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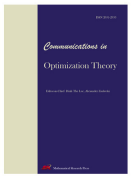

GLOWINSKI AND SPLITTING

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This paper is dedicated to the memory of Roland Glowinski,
with my respect and deep thanks

Abstract. This paper is devoted to recall several contributions to the numerical solution and control of PDE's that have origin in Roland Glowinski's activity. First, I will recall some results contained in works carried out under his direction in the first 80's. Then, operator-splitting methods primarily used to solve Navier-Stokes equations will be reviewed. In the final part of the paper, I will describe briefly several advances obtained in the last decades.

Keywords. Monge-Ampère equation; Navier-Stokes equations; Splitting methods; Semiconductor process modeling; Vortex rings.

2020 Mathematics Subject Classification. 35Q30, 35Q93.

1. INTRODUCTION

This paper is organized on the basis of two keywords:

- Of course, **Glowinski.**
- **Splitting.** This can be found in much of Glowinski's activity. In particular, we will recall operator splitting techniques for the solution of the Navier-Stokes and other equations.

I had the fortune to enjoy a beautiful period in INRIA-Rocquencourt, France, from 1979 to 1982.

My first contact with Glowinski was in November 1979. My Professor in Spain, A. Valle, had previously sent a letter asking him to accept me as PhD student and he had kindly said yes.

In France, the initial hard days were followed by happy days: first, the PhD Thesis months under Glowinski's direction, ended with the defense in 1981; then, a short postdoc period (1981–1982).

In fact, Glowinski and I did not gather together very often, not too much time per meeting. But now I feel that everything was extremely useful and fruitful.

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He taught me to distinguish important from unimportant things. For example, with him I understood that P_1 finite elements must be preferred to P_2 or P_3 elements as much as possible; I also grasped the relevance of the Inf-Sup condition; the need of a good choice of the *gradient,* that is, the scalar product in descent methods for extremal problems, etc.

I also learned to be insistent when trying to give an answer to a question and thus apply different viewpoints to solve a problem. For instance, it was a good idea to view nonlinear elliptic systems as optimal control probems for linear PDE's and, accordingly, to incorporate control techniques to the computation of the solution(s).

Furthermore, he was for me an excellent bridge to establish contacts with many other people: H. Berestycki, A. Marrocco, O. Pironneau, F. Hecht, ... And then F. Murat, J. Simon, J.-P. Puel and J.-M. Coron.

I am deeply grateful to all of them.

This paper is organized as follows.

In Section 2, I will describe briefly the first two problems I was involved in, both suggested by Glowinski during my PhD Thesis period.

Section 3 deals with splitting methods. First, we will explain some general ideas that motivate the techniques. Then, we will recall their applications to Navier-Stokes and similar problems: alternate direction methods of the Peaceman-Rachford, Strang and θ-scheme kinds; simultaneous (i.e parallel) direction methods in time and space, etc. Finally, we will mention recent applications to some fully nonlinear PDE's, more precisely the Monge-Ampère and the Gauss curvature equations.

2. The old times

This section is devoted to recall contributiona to the solution of two problems proposed to me by Glowinski.

2.1. **Vortex rings analysis and computation.** Let us consider the rectangle $\Omega = (0, L_1) \times (-L_2, L_2)$, let us assume that $W, \eta > 0$ and $F \in C^0(\mathbb{R})$ is a non-decreasing function satisfying

$$F(s) = 0 \ \ \forall s \leq 0 \ \text{ and } \ F(s) \leq Cs \ \ \forall s > 0.$$

Let us consider the following vortex ring problem, introduced in [18, 31]:

Find $u = u(\mathbf{x})$ and $K > 0$ such that

$$\begin{cases} -\Delta u = F(u - Wx_1 - K), \quad \mathbf{x} \in \Omega \\ u\big|_{\Gamma_0} = 0, \ \ \partial_n u\big|_{\Gamma_1} = 0 \\ \displaystyle\int_\Omega |\nabla u|^2 \, d\mathbf{x} = \eta. \end{cases} \tag{2.1}$$

Here, Γ_0 and Γ_1 respectively denote the vertical and horizontal edges of $\partial\Omega$ and $\partial_n u = \nabla u \cdot \mathbf{n}$ stands for the normal derivative of u.

The following interpretation is in order:

- $z := u - Wx_1 - K$ is the stream function of a plane *vortex ring.*
- The associated vortex ring region is given by $\{\mathbf{x} \in \Omega : u(\mathbf{x}) - Wx_1 - K > 0\}$.
- W and η are respectively the (vertical) speed and the kinetic energy of the ring.
- K must be viewed as the flux amount that flows between the axis $\{x_1 = 0\}$ and the ring.

The main assumption used in [18, 31] to deduce the model is that vorticity is parallel to stream potential. It is also interesting to exchange the roles of W and K in the problem, that is, to try to find u and W for given K.

Several theoretical and numerical results for (2.1) can be found in [4]. There are also other similar confinement problems arising in plasma physics for which similar results can be obtained; see [3, 12, 32] and others.

In particular, let us recall the following two algorithms for the computation of a solution:

ALG 1 (fixed-point):
(1) Choose u^0
(2) For given $n \geq 0$ and u^n, find u^{n+1} and K^{n+1} with

$$\begin{cases} -\Delta u^{n+1} = F(u^n - Wx_1 - K^{n+1}), \quad \mathbf{x} \in \Omega \\ u^{n+1}\big|_{\Gamma_0} = 0, \;\; \partial_n u^{n+1}\big|_{\Gamma_1} = 0 \\ \displaystyle\int_\Omega |\nabla u^{n+1}|^2 \, d\mathbf{x} = \eta \end{cases}$$

ALG 2 (least-squares + gradient descent):
(1) Choose u^0
(2) For given $n \geq 0$ and u^n, find u^{n+1} and $K^{n+1} = K(u^{n+1})$, with

$$u^{n+1} = u^n - \rho^n z^n, \;\; z^n = J'(u^n), \;\; \rho^n = \arg\min_{\rho \geq 0} J(u^n - \rho z^n)$$

Here, we have used the notation

$$J(v) := \frac{1}{2} \int_\Omega |(-\Delta)^{-1}(\Delta v + F(v - Wx_1 - K))|^2 \, d\mathbf{x}$$

and, for every v, $K(v)$ stands for the quantity K satisfying

$$\int_\Omega |\nabla(-\Delta)^{-1}(F(v - Wx_1 - K))|^2 \, d\mathbf{x} = \eta.$$

In practice, we use finite dimensional versions of ALG 1 and ALG 2. Thus, we introduce a traingulation $\mathscr{T}_h$ of Ω and the associated P_1-Lagrange finite element space

$$W_h := \{w_h \in C^0(\overline{\Omega}) : w_h|_K \in \mathbb{P}_1(K) \;\; \forall K \in \mathscr{T}_h, \;\; w_h|_{\Gamma_0} = 0\}.$$

For instance, at each step in ALG 1, we search for a couple $(u_h^{n+1}, K_h^{n+1}) \in W_h \times \mathbb{R}$ such that

$$\begin{cases} \displaystyle\int_\Omega \nabla u_h^{n+1} \cdot \nabla w_h \, d\mathbf{x} = \int_\Omega F(u_h^n - Wx_1 - K_h^{n+1}) \, w_h \, d\mathbf{x} \\ \displaystyle\int_\Omega u_h^{n+1} F(u_h^n - Wx_1 - K_h^{n+1}) \, d\mathbf{x} = \eta. \end{cases}$$

The results of some numerical experiments are shown in Fig. 1. The data are the following: $\Omega = (0, L_1) \times (-L_2, L_2)$ with $L_1 = L_2 = 5$; $F(s) = s_+$; $\eta = 10$ and several values of W.

2.2. **Semiconductor process and device modeling.** There are two typical systems related to semiconductor technology. They are usually identified as *process modeling* (the simulation of the fabrication process) and *device modeling* (the description of the electric behavior); see [29] for details.

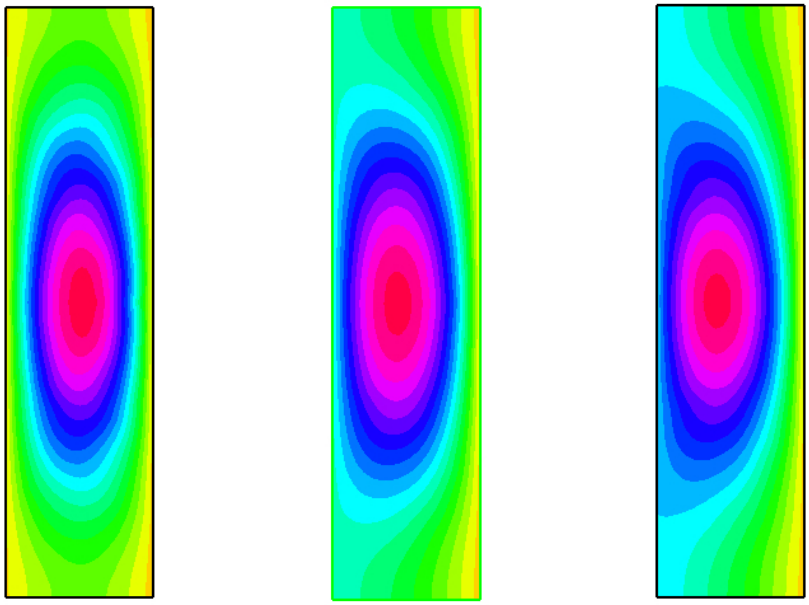


FIGURE 1. Vortex rings corresponding to $W = 0.005$, $W = 0.015$ and $W = 0.02$ (from left to right). Streamlines $u - Wx_1 - K = \text{Const}$. These experiments, as well as the following, have been performed with the help of the Freefem package (http://www.freefem.org//ff++).

