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GENERALIZATION OF THE ELVIS PROBLEM

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This paper is dedicated to the memory of a great friend and mathematician Jack Warga

Abstract. The (classical) Elvis problem refers to a particular type of minimum time problem in which the control dynamics are piece-wise constant and isotropic on two mediums separated by an interface. The somewhat impertinent nomenclature refers to an observation by Timothy Pennings [1] whose dog (named Elvis) enjoyed fetching an object thrown from the shore of Lake Michigan into the water. Elvis was observed to retrieve the object by going in a path that resembled how light would refract (according to Snell's Law) in isotropic mediums. The problem is first generalized to allow for anisotropic velocity sets that are closed, convex, bounded and with 0 in its interior. Tools of Convex Analysis are employed to characterize optimum movement. Further generalizations are then considered with potentially having faster movement on the interface and with more than two mediums.

Keywords. Anisotropic mediums; Convex Optimization; Fully Convex Control; Minimum time problems.

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1. Introduction

Suppose that $\mathrm{M}_0, \mathrm{M}_1 \subseteq \mathbb{R}^2$ are two open half-spaces with a common defining vector $\mathrm{n} \in \mathbb{R}^2$ and level $r \in \mathbb{R}$:

$$\mathrm{M}_0 := \mathrm{H}_{\mathrm{n}}^{<r} := \left\{ \mathrm{x} := \begin{pmatrix} x \\ y \end{pmatrix} : \, x, y \in \mathbb{R}, \, \langle \mathrm{n}, \mathrm{x} \rangle < r \right\}$$

(where $\langle \cdot, \cdot \rangle$ is the usual inner product) and $\mathrm{M}_1 := \mathrm{H}_{\mathrm{n}}^{>r}$ (similarly defined) with closures intersecting at $\Sigma := \mathrm{H}_{\mathrm{n}}^{=r} := \mathrm{H}_{\mathrm{n}}^{\leq r} \bigcap \mathrm{H}_{\mathrm{n}}^{\geq r}$. We use this notation later for half-spaces in $\mathbb{R}^n$, but first consider $n = 2$, $\mathrm{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $r = 0$ so that M_0 is the lower half-plane, M_1 the upper half-plane and Σ the x-axis (see Fig. 1).

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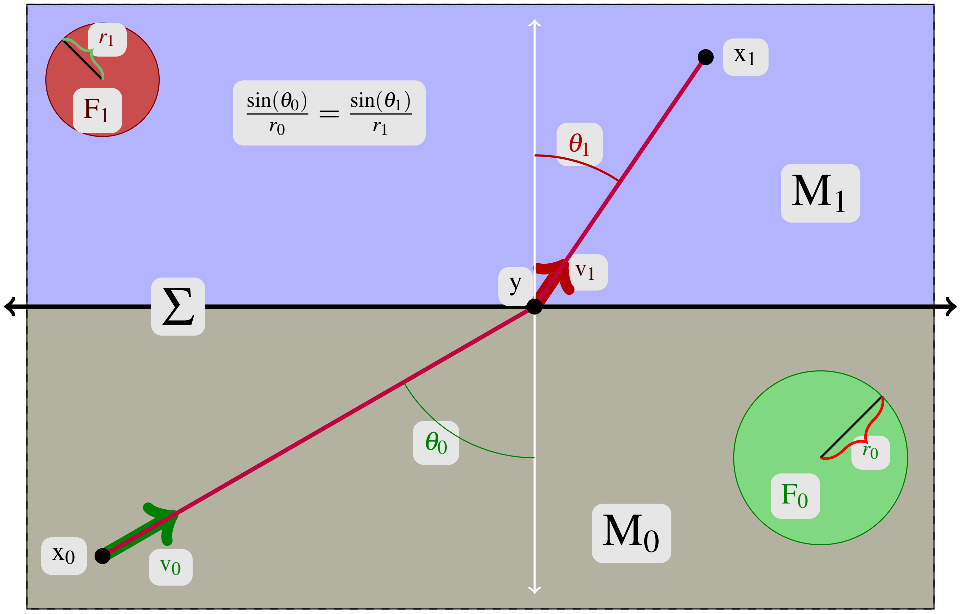

FIGURE 1. The angles of incidence and Snell's Law

1.1. **Classical Elvis problem.** Each half-space is endowed with a speed parameter r_i ($i = 0, 1$) that dictates how fast Elvis can move in that medium, and it is assumed Elvis can go that speed in any direction (the mediums are isotropic). Let $\mathrm{F}_0 := r_0\overline{\mathrm{B}}$ (where $\overline{\mathrm{B}}$ is the unit ball) and $\mathrm{F}_1 := r_1\overline{\mathrm{B}}$ be the respective velocity sets on M_0, M_1. Given $\mathrm{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathrm{M}_0$ and $\mathrm{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \mathrm{M}_1$, the classical Elvis problem is to find a feasible path connecting x_0 to x_1 in minimal time, whereby feasible means velocities of the path can be taken from F_0 while it is in M_0 and from F_1 while in M_1. The path has to cross the interface Σ at some point y. The shortest time it can leave x_0 and get to y is $T_0(\mathrm{y}) := \frac{1}{r_0}\|\mathrm{y} - \mathrm{x}_0\|$, and similarly the shortest time to move from y into M_1 and arrive at x_1 is $T_1(\mathrm{y}) := \frac{1}{r_1}\|\mathrm{x}_1 - \mathrm{y}\|$. The goal is then to minimize $T(\mathrm{y}) := T_0(\mathrm{y}) + T_1(\mathrm{y})$ over $\mathrm{y} \in \Sigma$. A reduction is possible since $\mathrm{y} = \begin{pmatrix} x \\ 0 \end{pmatrix}$ is on the x-axis, and the problem becomes an elementary calculus problem of minimizing

$$T(x) = \frac{1}{r_0}\sqrt{(x - x_0)^2 + y_0^2} + \frac{1}{r_1}\sqrt{(x_1 - x)^2 + y_1^2}$$

over $x \in \mathbb{R}$. Taking the derivative, setting it equal to 0, and solving for $\bar{x}$ produces the classical Snell's Law:

$$\frac{1}{r_0}\frac{\bar{x} - x_0}{\sqrt{(\bar{x} - x_0)^2 + y_0^2}} = \frac{1}{r_1}\frac{x_1 - \bar{x}}{\sqrt{(x_1 - \bar{x})^2 + y_1^2}} \quad \Rightarrow \quad \frac{\sin(\theta_0)}{r_0} = \frac{\sin(\theta_1)}{r_1},$$

where θ_0, θ_1 are the angles of incidence. See Figure 1. This condition is also sufficient for optimality of $\bar{x}$ due to the convexity of $T(\cdot)$, but one should note that Snell's Law by itself does not provide an obvious means to find the solution $\bar{x}$ – calculus did that.

1.2. **Generalized Elvis problem.** We first generalize this problem from balls to using so-called Elvis velocity sets $\mathrm{F} \subseteq \mathbb{R}^n$ that are nonempty, closed, convex, bounded, and contain 0 in its interior. Given $\mathrm{v} \in \mathbb{R}^n$ and such F, the shortest time to traverse from 0 to v using velocities from F is recorded by the gauge function

$$\gamma_{\mathrm{F}}(\mathrm{v}) := \inf\left\{t > 0 : \frac{1}{t}\mathrm{v} \in \mathrm{F}\right\} = \inf\{t > 0 : \mathrm{v} \in t\mathrm{F}\}.$$

With general Elvis velocity sets F_i associated to M_i and $\mathrm{x}_i \in \mathrm{M}_i$ ($i = 0, 1$), we consider the (anisotropic) problem

$$\inf\left[\gamma_{\mathrm{F}_0}(\mathrm{y}-\mathrm{x}_0) + \gamma_{\mathrm{F}_1}(\mathrm{x}_1-\mathrm{y})\right] \quad \text{over } \mathrm{y} \in \Sigma. \tag{$P_{\mathrm{x}_0,\mathrm{x}_1}$}$$

This is a convex optimization problem for which the tools of Convex Analysis are amply suited for a complete investigation. In fact, one may say this is a prototype of a Convex Optimization problem with the objective function and constraints separated in a manner that Fenchel Duality transparently applies. Related to our formulation is Gauge Optimization, introduced by Freund [2] and further developed by Friedlander et al. [3], but these do not appear to be directly relevant to our development since our objective function is the sum of gauges with an affine constraint.

Our main theorem provides optimality conditions for a given y to solve $(P_{\mathrm{x}_0,\mathrm{x}_1})$:

Theorem 1. A necessary and sufficient condition for $\mathrm{y} \in \mathbb{R}^n$ to solve $(P_{\mathrm{x}_0,\mathrm{x}_1})$ is the existence of $\zeta_0, \zeta_1 \in \mathbb{R}^n$ satisfying

$$\begin{aligned} \zeta_0 &\in \partial\gamma_{\mathrm{F}_0}(\mathrm{y}-\mathrm{x}_0), \\ -\zeta_1 &\in \partial\gamma_{\mathrm{F}_1}(\mathrm{x}_1-\mathrm{y}), \quad \text{and} \\ \zeta_0+\zeta_1 &\in -\mathrm{N}_{\Sigma}(\mathrm{y}). \end{aligned}$$

The current paper fills in details of the recent conference paper [9]. A full explanation of the notation is given in the next section, as well as a review of Convex Analysis (CA). We are attempting to make the paper self-contained except for five main theorems of CA which are stated as Theorems 2-6. Most of Proposition 2 is quoted from [10] but is proved here as well; it offers a variety of equivalent primal/dual charaterizations of the subgradients of gauge functions with their polars. Section 3 begins with a discussion of how $(P_{\mathrm{x}_0,\mathrm{x}_1})$ is related to a minimum time control problem, and the proof of Theorem 1 is contained in Section 3.2. A derivation of the classical Snell's Law based on Theorem 1 is provided in Section 3.3 followed in Section 3.4 by a slightly different interpretation of Snell's Law. Polars of Elvis velocity sets are also Elvis velocity sets and Section 4 explores how they can be computed in several circumstances. Section 5 considers special cases of $(P_{\mathrm{x}_0,\mathrm{x}_1})$ restricted to $\mathbb{R}^2$ and Section 6 discusses further generalizations.

2. Preliminaries in Convex Analysis

There are many good texts in Convex Analysis and Optimization [4, 5, 6, 7, 8]. Detailed proofs of Theorems 2-6 below can be found in these and many other texts. Here we offer a simplified introductory review that focuses mainly on bounded convex sets; in particular, issues involving relative interiors and recession properties will be deferred to future work.

The extended real numbers are denoted by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ and have natural arithmetic properties and order relations. The one convention that could be ambiguous but biases our approach to minimization problems and convexity is $\pm\infty \mp \infty = +\infty$. Only $0 \cdot (\pm\infty)$ remains undefined.

The set of all nonempty closed and convex sets of $\mathbb{R}^n$ is denoted by $\mathscr{C}$, and recall this means $\mathrm{S} \in \mathscr{C}$ if and only if $\mathrm{S} \neq \emptyset$ is closed and satisfies

$$\mathrm{x}_0, \mathrm{x}_1 \in \mathrm{S},\ 0 \leq \lambda \leq 1 \quad \Rightarrow \quad (1-\lambda)\mathrm{x}_0 + \lambda \mathrm{x}_1 \in \mathrm{S}. \tag{2.1}$$

Associated to any function $f(\cdot) : \mathbb{R}^n \to \overline{\mathbb{R}}$ is its effective domain $\mathrm{dom}(f) := \{\mathrm{x} : f(\mathrm{x}) < +\infty\}$ and its epigraph $\mathrm{epi}(f) := \{(\mathrm{x}, r)^\top : f(\mathrm{x}) \leq r\} \subseteq \mathbb{R}^{n+1}$. One says $f(\cdot)$ is (i) *lower semicontinuous* (lsc) if $\mathrm{epi}(f)$ is closed; (ii) *proper* if $\mathrm{epi}(f) \neq \emptyset$ and $\{\mathrm{x}\} \times \mathbb{R} \not\subseteq \mathrm{epi}(f)\ \forall \mathrm{x} \in \mathbb{R}^n$; and (iii) *convex* if $\mathrm{epi}(f)$ is a convex set of $\mathbb{R}^{n+1}$. The set of all lsc, proper, convex functions is denoted by $\mathscr{F}$.

Associated to any set $\mathrm{S} \subseteq \mathbb{R}^n$ is the indicator function $\mathrm{I_S}(\cdot) : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\mathrm{I_S}(\mathrm{x}) = \begin{cases} 0 & \text{if } \mathrm{x} \in \mathrm{S} \\ +\infty & \text{if } \mathrm{x} \notin \mathrm{S}. \end{cases}$$

One has $\mathrm{I_S}(\cdot) \in \mathscr{F}$ if and only if $\mathrm{S} \in \mathscr{C}$. The distance function $d_\mathrm{S}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is given by

$$d_\mathrm{S}(\mathrm{z}) = \inf\left\{ \|\mathrm{z} - \mathrm{s}\| : \mathrm{s} \in \mathrm{S} \right\}$$

and is Lipschitz continuous of rank one. The set of projections of $\mathrm{z} \in \mathbb{R}^n$ into S consists of

$$\mathrm{proj_S}(\mathrm{z}) := \left\{ \mathrm{s} \in \mathrm{S} : d_\mathrm{S}(\mathrm{z}) = \|\mathrm{s} - \mathrm{z}\| \right\}.$$

One has $\mathrm{S} \in \mathscr{C}$ if and only if $d_\mathrm{S}(\cdot) \in \mathscr{F}$ if and only if $\mathrm{proj_S}(\mathrm{z})$ is a singleton for all $\mathrm{z} \in \mathbb{R}^n$.

