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NEW FIXED POINT RESULTS FOR α-η-Θ_f TYPE FUZZY CONTRACTION

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Abstract. In the present paper, we initiate a new type of fuzzy contraction, namely the concept of fuzzy α-η-Θ_f-contraction and prove some fixed point results for such class of mappings in the setting of complete fuzzy metric spaces. We discuss certain significant consequences of our findings using different instances of admissible functions and auxiliary functions. The results combine, extend, and improve a number of earlier findings in the literature.

Keywords. Admissible functions; Contraction Mappings; Fixed Point Theory; Fuzzy metric spaces; α-η-Θ_f-contraction.

2020 Mathematics Subject Classification. 47H10, 54H25.

1. Introduction

One of the bases of study in functional analysis is fixed point theory, which provides a combination of mathematical techniques, ideas, and useful tools for the solution of numerous issues that arise from other disciplines of mathematics as well as several scientific and engineering specialties. The improvements in fixed point theory over the past 60 years or more have created a vibrant and stimulating area of research in modern mathematics that primarily serves the study of nonlinear phenomena. In fact, it is possible to examine the existence of solutions to a number of fundamental problems by presenting them as an equivalent fixed point problem. Practically, it is possible to convert the operator equation $\mathscr{F}x = 0$ into the fixed point equation $\mathscr{L}x = x$, where $\mathscr{L}$ is a self-mapping with the suitable domain.

Since the initial paper of Zadeh [28] in 1965, interest in fuzzy sets has been steadily increasing. As a result, significant theoretical and applied progress is made in the fields of logic, topology and analysis, with numerous applications in the realms of computer sciences and engineering. Kramosil and Michaelek [14] presented fuzzy metric spaces, George and Veeramani [6] refined the concept of fuzzy metric spaces in [14] and proved that each fuzzy metric yields

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Hausdorff topology. The approach for defining contractive mapping in fuzzy metric spaces is currently a significant theoretical advancement. In fact, the Banach and Edelstein theorems were first extended to fuzzy metric spaces by Grabiec [5] in 1988. Gregori and Sapena [7] offered the concept of fuzzy contractive mappings and showed numerous fixed point results for these mappings. As a notable enhancement, Mihet [17] generalized the idea of fuzzy contractive mappings and presented the notion of fuzzy ψ-contractive mappings. Wardowski [27] recently defined the idea of fuzzy $\mathscr{H}$-contractive mappings and used it to demonstrate several results. In order to unify and enrich various classical types of fuzzy contraction, Moussaoui *et al.* [19] (see also [20]) used the simulation function approach to initiate new types of fuzzy contractive principles and proved some new fixed point theorems. Recently, Saleh *et al.* [8] achieved some new fixed point results by initiating a new class of auxiliary functions $\Theta : (0,1) \to (0,1)$ which was motivated by the studies of Jleli *et al.* [13]. For more details about current achievements in metric and fuzzy metric fixed point theory as well as related techniques see (e.g [1, 2, 3, 7, 9, 12, 13, 15, 16, 18, 21, 22, 24, 27]).

2. Preliminaries

In order to make our study self-contained, we cover some fundamental notions in this section. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ will stand for the set of all positive integer numbers and the set of all real numbers, respectively.

Definition 2.1. [23] A continuous t-norm is a binary operator $* : [0,1] \times [0,1] \longrightarrow [0,1]$ satisfying the following conditions:

($\mathscr{N}_1$): $*$ is commutative and associative,
($\mathscr{N}_2$): $*$ is continuous,
($\mathscr{N}_3$): $\chi * 1 = \chi$ for all $\chi \in [0,1]$,
($\mathscr{N}_4$): $\chi * \phi \leq \sigma * \pi$ whenever $\chi \leq \sigma$ and $\phi \leq \pi$, for all $\chi, \phi, \sigma, \pi \in [0,1]$.

Example 2.2. The following ones are classical examples of continuous t-norm:

1)**:** $\chi *_P \phi = \chi . \phi$,
2)**:** $\chi *_L \phi = \max\{0, \chi + \phi - 1\}$,
3)**:** $\chi *_Z \phi = \min\{\chi, \phi\}$.

Definition 2.3. [6] The 3-tuple $(\Gamma, \nu, *)$ is said to be a fuzzy metric space if ν is an arbitrary set, $*$ is a continuous t-norm and ν is a fuzzy set on $\Gamma^2 \times (0, +\infty)$ satisfying:

($\mathscr{G}1$)**:** $\nu(x,y,t) > 0$,
($\mathscr{G}2$)**:** $\nu(x,y,t) = 1$ if and only if $x = y$,
($\mathscr{G}3$)**:** $\nu(x,y,t) = \nu(y,x,t)$,
($\mathscr{G}4$)**:** $\nu(x,z,t+s) \geq \nu(x,y,t) * \nu(y,z,s)$,
($\mathscr{G}5$)**:** $\nu(x,y,.) : (0,+\infty) \to [0,1]$ is continuous.

for all $x,y,z \in \Gamma$ and $s,t > 0$.