Let $\Omega \in \mathbb{R}^N$ be a parallelepiped, let $T > 0$ be given and let us set $Q := \Omega \times (0,T)$, $\Sigma := \partial\Omega \times (0,T)$. As before, we consider a partition $\{\Gamma_0, \Gamma_1\}$ of $\partial\Omega$. The semiconductor *process modeling* problem in Q is to find u and v with

$$\begin{cases} u_t - \nabla \cdot (D_{11}(u,v)\nabla u + D_{12}(u,v)\nabla v) = 0 & \text{in } Q \\ v_t - \nabla \cdot (D_{21}(u,v)\nabla u + D_{22}(u,v)\nabla v) = 0 & \text{in } Q \\ u = u_\Gamma(\mathbf{x}) & \text{on } \Gamma_0 \times (0,T) \\ D_{11}(u,v)\partial_n u + D_{12}(u,v)\partial_n v = 0 & \text{on } \Gamma_1 \times (0,T) \\ D_{21}(u,v)\partial_n u + D_{22}(u,v)\partial_n v = 0 & \text{on } \Sigma \times (0,T) \\ u|_{t=0} = v|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

The unknowns u and v must be viewed as concentrations of impurities (for instance As, B, ...).

On the other hand, in *device modeling* problems, we find similar systems, where u and v are now electron and hole concentrations, some semilinear right hand sides must be added and a new variable $\psi = (-\Delta)^{-1}(N + v - u)$ (the electric potential) and a new equation appear; see [30].

In the most simple model, the diffusion coefficients D_{ij} have the following structure:

$$D_{11} = a_{11}\left(1 + \frac{u}{\sqrt{(u-v)^2+4}}\right), \quad D_{12} = -a_{12}\frac{u}{\sqrt{(u-v)^2+4}} \tag{2.3}$$

and similar expressions hold for D_{22} and D_{21}.

For the numerical solution of (2.2), we first introduce an approximation in time; this leads to a family of stationary nonlinear problems of the form

$$\begin{cases} \beta u - \nabla\cdot\sigma_u = f(\mathbf{x}), \quad \sigma_u = D_{11}(u,v)\nabla u + D_{12}(u,v)\nabla v & \text{in } \Omega \\ \beta v - \nabla\cdot\sigma_v = g(\mathbf{x}), \quad \sigma_v = D_{21}(u,v)\nabla u + D_{22}(u,v)\nabla v & \text{in } \Omega \\ u = u_\Gamma(\mathbf{x}) & \text{on } \Gamma_0 \\ \sigma_u\cdot\mathbf{n} = 0 & \text{on } \Gamma_1, \\ \sigma_v\cdot\mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $\beta > 0$.

Then, the problems (2.4) are replaced by appropriate approximations in space. Specifically, as in Section 2.1, we introduce a triangulation $\mathscr{T}_h$ and the spaces

$$X_h := \{w_h \in C^0(\overline{\Omega}) : w_h|_K \in \mathbb{P}_1(K) \ \ \forall K \in \mathscr{T}_h\}$$

and

$$V_h := \{w_h \in X_h : w_h|_{\Gamma_0} = 0\}$$

and we search for a couple (u_h, v_h) with $u_h \in u_\Gamma + V_h$, $v_h \in X_h$ and

$$\begin{cases} \alpha\displaystyle\int_\Omega (u_h w_h + v_h z_h)\,d\mathbf{x} + \int_\Omega \begin{pmatrix} D_{11}(u_h,v_h) & D_{12}(u_h,v_h) \\ D_{21}(u_h,v_h) & D_{22}(u_h,v_h) \end{pmatrix} \begin{pmatrix} \nabla u_h \\ \nabla v_h \end{pmatrix} \cdot \begin{pmatrix} \nabla w_h \\ \nabla z_h \end{pmatrix} d\mathbf{x} \\ \qquad = \displaystyle\int_\Omega (f w_h + g z_h)\,d\mathbf{x} \\ \forall (w_h, z_h) \in V_h \times X_h. \end{cases} \tag{2.5}$$

The nonlinear finite dimensional system (2.5) can be solved, among other possibilities, by an extrapolated Gauss-seidel-Newton algorithm; see [7, 13, 28], where results on the existence and numerical approximation as well as realistic simulations can be found.

The results of an experiment are displayed in Fig. 2–4. There, the domain is a Silicium half-plate; the impurities are injected through the upper side on the left (on the right an oxide layer avoids penetration) and are assumed to e zero on the lower edge. The data are the following: $\Omega = (0,5)\times(0,5)$, $\Gamma_0 = [0,5]\times\{0,5\}$, $\Gamma_1 = \{0,5\}\times(0,5)$; $u_\Gamma(\mathbf{x}) = f_u(x_1)x_2/5$, with $f_u(x_1) = 1.76$ for $x_1 \le 2.3$, $f_u(x_1) = 1.76\cdot(12.5 - 5x_1)$ for $2.3 < x_1 < 2.5$ and $f_u(x_1) = 0$ elsewhere; the D_{ij} are as in (2.3), with $a_{11} = 1$, $a_{12} = a_{21} = 0.01$, $a_{22} = 0.05$.

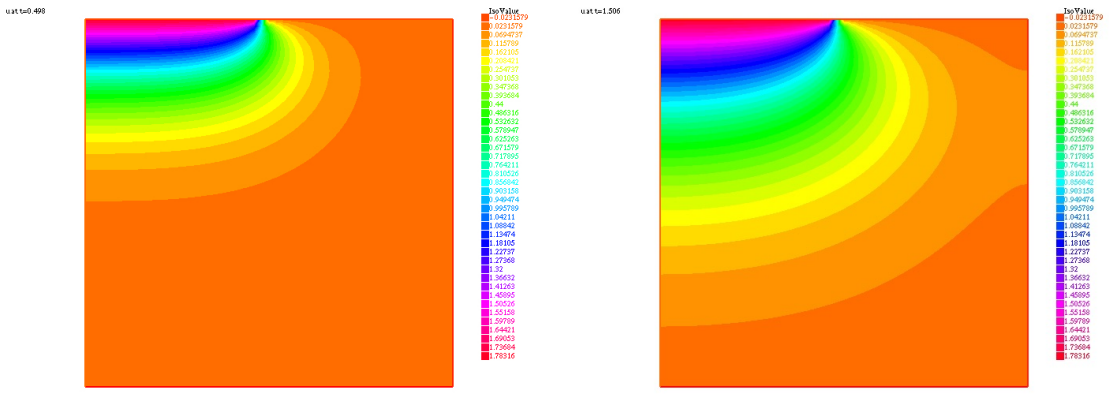


FIGURE 2. Final computed u. Curves $u =$ Const. for $t = 0.5$ (left) and $t = 1.5$ (right).

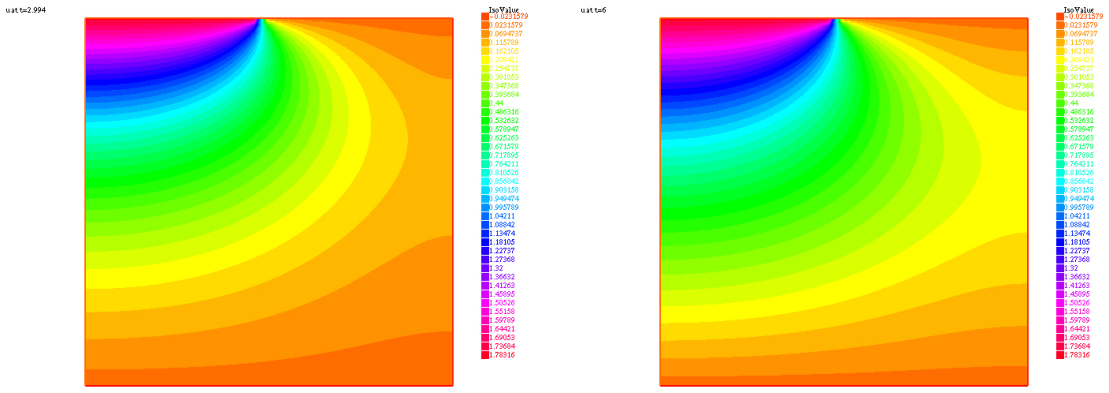


FIGURE 3. Final computed u. Curves $u = \text{Const.}$ for $t = 3$ (left) and $t = 6$ (right).