2.1. **The Separation Theorem.** The backbone of convex (perhaps *all*) optimization is

Theorem 2 (Separation Theorem)**.** Suppose $\mathrm{S} \in \mathscr{C}$ and $\mathrm{z} \notin \mathrm{S}$. Then there exists $\zeta \in \mathbb{R}^n$ satisfying

$$\sup\left\{ \langle \zeta, \mathrm{y} \rangle : \mathrm{y} \in \mathrm{S} \right\} < \langle \zeta, \mathrm{z} \rangle. \tag{2.2}$$

Suppose $\mathrm{S} \in \mathscr{C}$. The normal cone $\mathrm{N_S}(\mathrm{x})$ to S at $\mathrm{x} \in \mathrm{S}$ is the set of $\zeta \in \mathbb{R}^n$ satisfying

$$\langle \zeta, \mathrm{y} - \mathrm{x} \rangle \leq 0 \quad \forall \mathrm{y} \in \mathrm{S}. \tag{2.3}$$

If $\mathrm{x} \notin \mathrm{S}$, then $\mathrm{N_S}(\mathrm{x}) = \emptyset$ by definition. A standard proof of the Separation Theorem takes $\zeta := \mathrm{z} - \mathrm{x}$ where $\{\mathrm{x}\} = \mathrm{proj}_S(\mathrm{z})$. It is easy to see then that $\zeta \in \mathrm{N_S}(\mathrm{x})$ and the function $\mathrm{y} \mapsto \langle \zeta, \mathrm{y} \rangle$ is maximized over $\mathrm{y} \in \mathrm{S}$ at x. Moreover, the boundary is characterized:

$$\mathrm{x} \in \mathrm{bdry}(\mathrm{S}) \text{ if and only if } \mathrm{N_S}(\mathrm{x}) \neq \emptyset \text{ and } \neq \{0\}.$$

The closed convex hull $\overline{\mathrm{co}}\,(\mathrm{S})$ of any set $\mathrm{S} \subseteq \mathbb{R}^n$ is the smallest $\mathscr{C}$-type set containing S. The *inner* representation of $\overline{\mathrm{co}}\,(\mathrm{S})$ takes the form

$$\begin{aligned} \overline{\mathrm{co}}\,(\mathrm{S}) &= \mathrm{cl}\left\{ \sum_{i=0}^{n} \lambda_i \mathrm{x}_i : \left\{\mathrm{x}_i\right\}_{i=0}^{n} \subseteq \mathrm{S},\ \left\{\lambda_i\right\}_{i=0}^{n} \in \Lambda_n \right\} \\ &= \mathrm{cl}\left\{ \sum_{i=0}^{k} \lambda_i \mathrm{x}_i : k \in \mathbb{N},\ \left\{\mathrm{x}_i\right\}_{i=0}^{k} \subseteq \mathrm{S},\ \left\{\lambda_i\right\}_{i=0}^{k} \in \Lambda_k \right\}. \end{aligned} \tag{2.4}$$

Here $\Lambda_k := \left\{\{\lambda_i\}_{i=0}^k : \sum_{i=0}^k \lambda_i = 1, \lambda \geq 0\right\}$ refers to the unit k–simplex and Carathéodory's Theorem says there is no loss of generality using only $k = n$. The Separation Theorem provides a dual way to characterize $\overline{\text{co}}\,(\mathrm{S})$ with an *outer* representation:

$$\overline{\text{co}}\,(\mathrm{S}) = \bigcap\left\{\mathrm{H}_\zeta^{\leq r} :\ \zeta \in \mathbb{R}^n, r \in \mathbb{R} \text{ satisfy } \mathrm{S} \subseteq \mathrm{H}_\zeta^{\leq r}\right\}. \tag{2.5}$$

The associated normal concept for convex functions $f(\cdot) \in \mathscr{F}$ is the subgradient $\partial f(\mathrm{x}) \subseteq \mathbb{R}^n$ defined as

$$\partial f(\mathrm{x}) := \left\{\zeta \in \mathbb{R}^n :\ \begin{pmatrix} \zeta \\ -1 \end{pmatrix} \in \mathrm{N}_{\mathrm{epi}(f)}\begin{pmatrix} \mathrm{x} \\ f(\mathrm{x}) \end{pmatrix}\right\}.$$

The subgradient inequality writes this out as

$$\zeta \in \partial f(\mathrm{x}) \quad \Leftrightarrow \quad f(\mathrm{y}) \geq f(\mathrm{x}) + \langle \zeta, \mathrm{y} - \mathrm{x}\rangle \quad \forall\, \mathrm{y} \in \mathbb{R}^n. \tag{2.6}$$

If $f(\cdot) \in \mathscr{F}$ and $\mathrm{x} \notin \mathrm{dom}(f)$, then clearly $\partial f(\mathrm{x}) = \emptyset$. When $\mathrm{S} \in \mathscr{C}$, one has

$$\partial \mathrm{I}_\mathrm{S}(\mathrm{x}) = \mathrm{N}_\mathrm{S}(\mathrm{x}) \quad \forall\, \mathrm{x} \in \mathbb{R}^n.$$

2.2. **The Involutive Theorem.** The Legendre-Fenchel conjugate is defined by

$$f^*(\zeta) = \sup_{\mathrm{x} \in \mathbb{R}^n} \left\{\langle \zeta, \mathrm{x}\rangle - f(\mathrm{x})\right\} \tag{2.7}$$

and belongs to $\mathscr{F}$ when $f(\cdot)$ is proper. We are only interested here when $f(\cdot) \in \mathscr{F}$, in which case, $f^*(\cdot) \in \mathscr{F}$ and its conjugate $f^{**}(\cdot)$ returns to $f(\cdot)$:

Theorem 3 (Involutive Theorem)**.** If $f(\cdot) \in \mathscr{F}$, then for all $\mathrm{x} \in \mathbb{R}^n$, we have

$$f(\mathrm{x}) = f^{**}(\mathrm{x}) = \sup_{\zeta \in \mathbb{R}^n} \left\{\langle \zeta, \mathrm{x}\rangle - f^*(\zeta)\right\}. \tag{2.8}$$

Let $f(\cdot) \in \mathscr{F}$. For any $\mathrm{x}, \zeta \in \mathbb{R}^n$, one always has

$$f(\mathrm{x}) + f^*(\zeta) \geq \langle \zeta, \mathrm{x}\rangle.$$

The next result characterizes when equality holds.

Theorem 4. Suppose $f(\cdot) \in \mathscr{F}$ and x, $\zeta \in \mathbb{R}^n$. The following are equivalent:

(a) $\zeta \in \partial f(\mathrm{x})$;
(b) x attains the supremum of $\sup_{\mathrm{y} \in \mathbb{R}^n} \left\{\langle \zeta, \mathrm{y}\rangle - f(\mathrm{y})\right\}$;
(c) $\mathrm{x} \in \partial f^*(\zeta)$;
(d) ζ attains the supremum of $\sup_{\xi \in \mathbb{R}^n} \left\{\langle \xi, \mathrm{x}\rangle - f^*(\xi)\right\}$;
(e) $f(\mathrm{x}) + f^*(\zeta) = \langle \zeta, \mathrm{x}\rangle$.

2.3. **Optimization and Convex Calculus.** A feature of the subgradient inequality (2.6) is that a point x is a global minimum of $f(\cdot)$ if and only if $0 \in \partial f(\mathrm{x})$. A convex optimization problem has the form

$$\inf f(\mathrm{x}) \quad \text{subject to} \quad \mathrm{x} \in \mathrm{S}. \tag{P}$$

where $f(\cdot) \in \mathscr{F}$ and $\mathrm{S} \in \mathscr{C}$. We assume this is the case and in addition $\mathrm{dom}(f) = \mathbb{R}^n$. Then $(f + \mathrm{I}_\mathrm{S})(\cdot) \in \mathscr{F}$ and will have a global minimum at x (and thereby solve (P)) if and only if $0 \in \partial(f + \mathrm{I}_\mathrm{S})(\mathrm{x})$. We next see how a sum rule can be applied to obtain information directly involving the data $f(\cdot)$ and S. The following result holds in greater generality than stated here, but is all that we will (at first) require.

Theorem 5 (Sum Rule)**.** Suppose $f(\cdot), g(\cdot) \in \mathscr{F}$ and $\mathrm{dom}(f) = \mathbb{R}^n$. Then $(f+g)(\cdot) \in \mathscr{F}$ and $\zeta \in \partial\big(f+g\big)(\mathrm{x})$ if and only if there exists $\zeta_1 \in \partial f(\mathrm{x})$ and $\zeta_2 \in \partial g(\mathrm{x})$ satisfying $\zeta = \zeta_1 + \zeta_2$.

Corollary 1. A point x solves (P) if and only if there exists $\zeta \in \partial f(\mathrm{x})$ with $-\zeta \in \mathrm{N_S}(\mathrm{x})$.

Proof. We have seen x solves (P) if and only if $0 \in \partial(f + \mathrm{I_S})(\mathrm{x})$. The Sum Rule says this holds if and only if there exist $\zeta_1 \in \partial f(\mathrm{x})$ and $\zeta_2 \in \partial \mathrm{I_S}(\mathrm{x})$ with $\zeta_1 + \zeta_2 = 0$. The result follows since $\partial \mathrm{I_S}(\mathrm{x}) = \mathrm{N_S}(\mathrm{x})$. □

Another somewhat surprising result is the following.

Theorem 6. Suppose $f(\cdot) \in \mathscr{F}$ and $\mathrm{dom}(f) = \mathbb{R}^n$. For $\mathrm{x} \in \mathbb{R}^n$, if $\partial f(\mathrm{x}) = \{\zeta\}$ (a singleton), then $f(\cdot)$ is differentiable at x and $\zeta = \nabla f(\mathrm{x})$. Moreover, if $\partial f(\mathrm{x})$ is a singleton for all $\mathrm{x} \in \mathbb{R}^n$, then $f(\cdot)$ is continuously differentiable on $\mathbb{R}^n$.

2.4. **Polars; Gauge and support functions.** Suppose $\mathrm{S} \in \mathscr{C}$. The polar S° of S is defined by

$$\mathrm{S}^\circ := \big\{\zeta : \langle \zeta, \mathrm{y}\rangle \le 1 \;\forall \mathrm{y} \in \mathrm{S}\big\}.$$

One always has $0 \in \mathrm{S}^\circ$, $\mathrm{S}^\circ \in \mathscr{C}$ and $\mathrm{S} \subseteq \mathrm{S}^{\circ\circ}$. A less obvious fact is

Proposition 1. For any $\mathrm{S} \subseteq \mathbb{R}^n$.

$$\overline{\mathrm{co}}\big\{\{0\}, \mathrm{S}\big\} = \mathrm{S}^{\circ\circ} \quad \big(= \big\{\mathrm{y} : \langle \zeta, \mathrm{y}\rangle \le 1 \;\forall \zeta \in \mathrm{S}^\circ\big\}\big). \tag{2.9}$$

Proof. The inclusion $\subseteq$ in (2.9) is clear as just noted, so consider the opposite inclusion. Suppose $\mathrm{z} \notin \overline{\mathrm{co}}\big\{\{0\}, \mathrm{S}\big\} =: \tilde{\mathrm{S}}$, and set $\mathrm{x} := \mathrm{proj}_{\tilde{\mathrm{S}}}(\mathrm{z})$ and $\zeta := \mathrm{z} - \mathrm{x} \neq 0$. Recall (2.3) says $\mathrm{y} \mapsto \langle \zeta, \mathrm{y}\rangle$ is maximized over $\mathrm{y} \in \tilde{\mathrm{S}}$ at $\mathrm{y} = \mathrm{x}$.
<u>*Case 1*</u> $\langle \zeta, \mathrm{x}\rangle = 0$: Then $\langle \zeta, \mathrm{y}\rangle \le 0$ for all $\mathrm{y} \in \tilde{\mathrm{S}}$, and so $r\zeta \in \mathrm{S}^\circ$ for all $r > 0$. We make the choice $r = \frac{2}{\|\zeta\|^2}$, and thus $1 < 2 = \langle r\zeta, \zeta\rangle = \langle r\zeta, \mathrm{z}\rangle$. Therefore $\mathrm{z} \notin \mathrm{S}^{\circ\circ}$.
<u>*Case 2*</u> $\langle \zeta, \mathrm{x}\rangle \neq 0$: Then $\langle \zeta, \mathrm{x}\rangle > 0$ since $0 \in \tilde{\mathrm{S}}$. Set $r := 1/\langle \zeta, \mathrm{x}\rangle > 0$ and note $r\zeta \in N_{\tilde{\mathrm{S}}}(\mathrm{x})$. For all $\mathrm{y} \in \tilde{\mathrm{S}}$, we have

$$\langle r\zeta, \mathrm{y} - \mathrm{x}\rangle \le 0 \quad \Rightarrow \quad \langle r\zeta, \mathrm{y}\rangle \le \langle r\zeta, \mathrm{x}\rangle = 1$$

which implies $r\zeta \in \mathrm{S}^\circ$. On the other hand,

$$0 < \langle r\zeta, \zeta\rangle = \langle r\zeta, \mathrm{z} - \mathrm{x}\rangle \quad \Rightarrow \quad \langle r\zeta, \mathrm{z}\rangle > \langle r\zeta, \mathrm{x}\rangle = r\langle \zeta, \mathrm{x}\rangle = 1$$

and therefore again conclude $\mathrm{z} \notin \mathrm{S}^{\circ\circ}$.

The two cases combine to validate (2.9). It follows immediately from (2.9) that if $0 \in \mathrm{S}$, then $\mathrm{S} = \mathrm{S}^{\circ\circ}$. □

We consider two further assumptions that are "dual" to each other in the sense that

$$\begin{aligned} 0 \in \mathrm{int}(\mathrm{F}) \;&\Leftrightarrow\; \mathrm{F}^\circ \text{ is bounded, and similarly,} \\ \mathrm{F} \text{ is bounded } \;&\Leftrightarrow\; 0 \in \mathrm{int}(\mathrm{F}^\circ). \end{aligned} \tag{2.10}$$

The proofs of these facts are elementary. The collection of all nonempty, closed, convex, bounded sets containing 0 in the interior is denoted by $\mathscr{C}_0$. These are called *Elvis velocity sets*. It is clear that $\mathrm{F} \in \mathscr{C}_0 \Leftrightarrow \mathrm{F}^\circ \in \mathscr{C}_0$.

