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NON-OCCURRENCE OF THE LAVRENTIEV GAP FOR A BOLZA TYPE OPTIMAL CONTROL PROBLEM WITH STATE CONSTRAINTS AND NO END COST

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In memory of Professor Jack Warga on the occasion of his 100th birthday

Abstract. Let $\Lambda : [t,T]\times\mathbb{R}^n\times\mathbb{R}^m \to [0,+\infty[\cup\{+\infty\}$ be Borel. We consider the problem $(\mathscr{P})$ of minimizing an integral functional F of the form $F(y,u) := \int_I \Lambda(s,y(s),u(s))\,ds$ in the set of *admissible* pairs (y,u) such that $F(y,u) < +\infty$ and satisfy the following linear controlled dynamics, state and control constraints:

$$\begin{cases} y \in \mathrm{W}^{1,1}([t,T];\mathbb{R}^n),\\ y' = b(y)u \text{ a.e. in } [t,T],\ y(t) = X \in \mathbb{R}^n,\\ u \in L^1(I;\mathbb{R}^m),\ u(s)\in\mathscr{U} \text{ a.e. } s\in[t,T],\ y(s)\in\mathscr{S}\ \forall s\in[t,T]. \end{cases}$$

We prove that if Λ is radially convex on the control variable, locally Lipschitz in the time variable and a mild boundedness assumption (satisfied if Λ is locally bounded where it is finite), then there is a minimizing sequence of admissible pairs with bounded controls. In the calculus of variations ($b = 1$) this corresponds to the non-occurrence of the Lavrentiev phenomenon for the problem with an initial constraint.

Keywords. Approximation; Extended valued; Lipschitz; Minimizing sequence; Regularity; Radial convexity.

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1. Introduction

Consider the basic problem $(\mathscr{P})$ of the Calculus of Variations, of minimizing an "energy"

$$J(y) = \int_t^T \Lambda(s,y(s),y'(s))\,ds \qquad (\mathscr{P})$$

among the absolutely continuous functions $y : I = [t,T] \to \mathbb{R}^n$, with possibly none, or one, or two boundary conditions, where $\Lambda(s,y,v)$ is a Borel Lagrangian defined in $[t,T]\times\mathbb{R}^n\times\mathbb{R}^n$, with values in $[0,+\infty[\cup\{+\infty\}$. Allowing Λ to take the value $+\infty$ is not just a matter of generality, it actually enables to consider state or velocity constraints via the addition of indicator

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functions of sets. The Lavrentiev phenomenon occurs for such a problem when the infimum of ($\mathscr{P}$) cannot be achieved with Lipschitz functions sharing the same end-point conditions. Since Lipschitz functions are dense in the absolutely continuous ones, the phenomenon represents a lack of continuity of the functional with respect strong convergence. This fact has practical issues; indeed when the phenomenon occurs the classical finite elements methods is not able to catch the infimum of the functional. Though rare (see [26]), the occurrence of the phenomenon may occur even with polynomial Lagrangians that satisfy Tonelli's existence conditions (i.e., superlinear and convex in the last velocity variable), as shown by Ball and Mizel in [2].

There are some obvious ways to exclude the phenomenon: growth conditions from above (implying strong continuity of the functional), superlinear growth conditions from below implying in some cases a priori Lipschitz regularity of the minimizers, if any: we refer to [4, 6, 7, 14, 16, 22] for some results in this direction.

Our interest here is devoted to the cases where no growth conditions are involved and no minimizers are expected to exist. As in regularity theory, *autonomous* Lagrangians $\Lambda = \Lambda(x, v)$ stand on their own: a result by Alberti and Serra Cassano [1, Theorem 2.4], states that the Lavrentiev phenomenon does not occur under the mild local boundedness condition (B):

$$\forall K \geq 0 \quad \exists r_K > 0 \quad \Lambda \text{ is bounded on } B^n_K \times B^n_{r_K}, \tag{B}$$

