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DISTRIBUTIONAL SOLUTIONS FOR A DIRICHLET PROBLEM LOOSELY RELATED TO MEAN FIELD GAMES SYSTEMS

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Dedicated to Roland Glowinski, a cornerstone of Paris VI, where I met him and a lot of friends

Abstract. In this paper, the existence of distributional solutions of an elliptic system (loosely related to mean field games systems) is proved. The solutions belong to the Sobolev space $W_0^{1,q}(\Omega)$; in some cases $q=2$, in some cases $q<2$ and even $q=1$.

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1. INTRODUCTION

In this paper is studied the existence of distributional solutions (u,ψ) of the boundary value problem

$$\begin{cases} 0 \le u, \text{ in } \Omega, \\ u = 0 = \psi, \text{ on } \partial\Omega, \\ -\operatorname{div}(M(x)\nabla u) + u = -A \operatorname{div}(uM(x)\nabla\psi) + f(x), \text{ in } \Omega, \\ \psi \in W_0^{1,2}(\Omega) : -\operatorname{div}(M(x)\nabla\psi) + \psi = B\, M(x)\nabla\psi\nabla\psi + u^{\lambda-1}, \text{ in } \Omega. \end{cases} \tag{1}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, with $N>2$, $M(x)$ is a measurable, symmetric matrix such that, for $\alpha, \beta \in \mathbb{R}^+$,

$$\alpha|\xi|^2 \le M(x)\xi\xi, \quad |M(x)| \le \beta, \tag{2}$$

$$0 \le f(x) \in L^1(\Omega), \tag{3}$$

$$1 < \lambda < \frac{N}{N-2}, \tag{4}$$

$$0 < A \le B. \tag{5}$$

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The above boundary value problem (1) is an elliptic system loosely related to mean field games systems, where, roughly speaking, $A,B<0$. We refer to the papers [8], [7], for existence and regularity results for a mean field games type system, where a duality approach is used.

Even if we do not assume $A,B<0$, nevertheless it will still be possible to use a duality approach in the study of (1), in spite of the nonlinearity of our problem. On the other hand, in our framework, we cannot take advantage to the presence of the positive term $M(x)\nabla\psi\nabla\psi$ on the left hand side of the second equation, as in [8], [7], since, in (1), it is on the right hand side. For the basic theory of Mean Field Games see [9].

We point out that we prove that the solution ψ of the second equation belongs to $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$; whereas the solution u is less regular: u does not belong to $W_0^{1,2}(\Omega)$, but only to $W_0^{1,1}(\Omega)$ (Theorem 3.2) or $W_0^{1,\frac{N}{N-1}}(\Omega)$ (Theorem 3.1).

However, in the first case, $u\in W_0^{1,1}(\Omega)$ and not only in $BV(\Omega)$ (as usual in elliptic problems with L^1 estimates on the gradient of the solution).

2. Approximation of system (1) and preliminary results

We begin this section by proving that, thanks to Schauder's theorem, there exist solutions for a sequence of systems which approximate system (1).

Let $k\in\mathbb{R}^+$, $n\in\mathbb{N}$, $s\in\mathbb{R}$ and let $f(x)\geq 0$ be a function in $L^1(\Omega)$; define

$$T_k(s)=\max(-k,\min(s,k)), \qquad f_n(x)=\frac{f(x)}{1+\frac{1}{n}f(x)}.$$

Note that $0\leq f_n\leq f$. We consider the following system

$$\begin{cases} u_n\in W_0^{1,2}(\Omega):\ -\operatorname{div}(M(x)\nabla u_n)+u_n \\ \qquad\qquad = -A\operatorname{div}\left(\dfrac{u_n}{1+\frac{1}{n}|u_n|}\dfrac{M(x)\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}\right)+f_n(x) \\ \psi_n\in W_0^{1,2}(\Omega):\ -\operatorname{div}(M(x)\nabla\psi_n)+\psi_n=B\dfrac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}+T_{n^4}[|u_n|^{\lambda-1}] \end{cases}$$

The existence of (u_n,ψ_n) can be proved (thanks to the Schauder theorem) as in [8], [7]. The positivity $u_n(x)\geq 0$ follows by the results of [1] (see also [2]), so that we rewrite the above system as

$$\begin{cases} u_n\in W_0^{1,2}(\Omega):\ -\operatorname{div}(M(x)\nabla u_n)+u_n \\ \qquad\qquad = -A\operatorname{div}\left(\dfrac{u_n}{1+\frac{1}{n}u_n}\dfrac{M(x)\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}\right)+f_n(x) \\ \psi_n\in W_0^{1,2}(\Omega):\ -\operatorname{div}(M(x)\nabla\psi_n)+\psi_n=B\dfrac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}+T_{n^4}[u_n^{\lambda-1}]. \end{cases} \tag{6}$$