Two further $\mathscr{F}$-type functions associated with $\mathrm{F} \in \mathscr{C}$ are the gauge $\gamma_{\mathrm{F}}(\cdot)$ and support $\sigma_{\mathrm{F}}(\cdot)$ functions given by

$$\gamma_{\mathrm{F}}(\mathrm{v}) = \inf\left\{t > 0 : \ \frac{1}{t}\mathrm{v} \in \mathrm{F}\right\} = \inf\{t > 0 : \ \mathrm{v} \in t\mathrm{F}\}$$
$$\sigma_{\mathrm{F}}(\zeta) = \sup\big\{\langle \zeta, \mathrm{v}\rangle, \ \mathrm{v} \in \mathrm{F}\big\} = \mathrm{I}_{\mathrm{F}}^{*}(\zeta).$$

It is immediate both $\gamma_{\mathrm{F}}(\cdot)$ and $\sigma_{\mathrm{F}}(\cdot)$ are positive homogeneous, which means $\gamma_{\mathrm{F}}(t\mathrm{v}) = t\gamma_{\mathrm{F}}(\mathrm{v})$ for all $\mathrm{v} \in \mathbb{R}^n$, $t \geq 0$, and similarly for $\sigma_{\mathrm{F}}(\cdot)$. A few more observations are

(i) $\gamma_{\mathrm{F}^\circ}(\zeta) = \sigma_{\mathrm{F}}(\zeta) = (\mathrm{I}_{\mathrm{F}})^{*}(\zeta) \quad \forall \zeta \in \mathbb{R}^n$;
(ii) If $0 \in \mathrm{F}$, then $\gamma_{\mathrm{F}}(\mathrm{v}) = \sigma_{\mathrm{F}^\circ}(\mathrm{v}) \quad \forall \mathrm{v} \in \mathbb{R}^n$;
(iii) If $\mathrm{F} \in \mathscr{C}_0$, then $\mathrm{dom}\big(\gamma_{\mathrm{F}}\big) = \mathbb{R}^n = \mathrm{dom}\big(\gamma_{\mathrm{F}^\circ}\big)$ and $d(\mathrm{x},\mathrm{y}) := \gamma_{\mathrm{F}}(\mathrm{x}-\mathrm{y})$ defines a distance-like function on $\mathbb{R}^n$ but possibly without symmetry.

Indeed, (i) follows since

$$\begin{aligned}\gamma_{\mathrm{F}^\circ}(\zeta) = \inf\left\{t > 0 : \ \frac{1}{t}\zeta \in \mathrm{F}^\circ\right\} &= \inf\left\{t > 0 : \ \left\langle \frac{1}{t}\zeta, \mathrm{v}\right\rangle \leq 1 \ \forall \mathrm{v} \in \mathrm{F}\right\} \\ &= \inf\{t > 0 : \ \langle \zeta, \mathrm{v}\rangle \leq t \ \forall \mathrm{v} \in \mathrm{F}\} = \sup_{\mathrm{v}\in\mathrm{F}} \langle \zeta, \mathrm{v}\rangle \\ &= \sigma_{\mathrm{F}}(\zeta) = \sup_{\mathrm{v}\in\mathbb{R}^n} \big\{\langle \zeta, \mathrm{v}\rangle - \mathrm{I}_{\mathrm{F}}(\mathrm{v})\big\} = (\mathrm{I}_{\mathrm{F}})^{*}(\zeta).\end{aligned}$$

Part (ii) follows from part (i) by interchanging F and F°, which now is valid because (2.9) holds. The first part of (iii) is immediate since $0 \in \mathrm{int}(\mathrm{F})$ and $0 \in \mathrm{int}(\mathrm{F}^\circ)$, and so consider the last statement. Of course one may have $d(\mathrm{x},\mathrm{y}) \neq d(\mathrm{y},\mathrm{x})$, but the triangle inequality holds: For x, y, $\mathrm{z} \in \mathbb{R}^n$, we have (by homogeneity and convexity)

$$d(\mathrm{x},\mathrm{y}) = \gamma_{\mathrm{F}}(\mathrm{x}-\mathrm{y}) = 2\gamma_{\mathrm{F}}\left(\frac{\mathrm{x}-\mathrm{z}}{2} + \frac{\mathrm{z}-\mathrm{y}}{2}\right) \leq 2\left[\frac{\gamma_{\mathrm{F}}(\mathrm{x}-\mathrm{z})}{2} + \frac{\gamma_{\mathrm{F}}(\mathrm{z}-\mathrm{y})}{2}\right] = d(\mathrm{x},\mathrm{z}) + d(\mathrm{z},\mathrm{y}).$$

Our theory can be developed with bounded velocity sets $\mathrm{F} \in \mathscr{C}$ with $0 \in \mathrm{F}$ (rather than $0 \in \mathrm{int}(\mathrm{F})$ as in [11], and will be in future work. The generalizations discussed in later sections will allow optimum movement on the boundary and will necessarily require similar velocity sets in lower dimensions. As is common in CA theory, results valid on $\mathbb{R}^n$ have versions modified to relative affine spaces. Three advantageous properties of $\mathscr{C}_0$ accrue with the additional requirements (2.10), which justify singling them out for this introductory presentation. Firstly, the gauge functions are finite-valued and the boundary of F is easily identified as equaling $\{\mathrm{v} : \ \gamma_{\mathrm{F}}(\mathrm{v}) = 1\}$ (and similarly with F°); secondly, the Sum Rule can be invoked without any further restriction; and thirdly, the main tools contained in the following proposition can be stated without recourse to extended arithmetic and recession functions.

Proposition 2. Let $\mathrm{F} \in \mathscr{C}_0$. One has

$$\partial\gamma_{\mathrm{F}}(0) = \mathrm{F}^\circ \text{ and } \partial\gamma_{\mathrm{F}^\circ}(0) = \mathrm{F}.$$

For any nonzero v, $\zeta \in \mathbb{R}^n$, the following statements are equivalent:

(a) $\langle \zeta, \mathrm{v}\rangle = \gamma_{\mathrm{F}}(\mathrm{v})\gamma_{\mathrm{F}^\circ}(\zeta)$.
(b) $\frac{\mathrm{v}}{\gamma_{\mathrm{F}}(\mathrm{v})}$ maximizes $\mathrm{u} \to \langle \zeta, \mathrm{u}\rangle$ over $\mathrm{u} \in \mathrm{F}$.
(c) $\zeta \in \mathrm{N}_{\mathrm{F}}\left(\frac{\mathrm{v}}{\gamma_{\mathrm{F}}(\mathrm{v})}\right) = \partial\sigma_{\mathrm{F}}\left(\frac{\mathrm{v}}{\gamma_{\mathrm{F}}(\mathrm{v})}\right) = \partial\gamma_{\mathrm{F}^\circ}^{*}\left(\frac{\mathrm{v}}{\gamma_{\mathrm{F}}(\mathrm{v})}\right)$.

(d) $\frac{\zeta}{\gamma_{F^\circ}(\zeta)}$ maximizes $\xi \to \langle \xi, v\rangle$ over $\xi \in F^\circ$.
(e) $v \in N_{F^\circ}\left(\frac{\zeta}{\gamma_{F^\circ}(\zeta)}\right) = \partial\sigma_{F^\circ}\left(\frac{\zeta}{\gamma_{F^\circ}(\zeta)}\right) = \partial\gamma_F^*\left(\frac{\zeta}{\gamma_{F^\circ}(\zeta)}\right)$.
(f) $\frac{v}{\gamma_F(v)} \in \partial\gamma_{F^\circ}(\zeta)$.
(g) $\frac{\zeta}{\gamma_{F^\circ}(\zeta)} \in \partial\gamma_F(v)$.

Proof. Recall $\zeta \in F^\circ$ if and only if $1 \geq \langle \zeta, v\rangle$ for all $v \in F$, and this holds if and only if

$$1 \geq \sup_{0\neq v\in\mathbb{R}^n} \left\langle \zeta, \frac{v}{\gamma_F(v)} \right\rangle \quad \Leftrightarrow \quad \gamma_F(v) \geq \langle \zeta, v\rangle \ \forall v \in \mathbb{R}^n.$$

Since $\gamma_F(0) = 0$ and by the subgradient inequality (2.6), the last inequality is equivalent to $\zeta \in \partial\gamma_F(0)$. A similar argument shows $\partial\gamma_{F^\circ}(0) = F$.

Now suppose v, $\zeta \in \mathbb{R}^n$ are any nonzero elements of $\mathbb{R}^n$. The positive homogeneity of gauge functions implies

$$\gamma_F\left(\frac{v}{\gamma_F(v)}\right) = 1 = \gamma_{F^\circ}\left(\frac{\zeta}{\gamma_{F^\circ}(\zeta)}\right).$$

Therefore $\frac{v}{\gamma_F(v)} \in F$ and $\frac{\zeta}{\gamma_{F^\circ}(\zeta)} \in F^\circ$. Assume (a). We have (using the above Observation (i)) that

$$\left\langle \zeta, \frac{v}{\gamma_F(v)} \right\rangle = \gamma_{F^\circ}(\zeta) = \sigma_F(\zeta) = \sup_{u\in F} \langle \zeta, u\rangle$$

which says (b) holds. The steps are reversible, so in fact (a) and (b) are equivalent. Theorem 4(a) and (b) say these are also equivalent to $\zeta \in \partial I_F\left(\frac{v}{\gamma_F(v)}\right) = N_F\left(\frac{v}{\gamma_F(v)}\right)$, which is (c). The same reasoning with the roles of (v, F) and (ζ, F°) switched shows the equivalence with parts (d) and (e). Finally, Theorem 4(a) and (c) imply the equivalence of (c) and (f), and of (e) and (g). □

2.5. **Strict convexity and smooth boundary.** Notions of strict convexity (for sets and functions) can be stated more generally than is being presented here, but we keep it simple by restricting ourselves to Elvis velocity sets that have finite-valued gauge functions.

Let $F \in \mathscr{C}_0$. Then F is strictly convex provided $\forall\, v \in \mathrm{bdry}(F)$,

$$v = (1-\lambda)v_0 + \lambda v_1,\ v_0,\ v_1 \in F,\ 0 < \lambda < 1 \quad \Rightarrow \quad v = v_0 = v_1.$$

This means there are no "flat" spots on the boundary of F. This property is equivalent to

$$\gamma_F(v_0) = 1 = \gamma_F(v_1),\ v_0 \neq v_1 \quad \Rightarrow \quad \gamma_F(v_\lambda) < 1 \quad \forall \lambda \in (0,1)$$

where $v_\lambda := (1-\lambda)v_0 + \lambda v_1$.

A dual property is that F has smooth boundary, which means $\partial\sigma_{F^\circ}(\zeta)$ is a singleton for all $0 \neq \zeta \in \mathbb{R}^n$. Recall Theorem 6 which says this is equivalent to $\gamma_{F^\circ}(\cdot)$ being continuously differentiable away from 0. In conjunction with Proposition 2, this says for all $v \in \mathrm{bdry}(F)$, $N_F(v) = \mathbb{R}^+ \cdot \zeta$ is a single ray that is continuous in $v \neq 0$, hence the name "smooth boundary". It is a dual property because Observation (ii) implies

$$\begin{aligned} \text{F is strictly convex} &\ \Leftrightarrow\ F^\circ \text{ has smooth boundary} \\ \text{F has smooth boundary} &\ \Leftrightarrow\ F^\circ \text{ is strictly convex}\,. \end{aligned} \tag{2.11}$$

We will not need the corresponding differentiable/strict convex statements for $f(\cdot), f^*(\cdot) \in \mathscr{F}$.

3. The Generalized Elvis problem

3.1. **Reachable sets.** We now return to the Elvis problem and first briefly explain its relationship with control theory. Our preoccupation with generalized Elvis problems would be less interesting without this connection and motivation.

It is well-known that a controlled dynamic equation

$$\begin{aligned} \dot{\mathrm{x}}(t) &= f\big(t, \mathrm{x}(t), \mathrm{u}(t)\big) \quad \text{a.e. } t \in [t_0, t_1] \\ \mathrm{u}(t) &\in \mathrm{U}(t) \quad \text{a.e. } t \in [t_0, t_1] \\ \mathrm{x}(t_0) &= \mathrm{x}_0 \end{aligned}$$

can be equivalently reformulated as a *differential inclusion* (DI)

$$\begin{aligned} \dot{\mathrm{x}}(t) &\in F\big(t, \mathrm{x}(t)\big) \quad \text{a.e. } t \in [t_0, t_1] \\ \mathrm{x}(t_0) &= \mathrm{x}_0 \end{aligned} \tag{3.1}$$

where the data (DI) multifunction is $F(t,\mathrm{x}) = \big\{ f(t,\mathrm{x},\mathrm{u}) : \ \mathrm{u} \in \mathrm{U}(t) \big\}$. The trajectory $\mathrm{x}(\cdot)$ in both systems is absolutely continuous, and by equivalence we mean the state trajectories of the two systems coincide. The (DI) theory can thus be developed with sole attention on the state trajectories $\mathrm{x}(\cdot)$ and suppress mentioning the control variable u at all; this amplifies the importance of various data assumptions.

Of fundamental importance to studying (3.1) is the reachable set (from time t_0 to time $t_1 \geq t_0$ and from $\mathrm{x}_0 \in \mathbb{R}^n$) defined as

$$\mathrm{R}^{t_0,t_1}(\mathrm{x}_0) := \left\{ \mathrm{x}(t_1) \,\middle|\, \mathrm{x}(\cdot) \text{ satisfies (3.1)} \right\}.$$

Uniqueness theorems characterizing the time-parameterized multifunctions $\mathrm{x} \rightrightarrows \mathrm{R}^{t_0,t_1}(\mathrm{x})$ were first proven in [12] and alternatively and independently in [13] and [14]. There are also variations. Even for differential equations, however, an additional property on the data (like a Lipschitz assumption) is required to obtain uniqueness results. Similarly with differential inclusions, one usually requires the multifunction $x \rightrightarrows F(t,\mathrm{x})$ be Lipschitz with respect to the Hausdorff metric; another approach uses a weaker "one-sided" Lipschitz condition [15]. In particular, certain state-discontinuities of multifunctions $F(t,\cdot)$ are problematic with regard to uniqueness issues and in identifying the boundary of $\mathrm{R}^{t_0,t_1}(\mathrm{x}_0)$.