where B^k_r denotes the closed ball of $\mathbb{R}^k$ centered in the origin, of radius r: more precisely, given an admissible trajectory y there is a sequence of Lipschitz functions with the same initial value at t that approximates y in norm $W^{1,1}$ and in energy. Now, the occurrence of the phenomenon seems to depend strongly on which boundary conditions are taken into account in the desired Lipschitz approximation in norm and in energy. There are indeed examples, like Mazià's (Example 2.3), in which the phenomenon occurs when both end-point conditions have to be preserved, but it does anymore occur when one considers just one boundary condition. In this regard, it has been recently clarified that [1, Theorem 2.4] is valid *just* for autonomous problems with *one* end-point condition. This fact is thoroughly discussed in [21], where an example, kindly provided by Alberti to the author, exhibits an extended valued autonomous Lagrangian that satisfies (B), for which the Lavrentiev phenomenon occurs with prescribed boundary conditions at both end-points of the interval $[t, T]$. Moreover, when (B) fails, the phenomenon may occur even by considering just one end-point condition: an example involving a real valued autonomous Lagrangian was recently provided in [11] (see Example 2.4).

The method used in the proof of [1, Theorem 2.4] is based first in a Lusin type Lipschitz approximation of a given admissible trajectory, followed by a suitable reparametrization of the approximation. As a matter of fact, this approach does not preserve a state constraint of the form $y(I) \subset \mathscr{S} \subset \mathbb{R}^n$.

For *autonomous* Lagrangians, an alternative approach for obtaining Lipschitz approximating sequences keeping both end point conditions, was developed by Cellina and Ferriero in [9] and with Marchini in [10]. In [9] the authors consider extended valued Lagrangians and assume a slow growth condition, obtaining the existence of a equi-Lipschitz minimizing sequence; in [10] growth conditions are not assumed anymore, and Lagrangians are real valued. In both cases the authors subsume continuity of the Lagrangian and convexity in the velocity variable (not needed in [1, 21]). The technique is based just on a reparametrization technique and thus, differently from [1, 21], it preserves any given state constraint. An effort to weaken the assumptions of [9] and to extend the result to non-autonomous Lagrangians was carried on by the author in [19]: the main achievement is the existence of a equi-Lipschitz minimizing sequence under Clarke's slower growth condition introduced in [12] for possibly discontinuous Lagrangians, radially convex (instead of convex) in the last variable. In the author's attempt to extend [10] to non-autonomous and extended valued Lagrangians in [20], it appears that the case of just one end point conditions is much simpler to handle with respect to the problem with two end point conditions: we just mention for instance that neither growth conditions nor continuity at the boundary of the effective domain assumed in [9, 19] are not needed anymore. The purpose of this paper is to give a simplified and self-contained proof of the non-occurrence of the Lavrentiev gap in the framework of one end point problems, possibly non-autonomous, avoiding various complications that are present in [21] in the general case.

Actually, with respect to [21], we consider the more general framework of an optimal control problem

$$\min F(y,u)=\int_t^T \Lambda(s,y(s),u(s))\,ds$$

with a linear controlled dynamic of the form $y'=b(y)u$, where b is bounded on bounded sets: in this context the non-occurrence of the Lavrentiev phenomenon means the existence of a minimizing sequence of admissible pairs (y_k,u_k) where the controls u_k are bounded. This kind of systems was studied from the regularity perspective in [5, 7] and appears for instance in the problem of the geodesic's under Grushin's metric (see [18]). We consider the problem $(\mathscr{P})$ of minimizing F among the pairs (y,u) with $y\in W^{1,1}(I;\mathbb{R}^n), u\in L^1(I;\mathbb{R}^m)$ that satisfy a prescribed initial condition $y(t)=X$ and a possible *state constraint* $y(I)\subset\mathscr{S}\subset\mathbb{R}^n$ and u with values in a cone $\mathscr{U}\subset\mathbb{R}^m$. The main result states, under the assumption that $0<r\mapsto\Lambda(s,y,rv)$ is convex for every (s,y,v) and a classical local Lipschitz condition of Λ on the first variable, that there is no Lavrentiev phenomenon for $(\mathscr{P})$ whenever

- (B^w_Λ) For all $K>0$ there is $r_K>0$ such that Λ is bounded on $\big(I\times(B^n_K\cap\mathscr{S})\times(B^m_{r_K}\cap\mathscr{U})\big)\cap$ $\mathrm{Dom}(\Lambda)$.