Since in the first equation the right hand side terms are controlled by n^2 and n, we can say that $\|u_n\|_{L^\infty(\Omega)}\leq\bar{C}n^2$, $\|\psi_n\|_{L^\infty(\Omega)}\leq\bar{C}n^3$ (see [11]). Observe that $u_n^{\lambda-1}\leq(\bar{C}n^2)^{\lambda-1}$ and (consequence of (4)) that $2\lambda-2\leq 2\frac{N}{N-2}-2=\frac{4}{N-2}\leq 4$. Thus for n large enough, $T_{n^4}[(u_n)^{\lambda-1}]=$

$(u_n)^{\lambda-1}$ and the solutions of the above system are solutions of the following Dirichlet problem

$$\begin{cases} 0 \le u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \ \psi_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega): \\ -\operatorname{div}(M(x)\nabla u_n) + u_n = -A \operatorname{div}\left(\dfrac{u_n}{1+\frac{1}{n}u_n}\dfrac{M(x)\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}\right) + f_n(x). \\ -\operatorname{div}(M(x)\nabla\psi_n) + \psi_n = B\dfrac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2} + (u_n)^{\lambda-1}. \end{cases} \tag{7}$$

Moreover, a consequence of a result proved in [1] (and [2]) is that

$$\int_\Omega u_n \le \int_\Omega f_n \le \int_\Omega f \tag{8}$$

and a consequence of a result proved in [6] is that

$$\|\psi_n\|_{W_0^{1,2}(\Omega)} + \|\psi_n\|_{L^\infty(\Omega)} \le C_0 + C_0\|(u_n)^{\lambda-1}\|_{L^p(\Omega)}, \quad p > \frac{N}{2}. \tag{9}$$

REMARK 2.1. *The use of the truncation T_{n^4} (instead of T_n) in (6) simplifies the calculations (see [10]).*

3. EXISTENCE RESULTS

Even if our problem is nonlinear, in the proof of our theorems, we use a duality approach in the study of the existence of solutions of (1).

THEOREM 3.1. *Under the assumptions (2), (4),*

$$B > A \tag{10}$$

and

$$\int_\Omega f \log(1+f) \in L^1(\Omega), \tag{11}$$

there exist a weak solution (u,ψ) of the system (1), that is a solution of

$$\begin{cases} 0 \le u \in W_0^{1,\frac{N}{N-1}}(\Omega), \ \psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega): \\ \forall\, w \in \operatorname{Lip}(\Omega), \ \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega M(x)\nabla u \nabla w + \int_\Omega u w = A\int_\Omega u[M(x)\nabla\psi\nabla w] + \int_\Omega f(x)\, w(x), \\ \displaystyle\int_\Omega M(x)\nabla\psi\nabla\varphi + \int_\Omega \psi\varphi = B\int_\Omega [M(x)\nabla\psi\nabla\psi]\varphi + \int_\Omega u^{\lambda-1}\varphi. \end{cases} \tag{12}$$

PROOF. Note that the assumption (11) is slightly stronger than (3).

We use ψ_n as test function in the weak formulation of the first equation of (7) and use u_n as test function in the weak formulation of the second equation and we deduce that

$$B\int_\Omega \frac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}u_n + \int_\Omega u_n^\lambda = A\int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}\frac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2} + \int_\Omega f_n\,\psi_n \tag{13}$$

Now the equality (13) and $B > A$ imply

$$(B-A)\int_\Omega \frac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}u_n + \int_\Omega u_n^\lambda \le \|f\|_{L^1(\Omega)}\|\psi_n\|_{L^\infty(\Omega)}$$

and, thanks to (9), if $p > \frac{N}{2}$,

$$(B-A)\int_\Omega \frac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}u_n + \int_\Omega u_n^\lambda \le \|f\|_{L^1(\Omega)}\big[C_0 + C_0\|(u_n)^{\lambda-1}\|_{L^p(\Omega)}\big].$$

Since $\lambda < \frac{N}{N-2}$, it is true that $\frac{N}{2} < \lambda'$; then we take $p \in (\frac{N}{2}, \frac{\lambda}{\lambda-1})$ and we deduce that

$$\text{the sequence } \{u_n\} \text{ is bounded in } L^\lambda(\Omega) \tag{14}$$

and

$$\int_\Omega \frac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}u_n \le \frac{C_f}{(B-A)}\|f\|_{L^1(\Omega)} = C_1. \tag{15}$$