Our approach to the Elvis problem is to consider piece-wise constant multifunctions on two open half-space mediums $\mathrm{M}_0 = \mathrm{H}_\mathrm{n}^{<r}$, $\mathrm{M}_1 = \mathrm{H}_\mathrm{n}^{>r}$ with a common interface $\Sigma := \mathrm{cl}(\mathrm{M}_0) \bigcap \mathrm{cl}(\mathrm{M}_1)$. Assume $F(t,\mathrm{x}) \equiv \mathrm{F}_i \in \mathscr{C}_0$ when $\mathrm{x} \in \mathrm{M}_i$ and for the moment, leave its value on Σ undefined. (See Section 5.1 where this is discussed in detail). Such multifunctions have the simplest type of discontinuity, and for any autonomous problem (i.e. $F(\cdot)$ is independent of t), there is no loss in generality in taking $t_0 = 0$ in (3.1) and denoting $\mathrm{R}^{t_0,t_1}(\mathrm{x}_0)$ as just $\mathrm{R}^T(\mathrm{x}_0)$ (where $T := t_1 - t_0$).

We call a trajectory $\mathrm{x}(\cdot) : [0,T] \to \mathbb{R}^n$ *time-optimum* provided $\mathrm{x}(T) \in \mathrm{bdry}\big(\mathrm{R}^T(\mathrm{x}_0)\big)$. Since we are assuming $0 \in \mathrm{int}(\mathrm{F})$, this is equivalent to $\mathrm{x}(t) \in \mathrm{bdry}\big(\mathrm{R}^t(\mathrm{x}_0)\big)$ for all $0 \leq t \leq T$.

Suppose a medium M is open with a constant velocity set $\mathrm{F} \in \mathscr{C}_0$. Fix $\mathrm{x}_0 \in \mathrm{M}$, $\mathrm{v} \in \mathrm{F}$, and suppose $T > 0$ is sufficiently small so that $\mathrm{x}_0 + T\mathrm{F} \subseteq \mathrm{M}$. Then $\mathrm{x}(t) := \mathrm{x}_0 + t\mathrm{v}$ $(0 \leq t \leq T)$ is a trajectory and is time-optimum if and only if $\mathrm{v} \in \mathrm{bdry}(\mathrm{F})$. Moreover, all points in $\mathrm{R}^T(\mathrm{x}_0)$ can be reached by such a simple straight-line trajectory. To see this, suppose associated to $\mathrm{x}_1 \in \mathrm{R}^T(\mathrm{x}_0)$

is a trajectory $\mathrm{x}(\cdot) := [0,T] \to \mathrm{M}$ with $\mathrm{x}(T) = \mathrm{x}_1$ and $\dot{\mathrm{x}}(t) \in \mathrm{F}$ has exactly two values $\mathrm{v}_0 \neq \mathrm{v}_1$. Set $t_i = m\{t : \dot{\mathrm{x}}(t) = \mathrm{v}_i\}$ $(i = 0, 1)$ where $m(\cdot)$ is Lebesgue measure on $[0,T]$. Then

$$\mathrm{x}_1 = \mathrm{x}_0 + \int_0^T \dot{\mathrm{x}}(s)\,ds = \mathrm{x}_0 + t_0\mathrm{v}_0 + t_1\mathrm{v}_1 = \mathrm{x}_0 + T\left(\frac{t_0}{T}\mathrm{v}_0 + \frac{t_1}{T}\mathrm{v}_1\right) = \mathrm{x}_0 + T\mathrm{v}$$

where $\mathrm{v} := \left(\frac{t_0}{T}\mathrm{v}_0 + \frac{t_1}{T}\mathrm{v}_1\right)$. Of course v belongs to F by convexity. Hence if x_1 can be reached with a trajectory with two distinct velocities, it can be reached by a trajectory with only one velocity. If $\gamma_{\mathrm{F}}(\mathrm{v}) < 1$, then it will be able to be reached in a time strictly less than T by using the velocity $\mathrm{v}/\gamma_{\mathrm{F}}(\mathrm{v}) \in \mathrm{F}$. The argument can be modified to include any finite number of velocities, and since such functions are dense in $L^1[0,T]$, every point in $\mathrm{R}^T(\mathrm{x}_0)$ can be reached by a straight-line trajectory, as claimed.

In the case $\mathrm{x}_1 \in \mathrm{bdry}\left(\mathrm{R}^T(\mathrm{x}_0)\right)$, the straight-line trajectory $\mathrm{x}(t) = \mathrm{x}_0 + t\mathrm{v}$ realizing x_1 will be time-optimum, and that holds if and only if $\mathrm{v} \in \mathrm{bdry}(\mathrm{F})$. Moreover, it is unique if and only if $\partial\gamma_{\mathrm{F}^\circ}(\mathrm{v})$ is a singleton. Hence all straight-line time-optimum trajectories are unique if and only if F is strictly convex.

3.2. **The Problem and its optimality conditions – proof of Theorem 1.** So now we are given two half-spaces $\mathrm{M}_0 := \mathrm{H}_{\mathrm{n}}^{<r}$, $\mathrm{M}_1 := \mathrm{H}_{\mathrm{n}}^{>r}$ of $\mathbb{R}^n$ each having an associated Elvis velocity set F_0, $\mathrm{F}_1 \in \mathscr{C}_0$, and the intersection of their closures is the hyperplane $\Sigma := \mathrm{cl}(\mathrm{M}_0) \cap \mathrm{cl}(\mathrm{M}_1) = \mathrm{H}_{\mathrm{n}}^{=r}$. For given $\mathrm{x}_0 \in \mathrm{M}_0$ and $\mathrm{x}_1 \in \mathrm{M}_1$, the generalized Elvis problem $\left(P_{\mathrm{x}_0,\mathrm{x}_1}\right)$ is equivalent to

$$\inf_{\mathrm{y}\in\mathbb{R}^n}\left\{\gamma_{\mathrm{F}_0}(\mathrm{y}-\mathrm{x}_0) + \gamma_{\mathrm{F}_1}(\mathrm{x}_1-\mathrm{y}) + \mathrm{I}_\Sigma(\mathrm{y})\right\} \qquad \left(P_{\mathrm{x}_0,\mathrm{x}_1}\right)$$

One notes this matches Pennings' formulation if M_0 is the lower half-space, M_1 the upper, and $\mathrm{F}_0 = r_0\overline{\mathrm{B}}$, $\mathrm{F}_1 = r_1\overline{\mathrm{B}}$.

It is easy to see there exists at least one optimum solution y to $\left(P_{\mathrm{x}_0,\mathrm{x}_1}\right)$. Thus

$$0 \in \partial\left[\gamma_{\mathrm{F}_0}((\cdot)-\mathrm{x}_0) + \gamma_{\mathrm{F}_1}(\mathrm{x}_1-(\cdot)) + \mathrm{I}_\Sigma(\cdot)\right](\mathrm{y}).$$

Recall Observation (iii) says $\mathrm{dom}\left(\gamma_{\mathrm{F}_0}\right) = \mathbb{R}^n = \mathrm{dom}\left(\gamma_{\mathrm{F}_1}\right)$, and so the Sum Rule is applicable and produces vectors $\zeta_0, \zeta_1 \in \mathbb{R}^n$ satisfying

$$\zeta_0 \in \partial\left[\gamma_{\mathrm{F}_0}((\cdot)-\mathrm{x}_0)\right](\mathrm{y}), \qquad (3.2)$$

$$\zeta_1 \in \partial\left[\gamma_{\mathrm{F}_1}(\mathrm{x}_1-(\cdot))\right](\mathrm{y}), \quad \text{and} \qquad (3.3)$$

$$\zeta_0 + \zeta_1 \in -\partial\left(\mathrm{I}_\Sigma\right)(\mathrm{y}) = -\mathrm{N}_\Sigma(\mathrm{y}).$$

The following lemma interprets (3.2) and (3.3).

Lemma 7. Suppose $f(\cdot) \in \mathscr{F}$ and $\bar{\mathrm{x}} \in \mathbb{R}^n$.

(a) Let $g_1(\mathrm{x}) := f(\mathrm{x}-\bar{\mathrm{x}})$ and suppose $\mathrm{y} \in \mathrm{dom}(g_1)$. Then $g_1(\cdot) \in \mathscr{F}$ and

$$\partial g_1(\mathrm{y}) = \partial f(\mathrm{y}-\bar{\mathrm{x}}).$$

(b) Let $g_2(\mathrm{x}) := f(\bar{\mathrm{x}}-\mathrm{x})$ and suppose $\mathrm{y} \in \mathrm{dom}(g_2)$. Then $g_2(\cdot) \in \mathscr{F}$ and

$$\partial g_2(\mathrm{y}) = \left\{\zeta : -\zeta \in \partial f(\bar{\mathrm{x}}-\mathrm{y})\right\}.$$

Proof. Consider (a). It is clear that $g_1(\cdot) \in \mathscr{F}$. By (2.6) we have

$$\begin{aligned} \zeta \in \partial g_1(\mathrm{y}) &\Leftrightarrow g_1(\mathrm{x}) \geq g_1(\mathrm{y}) + \langle \zeta, \mathrm{x}-\mathrm{y}\rangle \quad \forall \mathrm{x} \in \mathbb{R}^n \\ &\Leftrightarrow f(\mathrm{x}-\bar{\mathrm{x}}) \geq f(\mathrm{y}-\bar{\mathrm{x}}) + \langle \zeta, \mathrm{x}-\mathrm{y}\rangle \quad \forall \mathrm{x} \in \mathbb{R}^n \\ &\Leftrightarrow f(\mathrm{z}) \geq f(\mathrm{y}-\bar{\mathrm{x}}) + \langle \zeta, \mathrm{z}-(\mathrm{y}-\bar{\mathrm{x}})\rangle \quad \forall \mathrm{z} \in \mathbb{R}^n \\ &\Leftrightarrow \zeta \in \partial f(\mathrm{y}-\bar{\mathrm{x}}). \end{aligned}$$

Now consider (b), and it is clear $g_2(\cdot) \in \mathscr{F}$. Again by (2.6), we have

$$\begin{aligned} \zeta \in \partial g_2(\mathrm{y}) &\Leftrightarrow g_2(\mathrm{x}) \geq g_2(\mathrm{y}) + \langle \zeta, \mathrm{x}-\mathrm{y}\rangle \quad \forall \mathrm{x} \in \mathbb{R}^n \\ &\Leftrightarrow f(\bar{\mathrm{x}}-\mathrm{x}) \geq f(\bar{\mathrm{x}}-\mathrm{y}) + \langle \zeta, \mathrm{x}-\mathrm{y}\rangle \quad \forall \mathrm{x} \in \mathbb{R}^n \\ &\Leftrightarrow f(\mathrm{z}) \geq f(\bar{\mathrm{x}}-\mathrm{y}) + \langle -\zeta, \mathrm{z}-(\bar{\mathrm{x}}-\mathrm{y})\rangle \quad \forall \mathrm{z} \in \mathbb{R}^n \\ &\Leftrightarrow -\zeta \in \partial f(\bar{\mathrm{x}}-\mathrm{y}). \end{aligned}$$

This is (b). □

Using the lemma on (3.2),(3.3) provides necessary and sufficient conditions for y to solve $(P_{\mathrm{x}_0,\mathrm{x}_1})$ as the existence of ζ_0, $\zeta_1 \in \mathbb{R}^n$ satisfying

$$\zeta_0 \in \partial\gamma_{\mathrm{F}_0}(\mathrm{y}-\mathrm{x}_0), \tag{3.4}$$

$$-\zeta_1 \in \partial\gamma_{\mathrm{F}_1}(\mathrm{x}_1-\mathrm{y}), \quad \text{and} \tag{3.5}$$

$$\zeta_0+\zeta_1 \in -\mathrm{N}_\Sigma(\mathrm{y}). \tag{3.6}$$

We now draw on Proposition 2(g) to deduce

$$\gamma_{\mathrm{F}_0^\circ}(\zeta_0) = 1 = \gamma_{\mathrm{F}_1^\circ}(-\zeta_1). \tag{3.7}$$

The Maximum Principle (Proposition 2(b)) also holds:

$$\mathrm{v} \mapsto \langle \zeta_0, \mathrm{v}\rangle \text{ is maximized over } \mathrm{v} \in \mathrm{F}_0 \text{ at } \mathrm{v} = \mathrm{v}_0 \tag{3.8}$$

$$\mathrm{v} \mapsto \langle -\zeta_1, \mathrm{v}\rangle \text{ is maximized over } \mathrm{v} \in \mathrm{F}_1 \text{ at } \mathrm{v} = \mathrm{v}_1, \tag{3.9}$$

where

$$\mathrm{v}_0 := \frac{\mathrm{y}-\mathrm{x}_0}{\gamma_{\mathrm{F}_0}(\mathrm{y}-\mathrm{x}_0)} \in \mathrm{F}_0 \quad \text{and} \quad \mathrm{v}_1 = \frac{\mathrm{x}_1-\mathrm{y}}{\gamma_{\mathrm{F}_1}(\mathrm{x}_1-\mathrm{y})} \in \mathrm{F}_1$$

are the velocities used while an optimum straight-line trajectory is in their respective regions.

3.3. **Derivation of the classical Snell's Law.** Although we derived Snell's Law from elementary calculus in Section 1, we next see how our optimality conditions can do the same thing.