The number $\nu(x,y,t)$ can be regarded as the degree of nearness of x and y with respect to the variable t.

Lemma 2.4. [5] *$\nu(x,y,.)$ is nondecreasing function for all x,y in Γ.*

Example 2.5. [6] Let (Γ, d) be a metric space, $\chi * \phi = \min(\chi, \phi)$ for all $\chi, \phi \in [0,1]$ and

$$\nu(x,y,t) = \frac{\lambda t^l}{\lambda t^l + pd(x,y)}\ , \lambda, p, l \in \mathbb{R}^+$$

Then $(\Gamma, \nu, *)$ is a fuzzy metric space.
Setting $\lambda = p = l = 1$, we obtain

$$\nu(x,y,t) = \frac{t}{t + d(x,y)}$$

we call this fuzzy metric induced by a metric d the standard fuzzy metric.

Example 2.6. [6] Let $\Gamma = \mathbb{R}$ and $\chi * \phi = \chi.\phi$ for all $\chi, \phi \in [0,1]$ and the mapping $\nu : \Gamma \times \Gamma \times (0, +\infty) \to [0,1]$ by

$$\nu(x,y,t) = \left[\exp\left(\frac{|x-y|}{t}\right)\right]^{-1} \text{ for all } x, y \in \Gamma, t > 0.$$

Then $(\Gamma, \nu, *)$ is a fuzzy metric space.

Definition 2.7. [6] Let $(\Gamma, \nu, *)$ be a fuzzy metric space.

(1) A sequence $\{y_j\} \subseteq \Gamma$ is said to be convergent or converges to $y \in \Gamma$ if $\lim_{j \to +\infty} \nu(y_j, y, t) = 1$ for all $t > 0$.
(2) A sequence $\{y_j\} \subseteq \Gamma$ is said to be a Cauchy sequence if for all $\varepsilon \in (0,1)$ and $t > 0$, there exists $j_0 \in \mathbb{N}$ such that $\nu(y_j, y_i, t) > 1 - \varepsilon$ for all $j, i \geq n_0$.
(3) A fuzzy metric space in which each Cauchy sequence is convergent is called a complete fuzzy metric space.

Definition 2.8. [7] Let $(\Gamma, \nu, *)$ be a fuzzy metric space. A mapping $\mathscr{L} : \Gamma \to \Gamma$ is said to be a fuzzy contractive mapping, if there exists $\tau \in (0,1)$ such that

$$\frac{1}{\nu(\mathscr{L}x, \mathscr{L}y, t)} - 1 \leq \tau\left(\frac{1}{\nu(x,y,t)} - 1\right), \tag{2.1}$$

for all $x, y \in \Gamma$ and $t > 0$.

Definition 2.9. [7] A sequence $\{x_n\}$ in a fuzzy metric space $(\Gamma, \nu, *)$ is said to be fuzzy contractive, if there exists $\tau \in (0,1)$ such that

$$\frac{1}{\nu(x_{n+1}, x_{n+2}, t)} - 1 \leq \tau\left(\frac{1}{\nu(x_n, x_{n+1}, t)} - 1\right)$$

for all $n \in \mathbb{N}$ and $t > 0$.

Gregori and Sapena then proved the following fixed point theorem.

Theorem 2.10. [7] *Let $(\Gamma, \nu, *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. If $\mathscr{L} : \Gamma \to \Gamma$ is a fuzzy contractive mapping then $\mathscr{L}$ has a unique fixed point.*

As a result of his study the following theorem was established by Tirado [26].

Theorem 2.11. [26] *Let $(\Gamma,\nu,*_L)$ be a complete fuzzy metric space and $\mathscr{L}:\Gamma\to\Gamma$ be a mapping such that*

$$1-\nu(\mathscr{L}x,\mathscr{L}y,t)\leq\tau\left(1-\nu(x,y,t)\right).$$

for all $x,y\in\Gamma$, $t>0$ and for some $\tau\in(0,1)$. Then $\mathscr{L}$ has a unique fixed point.

In 2020, Saleh *et al*. [8] brought in the concept of fuzzy Θ_f-contractive mappings, which was inspired by the results of Jleli *et al*. [13], by employing an auxiliary function $\Theta_f:(0,1)\to(0,1)$ fulfilling the following conditions

(Ω_1): Θ_f is non-decreasing,
(Ω_2): Θ_f is continuous,
(Ω_3): $\lim_{n\to+\infty}\Theta_f(\omega_n)=1$ if and only if $\lim_{n\to+\infty}\omega_n=1$, where $\{\omega_n\}$ is a sequence in $(0,1)$.

