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NECESSARY OPTIMALITY CONDITIONS OF DELAY PARAMETERS FOR THE NONLINEAR OPTIMIZATION PROBLEM WITH THE MIXED INITIAL CONDITION

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Dedicated to the memory of Professor Rafail Gabasov

Abstract. In this paper, the necessary conditions of optimality of delay parameters, of the initial vector, of the initial and control functions are proved for the nonlinear optimization problem with constant delays in the phase coordinates and controls. The necessary conditions are concretized for the optimization problem with the integral functional and fixed right end.

Keywords. Delay optimization problem; Mixed initial condition; Necessary optimality conditions; Nonlinear control system.

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1. Introduction

In the paper, an optimization problem is considered for the differential equation with delays in the phase coordinates and controls

$$\dot{x}(t) = (\dot{p}(t), \dot{q}(t))^T = f(t, x(t), p(t-\tau), q(t-\sigma), u(t), u(t-\theta)), \tag{1.1}$$
$$x(t) \in \mathbb{R}^n,\ t \in [t_0, t_1]$$

with the mixed initial condition

$$\begin{cases} x(t) = (p(t), q(t))^T = (\varphi(t), g(t))^T, t < t_0, \\ x(t_0) = (p(t_0), q(t_0))^T = (p_0, g(t_0))^T, \end{cases} \tag{1.2}$$

where T is the sign transposition.

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Condition (1.2) is called the mixed initial condition because it consists of two parts: the first part is $p(t)=\varphi(t), t<t_0, p(t_0)=p_0$, the discontinuous part, since in general $p(t_0)\neq p_0$; discontinuity at the initial moment may be related to the instant change in a dynamic process, for example changes of investment and environment etc; the second part is $q(t)=g(t), t\leq t_0$, the continuous part, since always $q(t_0)=g(t_0)$.

In this paper, for the optimization problem containing equation (1.1) and initial condition (1.2), and general boundary conditions

$$z^i(\tau,\sigma,\theta,p_0,x(t_1))=0, i=\overline{1,l}$$

and functional

$$z^0(\tau,\sigma,\theta,p_0,x(t_1))\rightarrow min$$

the necessary optimality conditions are proved: for delays τ,σ and θ; for the initial vector p_0; for the initial functions $\varphi(t)$ and $g(t)$; for the control $u(t)$.

Delay optimal control problems with the mixed initial condition, without optimization of delay parameters were considered in [1, 2]. The origin of the development of the optimal control theory with delay goes back to [3]. Many works have been devoted to the investigation of optimization problems with delay; see, e.g., [4]-[22] and the references therein.

The paper is organized as follows. In Section 2, the main theorem and its corollary are formulated. The main theorem is proved in Section 3 by the scheme given in [21, 22].

2. Statement of the Problem and Formulation of Main Results

Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$. Let $\tau_2>\tau_1>0$, $\sigma_2>\sigma_1>0$, $\theta_2>\theta_1>0$ be given numbers and let $I=[t_0,t_1]$, with $t_0+\tau_2<t_1; I_1=[\hat{\tau},t_0]$ and $I_2=[t_0-\theta_2,t_1]$, where $\hat{\tau}=t_0-\max\{\tau_2,\sigma_2\}$. Suppose that $P\subset\mathbb{R}^k, Q\subset\mathbb{R}^m, V\subset\mathbb{R}^r$ are convex and open sets with $k+m=n, x=(p,q)^T\in O=(P,Q)^T$.

Furthermore, let the n-dimensional function $f(t,x,p,q,u,v)$ be continuous on $I\times O\times P\times Q\times V^2$, and continuously differentiable with respect to x, p, q, u and v; there exists a number $L>0$ such that, for all $(t,x,p,q.u,v)\in I\times P\times Q\times V^2$,

$$|f(t,x,p,q,u,v)|+|f_x(\cdot)|+|f_p(\cdot)|+|f_q(\cdot)|+|f_u(\cdot)|+|f_v(\cdot)|\leq L.$$