Recall $n=2$, M_0 (resp. M_1) the lower (resp. upper) half-plane, Σ the x-axis, $\mathrm{F}_0 = r_0\overline{\mathrm{B}}$, $\mathrm{F}_1 = r_1\overline{\mathrm{B}}$, where $r_0, r_1 > 0$ are the speed parameters. Recall $\mathrm{F}_i^\circ = \frac{1}{r_i}\overline{\mathrm{B}}$ for $i=0,1$. By (3.7), the ζ_i's in (3.4)-(3.6) can be represented by an angle $\theta_i \in [0,2\pi)$ with

$$\zeta_0 = \frac{1}{r_0}\begin{pmatrix}\sin(\theta_0)\\ \cos(\theta_0)\end{pmatrix} \quad \text{and} \quad -\zeta_1 = \frac{1}{r_1}\begin{pmatrix}\sin(\theta_1)\\ \cos(\theta_1)\end{pmatrix}.$$

Conditions (3.8), (3.9) imply the optimum velocities are

$$\mathrm{v}_0 = r_0\begin{pmatrix}\sin(\theta_0)\\ \cos(\theta_0)\end{pmatrix} \quad \text{and} \quad \mathrm{v}_1 = -r_1\begin{pmatrix}\sin(\theta_1)\\ \cos(\theta_1)\end{pmatrix},$$

which geometrically explain the angles θ_0, θ_1 — see Figure 1. Now $\mathrm{N}_\Sigma(\mathrm{y})$ equals the cone that is the y-axis, and so (3.6) implies

$$\zeta_0+\zeta_1 \in \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{R} \right\}.$$

Therefore the first component of $\zeta_0+\zeta_1$ is equal to 0, or that

$$\frac{\sin(\theta_0)}{r_0} = \frac{\sin(\theta_1)}{r_1}.$$

3.4. **Interpretation of Snell's Law in $\mathbb{R}^2$.** The problem as stated in $(P_{\mathrm{x}_0,\mathrm{x}_1})$ had fixed initial point $\mathrm{x}_0 \in \mathrm{M}_0$ and terminal point $\mathrm{x}_1 \in \mathrm{M}_1$ and asked to find $\mathrm{y} \in \Sigma$ through which a trajectory would pass in least time. One can make a reduction, as is done classically with centered balls, if $\mathrm{bdry}(\mathrm{F})$ has a smooth convex parameterization: $\mathrm{F} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \ a \le x \le b \text{ and } -f(x) \le y \le f(x) \right\}$ where $f(\cdot) : [a,b] \to \mathbb{R}^+$ is continuous convex and smooth on (a,b). Or the roles of x and y can be switched, or modified nonsymmetric cases can be considered. If this is so, then the problem can be solved by elementary calculus. This is the case for ellipses whose centers lie on one of the axes. However, finding optimum points y more generally seems quite difficult since parameterizations and reductionist formulas are not available. We consider a simple algorithm in [16] based on our optimality conditions. Here we suggest a modified formulation of the problem which in essence finds all solutions passing through a point $y \in \Sigma$ that are optimal for a point $\mathrm{x}_1 \in \mathrm{M}_1$.

Turning the problem on its head, we start with a given x_0 and $\mathrm{y} \in \Sigma$ and note the least time to go from x_0 to y is $\gamma_{F_0}(\mathrm{y}-\mathrm{x}_0)$ (assuming this is the Fast-to-Slow scenario - see Section 5.2). The question is then how can (if possible) the trajectory enter M_1 with velocities from F_1 in such a way that the endpoint of the trajectory will remain on the boundary of the reachable set? If x_1 is such an endpoint, then y solves problem $(P_{\mathrm{x}_0,\mathrm{x}_1})$.

4. Calculating some polars

Before going into further details on solving generalized Elvis problems, this section presents techniques and examples to calculate the polar of a given $\mathrm{F} \in \mathscr{C}_0$. This can be surprisingly more difficult than it may first appear because gauge functions do not lend themselves to easy computation. However, support functions can be easily found since they themselves are defined as the value of a convex optimization problem. We can hence find the polar through using Observation (i). We provide details in Section 4.3.

4.1. **Norms.** If $\|\cdot\|$ is any norm on $\mathbb{R}^n$, then the dual space $\mathbb{R}^n$ associated with $(\mathbb{R}^n, \|\cdot\|)$ has a dual norm $\|\cdot\|^*$ defined by

$$\|\zeta\|^* = \sup_{\|\mathrm{x}\|=1} \left|\langle \zeta, \mathrm{x}\rangle\right|$$

If $\mathrm{F} := \left\{ \mathrm{x} \in \mathbb{R}^n : \ \|\mathrm{x}\| \le r \right\} =: r\mathscr{B}$ is an r-ball ($r>0$) centered at 0 in this norm, then

$$\mathrm{F}^\circ := \left\{ \zeta \in \mathbb{R}^n : \ \|\zeta\|^* \le \frac{1}{r} \right\}.$$

If $\mathrm{F} = \mathrm{x}_0 + r\overline{\mathrm{B}}$ (where $\|\mathrm{x}_0\| < r$) is a non-centered ball, then F° has no simple formula. If $\|\cdot\|$ is differentiable away from 0, then F° can be calculated by the methods provided in Section 4.3. At the opposite extreme, if F has the structure of a polytope (Section 4.4), then F° can be obtained with a simple formula. The unit balls in the 1 or ∞-norms are like this.

4.2. **Ellipsoids.** A filled-in centered ellipse in $\mathbb{R}^2$ with axis lengths a, b has the form

$$\mathrm{F} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} :\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}. \tag{4.1}$$

We'll see below that

$$\mathrm{F}^{\circ} := \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} :\ a^2\alpha^2 + b^2\beta^2 \le 1 \right\} \tag{4.2}$$

Similar statements for an ellipsoid in $\mathbb{R}^n$ hold as well:

$$\mathrm{F} := \left\{ \mathrm{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n :\ \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} \le 1 \right\} \quad \Rightarrow \quad \mathrm{F}^{\circ} = \left\{ \zeta = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix} \in \mathbb{R}^n :\ \sum_{i=1}^{n} a_i^2 \zeta_i^2 \le 1 \right\}$$

The formulas for polars of non-centered and rotated ellipsoids are considerably more complicated, but nonetheless can be calculated by similar means (at least in dimension 2) as demonstrated in the next section.

4.3. **Parameterized smooth manifolds.** Now suppose an Elvis velocity set has the form

$$\mathrm{F} := \left\{ \mathrm{x} \in \mathbb{R}^n :\ f(\mathrm{x}) \le 0 \right\} \tag{4.3}$$

where $f(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is C^1 and convex. The polar F° can be found (at least in principle) by recalling Observation (i) that says $\gamma_{\mathrm{F}^{\circ}}(\zeta) = \sigma_{\mathrm{F}}(\zeta)$. We have

$$\zeta \in \mathrm{F}^{\circ} \quad \Leftrightarrow \quad \sigma_F(\zeta) = \gamma_{\mathrm{F}^{\circ}}(\zeta) \le 1.$$

We can (in some instances) explicitly find $\sigma_F(\zeta)$ by solving the convex optimization problem

$$\inf_{\mathrm{v} \in \mathbb{R}^n} \langle -\zeta, \mathrm{v} \rangle \quad \text{subject to } \mathrm{v} \in \mathrm{F}. \tag{P_ζ}$$

and noting $\sigma_F(\zeta) = -\inf(P_\zeta)$.

4.3.1. *Ellipses in* $\mathbb{R}^2$. Consider the case where $f(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(\mathrm{x}) := f\begin{pmatrix} x \\ y \end{pmatrix} := \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$

($a \ne 0$, $b \ne 0$), and so F in (4.3) is the ellipse as in (4.1). Let $\zeta \in \mathbb{R}^2$. To calculate $\sigma_F(\zeta)$, we first find $\operatorname{argmin}\big(\sigma_F(-\zeta)\big)$ by solving the convex optimization problem $\inf\{\langle -\zeta, \mathrm{v} \rangle :\ \mathrm{v} \in F\}$. Written in coordinates $\zeta =: \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $\mathrm{v} =: \begin{pmatrix} x \\ y \end{pmatrix}$, this is

$$\inf \left[-\alpha x - \beta y\right] \quad \text{subject to} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \le 0.$$

The Lagrangian $L(\cdot,\cdot) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is

$$L\left(\begin{pmatrix} x \\ y \end{pmatrix}, \lambda\right) = -\alpha x - \beta y + \lambda \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right].$$

Necessary (and sufficient) conditions consist of the existence of $\lambda \geq 0$ and the vanishing of the gradient of $L(\cdot,\lambda)$ with respect to v:

$$0 = \nabla_{x,y} L\left(\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix},\lambda\right) = \left(-\alpha + \frac{2\bar{x}\lambda}{a^2}, -\beta + \frac{2\bar{y}\lambda}{b^2}\right) \Rightarrow \quad \frac{\alpha a^2}{2\bar{x}} = \lambda = \frac{\beta b^2}{2\bar{y}}$$

or that $\bar{y} = \frac{\beta b^2}{\alpha a^2}\bar{x}$. Using $\frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{b^2} = 1$ gives $\bar{x}^2 = \frac{a^4\alpha^2}{a^2\alpha^2+b^2\beta^2}$ and $\bar{y}^2 = \frac{b^4\beta^2}{a^2\alpha^2+b^2\beta^2}$. Hence

$$\sigma_F(\zeta) = \left\langle \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}\right\rangle = \frac{a^2\alpha^2}{\sqrt{a^2\alpha^2+b^2\beta^2}} + \frac{b^2\beta^2}{\sqrt{a^2\alpha^2+b^2\beta^2}} = \sqrt{a^2\alpha^2+b^2\beta^2}.$$

As mentioned previously, $\mathrm{F}^\circ = \left\{\zeta : \sigma_\mathrm{F}(\zeta) \leq 1\right\}$, and therefore (4.2) holds.

4.4. **Polytopes.** A polytope F is a bounded set of the form

$$\mathrm{F} = \bigcap_{i=1}^{k} \mathrm{H}_{\zeta_i}^{\leq r_i}$$

where $\zeta_1,\ldots,\zeta_k \in \mathbb{R}^n$ and $r_1,\ldots,r_k \in \mathbb{R}$. We can assume $\|\zeta_i\| = 1$ for each i by adjusting r_i. Note

$$0 \in \mathrm{F} \Leftrightarrow r_i \geq 0\, \forall i \quad \text{and} \quad 0 \in \mathrm{int}(\mathrm{F}) \Leftrightarrow r_i > 0\, \forall i.$$

A polytope F is finitely generated ([4], Theorem 19.1) when there are finite many points $\mathrm{v}_1,\ldots,\mathrm{v}_\ell$ (taken here as the extreme points of F) so that $\mathrm{F} = \overline{\mathrm{co}}\{\mathrm{v}_1,\ldots,\mathrm{v}_\ell\}$.

Theorem 8. Let F be a polytope with $0 \in \mathrm{int}(\mathrm{F})$, and suppose the external and internal representations are

$$\mathrm{F} = \bigcap_{i=1}^{k} \mathrm{H}_{\zeta_i}^{\leq r_i} = \overline{\mathrm{co}}\{\mathrm{v}_1,\ldots,\mathrm{v}_\ell\},$$

where each $\|\zeta_i\| = 1$, $r_i > 0$, and $\{\mathrm{v}_1,\ldots,\mathrm{v}_\ell\}$ is the set of extreme points. Then the polar F° has external and internal representations as

$$\mathrm{F}^\circ = \bigcap_{j=1}^{\ell} \mathrm{H}_{\mathrm{v}_j}^{\leq 1} = \overline{\mathrm{co}}\left\{\frac{1}{r_1}\zeta_1,\ldots,\frac{1}{r_k}\zeta_k\right\}. \tag{4.4}$$

Proof. Let S equal the right hand side of (4.4). Clearly for each $i = 1,\ldots,k$ we have for all $\mathrm{v} \in \mathrm{F}$ that

$$\langle\zeta_i,\mathrm{v}\rangle \leq r_i \quad \Leftrightarrow \quad \left\langle \frac{1}{r_i}\zeta_i,\mathrm{v}\right\rangle \leq 1$$

which implies $\frac{1}{r_i}\zeta_i \in \mathrm{F}^\circ$. That $S \subseteq \mathrm{F}^\circ$ follows from the convexity of F°. On the other hand, suppose $\xi \notin S$, and let $\mathrm{v} := \xi - \mathrm{proj}_S(\xi)$. Then there exists $1 \leq i_0 \leq k$ for which

$$\left\langle \frac{1}{r_{i_0}}\zeta_{i_0},\mathrm{v}\right\rangle = \sup_{\zeta\in S}\langle\zeta,\mathrm{v}\rangle < \langle\xi,\mathrm{v}\rangle. \tag{4.5}$$

Since $0 \in \mathrm{int}(\mathrm{F})$, v can be re-scaled to belong to F (replace v by $\mathrm{v}/\gamma_\mathrm{F}(\mathrm{v})$), and by so doing, (4.5) remains valid with the left side equal to 1. This shows $\xi \notin \mathrm{F}^\circ$ and we conclude $\mathrm{F}^\circ = S$. It is clear that S is contained in the intersection in (4.4), and also that the intersection is contained in F°. This implies the equality of all three sets as in (4.4). □

Corollary 2. With F as in Theorem 8, we have $0 \in \mathrm{int}\left(\overline{\mathrm{co}}\left\{\frac{1}{r_1}\zeta_1,\ldots,\frac{1}{r_k}\zeta_k\right\}\right)$.