Thus (up to a subsequence) there exists $u \in L^\lambda(\Omega)$ (recall that $\lambda > 1$) such that

$$u_n \text{ converges weakly in } L^\lambda(\Omega) \text{ to } u. \tag{16}$$

Since the sequence $\{u_n\}$ is bounded in $L^\lambda(\Omega)$, then

$$\text{the sequence } \{(u_n)^{\lambda-1}\} \text{ is bounded in } L^{\frac{\lambda}{\lambda-1}}(\Omega) \subset L^p(\Omega),\ p > \tfrac{N}{2}.$$

Thus (9) becomes

$$\|\psi_n\|_{W_0^{1,2}(\Omega)} + \|\psi_n\|_{L^\infty(\Omega)} \le C_f. \tag{17}$$

At this point we use that in [6] is also proved that (up to a subsequence)

$$\begin{cases} \text{there exists}\, \psi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega) \text{such that} \|\psi_n - \psi\|_{W_0^{1,2}(\Omega)} \to 0, \\ \psi_n(x) \text{ converges a.e. to } \psi(x), \\ \nabla\psi_n(x) \text{ converges a.e. to } \nabla\psi(x). \end{cases} \tag{18}$$

With respect to the sequence $\{u_n\}$, in the first equation of (7) we take $\dfrac{u_n}{1+u_n}$ as test function, we use Young inequality and we have (recall (8))

$$\alpha\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^2} \le \frac{\alpha}{2}\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^2} + \frac{1}{2\alpha}\int_\Omega |\nabla\psi_n|^2 + \int_\Omega f,$$

that is, by (17),

$$\frac{\alpha}{2}\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^2} \le \frac{1}{2\alpha}\int_\Omega |\nabla\psi_n|^2 + \int_\Omega f \le \widetilde{C}_f + \int_\Omega f.$$

In [2], is proved that the above estimate allow us to say that (up to a subsequence)

$$u_n(x) \to u(x) \text{ a.e.} \tag{19}$$

As a consequence of this a.e. convergence we can say that

$$\text{the sequence } \{u_n\} \text{ converges strongly to } u \text{ in } L^\sigma(\Omega),\ 1 \le \sigma < \lambda, \tag{20}$$

$$\text{the sequence } \{(u_n)^{\lambda-1}\} \text{ converges weakly to } u^{\lambda-1} \text{ in } L^{\frac{\lambda}{\lambda-1}}(\Omega). \tag{21}$$

In the next step, we will prove that it is possible to pass to the limit in $\frac{u_n}{1+\frac{1}{n}u_n}\frac{M(x)\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}$.

First of all, we prove the equi-integrability of the sequence $\left\{\frac{u_n}{1+\frac{1}{n}u_n}\frac{M(x)\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}\right\}$. Let E be a measurable subset of Ω and $t\in\mathbb{R}^+$. Then

$$\begin{aligned}\int_E \frac{u_n}{1+\frac{1}{n}u_n}\frac{|M(x)\nabla\psi_n|}{1+\frac{1}{n}|\nabla\psi_n|^2} &\le \beta\int_E u_n\frac{|\nabla\psi_n|}{1+\frac{1}{n}|\nabla\psi_n|^2}\\ &\le t\beta\int_{E\cap\{|\nabla\psi_n|\le t\}}u_n+\frac{\beta}{t\,\alpha}\int_{E\cap\{t<|\nabla\psi_n|\}}u_n\frac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}\\ &\le t\beta\int_E u_n+\frac{C_2\beta}{t\,\alpha},\end{aligned}$$

which implies (thanks to the strong L^1 convergence of u_n)

$$\lim_{|E|\to 0}\int_E \frac{u_n}{1+\frac{1}{n}u_n}\frac{|M(x)\nabla\psi_n|}{1+\frac{1}{n}|\nabla\psi_n|^2}=0,\quad\text{uniformly w.r.t. } n,$$

where we denote $|E|$ the Lebesgue mesure of a measurable subset E. In addition, in (18) and (19) are proved almost everywhere convergences. Hence, by Vitali convergence theorem,

$$\text{the sequence }\left\{\frac{u_n}{1+\frac{1}{n}u_n}\frac{M(x)\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}\right\}\text{ strongly converges in } L^1. \tag{22}$$

The last step concerns an estimate of the sequence $\{u_n\}$ in a Sobolev space.