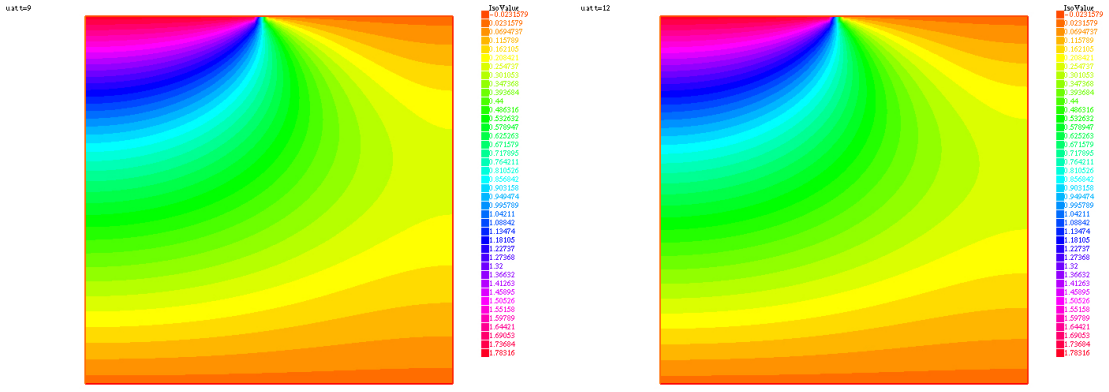


FIGURE 4. Final computed u. Curves $u = \text{Const.}$ for $t = 9$ (left) and $t = 12$ (right).

3. SPLITTING

Splitting techniques were used by Glowinski (in a broad sense) in connection with many problems. About 100 papers, many of them in collaboration with Bristeau, Périaux, Perrier, Pironneau, Dean, Pan, ... contain related results. They concern

- Domain decomposition methods,
- Fictitious domain and embedding domain methods,
- Numerical control (with J.-L. Lions, Carthel, He, ...),
- Operator-splitting for nonlinear PDE's and more precisely
  - For incompressible and compressible Navier-Stokes,
  - For many other PDE's: Bingham, Oldroyd, nonlinear waves, obstacle problems, Monge-Ampère, ...

The main ideas dealing with operator splitting methods can be explained as follows.

Assume that we want to compute (an approximato of) the solution $y:[0,T]\mapsto Y$ to an initial-value problem of the form

$$\begin{cases} y_t + A(y,t) = f(t), \ \ t\in[0,T] \\ y(0) = y_0 \end{cases} \tag{3.1}$$

where Y is a (finite or infinite dimensional) vector space, $A = A(y,t)$ is a time dependent (linear or nonlinear) operator on Y and $f:[0,T]\mapsto Y$ and $y_0\in Y$ are given.

It is assumed that A can be written in the form

$$A = A_1(y,t) + A_2(y,t)$$

and, also, that for some reason (3.1) it is much easier to work with A_1 or A_2 separately than to consider these operators together. Then, it seems reasonable to try to get advantage of this property.

Thus, for example, it can be interesting to introduce the following time-approximation scheme, where $\tau = T/M$ and M is a large integer:

(1) Take $y^0 = y_0$.
(2) Then, for any given $m = 0,1,\dots,M-1$ and y^m, do the following:
- Compute $y^{m+1/2}$, with

$$\frac{y^{m+1/2}-y^m}{\tau/2} + A_1(y^{m+1/2},t^{m+1/2}) + A_2(y^m,t^m) = f(t^{m+1/2}).$$

- Compute y^{m+1}, with

$$\frac{y^{m+1}-y^{m+1/2}}{\tau/2} + A_1(y^{m+1/2},t^{m+1/2}) + A_2(y^{m+1},t^{m+1}) = f(t^{m+1}).$$

This is a particular example of operator splitting method: the so called Peaceman-Rachford scheme. It produces the approximations $y^0, y^{1/2}, y^1, y^{3/2}, \dots$ of y at times $0, \tau/2, \tau, 3\tau/2, \dots$ In accordance with the previous assumption on A_1 and A_2, it can be of great interest for many problems involving ordinary and partial differential equations.

For a complete description of operator splitting methods, thier behavior and their applications in several contexts, see [21] and the references therein.

3.1. **Some alternating direction methods.** In this section, I will recall some applications of splitting techniques to the numerical integration of the Navier-Stokes system.

It is well known that these PDE's describe the behavior of a fluid under realistic conditions. They take the form

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{f}, \ \ \nabla\cdot\mathbf{u} = 0 \ \ \text{in} \ \ Q, \tag{3.2}$$

where again $Q = \Omega\times(0,T)$, $\Omega\subset\mathbb{R}^N$ is a non-empty bounded connected open set, $T>0$ and the unknowns are the velocity field $\mathbf{u} = (u_1,\dots,u_N)$ and the pressure p ($N=2$ or $N=3$).

Here, $\mathbf{f} = (f_1,\dots,f_N)$ must be viewed as a density of external forces and ν and ρ are positive constants respectively related to the kinematic viscosity and the mass density of the fluid. For simplicity, we will assume that (3.2) is completed with zero Dirichlet (no-slip) conditions on $\Sigma = \partial\Omega\times(0,T)$ and initial conditions for $\mathbf{u}$ at $t=0$.

For the numerical solution of (3.2), it is usual to first discretize or approximate in time and then approximate the resulting stationary problems in space, usually applying a finite element technique.

For the time discretization, a first splitting algorithm proposed by Glowinski to go from t^n to t^{n+1} was the following: first, solve in the spatial domain Ω the Burgers-like system

$$\left\{\begin{array}{l} \dfrac{1}{\tau/2}(\mathbf{u}^{n+1/2}-\mathbf{u}^n)-\nu\Delta\mathbf{u}^{n+1/2}+(\mathbf{u}^{n+1/2}\cdot\nabla)\mathbf{u}^{n+1/2}=\mathbf{f}^{n+1/2}-\dfrac{1}{\rho}\nabla p^n \\ \mathbf{u}^{n+1/2}|_{\partial\Omega}=0 \end{array}\right. \tag{3.3}$$

and then look for the solution to the Stokes-like problem

$$\left\{\begin{array}{l} \dfrac{1}{\tau/2}(\mathbf{u}^{n+1}-\mathbf{u}^{n+1/2})-\nu\Delta\mathbf{u}^{n+1}+\dfrac{1}{\rho}\nabla p^{n+1}=\mathbf{f}^{n+1}-(\mathbf{u}^{n+1/2}\cdot\nabla)\mathbf{u}^{n+1/2},\ \ \nabla\cdot\mathbf{u}^{n+1}=0 \\ \mathbf{u}^{n+1}|_{\partial\Omega}=0 \end{array}\right. \tag{3.4}$$

This can be viewed as an *alternate direction method* (ADI) of the Peaceman-Rachford type; see [20], Ch. 2, p. 52.

Some similar previous algorithms were introduced by Chorin, Témam and others in [8, 9, 34, 35]. They are known generically as *projection methods.* Note however that, in (3.3)–(3.4), nonlinearity is conserved. In fact, this is maybe the most representative aspect of Glowinski's work.

In a second step, (3.3) and (3.4) must be replaced by appropriate finite dimensional approximations. First, both systems can be reduced to collections of relatively simple Poisson-like problems of the form

$$\left\{\begin{array}{ll} -\alpha\Delta w+\beta w=F & \text{in } \Omega \\ w=0 & \text{on } \partial\Omega \end{array}\right. \tag{3.5}$$

where $\alpha,\beta>0$.

Thus, among many other possibilities, the nonlinear system (3.3) can be reformulated as a least-squares problem and related conjugate gradient algorithms can be applied. On the other hand, for the solution of (3.4), we can use *Lagrangian techniques* and iterative algorithms of the Uzawa or Arrow-Hurvicz kind. In both cases, the task is ultimately reduced to solve problems of the class (3.5).

Finally, the numerical solution of (3.5) can be accomplished with "classical" finite element methods.