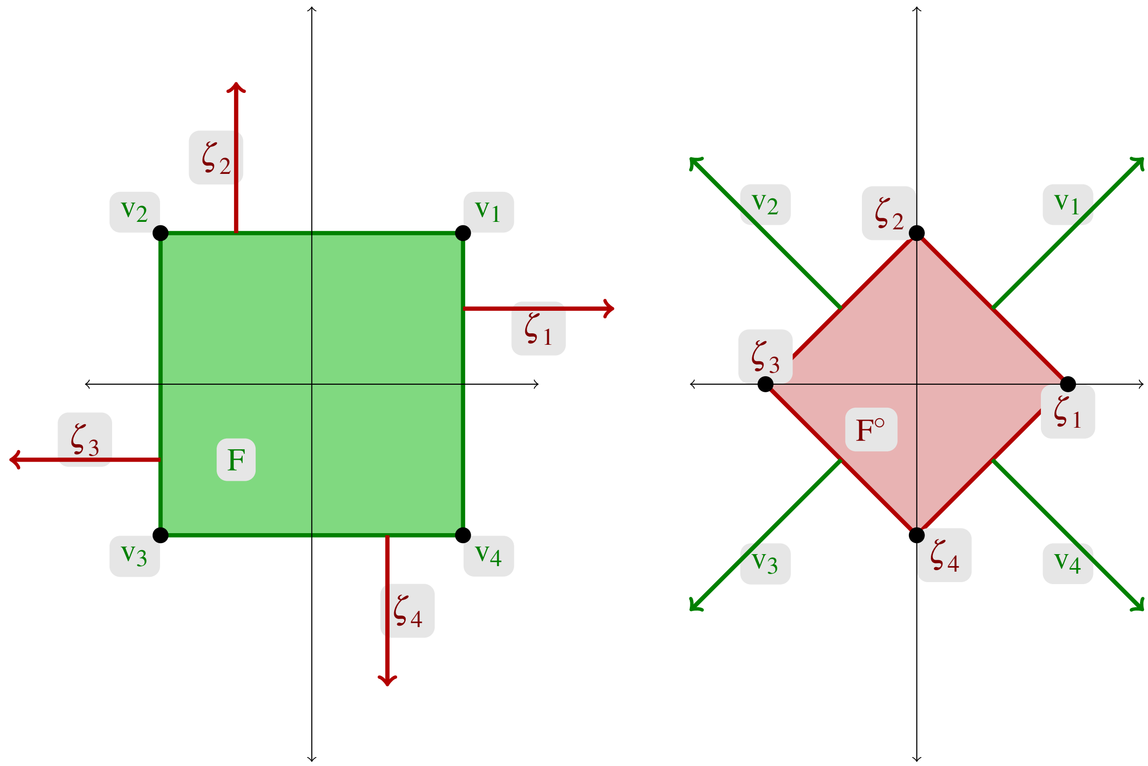

FIGURE 2. Duality between ∞ and 1 norms

4.4.1. *The infinity and one norms in* $\mathbb{R}^2$. We saw in Section 4.1 the unit ball of a norm has its polar equal the unit ball in the dual norm. In particular, the ∞ and one norms in $\mathbb{R}^2$ are polar to each other. This can also be seen from Theorem 8 as illustrated in the following simple example - see Figure 2.

$$\mathrm{F} = \overline{\mathrm{B}}_\infty = \bigcap_{i=1}^{4} \mathrm{H}_{\zeta_i}^{\leq 1} = \overline{\mathrm{co}}\left\{\mathrm{v}_j : \ j = 1,\dots,4\right\}, \text{ where}$$

$$\left\{\zeta_i : \ i = 1,\dots,4\right\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} \text{ and}$$

$$\left\{\mathrm{v}_j : \ j = 1,\dots,4\right\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

Then $\mathrm{F}^\circ = \overline{\mathrm{B}}_1 = \bigcap_{j=1}^{4} \mathrm{H}_{\mathrm{v}_j}^{\leq 1} = \overline{\mathrm{co}}\left\{\zeta_i : \ i = 1,\dots,4\right\}$.

5. EXAMPLES IN $\mathbb{R}^2$

We now restrict attention to $\mathbb{R}^2$ and let $\mathrm{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$. As in the classical Elvis problem, the two regions are the lower half-plane $\mathrm{M}_0 := \mathrm{H}_\mathrm{n}^{<0}$ and the upper half-plane $\mathrm{M}_1 := \mathrm{H}_\mathrm{n}^{>0}$, and the interface is the x-axis $\Sigma := \mathrm{H}_\mathrm{n}^{=0}$. The dynamics allow for general Elvis velocity sets $\mathrm{F}_0, \mathrm{F}_1 \in \mathscr{C}_0$.

Recall one main concern is determining which points lie on the boundary of the reachable set $\mathrm{R}^T(\mathrm{x}_0)$. We also desire every reachable set to be closed, which is equivalent to saying there is a time-optimum arc connecting any two points x_0, $\mathrm{x}_1 \in \mathbb{R}^2$. If $\mathrm{x}_0 \in \mathrm{M}_0$ and $\mathrm{x}_1 \in \mathrm{M}_1$, such an optimum arc always exists (we'll discuss this further below). If the terminal point x_1 instead lies in $\mathrm{cl}(\mathrm{M}_0)$ then other possibilities can occur. We shall introduce structural conditions on how an arc can move on Σ that are sufficient so that existence always holds.

We occasionally abuse notation in dealing with elements in Σ. The formal representation of an element $\mathrm{y} \in \Sigma \subseteq \mathbb{R}^2$ is $\mathrm{y} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, but when ordering or taking min/max of elements of Σ, we are only referring to their first component. This should be clear from the context. For example, if $\mathrm{z} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, the Σ-projection $\mathrm{proj}_\Sigma(\mathrm{z})$ may refer only to x instead of its technical definition as $\begin{pmatrix} x \\ 0 \end{pmatrix}$. Similar considerations are in effect for elements of the tangent space T_Σ.

5.1. **Discussion regarding interface velocity.** An issue deferred earlier was how movement on the interface Σ should be treated. There are several cases to consider with many possibilities and generalizations, and our point of view is influenced by distinguishing between the minimal time optimum control problem (that involves minimizing over an infinite dimensional space of arcs) and the reformulated convex optimization problem $(P_{\mathrm{x}_0,\mathrm{x}_1})$ (that reduced the problem to minimizing over a finite dimensional point $\mathrm{y} \in \Sigma$).

Recall the tangent space T_Σ at every $\mathrm{y} \in \Sigma$ has the form $\mathbb{R} \times \{0\}$, which of course is a copy of Σ itself. Suppose a constant velocity set $\mathrm{F}_\Sigma \subseteq \mathrm{T}_\Sigma$ is given and is Elvis relative to Σ. This means $\mathrm{F}_\Sigma = [\mathrm{v}_\Sigma^L, \mathrm{v}_\Sigma^R] \times \{0\}$ where $\mathrm{v}_\Sigma^{L,R} \in \mathbb{R}$ are Left and Right endpoints of an interval that satisfies $\mathrm{v}_\Sigma^L < 0 < \mathrm{v}_\Sigma^R$. The global definition of the velocity multifunction $\mathrm{F}(\cdot) : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is

$$\mathrm{F}(\mathrm{x}) := \begin{cases} \mathrm{F}_0 & \text{if } x \in \mathrm{M}_0 \\ \mathrm{F}_\Sigma & \text{if } \mathrm{x} \in \Sigma \\ \mathrm{F}_1 & \text{if } \mathrm{x} \in \mathrm{M}_1 \end{cases}$$

which has closed convex values but is not necessarily upper semicontinuous. The latter means the graph $\mathrm{gr}(\mathrm{F}) := \{(\mathrm{x}, \mathrm{v}) : \mathrm{v} \in \mathrm{F}(\mathrm{x})\}$ is not necessarily closed. This can put the data outside of the standard existence theorem for Differential Inclusions but nonetheless admits feasible arcs connecting any two points in $\mathbb{R}^2$.

The greatest horizontal velocity toward the Right (resp. Left) direction in M_i ($i = 0, 1$) is

$$\mathrm{v}_i^R := \max\{\mathrm{v} : \mathrm{v} \in \mathrm{F}_i \cap \mathrm{T}_\Sigma\} \quad (\text{resp. } \mathrm{v}_i^L := \min\{\mathrm{v} : \mathrm{v} \in \mathrm{F}_i \cap \mathrm{T}_\Sigma\}).$$

If $\mathrm{v}_\Sigma^R < \min\{\mathrm{v}_0^R, \mathrm{v}_1^R\}$, then for any choices of both x_0 and x_1 lying on Σ with $\mathrm{x}_0 < \mathrm{x}_1$, the min time problem connecting them will not have an optimum solution – a minimizing sequence could be defined slightly off Σ which limits to an arc lying on Σ that is not a trajectory of $F(\cdot)$. Hence it is natural to impose the structural condition

$$\mathrm{v}_\Sigma^L \leq \min\{\mathrm{v}_0^L, \mathrm{v}_1^L\} \quad \text{and} \quad \mathrm{v}_\Sigma^R \geq \max\{\mathrm{v}_0^R, \mathrm{v}_1^R\}. \tag{5.1}$$

This is the minimal requirement on the Σ-velocity set F_Σ that assures $\mathrm{R}^T(\mathrm{x}_0)$ will be closed for all $\mathrm{x}_0 \in \mathbb{R}^2$ and $T \geq 0$, but several situations need to be considered since the behavior of optimum

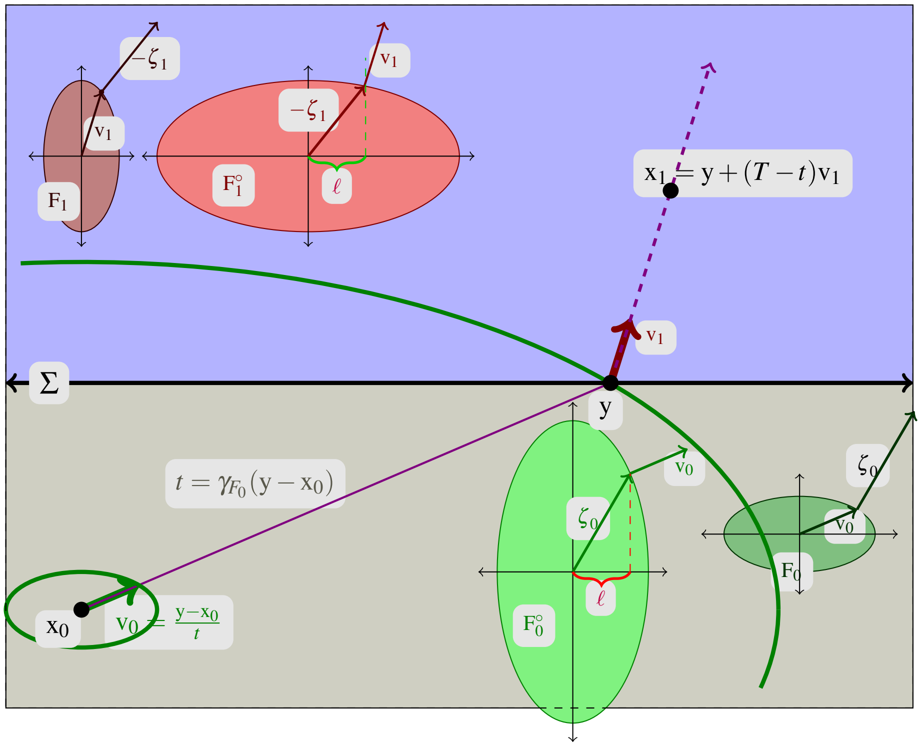

FIGURE 3. F-S scenario with centered ellipses

solutions will differ. An underlying assumption in all is that (5.1) holds. With $x_0 \in M_0$, a main issue is how a trajectory can move optimally into M_1.

5.2. **Fast-to-Slow horizontal movement.** We first consider in this section the simplest case where the interface maximal velocities equal $v_0^{L,R}$ and are larger in magnitude than $v_1^{L,R}$, which is the Fast-to-Slow (F-S) scenario. That is, the interface velocity F_Σ satisfies $v_\Sigma^{L,R} = v_0^{L,R}$, and which satisfies (5.1). A feasible trajectory $x(\cdot)$ can move horizontally on Σ with velocity belonging to F_Σ, although in this case an optimum trajectory need not do so; such a trajectory lies in $\mathrm{cl}\big(M_0\big)$, and its endpoint can be reached with a straight line trajectory in at least that time.

5.2.1. *Construction of optimum trajectories.* We describe a procedure to build optimum trajectories originating from $x_0 \in M_0$. For any $y \in \Sigma$, the least time to go from x_0 to y is $t := \gamma_{F_0}(y - x_0)$ and the optimum velocity is $v_0 := \frac{y - x_0}{t}$ (see Figure 3). The issue then is how to enter M_1 with a velocity $v_1 \in F_1$ so that for $T > t$ the point $x_1 := y + (T - t)v_1$ belongs to $\mathrm{bdry}\big(R^T(x_0)\big)$. Choose $\zeta_0 \in N_{F_0}(v_0)$ (note $\langle \zeta_0, n\rangle > 0$) normalized to satisfy $\gamma_{F_0^\circ}(\zeta_0) = 1$. There exists $\zeta_1 \in -\zeta_0 + N_\Sigma(y)$ with $\gamma_{F_1}(-\zeta_1) = 1$ and choose (any) $v_1 \in N_{F_1^\circ}(-\zeta_1)$ with $\langle n, v_1\rangle \geq 0$ and $\gamma_{F_1}(v_1) = 1$. The requirement $\langle n, v_1\rangle \geq 0$ is to ensure the trajectory will not re-enter M_0, and by Proposition 2, the other conditions are equivalent to $\gamma_{F_1^\circ}(-\zeta_1) = 1 = \gamma_{F_1}(v_1)$ and $-\zeta_1 \in N_{F_1}(v_1)$. For $T > t$, a point $x_1 \in y + (T - t)v_1$ is such that it belongs to $\mathrm{bdry}\big(R^T(x_0)\big)$, or equivalently, y solves $\big(P_{x_0,x_1}\big)$.