Example 2.12. [8] Let $\Theta_f:(0,1)\to(0,1)$ be a function defined by

$$\Theta_f(\omega)=1-\cos\left(\frac{\pi}{2}\omega\right),\text{ for all }\omega\in(0,1).$$

Example 2.13. [8] Let $\Theta_f:(0,1)\to(0,1)$ be a function defined by

$$\Theta_f(\omega)=e^{1-\frac{1}{\omega}},\text{ for all }\omega\in(0,1).$$

Gopal and Vetro extended the concept of α-admissible mappings to the framework of fuzzy metric space as follows.

Definition 2.14. [4] Let $(\Gamma,\nu,*)$ be a fuzzy metric space and let $\alpha:\Gamma\times\Gamma\times(0,+\infty)\to[0,+\infty)$ be a functions. We say that $\mathscr{L}:\Gamma\to\Gamma$ is α-admissible if, for all $x,y\in\Gamma$

$$\alpha(x,y,t)\geq 1\text{ implies }\alpha(\mathscr{L}x,\mathscr{L}y,t)\geq 1\text{ for all }t>0.$$

As per [10, 25], we employ the notion of admissible mapping in the form below.

Definition 2.15. [10, 25] Let $(\Gamma,\nu,*)$ be a fuzzy metric space and let $\alpha,\eta:\Gamma\times\Gamma\times(0,+\infty)\to[0,+\infty)$ be two functions. We say that $\mathscr{L}:\Gamma\to\Gamma$ is α-admissible with respect to η if, for all $x,y\in\Gamma$

$$\alpha(x,y,t)\geq\eta(x,y,t)\text{ implies }\alpha(\mathscr{L}x,\mathscr{L}y,t)\geq\eta(\mathscr{L}x,\mathscr{L}y,t)\text{ for all }t>0.$$

if we take $\alpha(x,y,t)=1$ for all $x,y\in\Gamma$ and $t>0$, then we say that $\mathscr{L}$ is an η-subadmissible mapping.

Definition 2.16. [11] Let $\alpha,\eta:\Gamma\times\Gamma\times(0,+\infty)\to[0,+\infty)$ be two functions. We say that $\mathscr{L}:\Gamma\to\Gamma$ is α-η-continuous mapping if for a given $x\in\Gamma$ and a sequence $\{x_n\}$ such that $x_n\to x\in\Gamma$ as $n\to+\infty$, $\alpha(x_n,x_{n+1},t)\geq\eta(x_n,x_{n+1},t)$ implies $\mathscr{L}x_n\to\mathscr{L}x$ as $n\to+\infty$.

3. Main results

In this part, we define the concept of fuzzy α-η-Θ_f-contraction and we also prove some fixed point theorems for this class of mappings in the framework of complete fuzzy metric spaces.

Definition 3.1. Let $(\Gamma,\nu,*)$ be a fuzzy metric space, $\mathscr{L}:\Lambda\to\Lambda$ a self mapping and $\alpha,\eta:\Gamma\times\Gamma\times(0,+\infty)\to[0,+\infty)$ be two functions. We say that $\mathscr{L}$ is a fuzzy α-η-Θ_f-contraction if

$$\begin{cases} 1>\nu(\mathscr{L}x,\mathscr{L}y,t) \\ \text{and } \alpha(x,y,t)\geq\eta(x,\mathscr{L}x,t) \end{cases} \text{implies } [\Theta_f(\nu(x,y,t))]^\tau\leq\Theta_f(\nu(\mathscr{L}x,\mathscr{L}y,t)) \tag{3.1}$$

for all $x,y\in\Gamma$ and $t>0$, where $\Theta_f\in\Omega$ and $\tau\in(0,1)$.

Theorem 3.2. *Let $(\Gamma,\nu,*)$ be a complete fuzzy metric space and let $\alpha,\eta:\Gamma\times\Gamma\times(0,+\infty)\to[0,+\infty)$ be two given functions and $\mathscr{L}:\Gamma\longrightarrow\Gamma$ be a fuzzy α-η-Θ_f-contraction such that*

(i) *$\mathscr{L}$ is α-admissible with respect to η;*
(ii) *there exists $x_0\in\Gamma$ such that $\alpha(x_0,\mathscr{L}x_0,t)\geq\eta(x_0,\mathscr{L}x_0,t)$;*
(iii) *$\mathscr{L}$ is α-η-continuous.*

Then, $\mathscr{L}$ has a fixed point.