A more ellaborate splitting scheme for time discretization follows some previous ideas of Strang [33]. It is the following: first, we go from t^n to $t^{n+1/4}$ by solving the Stokes-like problem

$$\left\{\begin{array}{l} \dfrac{1}{\tau/4}(\mathbf{u}^{n+1/4}-\mathbf{u}^n)-\nu\Delta\mathbf{u}^{n+1/4}+\dfrac{1}{\rho}\nabla p^{n+1/4}=\mathbf{f}^{n+1/4}-(\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\ \ \nabla\cdot\mathbf{u}^{n+1/4}=0 \\ \mathbf{u}^{n+1/4}|_{\partial\Omega}=0. \end{array}\right. \tag{3.6}$$

Then, we solve the nonlinear system

$$\left\{\begin{array}{l} \dfrac{1}{\tau/2}(\mathbf{u}^{n+3/4}-\mathbf{u}^{n+1/4})-\nu\Delta\mathbf{u}^{n+3/4}+(\mathbf{u}^{n+3/4}\cdot\nabla)\mathbf{u}^{n+3/4}=\mathbf{f}^{n+3/4}-\dfrac{1}{\rho}\nabla p^{n+1/4} \\ \mathbf{u}^{n+3/4}|_{\partial\Omega}=0. \end{array}\right. \tag{3.7}$$

Finally, to go from $t^{n+3/4}$ to t^{n+1}, we solve the second Stokes-like problem

$$\begin{cases} \dfrac{1}{\tau/4}(\mathbf{u}^{n+1}-\mathbf{u}^{n+3/4})-\nu\Delta\mathbf{u}^{n+1}+\dfrac{1}{\rho}\nabla p^{n+1}=\mathbf{f}^{n+1}-(\mathbf{u}^{n+3/4}\cdot\nabla)\mathbf{u}^{n+3/4}, \quad \nabla\cdot\mathbf{u}^{n+1}=0 \\ \mathbf{u}^{n+1}|_{\partial\Omega}=0. \end{cases} \tag{3.8}$$

Obviously, the numerical solution to the resulting systems can be achieved as before.

The convergence of splitting algorithms has been analyzed by several authors; see for instance [19, 25, 27]. In particular, in the case of (3.2), this was done in [16].

More precisely, assume that $\Omega \subset \mathbb{R}^N$ is a polygonal or polyhedrical domain and, again, that (3.2) is completed with homogeneous Dirichlet boundary conditions for $\mathbf{u}$ on Σ and the initial condition

$$\mathbf{u}|_{t=0}=\mathbf{u}_0 \ \text{ in } \Omega,$$

where $\mathbf{u}_0 \in L^2(\Omega)^N$, $\nabla\cdot\mathbf{u}_0=0$ in Ω and $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega$.

Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of Ω.

Assume that the approximations in time (3.3)–(3.4) and (3.6)–(3.8) are introduced for any small $\tau>0$ with the nonlinear terms in (3.3) and (3.7) modified respectively by

$$(\mathbf{u}^{n+1/2}\cdot\nabla)\mathbf{u}^{n+1/2}+\frac{1}{2}(\nabla\cdot\mathbf{u}^{n+1/2})\,\mathbf{u}^{n+1/2}$$

and

$$(\mathbf{u}^{n+3/4}\cdot\nabla)\mathbf{u}^{n+3/4}+\frac{1}{2}(\nabla\cdot\mathbf{u}^{n+3/4})\,\mathbf{u}^{n+3/4}.$$

Let the corresponding stationary problems be approximated in space for every small $h>0$ using standard *mixed* P_2-P_1 finite elements for $\mathbf{u}$ and p. Let us denote by $(\mathbf{u}_h^n, p_h^n)$ the corresponding solutions.

For fixed τ and h, we denote by $\mathbf{u}_{\tau,h}$ the piecewise linear continuous function satisfying

$$\mathbf{u}_{\tau,h}(t^n)=\mathbf{u}_h^n$$

for all n.

Then, the following result holds:

Theorem 3.1. *There exists a constant C_0 depending on $\mathbf{u}$, $\mathbf{f}$ and ν such that, if $\tau, h \to 0$ and τ and h are constrained to satisfy*

$$\frac{\tau}{h^2}\le C_0,$$

at least for a subsequence, one has weak convergence in $L^2(0,T;H_0^1(\Omega)^N)$ and strong convergence in $L^2(\Omega\times(0,T))^N$ of the $\mathbf{u}_{\tau,h}$ to a solution $\mathbf{u}$. Furthermore, if $N=2$ and $k/h^2\to 0$, the whole sequence $\{\mathbf{u}_{\tau,h}\}$ converges strongly in $L^2(0,T;H_0^1(\Omega)^N)$ to the unique solution.

A generalization of the Strang ADI algorithm introduced by Glowinski in [20] is as follows: fix $\theta\in(0,1/3]$; then, in order to go from t^n to t^{n+1}, we solve two Stokes problems and one Burgers problem according to the formulæ

$$\begin{cases} \dfrac{1}{\theta\tau}(\mathbf{u}^{n+\theta}-\mathbf{u}^{n})-\nu\Delta\mathbf{u}^{n+\theta}+\dfrac{1}{\rho}\nabla p^{n+\theta}=\mathbf{f}^{n+\theta}-(\mathbf{u}^{n}\cdot\nabla)\mathbf{u}^{n}, \quad \nabla\cdot\mathbf{u}^{n+\theta}=0 \\ \mathbf{u}^{n+\theta}|_{\partial\Omega}=0, \end{cases}$$

$$\begin{cases} \dfrac{1}{(1-2\theta)\tau}(\mathbf{u}^{n+1-\theta}-\mathbf{u}^{n+\theta})-\nu\Delta\mathbf{u}^{n+1-\theta}+(\mathbf{u}^{n+1-\theta}\cdot\nabla)\mathbf{u}^{n+1-\theta}=\mathbf{f}^{n+1-\theta}-\dfrac{1}{\rho}\nabla p^{n+1-\theta} \\ \mathbf{u}^{n+1-\theta}|_{\partial\Omega}=0, \end{cases}$$

$$\begin{cases} \dfrac{1}{\theta\tau}(\mathbf{u}^{n+1}-\mathbf{u}^{n+1-\theta})-\nu\Delta\mathbf{u}^{n+1}+\dfrac{1}{\rho}\nabla p^{n+1}=\mathbf{f}^{n+1}-(\mathbf{u}^{n+1-\theta}\cdot\nabla)\mathbf{u}^{n+1-\theta}, \quad \nabla\cdot!\mathbf{u}^{n+1}=0 \\ \mathbf{u}^{n+1}|_{\partial\Omega}=0. \end{cases}$$

This is usually known as the θ-scheme; see [20] for its description and analysis and related experiments.

The numerical results that can be obtained by applying these splitting techniques are remarkable. See Fig. 5 and 6 for the flow of a fluid around a cylinder at Reynolds number $\mathrm{Re}=800$. The data are the following: $\Omega=O\setminus\overline{B}$, with $O=B((0,0);3)\cup((0,4)\times(-3,3))$, $B=B((0,0);0.2)$; $T=50$; $\nu=0.00125$, $\rho=1$; $\mathbf{u}_0(\mathbf{x})=(cos(\pi/6),,\sin(\pi/6))$ near ∂O and $\mathbf{u}_0(\mathbf{x})=0$ near ∂B (initial data); $\mathbf{u}(\mathbf{x},t)=(cos(\pi/6),\sin(\pi/6))$ on $\partial O\times(0,T)$ (velocity field "at infinity"); $\mathbf{u}(\mathbf{x},t)=0$ on $\partial B\times(0,T)$ (velocity field on the cylinder surface).

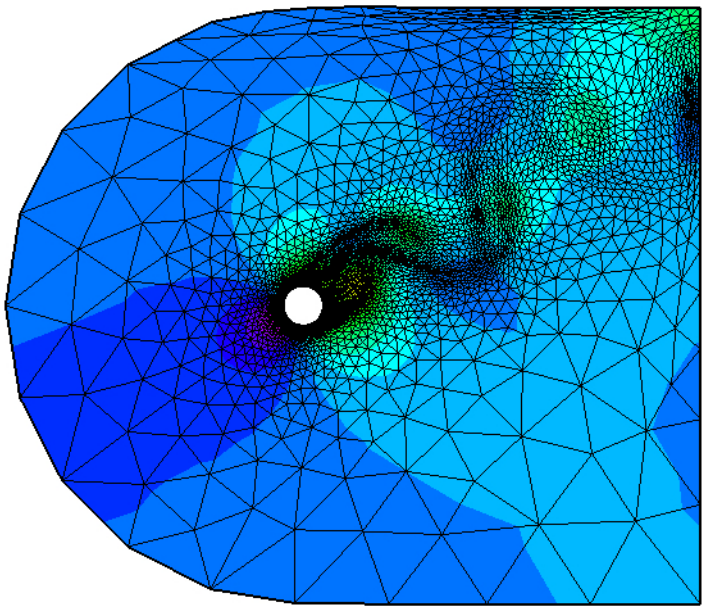


FIGURE 5. The flow around a body. $\mathrm{Re}=800$. Mesh and pressure.