5.2.2. *Example with centered ellipses.* Figure 3 shows an example where there is a symmetry across both x and y-axes, although Elvis velocity sets may have neither. Without the y symmetry, one may have fast-to-slow in one direction and slow-to-fast in the other. Without the x

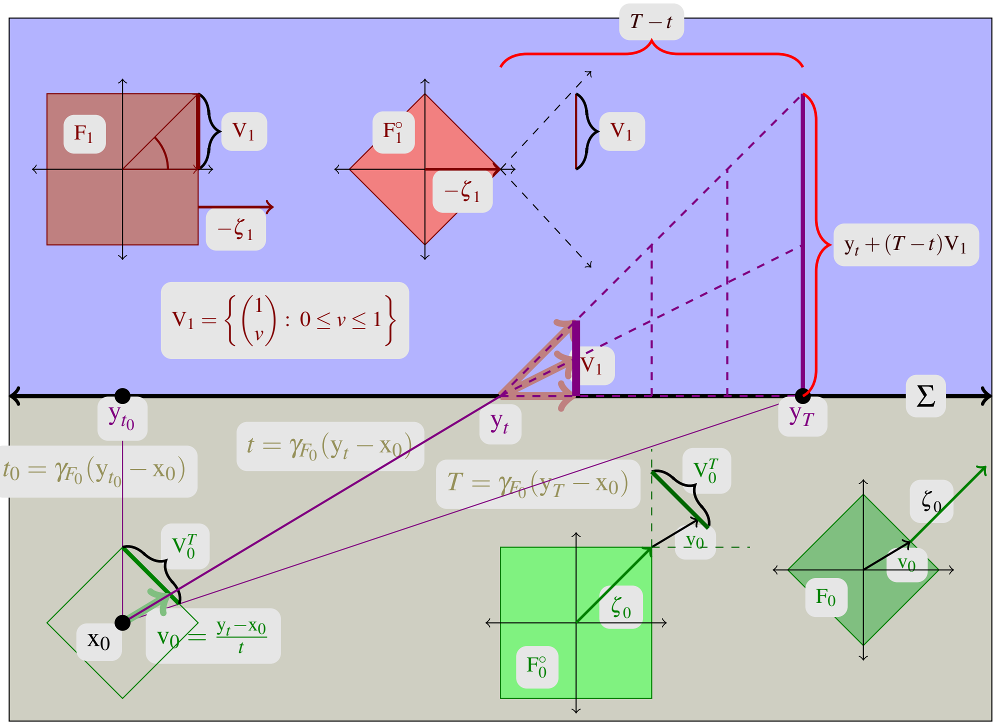

FIGURE 4. $F_0 = \overline{\mathrm{B}}_1$ and $F_1 = \overline{\mathrm{B}}_\infty$ and where $0 < t_0 \leq t \leq T$

symmetry, the critical velocity getting on Σ may differ substantially from the critical velocity getting off Σ. One reason to first consider the case $n = 2$ is that there are only two directions (Left and Right) to check behavior for getting on and off Σ. The situation in dimension $n > 2$ promises to be much more complicated and will be the topic of future research.

5.2.3. *Example without strict convexity.* An example without strict convexity is illustrated in Figure 4 where $\mathrm{F}_0 = \overline{\mathrm{B}}_1$ (the unit ball in the 1-norm) and $\mathrm{F}_1 = \overline{\mathrm{B}}_\infty$ (the unit ball in the ∞-norm). The points labelled as $\mathrm{y}_t + (T-t)\mathrm{V}_1$ are on $\mathrm{bdry}(\mathrm{R}^T(\mathrm{x}_0))$ and can be reached optimally at time T by passing through any y_t, $t_0 \leq t < T$ and then using velocities from V_1 for time $T-t$. The optimum velocities for time $0 \leq t$ capable of reaching y_T are labeled as V_0^T, and this set expands as T increases. The entire reachable set at time $T > t_0$ is not convex and equals

$$\mathrm{R}^T(\mathrm{x}_0) = \Big[\mathrm{y}_0 + \big[(T-t_0)\mathrm{F}_1 \bigcap \mathrm{cl}(\mathrm{M}_1)\big]\Big] \bigcup \Big[\big[\mathrm{x}_0 + T\mathrm{F}_0\big] \bigcap \mathrm{cl}(\mathrm{M}_0)\Big].$$

5.3. **Slow-to-Fast horizontal movement.** A Slow-to-Fast (S-F) scenario to the right (resp. left) is when $\mathrm{v}_0^R < \mathrm{v}_1^R$ (resp. $\mathrm{v}_0^L > \mathrm{v}_1^L$). We now analyze feasible movement on Σ to the right by allowing a trajectory to have the velocity v_1^R while on Σ.

5.3.1. *Restriction in classical setting.* In the classical setting (where $\mathrm{F}_i = r_i\overline{\mathrm{B}}$, $i = 0, 1$), this means $r_0 < r_1$ and Snell's Law says every optimum trajectory from $\mathrm{x}_0 \in \mathrm{M}_0$ hitting Σ is restricted by

$$\left|\sin(\theta_0)\right| \leq \left|\frac{r_0}{r_1}\sin(\theta_1)\right| < 1.$$

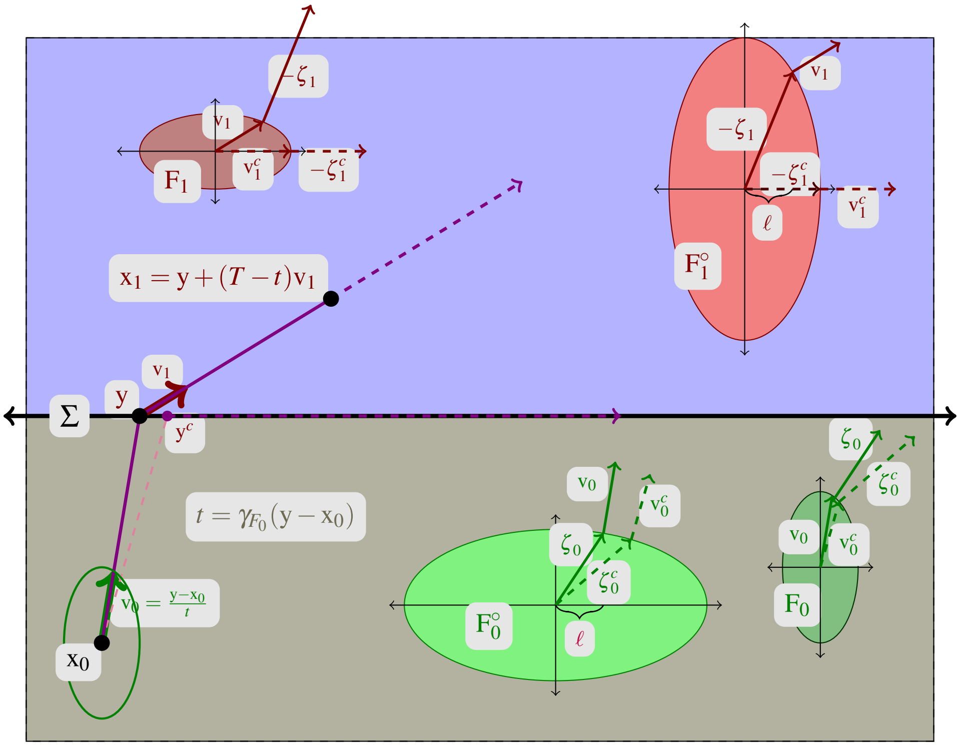

FIGURE 5. Slow to Fast scenario

The generalized Elvis problem has a corresponding condition that restricts the horizontal components $\left|\mathrm{proj}_{T_\Sigma}(\zeta_0)\right|$ of an optimum multiplier ζ_0.

5.3.2. *The critical velocity.* A velocity vector $v_0^c \in F_0$ with $\mathrm{proj}_{T_\Sigma}(v_0)$ maximal among all $v_0 \in F_0$ that satisfy (3.4)-(3.6) (for some $y \in \Sigma$, ζ_0, ζ_1) is called a *critical velocity*, and can be found as follows; see Figure 5.

Let $v_1^c \in F_1$ be so that $\mathrm{proj}_{T_\Sigma}(v_1^c) = v_1^R$. There exists ζ_1^c with $-\zeta_1^c \in N_{F_1}(v_1^c)$ and normalized to satisfy $\gamma_{F_1}(-\zeta_1^c) = 1$. Let $\ell := \mathrm{proj}_{T_\Sigma}(-\zeta_1^c)$, and choose $\zeta_0^c \in F_0$ satisfying $\gamma_{F_0}(\zeta_0^c) = 1$ and $\ell = \mathrm{proj}_{T_\Sigma}(\zeta_0^c)$. A critical velocity (to the right) is obtained by choosing $v_0^c \in N_{F_0^\circ}(\zeta_0^c)$ normalized to also satisfy $\gamma_{F_0}(v_0^c) = 1$. A similar construction produces one to the left if $v_0^L > v_1^L$.

Suppose $v_0^c \in F_0$ is critical (to the right) and $y^c := \begin{pmatrix} y^c \\ 0 \end{pmatrix} = x_0 + t_0 v_0^c \in \Sigma$ where $t_0 := \gamma_{F_0}(y^c - x_0)$. Then any point $x_1 := y_c + (T - t_0)v_1^c$ with $v_1^c \in \mathrm{bdry}(F_1) \bigcap T_\Sigma$ will belong to $\mathrm{bdry}\big(R^T(x_0)\big) \bigcap \Sigma$.

Suppose $y := \begin{pmatrix} y \\ 0 \end{pmatrix} \in \Sigma$ satisfies $\mathrm{proj}_\Sigma(x_0) \le y < y^c$ and $t := \gamma_{F_0}(y - x_0)$. Set $v_0 := \frac{y - x_0}{\gamma_{F_0}(y - x_0)}$. The same procedure described in Section 5.2.1 can be applied here to find $v_1 \in F_1$ that satisfies $x_1 := y + (T - t)v_1 \in \mathrm{bdry}\big(R^T(x_0)\big)$. See Figure 5.

5.3.3. *Re-entering* M_0 *optimally.* If a trajectory uses the critical velocity v_0^{on} to reach $y^c \in \Sigma$ at time t^{on} and then proceeds along Σ using v_1^c for time $t > t^{on}$, it will remain on $\mathrm{bdry}\big(R^t(x_0)\big)$. See Figures 6 and 7. It will also remain optimum by re-entering M_0 at time t^{off} with a critical velocity v_0^{off} for an additional time $T > t^{off}$ provided $T - t^{off} > 0$ is small. How small is determined by when this optimum trajectory first intersects $x_0 + TF_0$, the latter set being points directly

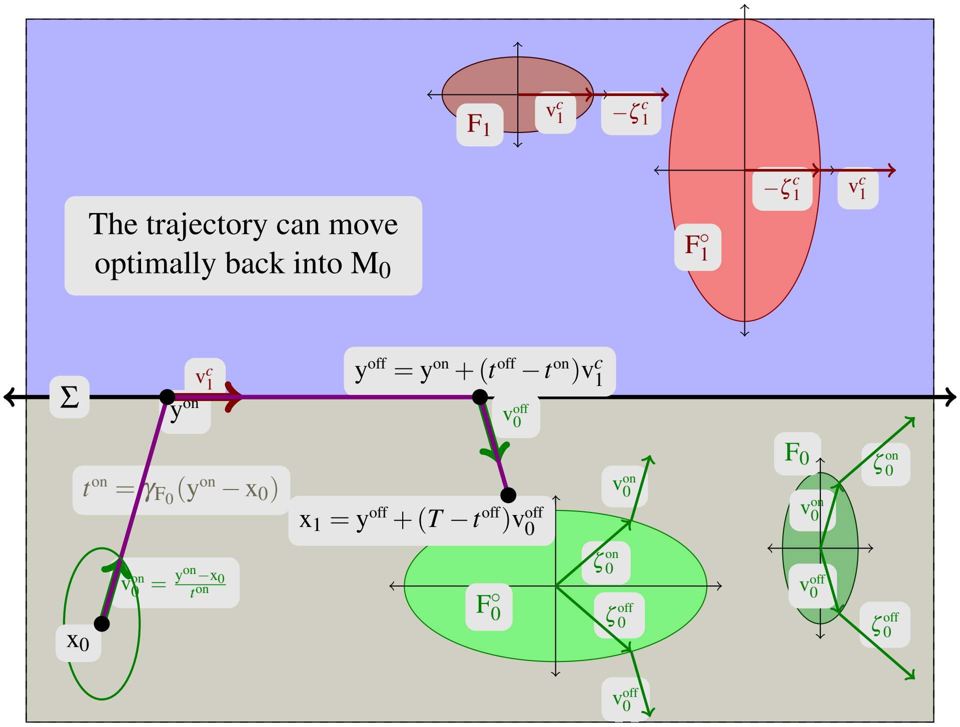

FIGURE 6. Re-entering M_0 optimally: $0 < t^{on} < t^{off} < T \leq \gamma_{F_0}(x_1 - x_0)$

reachable from x_0 at time T while staying in M_0. At this point, the trajectory can no longer be optimally prolonged: If the trajectory $x(\cdot)$ satisfies

$$x(T) \in \left[x_0 + TF_0\right] \bigcap \left[\text{bdry}\left(R^T(x_0)\right)\right]$$

and prolongs itself with either of the previously optimum velocities v_0^{off} or $\frac{x(T)-x_0}{\gamma_{F_0}(x(T)-x_0)}$, it will be absorbed into the interior of $R^t(x_0)$ for $t > T$. This behavior is similar to that which happens at a conjugate point in the Calculus of Variations. See Figure 6 where the symmetry of F_0 simplifies the calculation of v_0^{off} since it equals $-v_0^{on}$. Figure 7 uses translated ellipses and demonstrates the calculation of v_0^{off} more generally.

6. FURTHER GENERALIZATIONS.

We mentioned earlier that there are several advantages to requiring Elvis velocity sets to have $0 \in \text{int}(F)$. In our view and for the purpose of introducing CA to perhaps a new audience, this special case deserves direct analysis as we have given in this paper. However, as we now sketch pursuing greater generality, additional issues arise.