Proof. Let $x_0\in\Gamma$ such that $\alpha(x_0,\mathscr{L}x_0,t)\geq\eta(x_0,\mathscr{L}x_0,t)$ and define the sequence $\{x_n\}$ by

$$\mathscr{L}^n x_0=\mathscr{L}x_{n-1}$$

for all $n\geq1$. As $\mathscr{L}$ is α-admissible with respect to η, it follows that

$$\alpha(x_0,x_1,t)=\alpha(x_0,\mathscr{L}x_0,t)\geq\eta(x_0,\mathscr{L}x_0,t)=\eta(x_0,x_1,t)$$

Recursively, we derive

$$\alpha(x_n,\mathscr{L}x_n,t)=\alpha(x_n,x_{n+1},t)\geq\eta(x_n,x_{n+1},t),\ \text{for all } n\in\mathbb{N}. \tag{3.2}$$

If for some $m_0\in\mathbb{N}$, $x_{m_0}=x_{m_0+1}$, it follows that x_{m_0} is a fixed point of $\mathscr{L}$. Then, suppose that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$. Taking into account that $\mathscr{L}$ is a fuzzy α-η-Θ_f-contraction, we have

$$\begin{aligned} 1>\Theta_f(\nu(x_n,x_{n+1},t)) &\geq[\Theta_f(\nu(x_{n-1},x_n,t))]^\tau \\ &=[\Theta_f(\nu(x_{n-1},x_n,t)))]^\tau \\ &\geq[\Theta_f(\nu(x_{n-2},x_{n-1},t))]^{\tau^2} \\ &\vdots \\ &\geq[\Theta_f(\nu(x_0,x_1,t))]^{\tau^n}. \end{aligned}$$

Passing to the limit as $n\to+\infty$, we deduce

$$\lim_{n\to+\infty}\Theta_f(\nu(x_n,x_{n+1},t))=1.$$

Employing (Ω_3), we obtain

$$\lim_{n\to+\infty}\nu(x_n,x_{n+1},t)=1, \tag{3.3}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence $\{x_n\}$. On contrary, suppose that $\{x_n\}$ is not a Cauchy sequence. Thus, there exists $\varepsilon\in(0,1)$, $t_0>0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $m_k>n_k\geq k$ for all $k\in\mathbb{N}$ and

$$\nu(x_{m_k},x_{n_k},t_0)\leq1-\varepsilon. \tag{3.4}$$

Lemma 2.4 yields that

$$\nu(x_{m_k},x_{n_k},\frac{t_0}{2})\leq1-\varepsilon. \tag{3.5}$$

By choosing n_k as the lowest value satisfying (3.5), we obtain

$$\nu(x_{m_k-1},x_{n_k},\frac{t_0}{2}) > 1-\varepsilon. \tag{3.6}$$

Making use of (3.1) with $x = x_{m_k-1}$ and $x = x_{n_k-1}$, we have

$$\Theta_f(\nu(x_{m_k},x_{n_k},t_0) \geq [\Theta_f(\nu(x_{m_k-1},x_{n_k-1},t_0)))^{\tau} > \Theta_f(\nu(x_{m_{k-1}},x_{n_{k-1}},t_0)). \tag{3.7}$$

As Θ_f is nondecreasing, we conclude that

$$\nu(x_{m_k-1},x_{n_k-1},t_0) < \nu(x_{m_k},x_{n_k},t_0) \tag{3.8}$$

From (3.4), (3.6), (3.8), and ($\mathscr{G}4$), we have

$$\begin{aligned}1-\varepsilon &\geq \nu(x_{m_k},x_{n_k},t_0)\\ &> \nu(x_{m_k-1},x_{n_k-1},t_0)\\ &\geq \nu(x_{m_k-1},x_{n_k},\frac{t_0}{2}) * \nu(x_{n_k},x_{n_{k-1}},\frac{t_0}{2})\\ &> (1-\varepsilon) * \nu(x_{n_k},x_{n_k-1},\frac{t_0}{2}).\end{aligned}$$

Letting $k \to +\infty$ in both sides of the last inequality, and using (3.3), we get

$$\lim_{k\to+\infty} \nu(x_{m_k},x_{n_k},t_0) = \lim_{k\to+\infty} \nu(x_{m_k-1},x_{n_k-1},t_0) = 1-\varepsilon. \tag{3.9}$$

Again, taking the limit as $k \to +\infty$ in (3.7), taking into account the continuity of Θ_f and (3.9), we obtain

$$[\Theta_f(1-\varepsilon)]^{\tau} \leq \Theta_f(1-\varepsilon),$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. As $(X,M,*)$ is a complete fuzzy metric space, there exists $\omega \in \Gamma$ such that $x_n \to \omega$ as $n \to +\infty$. Now, as $\mathscr{L}$ is α-η-continuous and $\alpha(x_{n-1},x_n,t) \geq \eta(x_{n-1},x_n,t)$, one has

$$\nu(\omega,\mathscr{L}\omega,t) = \lim_{n\to+\infty} \nu(x_n,\mathscr{L}x_n,t) = \lim_{n\to+\infty} \nu(x_n,x_{n+1},t) = \nu(\omega,\omega,t) = 1,$$

which means that ω is a fixed point of $\mathscr{L}$. □

Theorem 3.3. *Let $(\Gamma,\nu,*)$ be a complete fuzzy metric space. Let $\alpha,\eta : \Gamma\times\Gamma\times(0,+\infty) \to [0,+\infty)$ be two given functions and $\mathscr{L} : \Gamma \longrightarrow \Gamma$ be a fuzzy α-η-Θ_f-contraction such that*