Splitting techniques have also been applied with success to many other nonlinear PDE systems. For instance, they work very well for the variable density Navier-Stokes equations

$$\begin{cases} \rho_t+\nabla\cdot(\rho\mathbf{u})=0 \\ \rho(\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u})-\mu\Delta\mathbf{u}+\nabla p=\rho\mathbf{f} \\ \nabla\cdot\mathbf{u}=0, \end{cases}$$

where the mass density ρ is unknown.

Thus, we have displayed in Fig. 7 the evolution of the density in a stratification phenomenon: starting from an unnatural initial data for ρ, we see that, under the action of gravity, heavy

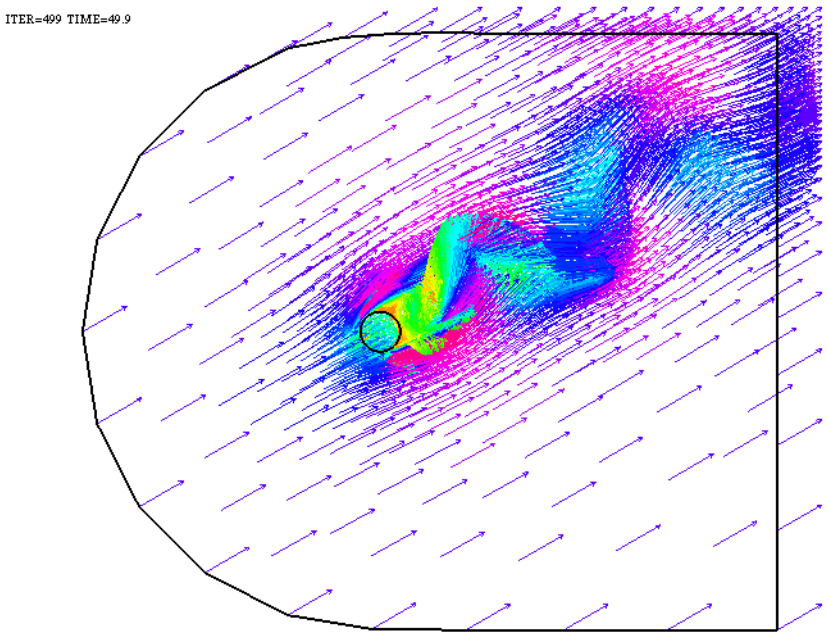


FIGURE 6. The flow around a body. Re = 800. Velocity field.

particles go down and lightweight particles climb to reach a stationary state after some time. The data are the following: $\Omega = (0,1)\times(0,3)$; $\mu = 0.025$; $\mathbf{f}(\mathbf{x}) = (0,-10)$; $\mathbf{u}_0(\mathbf{x}) = 0$ and $\rho_0(\mathbf{x}) = 1+33x_2$ (initial data); $\mathbf{u}(\mathbf{x},t) = 0$ on $\partial\Omega\times(0,T)$.

More details on the behavior of the solutions to the variable density Navier-Stokes equations can be found in [15].

3.2. **Simultaneous direction methods and parallelization.** The previous splitting schemes have been modified in other more recent works to incorporate parallelization issues.

More precisely, with appropriate variants it is possible to reduce the solution to (3.2) to a large family of elementary otdinary differential problems that can theoretically be solved in parallel as long as suitable hardware means are at our disposal.

The process can be the following:

(1) Parallelization in the time variable — Now, to go from t^n to t^{n+1}, we solve in parallel the nonlinear problem

$$\begin{cases} \dfrac{1}{\tau}(\mathbf{u}^{n,a}-\mathbf{u}^n) - \nu\Delta\mathbf{u}^{n,a} + (\mathbf{u}^{n,a}\cdot\nabla)\mathbf{u}^{n,a} = \mathbf{f}^{n+1} - \dfrac{1}{\rho}\nabla p^n \\ \mathbf{u}^{n,a}|_{\partial\Omega} = 0 \end{cases} \tag{3.9}$$

and the linear system

$$\begin{cases} \dfrac{1}{\tau}(\mathbf{u}^{n,b}-\mathbf{u}^n) - \nu\Delta\mathbf{u}^{n,b} + \dfrac{1}{\rho}\nabla p^{n,b} = \mathbf{f}^{n+1} - (\mathbf{u}^{n,a}\cdot\nabla)\mathbf{u}^{n,a}, \;\; \nabla\cdot\mathbf{u}^{n,b} = 0 \\ \mathbf{u}^{n,b}|_{\partial\Omega} = 0 \end{cases} \tag{3.10}$$

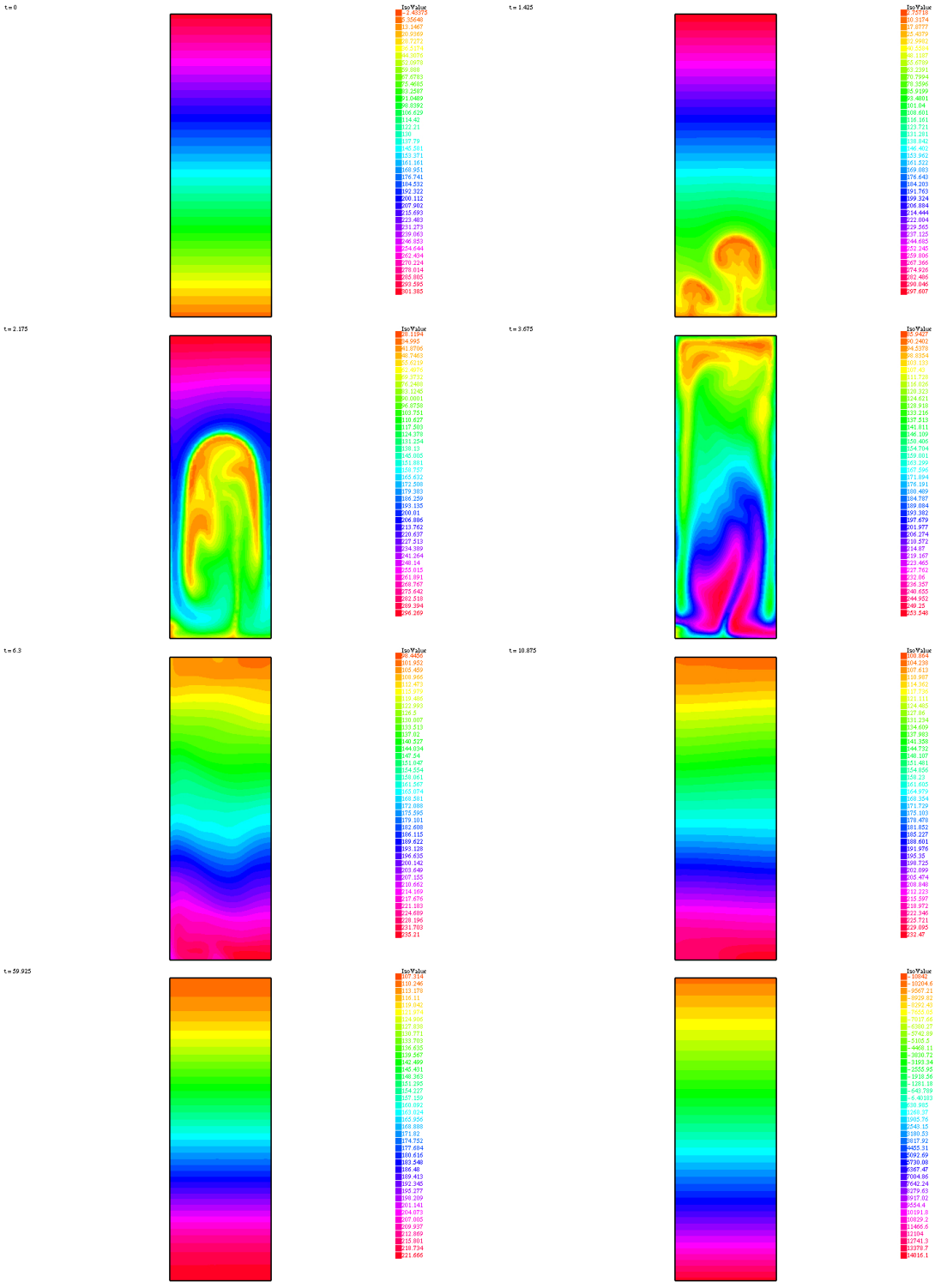


FIGURE 7. Evolution of the density along the time interval $(0,60)$. From left to right and top to bottom: $\rho(\cdot,t)$ at times $t = 0$ (initial state), $1.15, 2.175, 3.75, 6.3, 10.875, 59.925$ and, finally, p at time $t = 59.925$.

and then we proceed with the coordination step

$$\mathbf{u}^{n+1} = \frac{1}{2}(\mathbf{u}^{n,a} + \mathbf{u}^{n,b}), \quad p^{n+1} = p^{n,b}. \tag{3.11}$$

(2) Parallelization in the space variable — As before, the previous Burgers-like and Stokes-like problems can be reduced to families of scalar Poisson problems (3.5). Each of them can be decomposed to give a collection of ordinary differential problems and a *simultaneous directions* algorithm allows to compute a numerical solution.