(Here $\mathrm{Dom}(\Lambda)$ is the *effective domain* of Λ, i.e., set where Λ is finite). With respect to condition (B) introduced in [1], (B^w_Λ) is more suitable for extended valued Lagrangians, since it requires a boundedness property *inside* $\mathrm{Dom}(\Lambda)$ and does not forces, like (B), $\mathrm{Dom}(\Lambda)$ to contain rectangles. The additional radial convexity assumption required here allows, with respect to [1, Theorem 2.4], to build a Lipschitz approximating sequence that safeguards the given state constraint on the trajectories.

The main novelty with respect to the literature, is that we do not make use of regularity assumptions on the Lagrangian with respect to the state variable, no growth conditions nor convexity in the control variable are involved and that we take into account non-autonomous Lagrangians. The method is constructive: an explicit Lipschitz approximating sequence of a given trajectory is explicitly built in Example 5.1.

2. Basic assumptions, the gap and the phenomenon

2.1. **Basic Assumptions.** Let $n\in\mathbb{N}, n\geq 1$. The functional F (sometimes referred as to the "energy") is defined by

$$F(y,u):=\int_I \Lambda(s,y(s),u(s))\,ds.$$

We consider the problem $(\mathscr{P})$ of minimizing F in the set $\mathscr{A}$ of *admissible* pairs (y,u) such that $F(y,u)<+\infty$ and:

$$\begin{cases} y\in \mathrm{W}^{1,1}([t,T];\mathbb{R}^n)\\ y'=b(y)u \text{ a.e. in } [t,T],\, y(t)=X\in\mathbb{R}^n\\ u\in L^1(I;\mathbb{R}^m),\, u(s)\in\mathscr{U} \text{ a.e. } s\in[t,T],\, y(s)\in\mathscr{S}\ \forall s\in[t,T].\end{cases} \tag{D}$$

When $b(\cdot)$ is the identity map, then $y'=u$ and the problem concerns the calculus of variations: in that case in the examples we will write $F(y)$ instead of $F(y,y')$.

Basic Assumptions. We assume the following conditions.

- $I=[t,T]$ is a closed, bounded interval of $\mathbb{R}$;
- $\Lambda: I\times\mathbb{R}^n\times\mathbb{R}^m\to[0,+\infty[\cup\{+\infty\}$ is Lebesgue-Borel measurable in $(s,(y,u))$, i.e., measurable with respect to the σ-algebra generated by the products of Lebesgue measurable subsets of I (for t) and Borel measurable subsets of $\mathbb{R}^n\times\mathbb{R}^n$ (for (y,u)): this guarantees that if $y,u: I\to\mathbb{R}^n$ are measurable then $s\mapsto\Lambda(s,y(s),u(s))$ is measurable (see [13, Proposition 6.34]).
- The **control** $u:[t,T]\mapsto\mathbb{R}^m$ is in $L^1(I;\mathbb{R}^m)$;
- $b:\mathbb{R}^n\to L(\mathbb{R}^m;\mathbb{R}^n)$ (the space of linear functions from $\mathbb{R}^m$ to $\mathbb{R}^n$) is a Borel function, bounded on bounded sets;
- The **state constraint set** $\mathscr{S}$ is a nonempty subset of $\mathbb{R}^n$;
- The **control set** $\mathscr{U}\subset\mathbb{R}^m$ is a cone, i.e. if $u\in\mathscr{U}$ then $\lambda u\in\mathscr{U}$ whenever $\lambda>0$.
- There is at least an admissible pair.

