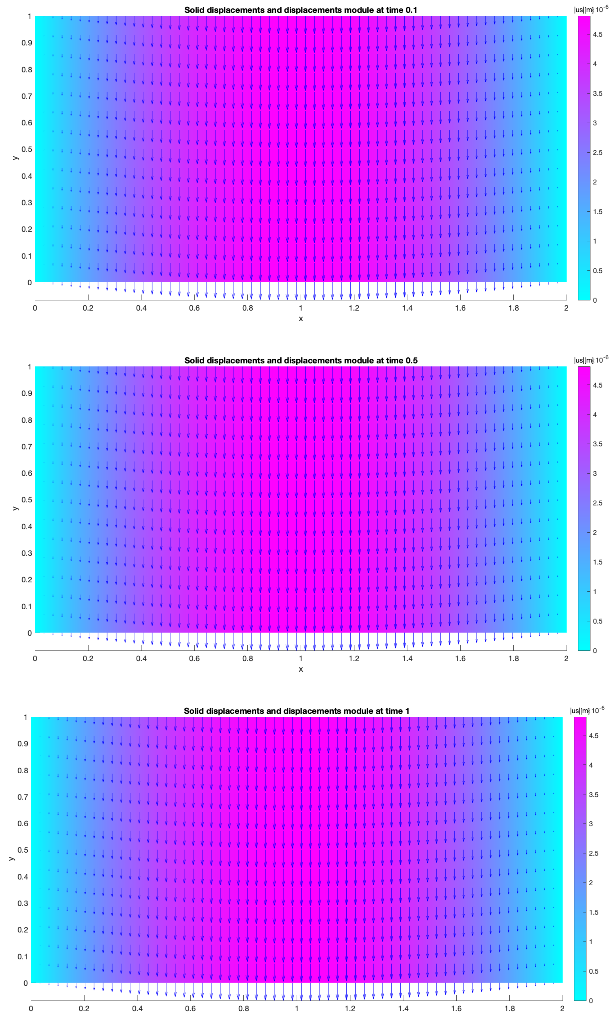

FIGURE 6. Test 2: Displacement at times $t = 0.1$, $t = 0.5$ and $t = 1$.

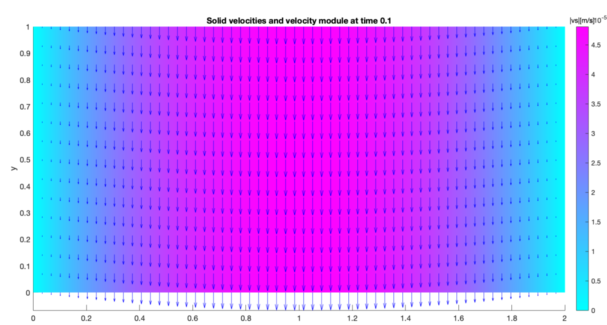

FIGURE 7. Test 2: Solid velocity at time $t = 0.1$

In both tests, we can observe that solid displacements are very small and practically identical, and solid velocities are negligible after time 0.2. We also observe that solid velocities are much smaller than fluid velocities.