Denote by $C^1_\varphi(I_1,\mathbb{R}^k)$ the space of continuous differentiable functions $\varphi: I_1\rightarrow\mathbb{R}^k$. Let us introduce the sets

$$\Phi=\{\varphi\in C^1_\varphi(I_1,\mathbb{R}^k):\varphi(t)\in K,\ t\in I_1\},\ G=\{g\in C^1_g(I_1,\mathbb{R}^m):g(t)\in M,\ t\in I_1\},$$

$$\Omega=\{C^1_u(I_2,\mathbb{R}^r):u(t)\in U,\ t\in I_2\},$$

where $K\subset P$, $M\subset Q$ and $U\subset V$ are convex and compact sets. To any element

$$w=(\tau,\sigma,\theta,p_0,\varphi,g,u)\in W=(\tau_1,\tau_2)\times(\sigma_1,\sigma_2)\times(\theta_1,\theta_2)\times P_0$$

$$\times\Phi\times G\times\Omega,$$

where $P_0\subset P$ is a convex and compact set, we assign the nonlinear control differential equation with delays in the phase coordinates and controls

$$\dot{x}(t)=f(t,x(t),p(t-\tau),q(t-\sigma),u(t),u(t-\theta)),t\in I \tag{2.1}$$

with the mixed initial condition

$$\begin{cases} x(t) = (p(t), q(t))^T = (\varphi(t), g(t))^T, t \in [\hat{\tau}, t_0), \\ x(t_0) = (p_0, g(t_0))^T. \end{cases} \tag{2.2}$$

Definition 2.1. Let $w = (\tau, \sigma, \theta, p_0, \varphi, g, u) \in W$. A function $x(t) = x(t;w) \in O, t \in [\hat{\tau}, t_1]$ is called a solution to equation (2.1) with initial condition (2.2) or a solution corresponding to the element w if it satisfies condition (2.1) and is absolutely continuous on the interval I and satisfies equation (2.1) almost everywhere on I.

It can be proved that for every element $w \in W$ there exits unique solution $x(t;w)$ defined on interval I and it is continuous with respect to w; see [22].

Let the scalar-valued functions $z^i(\tau, \sigma, \theta, p, x)$, $i = \overline{0,l}$, be continuously differentiable on $(\tau_1, \tau_2) \times (\sigma_1, \sigma_2) \times (\theta_1, \theta_2) \times P \times O$.

Definition 2.2. An element $w = (\tau, \sigma, \theta, p_0, \varphi, g, u) \in W$ is said to be admissible if the corresponding solution $x(t) = x(t;w)$ satisfies the boundary conditions

$$z^i(\tau, \sigma, \theta, p_0, x(t_1)) = 0, i = \overline{1,l}. \tag{2.3}$$

Denote by W_0 the set of admissible elements.

Definition 2.3. An element $w_0 = (\tau_0, \sigma_0, \theta_0, p_{00}, \varphi_0, g_0, u_0) \in W_0$ is said to be optimal if for an arbitrary element $w \in W_0$ the inequality

$$z^0(\tau_0, \sigma_0, \theta_0, p_{00}, x_0(t_1)) \leq z^0(\tau, \sigma, \theta, p_0, x(t_1)) \tag{2.4}$$

holds, where $x_0(t) = x(t; w_0)$, $x(t) = x(t;w)$.

(2.1)-(2.4) is called the optimization problem of the delays with the mixed initial condition.

Theorem 2.1. *Let w_0 be an optimal element and $x_0(t) = (p_0(t), q_0(t))^T$ be the corresponding solution. There exist a vector $\pi = (\pi_0, ..., \pi_l) \neq 0$, with $\pi_0 \leq 0$, and a solution $\psi(t) = (\psi_1(t), ..., \psi_n(t))$ of the equation*

$$\dot{\psi}(t) = -\psi(t) f_x[t] - \psi(t+\tau_0)\Big(f_p[t+\tau_0], \Theta_{n\times m}\Big) - \psi(t+\sigma_0)\Big(\Theta_{n\times k}, f_q[t+\sigma_0]\Big) \tag{2.5}$$

with the initial condition

$$\psi(t_1) = \pi Z_{0x}, \ \ \psi(t) = 0, t > t_1 \tag{2.6}$$

where $\Theta_{n\times m}$ is the $n \times m$ zero matrix and

$$Z = (z^0, ..., z^l)^T, Z_{0x} = \frac{\partial Z(\tau_0, \sigma_0, \theta_0, p_{00}, x_0(t_1))}{\partial x},$$

$$f_p[t] = f_p(t, x_0(t), p_0(t-\tau_0), q_0(t-\sigma_0), u_0(t), u_0(t-\theta_0)),$$

such that the following conditions hold:

1) the condition for the delay τ_0 $\pi Z_{0\tau} = \psi(t_0+\tau_0) f_1 + \int_{t_0}^{t_1} \psi(t) f_p[t] \dot{p}_0(t-\tau_0) dt$, where

$$f_1 = f(t_0+\tau_0, x_0(t_0+\tau_0), p_{00}, q_0(t_0+\tau_0-\sigma_0), u_0(t_0+\tau_0), u_0(t_0+\tau_0-\theta_0))$$
$$-f(t_0+\tau_0, x_0(t_0+\tau_0), \varphi_0(t_0), q_0(t_0+\tau_0-\sigma_0), u_0(t_0+\tau_0), u_0(t_0+\tau_0-\theta_0));$$

2) the condition for the delay σ_0 $\pi Z_{0\sigma} = \int_{t_0}^{t_1} \psi(t) f_q[t] \dot{q}_0(t-\sigma_0) dt$;

3) the condition for the delay θ_0 $\pi Z_{0\theta} = \int_{t_0}^{t_1} \psi(t) f_v[t] \dot{u}_0(t-\theta_0) dt$;

4) the condition for the vector p_{00},

$$\Big(\pi Z_{0p}+(\psi_1(t_0),...,\psi_k(t_0))\Big)p_{00}=\max_{p_0\in P_0}\Big(\pi Z_{0p}+(\psi_1(t_0),...,\psi_k(t_0))\Big)p_0;$$

5) the condition for the initial function $\varphi_0(t)$,

$$\int_{t_0-\tau_0}^{t_0}\psi(t+\tau_0)f_p[t+\tau_0]\varphi_0(t)dt=\max_{\varphi\in\Phi}\int_{t_0-\tau_0}^{t_0}\psi(t+\tau_0)f_p[t+\tau_0]\varphi(t)dt;$$

6) the condition for the initial function $g_0(t)$,

$$(\psi_{k+1}(t_0),...,\psi_n(t_0))g_0(t_0)+\int_{t_0-\sigma_0}^{t_0}\psi(t+\sigma_0)f_q[t+\sigma_0]g_0(t)dt$$

$$=\max_{g\in G}\Big[(\psi_{k+1}(t_0),...,\psi_n(t_0))g(t_0)+\int_{t_0-\sigma_0}^{t_0}\psi(t+\sigma_0)f_q[t+\sigma_0]g(t)dt\Big];$$

7) the condition for the control function $u_0(t)$,

$$\int_{t_0}^{t_1}\psi(t)\Big[f_u[t]u_0(t)+f_v[t]u_0(t-\theta_0)\Big]dt=\max_{u\in\Omega}\int_{t_0}^{t_1}\psi(t)\Big[f_u[t]u(t)$$

$$+f_v[t]u(t-\theta_0)\Big]dt.$$

Theorem 2.1 on the bases of the variation formula of solution [23] will be proved by the scheme given in [21, 22].

Now we consider the optimization problem with the integral functional

$$\dot{x}(t)=f(t,x(t),p(t-\tau),q(t-\sigma),u(t),u(t-\theta)),t\in I,$$

$$x(t)=(\varphi(t),g(t))^T,t\in[\hat{\tau},t_0),x(t_0)=(p_0,g(t_0))^T,\ x(t_1)=x_1,$$

$$\int_{t_0}^{t_1}f^0(t,x(t),p(t-\tau),q(t-\sigma),u(t),u(t-\theta))dt\to\min.$$

Here $f^0(t,x,p,q,u,v)$ is a scalar-valued function continuous on $I\times O\times P\times Q\times V^2$, and continuously differentiable with respect to $x,\ p,\ q,\ u$ and v; $\varphi(t)\in\Phi$ and $g(t)\in G$ are fixed initial functions; $p_0\in P$, and $x_1\in O$ are fixed points.