2.2. **Notation.** We introduce the main recurring notation:

- The closed ball of $\mathbb{R}^k$ centered in the origin of radius $r\geq 0$ is denoted by B^k_r;
- The Lebesgue measure of a subset A of $I=[t,T]$ is $|A|$ (no confusion can occur with the Euclidean norm);
- The norm of a linear map $T:\mathbb{R}^m\to\mathbb{R}^n$ is denoted by $\|T\|_{L(\mathbb{R}^m;\mathbb{R}^n)}$;
- The characteristic function of a set A is χ_A;
- If $x\in\mathbb{R}$, we denote by x^+ its positive part, by x^- its negative part;
- $L^1(I;\mathbb{R}^k)$ is the set of Lebesgue integrable functions defined on I with values in $\mathbb{R}^k$;
- $L^\infty(I;\mathbb{R}^k)$ is the set of the essentially bounded functions defined on I with values in $\mathbb{R}^k$;
- If $y:I\to\mathbb{R}^k$ we denote by $y(I)$ its image, by $\|y\|_\infty$ its sup-norm and by $\|y\|_1$ its norm in $L^1(I;\mathbb{R}^k)$;
- $\mathrm{Lip}(I;\mathbb{R}^k)=\{y:I\to\mathbb{R}^k,\, y \text{ Lipschitz}\}$; if $k=1$ we simply write $\mathrm{Lip}(I)$.

2.3. **Lavrentiev gap at a function and Lavrentiev phenomenon.** The non-occurrence of the Lavrentiev gap at an admissible pair (y,u) means that one can approximate (y,u) in norm and energy with a sequence of admissible pairs with bounded controls.

Definition 2.1 (Lavrentiev gap at $(y,u)\in\mathscr{A}$). Let $(y,u)\in\mathscr{A}$. We say that the *Lavrentiev gap* does not occur at (y,u) for $(\mathscr{P})$ if there exists a sequence $(y_k,u_k)_{k\in\mathbb{N}}$ of functions in $\mathrm{Lip}(I;\mathbb{R}^n)\times L^\infty(I;\mathbb{R}^m)$ satisfying:

(1) $\forall k\in\mathbb{N}\quad (y_k,u_k)\in\mathscr{A}$;
(2) $\displaystyle\limsup_{k\to+\infty} F(y_k,u_k)\leq F(y,u)$ (approximation in *energy*);

(3) $(y_k, u_k) \to (y, u)$ in $L^\infty(I;\mathbb{R}^n) \times L^1(I;\mathbb{R}^m)$ as $k \to +\infty$ (approximation in *norm*).

We will refer to $(y_k, u_k)_k$ as to an *approximating sequence*, both in energy and in norm.
We say that the *Lavrentiev phenomenon* does not occur for $(\mathscr{P})$ if

$$\inf_{(y,u)\in\mathscr{A}} F(y) = \inf_{\substack{(y,u)\in\mathscr{A} \\ u\in L^\infty(I;\mathbb{R}^m)}} F(y). \tag{2.1}$$

Remark 2.2. Of course, the non-occurrence of the gap at every admissible pair (y, u) implies the non-occurrence of the phenomenon for the same variational problem. This is actually the way the non-occurrence of the phenomenon is proved in the literature, unless one is aware that minimizers are Lipschitz. An exception, where minimizing sequences play a role in the proof, is given by Mingione in [15] for the multidimensional vectorial case.

The following celebrated example motivates the need to distinguish problems with just one end point condition from problems with conditions at both end points.

Example 2.3 (Manià, [17])**.** Consider the problem of minimizing

$$F(y) = \int_0^1 (y^3 - s)^2 (y')^6 \, ds : y \in W^{1,1}(I), y(0) = 0, y(1) = 1. \tag{$\mathscr{P}_{0,1}$}$$

Then $y_*(s) := s^{1/3}$ is a minimizer and $F(y_*) = 0$. Not only y_* is not Lipschitz; it turns out (see [8, §4.3]) that the Lavrentiev phenomenon occurs, i.e.,

$$0 = \min F = F(y_*) < \inf\{F(y) : y \in \mathrm{Lip}([0,1]), y(0) = 0, y(1) = 1\}.$$