6.1. **The interface is a highway.** One generalization of the Elvis problem is to allow for the interface Σ to act like a highway in which movement can be faster than the surrounding mediums. Our presentation in Section 2 of CA has to be modified so that Elvis velocity sets can have lower dimension and there is a natural way to do this using relative interiors. We plan to publish rigorous details elsewhere, but sketch the main idea here in dimension two.

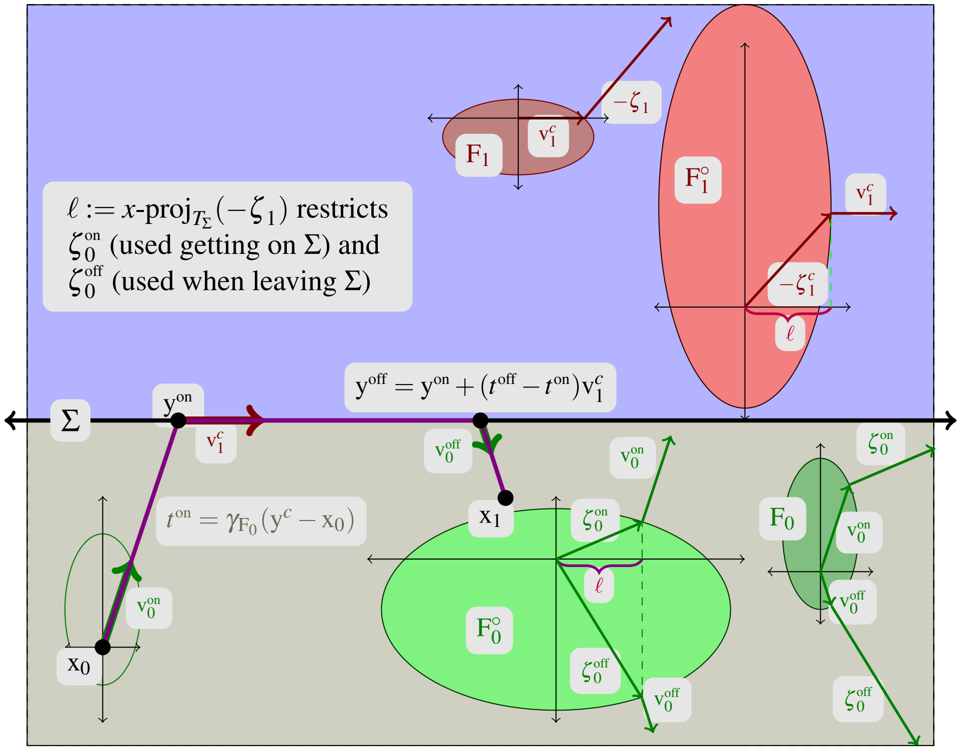

FIGURE 7. Re-entering M_0 optimally with translated ellipses

Suppose $F_\Sigma \subseteq T_\Sigma$ is an Elvis velocity set relative to Σ, which recall means $F_\Sigma = [v_\Sigma^L, v_\Sigma^R] \times \{0\}$. The "highway" property is the case where strict inequalities in (5.1) hold; that is,

$$v_\Sigma^L < \min\{v_0^L, v_1^L\} < 0 < \max\{v_0^R, v_1^R\} < v_\Sigma^R. \tag{6.1}$$

The polar $F_\Sigma^\circ \subseteq \mathbb{R}^2$ is $\left[\frac{1}{v_\Sigma^L}, \frac{1}{v_\Sigma^R}\right] \times \mathbb{R}$, but only the part on the x-axis is relevant here.

Suppose $x_0 \in M_0$ and $x_1 \in M_1$ is sufficiently to the right so that $\text{proj}_\Sigma(x_1) > y^c$. The right polar value $\frac{1}{v_\Sigma^R}$ plays the role that ℓ played in earlier examples. It first restricts how a right-moving optimum trajectory enters Σ with critical velocity $v_0^{on} \in F_0$. The trajectory can then move optimally on Σ with velocity v_Σ^R. By so doing there are options at any future time to move into M_1 and stay optimum with a critical velocity $v_1^c \in F_1$ or re-enter M_0 optimally for small additional time with a critical velocity $v_0^{off} \in F_0$. See Figure 8. The generalized Snell's Law (3.6) applied twice (first getting on Σ and then getting off Σ) is saying

$$\ell := \frac{1}{v_\Sigma^R} = \text{proj}_{T_\Sigma}(\zeta_0^{on}) = \text{proj}_{T_\Sigma}(\zeta_0^{off}) = \text{proj}_{T_\Sigma}(-\zeta_1^c).$$

One has $y^{off} := y^{on} + (T - t^{on})v_\Sigma^R \in \text{bdry}(R^T(x_0))$ for $T > t^{on}$, and y^{off} remains on Σ. A trajectory can enter M_1 optimally at any time $t^{off} > t^{on}$ by using a critical velocity v_1^c from F_1. In this case $x_1 := y^{off} + (T - t^{off})v_1^c$ belongs to $\text{bdry}(R^T(x_0))$ for all $T > t^{off}$. Alternatively, it can re-enter M_0 optimally by using the critical velocity $v_0^{off} \in F_0$ at least for a short while. Specifically, $x_1 := y^{off} + (T - t^{off})v_0^{off}$ belongs to $\text{bdry}(R^T(x_0))$ for all $t^{off} < T \leq \gamma_{F_0}(x_1 - x_0)$.

6.2. **More than two regions.** It is interesting to generalize to more than two regions, but additional requirements are needed for our approach to be applicable and be effective. We shall

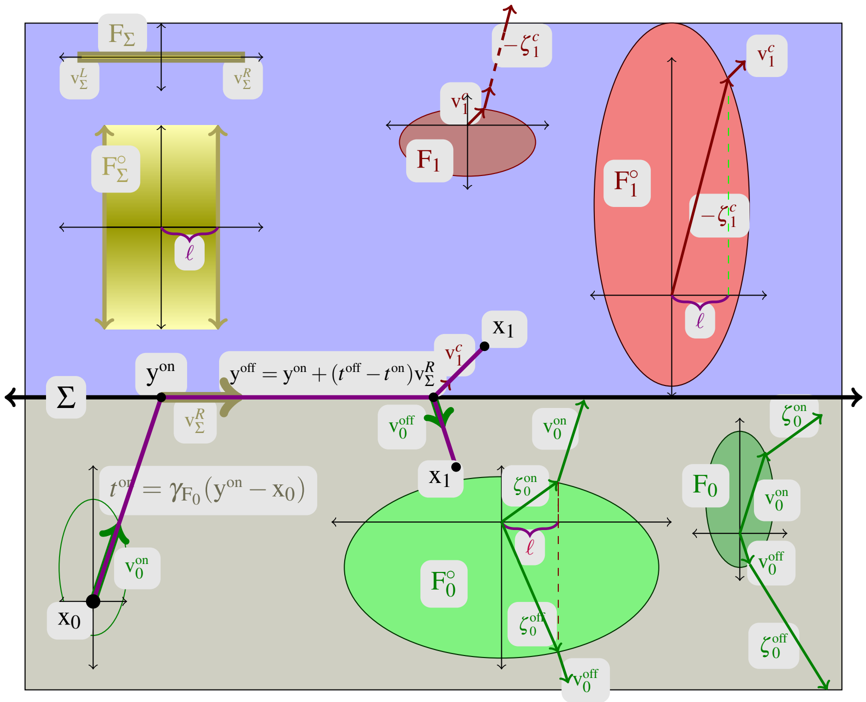

FIGURE 8. Σ is a highway

address this problem in complete detail in future work, but here restrict attention to dimension $n = 2$ with mediums the four quadrants

$$\mathrm{M}_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x < 0, y > 0 \right\} \qquad \mathrm{M}_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x > 0, y > 0 \right\}$$
$$\mathrm{M}_3 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x < 0, y < 0 \right\} \qquad \mathrm{M}_4 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x > 0, y < 0 \right\}$$

endowed with an Elvis velocity set $\mathrm{F}_i \in \mathscr{C}_0$. The boundaries Σ_{ij} are labelled and defined by

$$\Sigma_{ji} = \Sigma_{ij} := \mathrm{cl}(\mathrm{M}_i) \bigcap \mathrm{cl}(\mathrm{M}_j) \qquad (1 \le i, j \le 4).$$

We illustrate the problem by taking the initial point x_0 to lie in M_3 and the terminal point x_1 in M_1. There are now two ways an optimum trajectory can go from x_0 to x_1 – either going through $\mathrm{cl}(\mathrm{M}_2)$ or through $\mathrm{cl}(\mathrm{M}_4)$. We can find the optimum path over all those paths that go through $\mathrm{cl}(\mathrm{M}_2)$ by solving

$$\inf_{\mathrm{y}_1, \mathrm{y}_2 \in \mathbb{R}^n} \left\{ \gamma_{\mathrm{F}_3}(\mathrm{y}_1 - \mathrm{x}_0) + \gamma_{\mathrm{F}_2}(\mathrm{y}_2 - \mathrm{y}_1) + \gamma_{\mathrm{F}_1}(\mathrm{x}_1 - \mathrm{y}_2) + \mathrm{I}_{\Sigma_{23}}(\mathrm{y}_1) + \mathrm{I}_{\Sigma_{12}}(\mathrm{y}_2) \right\}. \qquad (P_{\mathrm{x}_0,\mathrm{x}_1}^{3\to 2\to 1})$$

Similarly we can find the optimum path over all those paths that go through $\mathrm{cl}(\mathrm{M}_4)$ by solving

$$\inf_{\mathrm{y}_1, \mathrm{y}_2 \in \mathbb{R}^n} \left\{ \gamma_{\mathrm{F}_3}(\mathrm{y}_1 - \mathrm{x}_0) + \gamma_{\mathrm{F}_4}(\mathrm{y}_2 - \mathrm{y}_1) + \gamma_{\mathrm{F}_1}(\mathrm{x}_1 - \mathrm{y}_2) + \mathrm{I}_{\Sigma_{34}}(\mathrm{y}_1) + \mathrm{I}_{\Sigma_{14}}(\mathrm{y}_2) \right\}. \qquad (P_{\mathrm{x}_0,\mathrm{x}_1}^{3\to 4\to 1})$$

Our notation $3 \to i \to 1$ $(i = 2, 4)$ is to specify the path-regions that a trajectory traverses. A third option is the path $3 \to 1$, but it has only one feasible (and therefore optimum) solution. It could be globally an optimum, but how can one tell?

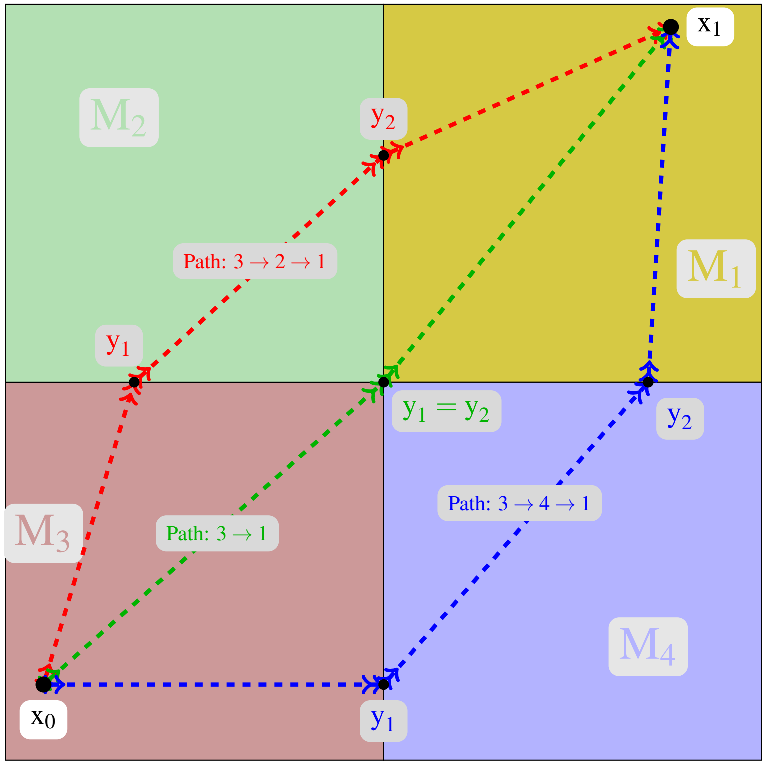

FIGURE 9. Four regions

Optimality conditions can be formulated for each problem similar to (3.4)-(3.6). The only potential issue in applying our techniques is the application of the sum rule, but nonetheless this is valid since 0 belongs to the intersection of the relative interiors; see [4], Theorem 23.8. The point is, however, we want to solve the problem of going from x_0 to x_1 in the best time any which way (i.e. globally). The desired solution must solve both $\left(P^{3\to2\to1}_{\mathrm{x}_0,\mathrm{x}_1}\right)$ and $\left(P^{3\to4\to1}_{\mathrm{x}_0,\mathrm{x}_1}\right)$ simultaneously if it is feasible for both.

The possibility that $\mathrm{y}_1 = \mathrm{y}_2$ in either problem is not ruled out, and if satisfied by both means the globally optimum trajectory goes through the origin. The path $3 \to 1$ admits a feasible trajectory for both but this path is not by itself an interesting restriction since there is only one path that goes directly from M_3 to M_1. This suggests that optimality conditions should be formulated with a partial ordering on paths with regard to feasibility.

The global optimization problem has roughly the structure of a bi-level optimization problem but in which the upper level is a discrete problem (feasible paths are determined) and the lower level a convex one (once the path is specified, our approach solves it). We have not encountered such problems elsewhere in the literature.

7. CONCLUSION

We presented an introduction to Convex Analysis based on solving generalized Elvis problems. Our approach required Elvis velocity sets F (in addition to being closed and convex) to be bounded with $0 \in \mathrm{int}(\mathrm{F})$. The latter two assumptions are dual to each other in the sense that F satisfies them if and only if its polar F° also satisfies them. This suggests duality theory can be further developed within this basic set-up. Further generalizations were also sketched proposing future research directions.

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