Evidently, the above considered problem is equivalent to the following problem

$$\dot{x}^0(t)=f^0(t,x(t),p(t-\tau),q(t-\sigma),u(t),u(t-\theta)),$$

$$\dot{x}(t)=f(t,x(t),p(t-\tau),q(t-\sigma),u(t),u(t-\theta)),$$

$$x^0(t_0)=0,x(t)=(\varphi(t),g(t))^T,t\in[\hat{\tau},t_0),x(t_0)=(p_0,g(t_0))^T,\ x(t_1)=x_1,$$

$$x^0(t_1)\to min,$$

which is a particular case of the problem (2.1)-(2.4). Therefore, Theorem 2.2 formulated below is a simple corollary of Theorem 2.1. Let us introduce the function $\hat{f}=(f_0,f)^T$.

Theorem 2.2. *Let* $(\tau_0,\sigma_0,\theta_0,u_0)$ *be an optimal element and* $x_0(t)=(p_0(t),q_0(t))^T$ *be the corresponding solution. There exists a nontrivial solution* $\hat{\psi}(t)=(\psi_0(t),\psi_1(t),\dots,\psi_n(t))=(\psi_0(t),\psi(t)),t\in I$ *with* $\psi_0(t)\equiv const\le 0$, *of the equation*

$$\dot{\psi}(t)=-\hat{\psi}(t)\hat{f}_x[t]-\hat{\psi}(t+\tau_0)\Big(\hat{f}_p[t+\tau_0],\Theta_{(n+1)\times m}\Big)-\hat{\psi}(t+\sigma_0)\Big(\Theta_{(n+1)\times k},\hat{f}_q[t+\sigma_0]\Big)$$

$$\hat{\psi}(t)=0,t>t_1$$

such that the following conditions hold:

8) the condition for the delay τ_0 $\hat{\psi}(t_0+\tau_0)\hat{f}_1+\int_{t_0}^{t_1}\hat{\psi}(t)\hat{f}_p[t]\dot{p}_0(t-\tau_0)dt=0$; where

$$\hat{f}_1=\hat{f}(t_0+\tau_0,x_0(t_0+\tau_0),p_{00},q_0(t_0+\tau_0-\sigma_0),u_0(t_0+\tau_0),u_0(t_0+\tau_0-\theta_0))$$
$$-\hat{f}(t_0+\tau_0,x_0(t_0+\tau_0),\varphi_0(t_0),q_0(t_0+\tau_0-\sigma_0),u_0(t_0+\tau_0),u_0(t_0+\tau_0-\theta_0)),$$
$$\hat{f}_p[t]=\hat{f}_p(t,x_0(t),p_0(t-\tau_0),q_0(t-\sigma_0),u_0(t),u_0(t-\theta_0));$$

9) the condition for the delay σ_0 $\int_{t_0}^{t_1}\hat{\psi}(t)\hat{f}_q[t]\dot{q}_0(t-\sigma_0)dt=0$;
10) the condition for the delay θ_0 $\int_{t_0}^{t_1}\hat{\psi}(t)\hat{f}_v[t]\dot{u}_0(t-\theta_0)dt=0$;
11) the condition for the control function $u_0(t)$,

$$\int_{t_0}^{t_1}\hat{\psi}(t)\Big[\hat{f}(t)u_0(t)+\hat{f}_v(t)u_0(t-\theta_0)\Big]dt=\max_{u\in\Omega}\int_{t_0}^{t_1}\hat{\psi}(t)\Big[\hat{f}_u[t]u(t)$$
$$+\hat{f}_v(t)v(t-\theta_0)\Big]dt.$$

3. Proof of Theorem 2.1

On the convex set $\Pi=\mathbb{R}_+\times W$, where $\mathbb{R}_+=[0,\infty)$, let us define the mapping

$$Q:\Pi\to\mathbb{R}^{1+l} \tag{3.1}$$

by the formula

$$Q(\varsigma)=(Q^0(\varsigma),...,Q^l(\varsigma))^T=Z(\tau,\sigma,\theta,p_0,x(t_1;w))+(\xi,0...,0)^T,\varsigma=(\xi,w)\in\Pi.$$

It is clear that

$$Q^0(\varsigma_0)\le Q^0(\varsigma),Q^i(\varsigma)=0,i=\overline{1,l},\forall\varsigma\in\mathbb{R}_+\times W_0\subset\Pi,$$

where $\varsigma_0=(0,w_0)$.