We use $\log(1+u_n)$ as test function in the first equation of (7) and we have

$$\alpha\int_\Omega\frac{|\nabla u_n|^2}{1+u_n}\le A\int_\Omega\frac{u_n}{(1+\frac{1}{n}u_n)\sqrt{1+u_n}}\frac{|M(x)\nabla\psi_n|}{1+\frac{1}{n}|\nabla\psi_n|^2}\frac{|\nabla u_n|}{\sqrt{1+u_n}}+\int_\Omega f_n(x)\,\log(1+u_n).$$

Then the Young inequality and $st\le e^s-1+t\,\log(1+t),\ s,t\in\mathbb{R}^+$, yield

$$\begin{aligned}\frac{\alpha}{2}\int_\Omega\frac{|\nabla u_n|^2}{1+u_n}&\le C_3\int_\Omega\frac{u_n^2}{1+u_n}\frac{M(x)\nabla\psi_n\nabla\psi_n}{(1+\frac{1}{n}|\nabla\psi_n|^2)^2}+\int_\Omega f(x)\,\log(1+u_n)\\ &\le C_3\int_\Omega u_n\frac{M(x)\nabla\psi_n\nabla\psi_n}{1+\frac{1}{n}|\nabla\psi_n|^2}+\int_\Omega f\log(1+f)+\int_\Omega u_n.\end{aligned}$$

Here we recall (15) and we have

$$\frac{\alpha}{2}\int_\Omega\frac{|\nabla u_n|^2}{1+u_n}\le C_4+\int_\Omega f\log(1+f)+\int_\Omega f.$$

Here we follow [4]. Recall that $1^*=\frac{N}{N-1}$ and that $(1^*)^*=\frac{N}{N-2}=\frac{1^*}{2-1^*}$. Then we have (using the Hölder inequality with exponents $\frac{2}{1^*}$ and $\frac{2}{2-1^*}$)

$$\int_\Omega|\nabla u_n|^{\frac{N}{N-1}}=\int_\Omega\frac{|\nabla u_n|^{1^*}}{(1+u_n)^{\frac{1^*}{2}}}(1+u_n)^{\frac{1^*}{2}}$$

$$\le C_5\left[1+\int_\Omega f\log(1+f)+\int_\Omega f\right]^{\frac{1^*}{2}}\left[\int_\Omega(1+u_n)^{\frac{1^*}{2-1^*}}\right]^{\frac{2-1^*}{2}}$$

which implies ($\mathscr{S}$ is the Sobolev constant)

$$\mathscr{S}\left[\int_\Omega u_n^{\frac{N}{N-2}}\right]^{\frac{N-2}{N}}\le\left[\int_\Omega|\nabla u_n|^{\frac{N}{N-1}}\right]^{\frac{N-1}{N}}\le C_6\left[\int_\Omega(1+u_n)^{\frac{N}{N-2}}\right]^{\frac{N-2}{2N}}.$$

Since $\frac{N-2}{2N} < \frac{N-2}{N}$, we deduce the boundedness of the sequence $\{u_n\}$ in $L^{\frac{N}{N-2}}$ (which is stronger than (14) because of the assumption (4)) and of $\{\nabla u_n\}$ in $L^{\frac{N}{N-1}}$. Thus now we have

$$\nabla u_n \text{ converges weakly in } L^{\frac{N}{N-1}} \text{ to } \nabla u. \tag{23}$$

Conclusion - (18), (21), (22), (23), allow us to pass to the limit in the weak form of the first and of the second equation of (7) and we prove the existence of a solution (u,ψ) of (1). $\square$

In the next theorem, we are in the borderline case $A = B$ and we prove the existence of solutions in the particular case $N = 3$.

We will show that the sequence $\{u_n\}$ is bounded in $W_0^{1,1}(\Omega)$. Nevertheless our proof yields solutions u not only in $BV(\Omega)$ (as usual in elliptic problems with L^1 estimates on the gradient of the solution) but $u \in W_0^{1,1}(\Omega)$.