Specifically, let us fix $\omega \in (0,2)$ and $\kappa \in (0,1)$, let us introduce the notation $Pw := -\alpha\Delta w + \beta w$ and set $P = P_1 + \cdots + P_N$ with $P_i := -\alpha\partial_i + \beta/N$. Then, we start from U^0 and, for each $m \geq 0$, we solve in parallel until convergence the two-point boundary value problems

$$\begin{cases} (1+\kappa P_n)U^{m+1,n} = (1-\kappa\sum_{j\neq n} P_j)U^m + \kappa f \\ U^{m+1,n} = 0 \ \text{ at the endpoints} \end{cases} \tag{3.12}$$

for $n = 1, \ldots N$, together with the coordination steps

$$U^{m+1} = \frac{\omega}{N}\sum_{n=1}^{N} U^{m+1,n} + (1-\omega)U^m. \tag{3.13}$$

A complete formulation, a convergence analysis and several satisfactory numerical experiments can be found in [1, 10, 11].

In particular, the following result is proved in [11]:

Theorem 3.2. *The conclusions in Theorem 3.1 still hold for the time approximation* (3.9)–(3.11) *with the nonlinear term modified by*

$$(\mathbf{u}^{n,a}\cdot\nabla)\mathbf{u}^{n,a} + \frac{1}{2}(\nabla\cdot\mathbf{u}^{n,a})\,\mathbf{u}^{n,a}$$

and the spatial approximation given by (3.12)–(3.13).

It can be shown that, even with a moderate number of processors, there is a considerable computational gain of this method. For instance, for the 2D cavity problem with Re = 4000, the *parallel efficiency* associated to an experiment with 8 processors is 0.6. This means that the computational time is around $1/5$ of the time corresponding to the (sequential) Peaceman-Rachford scheme (3.3)–(3.4).

Again, the scheme leads to very good numerical results. Some of them are shown in Fig. 8–12.

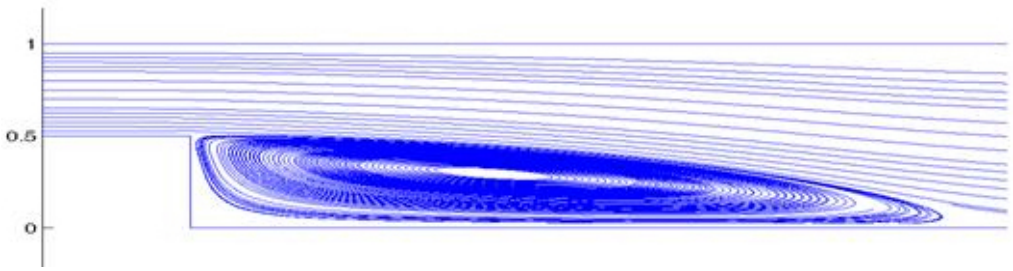


FIGURE 8. The streamlines for the step test, with $Re = 191$.

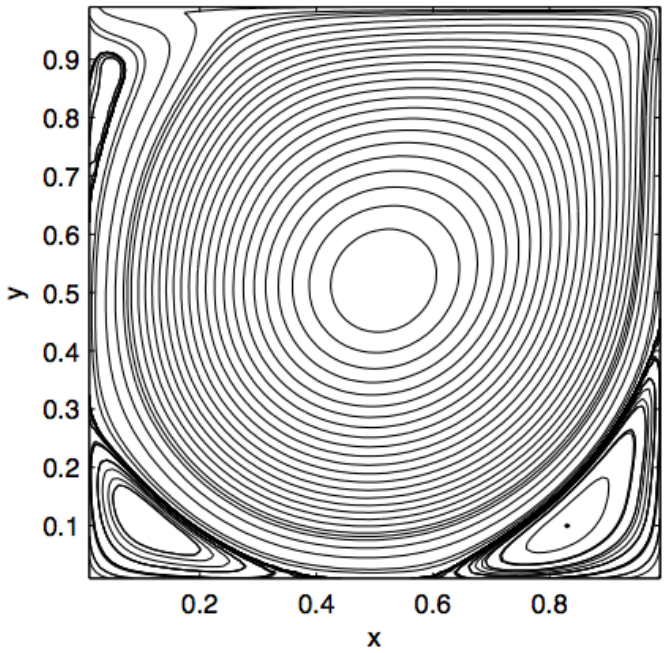


FIGURE 9. The 2D cavity test with $Re = 4000$ — Streamlines. Upper velocity is parallel to $\mathbf{e}_1$.

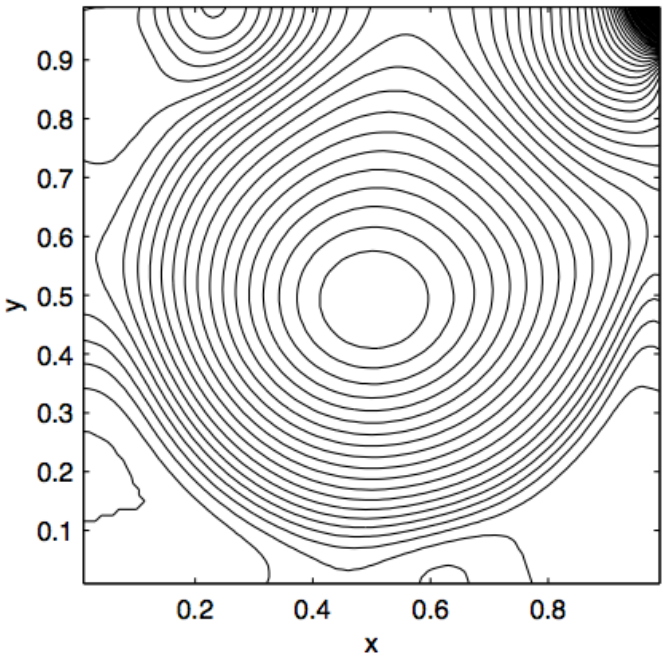


FIGURE 10. The 2D cavity test with $Re = 4000$ — Isobars. Upper velocity is parallel to $\mathbf{e}_1$.

3.3. **Splitting and Monge-Ampère equations.** In this section, I will recall a numerical method introduced by Glowinski to solve the stationary *Monge-Ampère* equation in two and three dimensions, see [22, 23].

It will be seen that one of the main ideas is to reformulate the problem as the search of a time-independent solution to a nonlinear evolution system and, then, perform suitable splitting.

The problem under consideration is the following:

$$\begin{cases} \det \mathbf{D}^2 u = f(\mathbf{x}) & \text{in } \Omega \\ u = g(\mathbf{x}) & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Once more, $\Omega \subset \mathbb{R}^N$ is a bounded connected open set ($N = 2$ or $N = 3$) and, for instance, $f \in L^2(\Omega)$ and $g \in H^{3/2}(\partial\Omega)$, with $f \geq 0$ and $g \geq 0$.

The Monge-Ampère equation can be applied to the resolution of problems in differential geometry, optimal transportation, theoretical physics, etc.; see for instance [2, 17, 24], where the existence and regularity properties of the solutions have also been discussed. Several attemps to produce efficient numerical approximations can be found in [6, 36].

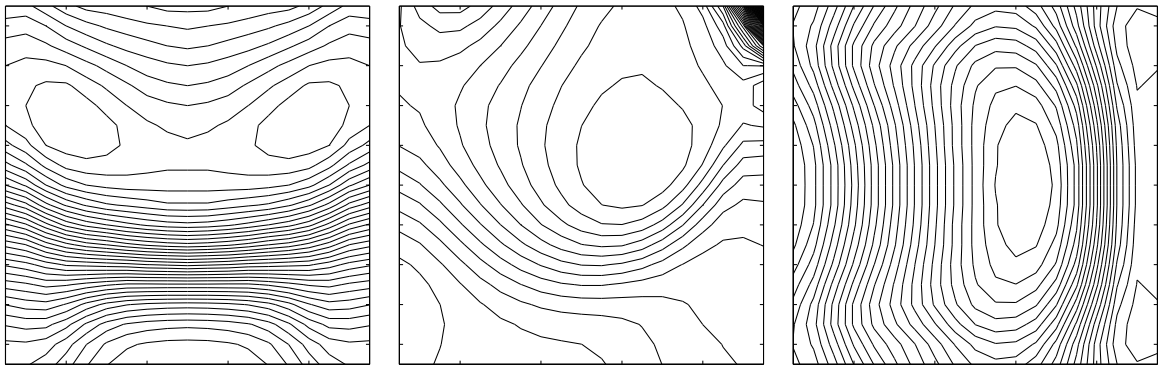
In order to solve (3.14), the first step is to rewrite the problem in the equivalent form

$$\begin{cases} -\nabla \cdot \big(\mathbf{cof}\,(\mathbf{D}^2 u)\nabla u\big) = -2f(\mathbf{x}) & \text{in } \Omega \\ u = g(\mathbf{x}) & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

where $\mathbf{cof}\,(\mathbf{D}^2 u)$ denotes the *co-factor matrix* of $\mathbf{D}^2 u$.