Thus, the point $\varsigma_0=(0,w_0)\in\Pi$ is a critical (see [21, 22]) since $Q(\varsigma_0)\in\partial Q(\Pi)$. Moreover, mapping (3.1) is continuous (see [22]).

There exist numbers $\varepsilon_0>0$ and $\alpha>0$ such that, for an arbitrary $\varepsilon\in(0,\varepsilon_0)$ and

$$\delta\varsigma=(\delta\xi,\delta w)\in\Psi_{\varsigma_0}:=[0,\alpha)\times\Psi_{w_0}\subset\Pi-\varsigma_0=\{\varsigma-\varsigma_0:\ \forall\varsigma\in\Pi\},$$

where

$$\delta w=(\delta\tau,\delta\sigma,\delta\theta,\delta p_0,\delta\varphi,\delta g,\delta u),$$
$$\Psi_{w_0}=(-\alpha,\alpha)\times(-\alpha,\alpha)\times(-\alpha,\alpha)\times[P_0-p_0]$$
$$\times[\Phi-\varphi_0]\times[G-g_0]\times[\Omega-u_0],$$
$$\varsigma_0+\varepsilon\delta\varsigma\in\Pi.$$

On the basis of the variation formula of solutions [23], we have

$$\Delta x(t_1;\varepsilon\delta w):=x(t_1;w_0+\varepsilon\delta w)-x_0(t_1)=\varepsilon\delta x(t_1;\delta w)+o(\varepsilon\delta w),$$
$$\forall(\varepsilon,\delta w)\in(0,\varepsilon_0)\times\Psi_{w_0},$$

where

$$\delta x(t_1;\delta\mu)=Y(t_0;t_1)\Big((\delta p_0,\Theta_{m\times1})^T+(\Theta_{k\times1},\delta g(t_0))^T\Big)$$
$$-\Big\{Y(t_0+\tau_0;t_1)f_1+\int_{t_0}^{t_1}Y(s;t_1)f_p[s]\dot{p}_0(s-\tau_0)ds\Big\}\delta\tau$$
$$-\Big\{\int_{t_0}^{t_1}Y(s;t_1)f_q[s]\dot{q}_0(s-\sigma_0)ds\Big\}\delta\sigma+\int_{t_0-\tau_0}^{t_0}Y(s+\tau_0;t_1)f_p[(s+\tau_0]\delta\varphi(s)ds$$

$$+\int_{t_{00}-\sigma_0}^{t_{00}} Y(s+\sigma_0;t)f_q[s+\sigma_0]\delta g(s)ds - \Big\{\int_{t_{00}}^{t} Y(s;t)f_v[s]\dot u_0(s-\theta_0)ds\Big\}\delta\theta$$
$$+\int_{t_{00}}^{t_1} Y(s;t_1)\Big[f_u[s]\delta u(s)+f_v[s]\delta u(s-\theta_0)\Big]ds; \qquad (3.2)$$

and

$$\lim_{\varepsilon\to 0}\frac{o(\varepsilon\delta w)}{\varepsilon}=0 \text{ uniformly for } \delta w\in\Psi_{w_0};$$

$Y(t;t_1)$ is the $n\times n-$ matrix function satisfying the linear differential equation with advanced argument

$$\frac{d}{dt}Y(t;t_1)=-Y(t;t_1)f_x[t]-Y(t+\tau_0;t_1)\Big(f_p[t+\tau_0],\Theta_{n\times m}\Big)-Y(t+\sigma_0;t_1)\Big(\Theta_{n\times k},f_q[t+\sigma_0]\Big)$$

and the condition

$$Y(t;t_1)=\begin{cases}E \; for\, t=t_1,\\ \Theta_{n\times n}\; for\, t>t_1,\end{cases}$$

where E is the identity matrix.