However, as it is noticed in [8], the situation changes drastically if one allows to vary the initial boundary condition along the sequence $(y_k)_k$. Indeed it turns out that the sequence $(y_k)_k$, where each y_k is obtained by truncating y_* at $1/k$, $k \in \mathbb{N}_{\geq 1}$, as follows (see Fig. 1):

$$y_k(s) := \begin{cases} 1/k^{1/3} & \text{if } s \in [0, 1/k], \\ s^{1/3} & \text{otherwise,} \end{cases}$$

is a sequence of Lipschitz functions satisfying

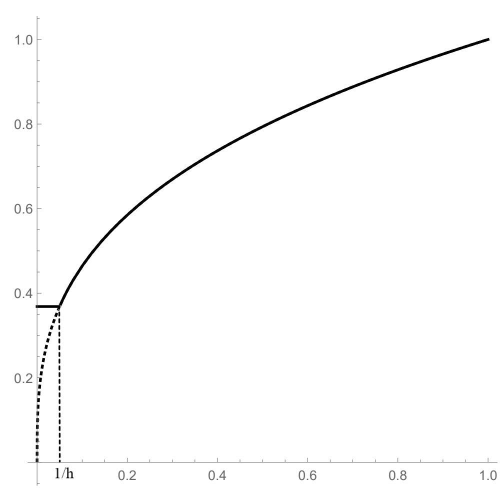

FIGURE 1. The function y_k in Example 2.3.

$$y_k(1) = y(1) = 1, \quad F(y_k) \to F(y_*), \quad y_k \to y_* \text{ in } W^{1,1}([0,1]).$$

Therefore, no Lavrentiev phenomenon occurs for the variational problem

$$\min F(y) = \int_0^1 (y^3 - s)^2 (y')^6 \, ds : y \in W^{1,1}(I), \; y(1) = 1. \tag{2.2}$$

Example 2.4. The Lavrentiev phenomenon may occur even for problems with one end-point condition. Let

$$\Lambda(y,v) := \begin{cases} \left(v^2 - \dfrac{1}{4y^2}\right)^2 v^2 & \text{if } y \neq 0, \\ 1 & \text{if } y = 0. \end{cases}$$

It can be easily checked that $y_*(s) := \sqrt{s}$ is a minimizer of the problem

$$\min F(y) := \int_0^1 \Lambda(y(s), y'(s)) \, ds, \quad y(0) = 0$$

and that $F(y_*) = 0$. However, it is shown in [11] that $F(y) = +\infty$ for any $y \in \operatorname{Lip}([0,1])$ such that $y(0) = 0$ and $F(y) < 1$. Notice that Λ violates Condition (B) in [1, Theorem 2.4].

2.4. **Condition (S).** We consider the following local Lipschitz condition (S) on the first variable of Λ.

Condition (S). For every $K \geq 0$ of $\mathbb{R}^n$ there are $\kappa, \beta \geq 0, \gamma \in L^1[t,T]$, $\varepsilon_* > 0$ satisfying, for a.e. $s \in I$

$$|\Lambda(s_2,y,v) - \Lambda(s_1,y,v)| \leq \big(\kappa\Lambda(s,y,v) + b(y)v + \gamma(s)\big) \, |s_2 - s_1| \tag{2.3}$$

whenever $s_1, s_2 \in [s - \varepsilon_*, s + \varepsilon_*] \cap I$, $y \in B^n_K$, $v \in \mathbb{R}^m$, $(s,y,v) \in \operatorname{Dom}(\Lambda)$.
Notice that, if (y,u) is admissible, then

$$\kappa\Lambda(s,y(s),u(s)) + |u(s)| + \gamma(s) \in L^1(I).$$

Remark 2.5. Condition (S) is fulfilled if $\Lambda = \Lambda(y,v)$ is autonomous.

2.5. **Structure assumptions.** We require here some additional conditions on Λ.