Then, with the help of a small parameter $\varepsilon > 0$, the system is approximated by a *regularized* mixed problem for u and $\mathbf{p} := \mathbf{D}^2 u$:

$$\begin{cases} -\nabla \cdot ((\varepsilon \mathbf{I} + \mathbf{cof}\,(\mathbf{p}))\nabla u) = -2f(\mathbf{x}) & \text{in } \Omega \\ u = g(\mathbf{x}) & \text{on } \partial\Omega \\ \mathbf{p} - \mathbf{D}^2 u = 0 & \text{in } \Omega. \end{cases} \tag{3.16}$$



(a) $x_1 = 0.5$ (b) $x_2 = 0.5$ (c) $x_3 = 0.5$

FIGURE 11. The 3D cavity test with $Re = 1000$ — Isobars. Upper velocity is parallel to $\mathbf{e}_1$.

To solve (3.16), the following time-dependent analog is considered:

$$\begin{cases} u_t - \nabla \cdot ((\varepsilon \mathbf{I} + \mathbf{cof}\,(\mathbf{p}))\nabla u) = -2f(\mathbf{x}) & \text{in } \Omega \times (0, +\infty) \\ u = g(\mathbf{x}) & \text{on } \partial\Omega \times (0, +\infty) \\ \mathbf{p}_t + \gamma(\mathbf{p} - \mathbf{D}^2 u) = 0 & \text{in } \Omega \times (0, +\infty) \\ u|_{t=0} = u_0, \;\; \mathbf{p}|_{t=0} = \mathbf{p}_0 & \text{in } \Omega, \end{cases} \tag{3.17}$$

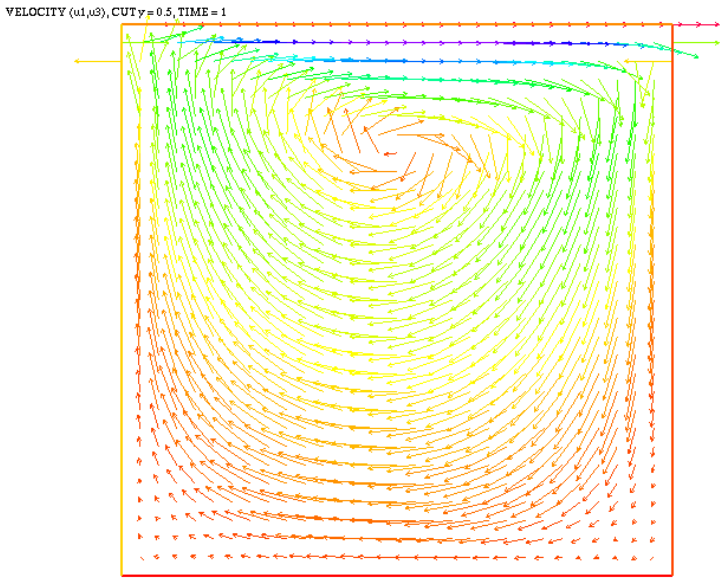


FIGURE 12. The 3D cavity test with $Re = 1000$ — Velocity field. Upper velocity is parallel to $\mathbf{e}_1$.

with carefully chosen $\gamma > 0$, u_0 and $\mathbf{p}_0$.

Finally, a splitting method is applied to integrate in time and compute a stationary solution. More precisely, to go from t^n to t^{n+1}, we first solve the linear elliptic problem

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} - \nabla\cdot\left((\varepsilon\mathbf{I}+\mathbf{cof}\,(\mathbf{p}^n))\nabla u^{n+1}\right) = -2f(\mathbf{x}) & \text{in } \Omega \\ u^{n+1} = g(\mathbf{x}) & \text{on } \partial\Omega \end{cases} \tag{3.18}$$

and then we take

$$\mathbf{p}^{n+1} = \mathbb{P}_+\left(e^{-\gamma\tau}\mathbf{p}^n + (1-e^{-\gamma\tau})\mathbf{D}^2u^{n+1}\right), \tag{3.19}$$

where $\mathbb{P}_+$ is a pointwise projector on the convex cone of symmetric positive semi-definite $N\times N$ matrices.

The reader can find in [22] and [23] details on the spatial approximation of these problems. Several numerical experiments can also be found there.

These arguments can be adapted to the numerical solution of other fully nonlinear elliptic equations. One of them is the so called *Gauss curvature problem*

$$\begin{cases} \det\mathbf{D}^2u = K(\mathbf{x})(1+|\nabla u|^2)^{1+N/2} & \text{in } \Omega \\ u = g(\mathbf{x}) & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

Here, the motivation is the search of a $N+1$-dimensional manifold described by an equation of the form $x_{N+1} = u(\mathbf{x})$ whose Gaussian curvature $K = K(\mathbf{x})$ and boundary shape are prescribed. For some theoretical and numerical results, see [5, 26].

Steps similar to those above, lead to the following iterates: to go from t^n to t^{n+1}, first solve

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} - \nabla\cdot\left((\varepsilon\mathbf{I}+\mathbf{cof}\,(\mathbf{p}^n))\nabla u^{n+1}\right) = -2K(\mathbf{x})(1+|\nabla u^{n+1}|^2)^{1+N/2} & \text{in } \Omega \\ u^{n+1} = g(\mathbf{x}) & \text{on } \partial\Omega \end{cases} \tag{3.21}$$

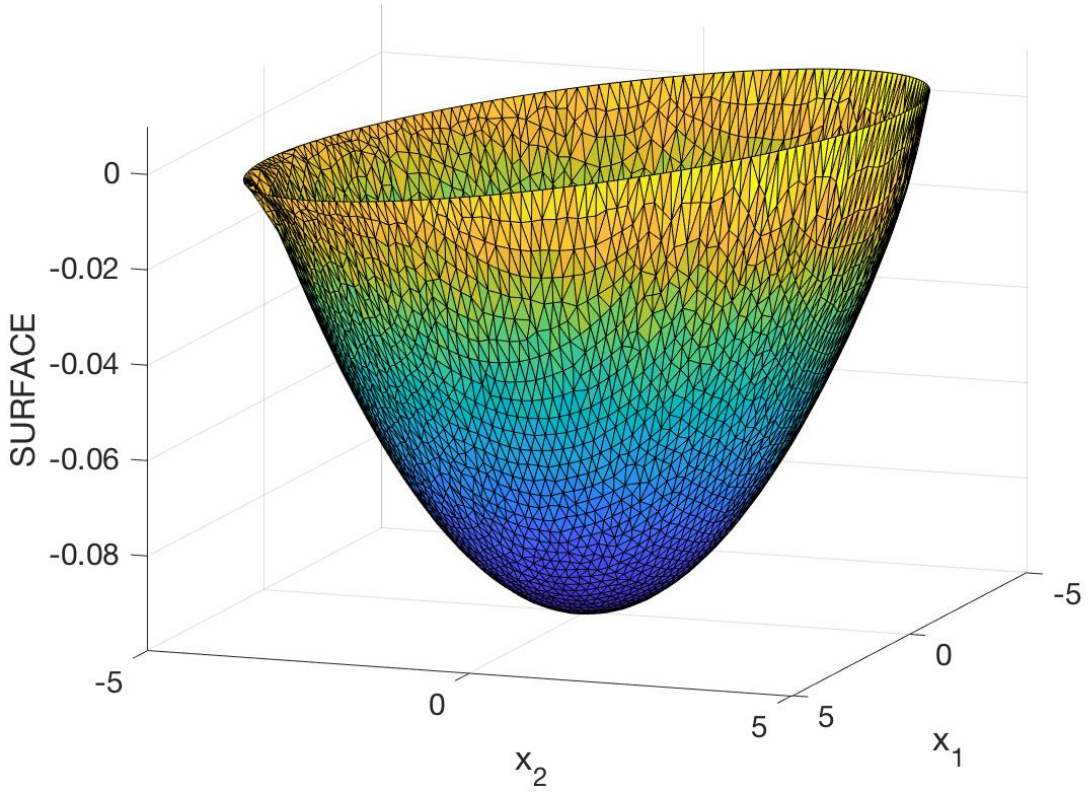


FIGURE 13. The computed solution to (3.20) in the ball $B((0,0);5)$ with $K(\mathbf{x}) = K_0(25-|\mathbf{x}|^2)$, $g(\mathbf{x}) = g_0 x_2$, $K_0 = 10^{-4}$ and $g_0 = 2\cdot 10^{-3}$.

and then take $\mathbf{p}^{n+1}$ as in (3.19).