- **(i)** *$\mathscr{L}$ is α-admissible with respect to η;*
- **(ii)** *there exists $x_0 \in \Gamma$ such that $\alpha(x_0,\mathscr{L}x_0,t) \geq \eta(x_0,\mathscr{L}x_0,t)$;*
- **(iii)** *if $\{x_n\}$ is a sequence in Γ such that $\alpha(x_n,x_{n+1},t) \geq \eta(x_n,x_{n+1},t)$ for all $n\in\mathbb{N}$, $t>0$ and $x_n \to x \in \Gamma$ as $n \to +\infty$, then either $\alpha(\mathscr{L}x_n,x,t) \geq \eta(\mathscr{L}x_n,\mathscr{L}^2x_n,t)$ or $\alpha(\mathscr{L}^2x_n,x,t) \geq \eta(\mathscr{L}^2x_n,\mathscr{L}^3x_n,t)$ for all $n\in\mathbb{N}$.*

Then, $\mathscr{L}$ has a fixed point.

Proof. Let $x_0 \in \Gamma$ such that $\alpha(x_0,\mathscr{L}x_0,t) \geq \eta(x_0,\mathscr{L}x_0,t)$. By following the same lines of the proof of Theorem 3.2, we obtain

$$\alpha(x_n,\mathscr{L}x_n,t) = \alpha(x_n,x_{n+1},t) \geq \eta(x_n,x_{n+1},t)$$

for all $n\in\mathbb{N}$, where $\mathscr{L}x_n = x_{n+1}$ and $x_n \to x \in \Gamma$ as $n \to +\infty$. From (iii), we have $\alpha(\mathscr{L}x_n,x,t) \geq \eta(\mathscr{L}x_n,\mathscr{L}^2x_n,t)$ or $\alpha(\mathscr{L}^2x_n,x,t) \geq \eta(\mathscr{L}^2x_n,\mathscr{L}^3x_n,t)$ for all $n\in\mathbb{N}$. which means $\alpha(x_{n+1},x,t) \geq$

$\eta(x_{n+1},x_{n+2},t)$ or $\alpha(x_{n+2},x,t) \geq \eta(x_{n+2},x_{n+3},t)$. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n_k},x,t) \geq \eta(x_{n_k},x_{n_k+1},t) = \eta(x_{n_k},\mathscr{L}x_{n_k},t) \tag{3.10}$$

From (3.10), we get

$$\Theta_f(\nu(\mathscr{L}x_{n_k},\mathscr{L}x,t) \geq [\Theta_f(\nu(x_{n_k},x,t)))^{\tau} > \Theta_f(\nu(x_{n_k},x,t)). \tag{3.11}$$

Since Θ_f is non-decreasing, we conclude

$$\nu(x_{n_k+1},\mathscr{L}x,t) > \nu(x_{n_k},x,t). \tag{3.12}$$

By passing to the limit as $n \to +\infty$ in (3.12), we obtain $\nu(x,\mathscr{L}x,t) = 1$, that is, $\mathscr{L}x = x$. □

The following criterion will be taken into account in order to ensure the uniqueness of the fixed point of an α-η-Θ_f-fuzzy contraction.

(**U**) For all $x,y \in \mathscr{F}\mathscr{P}(\mathscr{L})$, we have $\alpha(x,y,t) \geq \eta(x,y,t)$, where $\mathscr{F}\mathscr{P}(\mathscr{L})$ represents the set of fixed points of $\mathscr{L}$.

Theorem 3.4. *Adding hypothesis* (**U**) *to the assumptions of Theorem 3.2 and Theorem 3.3, we obtain the uniqueness of the fixed point of $\mathscr{L}$.*

Proof. We argue by contradiction, suppose that $\omega,\varpi \in \Gamma$ are two distinct fixed points. Thus, $\nu(\omega,\varpi,t) < 1$ for all $t > 0$. By the condition (**U**), we obtain

$$\alpha(\omega,\varpi,t) \geq \eta(\omega,\varpi,t).$$

Hence

$$\Theta_f(\nu(\mathscr{L}\omega,\mathscr{L}\varpi,t)) = \Theta_f(\nu(\omega,\varpi,t)) \geq [\Theta(\nu(\omega,\varpi,t))]^{\tau},$$

Which is a contradiction with the fact that $\tau < 1$. Thus, the fixed point of $\mathscr{L}$ is unique. □

Corollary 3.5. *Let $(\Gamma,\nu,*)$ be a complete fuzzy metric space and $\mathscr{L}:\Gamma \to \Gamma$ be a self mapping such that*

$$\begin{cases} \alpha(x,y,t) \geq \eta(x,y,t) \\ \text{and } 1 > \nu(\mathscr{L}x,\mathscr{L}y,t) \end{cases} \text{implies } [\nu(x,y,t)]^{\tau} \leq \nu(\mathscr{L}x,\mathscr{L}y,t)$$

for all $x,y \in \Gamma$ and $t > 0$, and suppose

(**i**) *$\mathscr{L}$ is α-admissible;*
(**ii**) *there exists $x_0 \in \Gamma$ such that $\alpha(x_0,\mathscr{L}x_0,t) \geq \eta(x_0,\mathscr{L}x_0,t)$;*
(**iii**) *$\mathscr{L}$ is α-η-continuous.*

Then, $\mathscr{L}$ has a fixed point.