Radial Convexity Assumption (RC). For a.e. $s \in I$, for all $(y,v) \in \mathscr{S} \times \mathscr{U}$ with $(s,y,v) \in \operatorname{Dom}(\Lambda)$,

$$0 < r \mapsto \Lambda(s,y,rv) \text{ is convex.} \tag{RC}$$

Remark 2.6. Assumption (RC) implies that, for every (s,y,v) in $\operatorname{Dom}(\Lambda) \cap (I \times \mathscr{S} \times \mathscr{U})$ there is a *convex subdifferential* for $0 < r \mapsto \Lambda(s,y,rv)$ at $r = 1$, namely a real number Q such that

$$\forall r > 0 \qquad \Lambda(s,y,rv) - \Lambda(s,y,v) \geq Q(r-1).$$

We shall denote by $\partial_r \Lambda(s,y,rv)_{r=1}$ the set of these subdifferentials, i.e., the *convex subgradient* of $0 < r \mapsto \Lambda(s,y,rv)$ at $r = 1$. It is easy to realize (see, for instance, [4, 25]) that the Radial Convexity Assumption (RC) is equivalent, at every $(s,y,v) \in \operatorname{Dom}(\Lambda)$, to the convexity of the map

$$0 < \mu \mapsto \Lambda\left(s, y, \frac{v}{\mu}\right)\mu.$$

In this case, if $P(s,y,v) \in \partial_\mu \left[\Lambda\left(s,y,\frac{v}{\mu}\right)\mu\right]_{\mu=1}$, we have

$$\forall \mu > 0 \quad \Lambda\left(s,y,\frac{v}{\mu}\right)\mu - \Lambda(s,y,v) \geq P(s,y,v)(\mu-1). \tag{2.4}$$

Notice that

$$P(s,y,v) \in \partial_\mu \left[\Lambda\left(s,y,\frac{v}{\mu}\right)\mu\right]_{\mu=1} \Leftrightarrow Q(s,y,v) := \Lambda(s,y,v) - P(s,y,v) \in \partial_r \Lambda(s,y,rv)_{r=1}$$

and $P(s,y,v)$ represents the intersection with the ordinate axis in the half-plane $\{rv, r>0\} \times \{w = \Lambda(s,y,rv) : r > 0\}$ of the tangent line to $r \mapsto \Lambda(s,y,rv)$ at $r=1$ (see Fig. 2). In the smooth

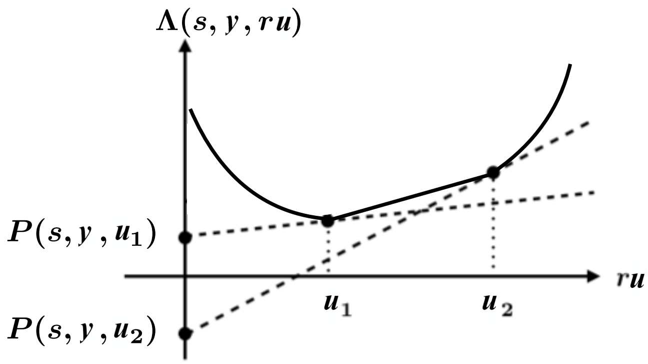

FIGURE 2. Interpretation of $P(s,z,u) \in \partial_\mu \left[\Lambda\left(s,z,\frac{u}{\mu}\right)\mu\right]_{\mu=1}$

case,

$$Q(s,y,v) \in \partial_r \Lambda(s,y,rv)_{r=1} \Rightarrow Q(s,y,v) = v \cdot \nabla_v \Lambda(s,y,v),$$

$$P(s,y,v) \in \partial_\mu \left[\Lambda\left(s,y,\frac{v}{\mu}\right)\mu\right]_{\mu=1} \Rightarrow P(s,y,v) = \Lambda(s,y,v) - v \cdot \nabla_v \Lambda(s,y,v).$$

In that case $P(s,y,v)$ is the ordinate of the intersection of the tangent hyperplane to $u \mapsto z = \Lambda(s,y,u)$ at $(v, \Lambda(s,y,v))$ (see Fig. 3).