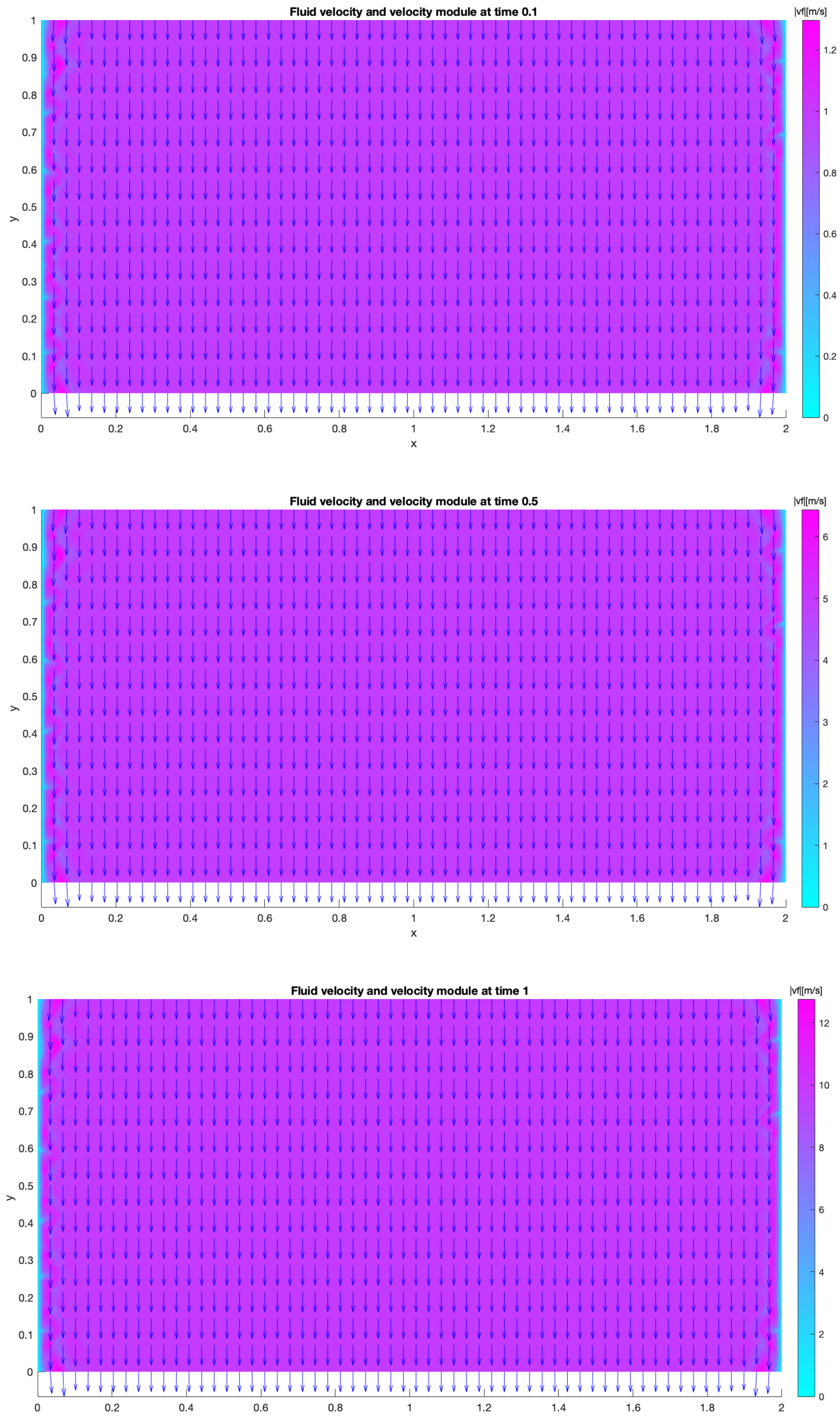

FIGURE 8. Test 2: Fluid velocity at times $t = 0.1$, $t = 0.5$ and $t = 1$.

Remark 6.1. In Figure 10, we display the average CPU time (in seconds) for the monolithic algorithm (4.4) and the decoupled algorithm (5.2) for different meshes. There, *nt* denotes the number of triangles in the mesh. We used MUMPS to solve the linear systems in FreeFem++,