Proof. The conclusion can be drawn from Theorem 3.2 by defining $\Theta_f(\omega) = \omega$ for all $\omega \in (0,1)$.

□

Corollary 3.6. *Let $(\Gamma,\nu,*)$ be a complete fuzzy metric space and $\mathscr{L}:\Gamma \to \Gamma$ be a self mapping such that*

$$\begin{cases} \alpha(x,y,t) \geq 1 \\ \text{and } 1 > \nu(\mathscr{L}x,\mathscr{L}y,t) \end{cases} \text{implies } [\Theta_f(\nu(x,y,t))]^{\tau} \leq \Theta_f(\nu(\mathscr{L}x,\mathscr{L}y,t))$$

for all $x,y \in \Gamma$ and $t > 0$, where $\Theta_f \in \Omega$ and $\tau \in (0,1)$. and

(**i**) *$\mathscr{L}$ is α-admissible;*
(**ii**) *there exists $x_0 \in \Gamma$ such that $\alpha(x_0, \mathscr{L}x_0, t) \geq \eta(x_0, \mathscr{L}x_0, t)$;*
(**iii**) *$\mathscr{L}$ is α-η-continuous.*

Then, $\mathscr{L}$ has a fixed point.

Proof. The conclusion can be drawn from Theorem 3.2 by defining $\eta(x,y,t) = 1$ for all $x, y \in \Gamma$. □

Corollary 3.7. [8] *Let $(\Gamma, \nu, *)$ be a complete fuzzy metric space and $\mathscr{L} : \Gamma \to \Gamma$ be a self mapping such that for all $x, y \in \Gamma$ with $\nu(\mathscr{L}x, \mathscr{L}y, t) < 1$ we have*

$$\Theta_f(\nu(\mathscr{L}x, \mathscr{L}y, t)) \geq \left[\Theta_f(\nu(x,y,t))\right]^{\tau},$$

Then $\mathscr{L}$ has a fixed point.

Proof. The conclusion can be drawn from Theorem 3.2 by defining $\eta(x,y,t) = \alpha(x,y,t) = 1$ for all $x, y \in \Gamma$. □

Corollary 3.8. *Let $(\Gamma, \nu, *)$ be a complete fuzzy metric space and $\mathscr{L} : \Gamma \longrightarrow \Gamma$ be a mapping satisfying:*

$$\begin{cases} 1 > \nu(\mathscr{L}x, \mathscr{L}y, t) \text{ and} \\ \alpha(x,y,t) \geq \eta(x,y,t) \end{cases} \text{implies } \left[1 + \sin\left(\frac{\pi}{2}(\nu(x,y,t) - 1)\right)\right]^{\tau} \leq 1 + \sin\left(\frac{\pi}{2}(\nu(\mathscr{L}x, \mathscr{L}y, \gamma) - 1)\right)$$

and

(**i**) *$\mathscr{L}$ is α-admissible with respect to η;*
(**ii**) *there exists $x_0 \in \Gamma$ such that $\alpha(x_0, \mathscr{L}x_0, t) \geq \eta(x_0, \mathscr{L}x_0, t)$;*
(**iii**) *$\mathscr{L}$ is α-η-continuous.*

Then $\mathscr{L}$ has a fixed point.

Proof. The proof follows from Theorem 3.2 by taking $\Theta_f(\omega) = 1 + \sin\left(\frac{\pi}{2}(\omega - 1)\right)$ for all $\omega \in (0,1)$. □

Example 3.9. Let $\Gamma = [0,1]$ endowed with the fuzzy metric $\nu : \Gamma \times \Gamma \times (0, +\infty) \to [0,1]$ defined by $\nu(x,y,t) = \frac{t}{t + |x-y|}$ for all $x, y \in \Gamma$, $t > 0$ and $*$ is the minimum t-norm. We define $\mathscr{L} : \Gamma \longrightarrow \Gamma$ by

$$\mathscr{L}x = \begin{cases} \frac{x}{5(x+1)} & \text{if } x \in [0, \frac{1}{2}], \\ \\ x & \text{otherwise,} \end{cases}$$

and $\Theta_f : (0,1) \to (0,1)$ by

$$\Theta_f(\omega) = e^{1 - \frac{1}{\omega}}$$

Also define the functions $\alpha, \eta : \Gamma \times \Gamma \times (0, +\infty) \longrightarrow [0, +\infty)$ by

$$\alpha(x,y,t) = \begin{cases} 2 + xy & \text{if } x, y \in [0, \frac{1}{2}], \\ 0 & \text{otherwise,} \end{cases}$$

$$\eta(x,y,t) = \begin{cases} 1 + xy & \text{if } x, y \in [0, \frac{1}{2}], \\ 9 & \text{otherwise,} \end{cases}$$