THEOREM 3.2. *Under the assumptions $B = A > 0$, $N = 3$, (2), (3),*

$$2 \le \lambda < \frac{N}{N-2} = 3, \tag{24}$$

there exist a weak solution (u,ψ) of the system (1), that is

$$\begin{cases} 0 \le u \in W_0^{1,1}(\Omega), \; \psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \\ \forall\, w \in \mathrm{Lip}(\Omega), \; \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega M(x)\nabla u \nabla w + \int_\Omega u w = A \int_\Omega u[M(x)\nabla\psi\nabla w] + \int_\Omega f(x)\, w(x), \\ \displaystyle\int_\Omega M(x)\nabla\psi\nabla\varphi + \int_\Omega \psi\varphi = B\int_\Omega [M(x)\nabla\psi\nabla\psi]\varphi + \int_\Omega u^{\lambda-1}\varphi. \end{cases} \tag{25}$$

PROOF. Even if $A = B$, the first part of the proof of the previous theorem still holds; in particular (14), (18), (19), (20) and (21) remain valid.

Note that (8) implies, for $k \in \mathbb{R}^+$,

$$|\{x : k < u_n(x)\}| \le \frac{1}{k}\int_\Omega f.$$

Now we improve the estimate on $\int_\Omega \frac{|\nabla u_n|^2}{(1+u_n)^2}$ proved in the previous theorem.

If in the first equation of (7) we take $\left[\frac{u_n}{1+u_n} - \frac{k}{1+k}\right]^+$, $k \ge 0$, as test function and we use Young inequality, we have (recall (8))

$$\alpha \int_{\{k<u_n\}} \frac{|\nabla u_n|^2}{(1+u_n)^2} \le \frac{\alpha}{2}\int_{\{k<u_n\}} \frac{|\nabla u_n|^2}{(1+u_n)^2} + C_7\int_{\{k<u_n\}} |\nabla\psi_n|^2 + \int_{\{k<u_n\}} f$$

and

$$\frac{\alpha}{2}\int_{\{k<u_n\}} \frac{|\nabla u_n|^2}{(1+u_n)^2} \le C_7 \int_{\{k<u_n\}} |\nabla\psi_n|^2 + \int_{\{k<u_n\}} f = \omega_n(k), \; k \ge 0. \tag{26}$$

The convergence (18) and $f \in L^1(\Omega)$ imply that

$$\text{for } k \text{ large, } \omega_n(k) \text{ is small, uniformly with respect to } n. \tag{27}$$

Moreover, if $\lambda \ge 2$, the sequence $\{u_n\}$ is bounded in $L^2(\Omega)$ so that (up to a subsequence) there exists $u \in L^2(\Omega)$ such that

$$u_n \text{ converges weakly in } L^2(\Omega) \text{ to } u. \tag{28}$$

Now we observe that (using the Hölder inequality)

$$\int_{\{k<u_n\}} |\nabla u_n| = \int_{\{k<u_n\}} \frac{|\nabla u_n|}{1+u_n}\,(1+u_n)$$

$$\leq \left[C_8 \int_{\{k<u_n\}} |\nabla \psi_n|^2 + C_9 \int_{\{k<u_n\}} f\right]^{\frac{1}{2}} C_{10}.$$

Furthermore a result proved in [1] can be read in the first equation as

$$\int_\Omega |\nabla T_k(u_n)|^2 \leq \frac{k^2}{\alpha^2} \int_\Omega A^2\beta^2 |\nabla \psi_n|^2 + k\,\frac{2}{\alpha}\int_\Omega f \leq k^2 \tilde{C}_f + k\,\frac{2}{\alpha}\int_\Omega f,$$

so that, for every measurable subset $E \subset \Omega$ we have (we use the Hölder inequality)

$$\int_E |\nabla u_n| = \int_E |\nabla T_k(u_n)| + \int_{\{k<u_n\}} |\nabla u_n|$$

$$\leq |E|^{\frac{1}{2}} \left[k^2 C_f + k\,\frac{2}{\alpha}\int_\Omega f\right]^{\frac{1}{2}} + [\omega_n(k)]^{\frac{1}{2}} C_{11},$$

so that

$$\lim_{|E|\to 0} \int_E |\nabla u_n| \leq [\omega_n(k)]^{\frac{1}{2}} C_{12},$$

that is, thanks to (27),

$$\lim_{|E|\to 0} \int_E |\nabla u_n| = 0, \quad \text{uniformly w.r.t. } n, \tag{29}$$

which proves the equi-integrability of the sequence $\{\nabla u_n\}$.

This equi-integrability and the convergence (28) give (result proved in [3], see also [5]) that $u \in W_0^{1,1}(\Omega)$ and

$$\nabla u_n \text{ converges weakly in } L^1 \text{ to } \nabla u. \tag{30}$$

Conclusion - (18), (21), (28), and (30) allow us to pass to the limit in the weak form of the first and of the second equation of (7) and we prove the existence of solutions of (1). □

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