Now we calculate a differential of the mapping (3.1) at the point ς_0. Note that

$$Q(\varsigma_0+\varepsilon\delta\varsigma)-Q(\varsigma_0)=Z(\tau_0+\varepsilon\delta\tau,\sigma_0+\varepsilon\delta\sigma,\theta_0+\varepsilon\delta\theta,p_{00}+\varepsilon\delta p_0,x(t_1;w_0+\varepsilon\delta w))$$
$$-Z(\tau_0,\sigma_0,\theta_0,p_{00},x_0(t_1))+\varepsilon(\delta\xi,0...,0)^T,\ \varepsilon\in(0,\varepsilon_0),\delta w\in\Psi_{w_0}.$$

We introduce the notation

$$Z[\varepsilon;s]=Z(\tau_0+\varepsilon s\delta\tau,\sigma_0+\varepsilon s\delta\sigma,\theta_0+\varepsilon s\delta\theta,p_{00}+\varepsilon s\delta p_0,x_0(t_1)+s\Delta x(t_1;\varepsilon\delta w))$$

Let us transform the difference

$$Z(\tau_0+\varepsilon\delta\tau,\sigma_0+\varepsilon\delta\sigma,\theta_0+\varepsilon\delta\theta,p_{00}+\varepsilon\delta p_0,x(t_1;w_0+\varepsilon\delta w))$$
$$-Z(\tau_0,\sigma_0,\theta_0,p_{00},x_0(t_1))=\int_0^1\frac{d}{ds}Z[\varepsilon;s]ds$$
$$=\int_0^1\Big[\varepsilon\Big(Z_\tau[\varepsilon;s]\delta\tau+Z_\sigma[\varepsilon;s]\delta\sigma+Z_\theta[\varepsilon;s]\delta\theta+Z_p[\varepsilon;s]\delta p_0\Big)+Z_x[\varepsilon;s]\Delta x(t_1;\varepsilon\delta w)\Big]ds$$
$$=\int_0^1\Big[\varepsilon\Big(Z_\tau[\varepsilon;s]\delta\tau+Z_\sigma[\varepsilon;s]\delta\sigma+Z_\theta[\varepsilon;s]\delta\theta+Z_p[\varepsilon;s]\delta p_0+Z_x[\varepsilon;s]\delta x(t_1;\varepsilon\delta w)\Big)$$
$$+Z_x[\varepsilon;s]o(\varepsilon\delta w)\Big]ds=\varepsilon\Big[Z_{0\tau}\delta\tau+Z_{0\sigma}\delta\sigma+Z_{0\theta}\delta\theta+Z_{0p}\delta p_0+Z_{0x}\delta x(t_1;\delta w)\Big]+\gamma(\varepsilon\delta w),$$

where

$$\gamma(\varepsilon\delta w)=\varepsilon\int_0^1\Big\{[Z_\tau[\varepsilon;s]-Z_{0\tau}]\delta\tau+[Z_\sigma[\varepsilon;s]-Z_{0\sigma}]\delta\sigma+[Z_\theta[\varepsilon;s]-Z_{0\theta}]\delta\theta$$
$$+[Z_p[\varepsilon;s]-Z_{0p}]\delta p_0+[Z_x[\varepsilon;s]-Z_{0x}]\delta x(t_1;\delta w)+Z_x[\varepsilon;s]\frac{o(\varepsilon\delta w)}{\varepsilon}\Big\}ds.$$

It is easy to see that

$$\lim_{\varepsilon\to 0}[Z_\tau[\varepsilon;s]-Z_{0\tau}]=0,\lim_{\varepsilon\to 0}[Z_\sigma[\varepsilon;s]-Z_{0\sigma}]=0,\lim_{\varepsilon\to 0}[Z_\theta[\varepsilon;s]-Z_{0\theta}]=0,$$

$$\lim_{\varepsilon\to 0}[Z_p[\varepsilon;s]-Z_{0p}]=0, \lim_{\varepsilon\to 0}[Z_x[\varepsilon;s]-Z_{0x}]=0.$$

Therefore, $\gamma(\varepsilon\delta\varsigma)=o(\varepsilon\delta w)$ and then $Q(\varsigma_0+\varepsilon\delta\varsigma)-Q(\varsigma_0)=\varepsilon dQ_{\varsigma_0}(\delta\varsigma)+o(\varepsilon\delta\varsigma)$, where $o(\varepsilon\delta v):=o(\varepsilon\delta w)$ and differential $dQ_{\varsigma_0}(\delta\varsigma)$ of the mapping (3.1) has the form

$$dQ_{\varsigma_0}(\delta\varsigma)=Z_{0\tau}\delta\tau+Z_{0\sigma}\delta\sigma+Z_{0\theta}\delta\theta+Z_{0p}\delta p_0+Z_{0x}\delta x(t_1;\delta w)+(\delta\xi,0,...,0)^T.$$