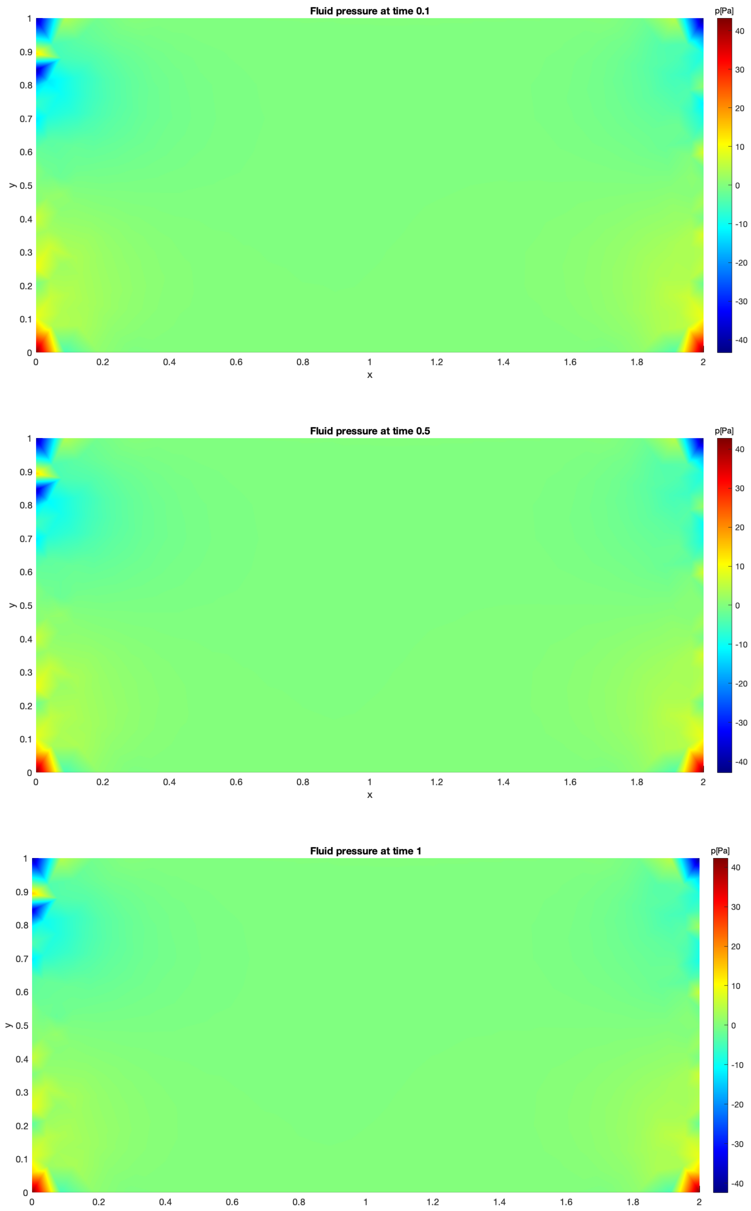

FIGURE 9. Test 2: Fluid pressure at times $t = 0.1$, $t = 0.5$ and $t = 1$.

and each algorithm is run for 10 time steps. We can observe that the decoupled algorithm always performs better than the monolithic algorithm. The differences in CPU time are smaller for coarser meshes; for the largest mesh we tested, with 710788 triangles, we save approximately $13\,s$ per time step, which amounts to a $25,26\%$ of time saving. These results were obtained with a MacBook Pro M2 with 8 kernels.

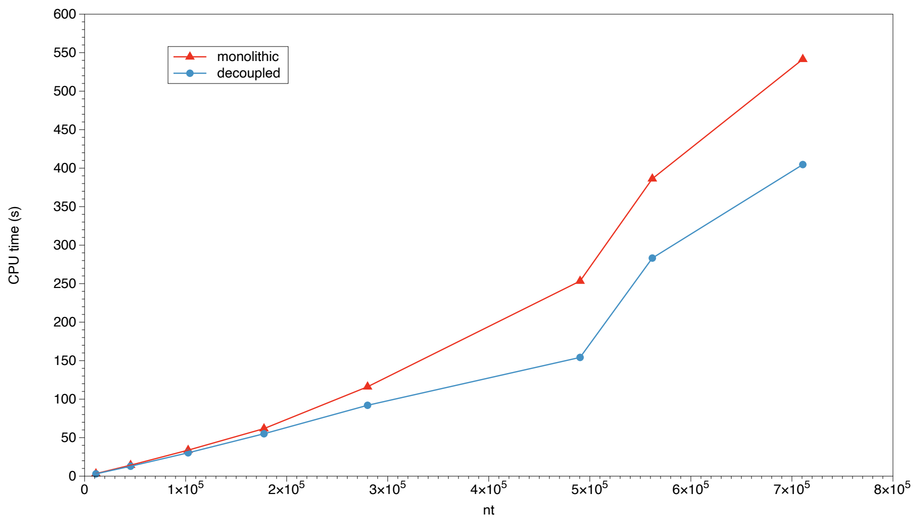

FIGURE 10. CPU time (in seconds) for the monolithic and decoupled algorithms.

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