First, we show the α-admissibility of $\mathscr{L}$ with respect to η. Let $x,y \in \Gamma$, observe that $\alpha(x,y,t) \geq \eta(x,y,t)$ for all $t>0$ if and only if $x,y \in [0,\frac{1}{2}]$. Hence, $\mathscr{L}x, \mathscr{L}y \in [0,\frac{1}{2}]$, thats is, $\alpha(\mathscr{L}x,\mathscr{L}y,t) \geq \eta(\mathscr{L}x,\mathscr{L}y,t)$. Clearly, for any $x_0 \in [0,\frac{1}{2}]$, we have $\alpha(x_0,\mathscr{L}x_0,t) \geq \eta(x_0,\mathscr{L}x_0,t)$ for all $t>0$. Next, if $\{x_n\}$ is a sequence in Γ such that $\alpha(x_n,x_{n+1},t) \geq \eta(x_n,x_{n+1},t)$ for all $n \in \mathbb{N}$, $t>0$ and $x_n \to x \in \Gamma$ as $n \to +\infty$, it is easy to see that $\{x_n\} \subset [0,\frac{1}{2}]$, $x \in [0,\frac{1}{2}]$ and $\mathscr{L}x_n \to \mathscr{L}x \in \Gamma$, thus $\mathscr{L}$ is α-η-continuous.
For all $x,y \in \Gamma$ with $\nu(\mathscr{L}x,\mathscr{L}y,t) < 1$ and $\tau = \frac{1}{2}$, we have

$$\begin{aligned}
1-\frac{1}{\nu(\mathscr{L}x,\mathscr{L}y,t)} &= 1-\frac{t+|\mathscr{L}x-\mathscr{L}y|}{t} \\
&= \frac{-|\mathscr{L}x,\mathscr{L}y|}{t} \\
&= \frac{-|\frac{x}{5(x+1)}-\frac{y}{5(y+1)}|}{t} \\
&= \frac{1}{5(x+1)(y+1)}.(-\frac{|x-y|}{t}) \\
&> \frac{-|x-y|}{2t} \\
&= \tau(\frac{1}{\nu(x,y,t)}-1)
\end{aligned}$$

Hence,

$$\begin{aligned}
\Theta_f(\nu(\mathscr{L}x,\mathscr{L}y,t)) &= e^{1-\frac{1}{\nu(\mathscr{L}x,\mathscr{L}y,t)}} \\
&> [e^{1-\frac{1}{\nu(x,y,t)}}]^\tau = [\Theta_f(\nu(x,y,t))]^\tau.
\end{aligned}$$

Therefore, $\mathscr{L}$ fulfil all the hypothesis of Theorem 3.4 and $\mathscr{L}$ has a unique fixed point $x=0$. However, by considering $x,y \in (\frac{1}{2},1]$, there is no $\tau \in (0,1)$ satisfying the contractive condition (2.1), which means that the result established by Gregori and Sapena [7] is not applicable in this case.

4. Applications

To study the existence and uniqueness of a solution to the mentioned first order periodic differential problem (4.1) , we employ theorem 3.4 in this part.

$$(\mathscr{P}) : \begin{cases} \frac{du(s)}{ds} = g(s,u(s)), \quad s \in [0,T], \\ \\ u(0) = u(T), \end{cases} \tag{4.1}$$

$g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous mapping and $T \geq 0$. Denotes by $\Gamma = \mathscr{C}([0,T],\mathbb{R})$ the space of continuous mappings $u : [0,T] \to \mathbb{R}$ and $\mathscr{M} : \Gamma \times \Gamma \to \mathbb{R}$ the metric given by $\mathscr{M}(u,w) = \sup_{s\in[0,T]} |u(s)-w(s)|$ for all $u,w \in \Gamma$. Define the function $\Theta_f : (0,1) \to (0,1)$ by

$$\Theta_f(\omega) = e^{1-\frac{1}{\omega}}$$

with $\eta(u,w,t)=\alpha(u,w,t)=1$ for all $u,w\in\Gamma$ and consider the fuzzy metric $\nu:\Gamma\times\Gamma\times(0,+\infty)\to[0,1]$ defined by

$$\nu(u,\omega,t)=\frac{t}{t+\mathscr{M}(u,\omega)}$$

and the product t-norm $*_p$, hence $(\Gamma,\nu,*)$ is a complete fuzzy metric space. Note that, the problem $(\mathscr{P})$ may be expressed as

$$\begin{cases}\frac{du(s)}{ds}+\delta u(s)=g(s,u(s))+\delta u(s),\\ \\ u(0)=u(T),\end{cases}$$