From relation (3.2), we have

$$\begin{aligned}dQ_{v_0}(\delta v)=&\Big[Z_{0\tau}-Z_{0x}Y(t_0+\tau_0;t_1)f_1-\int_{t_0}^{t_1}Z_{0x}Y(t;t_1)f_p[t]\dot p_0(t-\tau_0)dt\Big]\delta\tau\\
&\int_{t_0-\tau_0}^{t_0}Z_{0x}Y(t+\tau_0;t_1)f_p[t+\tau_0]\delta\varphi(t)dt+\Big[Z_{0\sigma}-\int_{t_0}^{t_1}Z_{0x}Y(t;t_1)f_q[t]\dot q_0(t-\sigma_0)dt\Big]\delta\sigma\\
&+Z_{0p}\delta p_0+Z_{0x}Y(t_0;t_1)(\delta p_0,\Theta_{m\times 1})^T+\Big[Z_{0x}Y(t_0;t_1)(\Theta_{k\times 1},\delta g(t_0))^T\\
&+\int_{t_0-\sigma_0}^{t_0}Z_{0x}Y(t+\sigma_0;t_1)f_q[t+\sigma_0]\delta g(t)dt+\Big[Z_{0\theta}-\int_{t_0}^{t_1}Z_{0x}Y(t;t_1)f_v[t]\dot u_0(t-\theta_0)dt\Big]\delta\theta\\
&+\int_{t_0}^{t_1}Z_{0x}Y(t;t_1)\Big\{f_u[t]\delta u(t)+f_v[t]\delta u(t-\theta_0)\Big\}dt+(\delta\xi,0,...,0)^T.\end{aligned}\tag{3.3}$$

From the necessary condition of criticality [21, 22] it follows that there exists a vector $\pi=(\pi_0,...,\pi_l)\neq 0$ such that

$$\begin{aligned}\pi dQ_{\varsigma_0}(\delta\varsigma)\leq 0,\forall\ \delta v\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times[P_0-p_{00}]\times[\Phi-\varphi_0]\\ \times[G-g_0]\times[\Omega-u_0]\end{aligned}\tag{3.4}$$

Introduce the function

$$\psi(t)=\pi Z_{0x}Y(t;t_1)\tag{3.5}$$

as is easily seen, it satisfies equation (2.5) and condition (2.6). Taking into account (2.6) and (3.5) from inequality (3.4), we obtain

$$\begin{aligned}&\Big[\pi Z_{0\tau}-\psi(t_0+\tau_0)f_1-\int_{t_0}^{t_1}\psi(t)f_p[t]\dot p_0(t-\tau_0)dt\Big]\delta\tau\\
&\int_{t_0-\tau_0}^{t_0}\psi(t+\tau_0)f_p[t+\tau_0]\delta\varphi(t)dt+\Big[\pi Z_{0\sigma}-\int_{t_0}^{t_1}\psi(t)f_q[t]\dot q_0(t-\sigma_0)dt\Big]\delta\sigma\\
&+\Big[\pi Z_{0p}+(\psi_1(t_0),...,\psi_k(t_0))\Big]\delta p_0+\Big[(\psi_{k+1}(t_0),...,\psi_n(t_0))\delta g(t_0)\\
&+\int_{t_0-\sigma_0}^{t_0}\psi(t+\sigma_0)f_q[t+\sigma_0]\delta g(t)dt\Big]+\Big[\pi Z_{0\theta}-\int_{t_0}^{t_1}\psi(t)f_v[t]\dot u_0(t-\theta_0)dt\Big]\delta\theta\\
&+\int_{t_0}^{t_1}\psi(t)\Big\{f_u[t]\delta u(t)+f_v[t]\delta u(t-\theta_0)\Big\}dt+(\delta\xi,0,...,0)^T\leq 0,\end{aligned}\tag{3.6}$$