A rather natural way to compute the solution to (3.21) is to consider an equivalent least-squares reformulation and carry out (for example) conjugate gradient iterates. This reduces the task to the solution of a (large) family of linear elliptic problems that can be approximated as in the previous sections, in a completely standard way. Some numerical results obtained this way are exhibited in Fig 13 and 14. A more complete analysis, with 2D and 3D experiments, will appear in [14].

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REFERENCES

[1] Albarreal I.I. and others. Time and space parallelization of the Navier-Stokes equations, Comput. Appl. Math. 24 (2005) 417–438.
[2] Aubin, T. Nonlinear analysis on manifolds. Monge-Ampère equations, Fundamental Principles of Mathematical Sciences, 252. Springer-Verlag, New York, 1982.
[3] Berestycki, H., Brézis, H. On a free boundary problem arising in plasma physics, Nonlinear Anal. 4 (1980) 415–436.

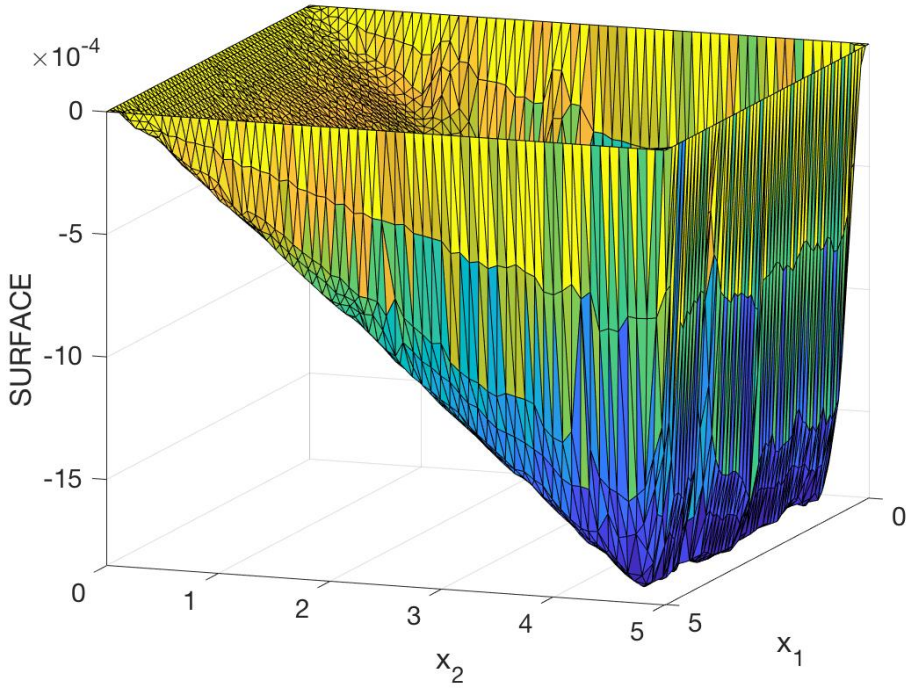

FIGURE 14. The computed solution to (3.20) in the square $(0,5)\times(0,5)$ with $K(\mathbf{x}) = K_0 x_2$, $K_0 = 10^{-5}$ and $g(\mathbf{x}) = 0$.

[4] Berestycki, H., Fernández-Cara, E., Glowinski, R. A numerical study of some questions in vortex rings theory, RAIRO Anal. Numér. 18 (1984) 7–85.
[5] Böhmer, K. Numerical methods for nonlinear elliptic differential equations. A synopsis, Numerical Mathematics and Scientific Computation. Oxford University Press, Oxford, 2010.
[6] Bouchiba, M. Numerical solution of Monge-Ampère equation, Int. J. Appl. Math. 20 (2007) 1–10.
[7] E. Caquot, E., Fernández-Cara, E., Marrocco, A. Résolution numérique d'un problème non linéaire lié à la diffusion d'impuretés dans du silicium, Proc. of the 5th Symposium on computing methods in science and engeneering, Versailles, Dec. 14–18, 1981.
[8] Chorin, A.J. A numerical method for solving incompressible viscous flow problems, J. Comput. Phys. 2 (1967) 12–26.
[9] Chorin, A.J. Numerical solution of the Navier-Stokes equations, Math. Comp. 22 (1968) 745–762.
[10] Cruz-Soto and others. A parallel algorithm for solving the incompressible Navier-Stokes problem, Comput. Math. Appl. 25 (1993) 51–58.
[11] Cruz-Soto and others. A convergence result for a parallel algorithm for solving the Navier-Stokes equations, Comput. Math. Appl. 35 (1998) 71–88.
[12] Cipolatti, R., Damlamian, A. Nonconvex duality for a free boundary problem: the plasma equilibrium, Contributions to nonlinear partial differential equations (Madrid, 1981), 130–145, Res. Notes in Math., 89, Pitman, Boston, MA, 1983.
[13] Fernández-Cara, E.. Sur l'existence de solutions d'un problème d'évolution apparaissant en physique des semi-conducteurs, RR-0079, INRIA. 1981 [inria-00076482].
[14] Fernández-Cara, E. An operator-splitting method for the numerical solution of the 2D and 3D Minkowski problems, in preparation.
[15] Fernández-Cara, E. Motivation, analysis and control of the variable density Navier-Stokes equations, Discrete Contin. Dyn. Syst. Ser. S 5 (2012) 1021–1090.

[16] Fernández-Cara, E., Marín-Beltrán, M. The convergence of two numerical schemes for the Navier-Stokes equations, Numer. Math. 55 (1989) 33–60.
[17] Figalli, A. The Monge-Ampère equation and its applications, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2017.
[18] Fraenkel, L.E.; Berger, M.S. A global theory of steady vortex rings in an ideal fluid, Acta Math. 132 (1974), 13–51.
[19] Gabay, D. Applications of the method of multipliers to variational inequalities, In: "Augmented Lagrangian methods. Applications to the numerical solution of boundary value problems", Fortin, M. and Glowinski, R. eds., Studies in Mathematics and its Applications, 15. North-Holland Publishing Co., Amsterdam, 1983.
[20] Glowinski, R. Finite element methods for incompressible viscous flow, Handbook of numerical analysis, Handbook of numerical analysis, Vol. IX, 3–1176, North-Holland, Amsterdam, 2003.
[21] Glowinski, R., Osher, Stanley J.; Yin, W. Splitting methods in communication, imaging, science, and engineering, Sci. Comput., Springer, Cham, 2016.
[22] Glowinski, R. and others. A finite element/operator-splitting method for the numerical solution of the two dimensional elliptic Monge-Ampère equation, J. Scientific Computing 79 (2019) 1–47.
[23] Glowinski, R. and others. A finite element/operator-splitting method for the numerical solution of the three dimensional elliptic Monge-Ampère equation, J. Sci. Computing 781 (2019) 2271-2302.
[24] Gutiérrez, C.E. The Monge-Ampère equation, Second edition, Progress in Nonlinear Differential Equations and their Applications, 89. Birkhäuser/Springer, Cham, 2016.
[25] Layton, W.J., Maubach, J.M.; Rabier, P.J. Parallel algorithms for maximal monotone operators of local type, Numer. Math. 71 (1995) 29–58.
[26] Lin, M., Trudinger, N.S. The Dirichlet problem for the prescribed curvature quotient equations, Topol. Methods Nonlinear Anal. 3 (1994), 307–323.
[27] Lions, P.-L., Mercier, B. Splitting algorithms for the sum of two nonlinear operators, SIAM J. Numer. Anal. 16 (1979) 964–979.
[28] Marrocco, A. Simulations numériques dans la fabrication des circuits a semi-conducteurs (Process modelling), RR-0305, INRIA. 1984 [inria-00076252].
[29] Mock, M.S. Analysis of mathematical models of semiconductor devices, Advances in Numerical Computation Series, 3. Boole Press, Dún Laoghaire, 1983.
[30] Mock, M.S. Some recent results and open questions in numerical simulation of semiconductor devices. Computing methods in applied sciences and engineering, VI (Versailles, 1983), 713–728, North-Holland, Amsterdam, 1984.
[31] Norbury, J. A steady vortex ring close to Hill's spherical vortex, Proc. Cambridge Philos. Soc. 72 (1972), 253–284.
[32] Puel, J.-P. Un problème de valeur propre non linéaire et de frontière libre, Journées d'Analyse Non Linéaire (Proc. Conf., Besançon, 1977), pp. 188–204, Lecture Notes in Math., 665, Springer, Berlin, 1978.
[33] Strang, G. On the construction and comparison of difference schemes, SIAM J. Numer. Anal. 5 (1968) 506–517.
[34] Témam, R. Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires I, Arch. Rational Mech. Anal. 32 (1969) 135–153.
[35] Témam, R. Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires LI, Arch. Rational Mech. Anal. 33 (1969) 377–385.
[36] Wang, Ch. Numerical and theoretical results for the real Monge-Ampère equations, Ph. D. Thesis, University of Connecticut, ProQuest LLC, Ann Arbor, MI, 1995.