$s\in[0,T]$, $0<\delta$. The problem above is equivalent to the following integral equation

$$u(s)=\int_0^T\mathscr{G}(s,\ell)\big(g(\ell,u(\ell)+\delta u(\ell)\big)d\ell \tag{4.2}$$

for all $u\in\Gamma$, with $\mathscr{G}:[0,T]\times[0,T]\to\mathbb{R}$ is given by

$$\mathscr{G}(s,\ell)=\begin{cases}\frac{e^{\delta(T+\ell-s)}}{e^{\delta T}-1}, & 0\leq\ell\leq s\leq T,\\ \frac{e^{\delta(\ell-s)}}{e^{\delta T}-1}, & 0\leq s\leq\ell\leq T.\end{cases}$$

with $\delta\int_0^T\mathscr{G}(s,\ell)d\ell=1$.

Theorem 4.1. *Suppose that for all $u,\omega\in\Gamma$ and $\ell\in[0,T]$:*

$$|g(\ell,u(\ell)+\delta u(\ell)-(g(\ell,\omega(\ell)+\delta\omega(\ell))|\leq\delta(\frac{|u(\ell)-\omega(\ell)|}{2}). \tag{4.3}$$

Then 4.1 has a unique solution.

Proof. Let $\mathscr{L}:\Gamma\to\Gamma$ be the integral operator given by

$$\mathscr{L}u(\ell)=\int_0^T\mathscr{G}(s,\ell)\big(g(\ell,u(\ell)+\delta u(\ell)\big)d\ell$$

for all $u\in\Gamma$. Notice that, u is a fixed point of $\mathscr{L}$ is equivalent to say that u is a solution of the equation (4.1). Let $u,\omega\in\Gamma$, by (4.3), we have

$$\begin{aligned}\mathscr{M}(\mathscr{L}u,\mathscr{L}\omega)&=\sup_{s\in[0,T]}|\mathscr{L}u(s)-\mathscr{L}\omega(s)|\\&\leq\sup_{s\in[0,T]}\int_0^T\mathscr{G}(s,\ell)\big(g(\ell,u(\ell)+\delta u(\ell)\big)d\ell\\&\leq\delta\sup_{s\in[0,T]}\int_0^T\mathscr{G}(s,\ell)(\frac{|u(\ell)-\omega(\ell)|}{2})d\ell.\end{aligned}$$

Considering that $\frac{|u(\ell)-\omega(\ell)|}{2}\leq\sup_{\ell\in[0,T]}\frac{|u(\ell)-\omega(\ell)|}{2}=\frac{\mathscr{M}(u,\omega)}{2}$ and $\delta.\int_0^T\mathscr{G}(s,\ell)d\ell=1$, it follows that

$$\begin{aligned}\mathscr{M}(\mathscr{L}u,\mathscr{L}\omega)&\leq\delta.\frac{\mathscr{M}(u,\omega)}{2}\sup_{s\in[0,T]}\int_0^T\mathscr{G}(s,\ell)d\ell\\&=\frac{\mathscr{M}(u,\omega)}{2}.\end{aligned}$$

Using the last inequality, we get

$$\begin{aligned}1-\frac{1}{\nu(\mathscr{L}u,\mathscr{L}w,t)} &= \frac{-\mathscr{M}(u,\omega)}{t}\\ &= \frac{-\mathscr{M}(\mathscr{L}u,\mathscr{L}\omega)}{t}\\ &\geq \frac{-\mathscr{M}(u,\omega)}{2t}\\ &= \frac{1}{2}[1-\frac{1}{\nu(u,w,t)}].\end{aligned}$$

Taking into account the expression of the function Θ_f, we obtain

$$\begin{aligned}\Theta_f(\nu(\mathscr{L}x,\mathscr{L}y,t)) &= e^{1-\frac{1}{\nu(\mathscr{L}x,\mathscr{L}y,t)}}\\ &\geq [e^{1-\frac{1}{\nu(x,y,t)}}]^{\frac{1}{2}} = [\Theta_f(\nu(x,y,t))]^{\frac{1}{2}}.\end{aligned}$$

As a result, Theorem 3.4's assumptions are fulfilled. As a consequence, $\mathscr{L}$ possesses an unique fixed point in Γ that is an unique solution of (4.1). □

5. Conclusion

We initiated the notion of α-η-Θ_f-fuzzy contraction in the context of fuzzy metric spaces by including the idea of α-admissibility with the control function Θ_f and developed some fixed point results regarding the existence and uniqueness of fixed point for such contractions. It is important to mention that by appropriately integrating a number of instances of the functions Θ_f, α and η, we may specialize and bring a wide range of possible consequences from our main findings. The results might open up possibilities for new directions in fuzzy metric fixed point research, in addition to the study of coincidence and common fixed point in a more extended framework, such as fuzzy b-metric spaces, partially ordered fuzzy metric spaces, and other general structures. .

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