$$\forall\delta\xi\in\mathbb{R}_+,\ \forall\delta\tau\in\mathbb{R},\ \forall\delta\sigma\in\mathbb{R},\ \forall\delta\theta\in\mathbb{R},\ \delta p_0\in P_0-p_{00},\ \delta\varphi\in\Phi-\varphi_0,$$

$$\delta g \in G - g_0,\ \delta u \in \Omega - u_0.$$

Let $\delta\tau = \delta\sigma = \delta\theta = 0, \delta p_0 = 0, \delta\varphi = 0, \delta g = 0$, and $\delta u = 0$ in (3.6). Then $\pi_0 \delta\xi \le 0$, $\delta\xi \in \mathbb{R}_+$. This implies $\pi_0 \le 0$. Letting $\delta\xi = \delta\sigma = \delta\theta = 0, \delta p_0 = 0, \delta\varphi = 0, \delta g = 0$, and $\delta u = 0$ in (3.6), we have

$$\Big[\pi Z_{0\tau} - \psi(t_0+\tau_0)f_1 - \int_{t_0}^{t_1} \psi(t) f_p[t]\dot{p}_0(t-\tau_0)dt\Big]\delta\tau \le 0.$$

Taking into account that $\delta\tau \in \mathbb{R}$, we obtain condition *1)*. Let $\delta\xi = \delta\tau = \delta\theta = 0, \delta p_0 = 0, \delta\varphi = 0, \delta g = 0$, and $\delta u = 0$ in (3.6). Then we have

$$\Big[\pi Z_{0\sigma} - \int_{t_0}^{t_1} \psi(t) f_q[t]\dot{q}_0(t-\sigma_0)dt\Big]\delta\sigma \le 0.$$

Taking into account that $\delta\sigma \in \mathbb{R}$, we obtain the condition *2)*. Letting $\delta\xi = \delta\tau = \delta\sigma = 0, \delta p_0 = 0, \delta\varphi = 0, \delta g = 0$, and $\delta u = 0$ in (3.6), we have

$$\Big[\pi Z_{0\theta} - \int_{t_0}^{t_1} \psi(t) f_v[t]\dot{u}_0(t-\theta_0)dt\Big]\delta\theta \le 0.$$

Taking into account that $\delta\theta \in \mathbb{R}$, we obtain condition *3)*. Let $\delta\xi = \delta\tau = \delta\sigma = \delta\theta = 0, \delta\varphi = 0, \delta g = 0$, and $\delta u = 0$ in (3.6). It follows that

$$\Big[\pi Z_{0p} + (\psi_1(t_0), ..., \psi_k(t_0))\Big]\delta p_0 \le 0.$$

Taking into account that $\delta p_0 \in P_0 - p_{00} = \{p - p_{00} :\ p \in P_0\}$, we obtain condition *4)*. Let $\delta\xi = \delta\tau = \delta\sigma = \delta\theta = 0, \delta p_0 = 0, \delta g = 0$, and $\delta u = 0$ in (3.6). It follows that

$$\int_{t_0-\tau_0}^{t_0} \psi(t+\tau_0) f_p[t+\tau_0]\delta\varphi(t)dt \le 0.$$

Taking into account that $\delta\varphi \in \Phi - \varphi_0$, we obtain condition *5)*. Let $\delta\xi = \delta\tau = \delta\sigma = \delta\theta = 0, \delta p_0 = 0, \delta\varphi = 0$, and $\delta u = 0$ in (3.6). It follows that

$$\Big[(\psi_{k+1}(t_0), ..., \psi_n(t_0))\delta g(t_0) + \int_{t_0-\sigma_0}^{t_0} \psi(t+\sigma_0) f_q[t+\sigma_0]\delta g(t)dt\Big] \le 0.$$

Taking into account that $\delta g \in G - g_0$, we obtain condition *6)*. Let $\delta\xi = \delta\tau = \delta\sigma = \delta\theta = 0, \delta p_0 = 0, \delta\varphi = 0$, and $\delta\varphi = 0$ in (3.6). It follows that

$$\int_{t_0}^{t_1} \psi(t)\Big\{f_u[t]\delta u(t) + f_v[t]\delta u(t-\theta_0)\Big\}dt \le 0.$$

Finally, taking into account that $\delta u \in \Omega - u_0$, we obtain condition *7)*.

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