Structure of the effective domain. When Λ is extended valued, we assume the following conditions on the effective domain, given by

$$\mathrm{Dom}(\Lambda) := \{(s,y,v) \in I \times \mathbb{R}^n \times \mathbb{R}^n : \Lambda(s,y,v) < +\infty\}.$$

(1) $\mathrm{Dom}\,\Lambda = I \times D_\Lambda$ for some $D_\Lambda \subset \mathbb{R}^n \times \mathbb{R}^n$;
(2) For every $y \in \mathbb{R}^n$ the y-section of D_Λ is strictly star-shaped on the variable v with respect to the origin, i.e.,

$$\forall (y,v) \in D_\Lambda, \forall r \in]0,1] \qquad (y, rv) \in D_\Lambda. \tag{2.5}$$

Remark 2.7. Condition 1 is fulfilled if Λ is a product of functions of the form $\Lambda(s,y,v) = a(s)L(y,v)$.

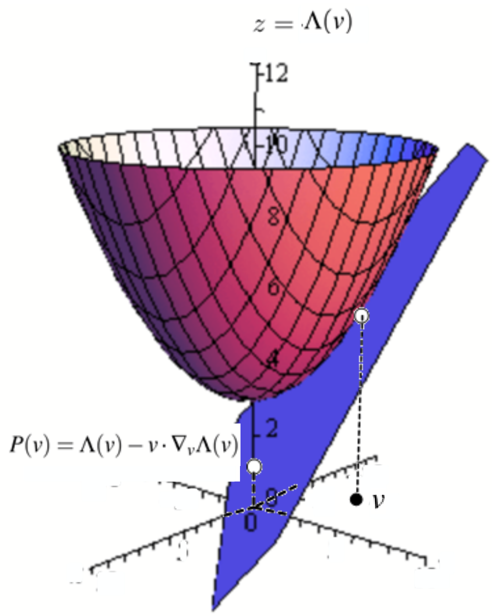

FIGURE 3. Interpretation of $P(v) = \Lambda(v) - v \cdot \nabla_v \Lambda(v)$

3. NON-OCCURRENCE OF THE LAVRENTIEV GAP AND PHENOMENON.

3.1. **Non-occurrence of the Lavrentiev gap.** The main result of the paper concerns the avoidance of the gap for $(\mathscr{P})$.

Theorem 3.1 (**Non-occurrence of the Lavrentiev gap at** $(y,u) \in \mathscr{A}$)**.** *Let* $(y,u) \in \mathscr{A}$ *be such that* $F(y,u) < +\infty$*. In addition to the Basic Assumptions, Condition* (S)*, the Radial Convexity Assumption* (RC) *on* Λ*, suppose that:*

($\mathrm{B}^w_{y,\Lambda}$) *There is* $r_y > 0$ *such that* Λ *is bounded on* $\big(I \times y(I) \times (B^m_{r_y} \cap \mathscr{U})\big) \cap \mathrm{Dom}(\Lambda)$.

Then there is no Lavrentiev gap for $(\mathscr{P})$ *at* (y,u)*. Moreover the approximating sequence of trajectories* $(y_k)_k$ *in Definition 2.1 has the following properties:*

(1) Each y_k *is Lipschitz and a* Lipschitz reparametrization of y*, i.e.,*

$$y_k = y \circ \psi_k$$

for a suitable injective and Lipschitz map $\psi_k : I \to I$*;*

(2) If b *is Lipschitz then* $y_k' \to y'$ *in* $L^1(I;\mathbb{R}^n)$ *so that* $y_k \to y$ *in* $W^{1,1}(I;\mathbb{R}^n)$.

Remark 3.2. In the case where $b = 1$ the optimal control problem $(\mathscr{P})$ is a particular case of the basic problem of the Calculus of Variations with a prescribed initial condition $y(t) = X$. In that framework Theorem 3.1 yields the convergence of y_k to y in $W^{1,1}(I;\mathbb{R}^n)$.

3.2. **Non-occurrence of the Lavrentiev phenomenon.** As a consequence of Theorem 3.1, we obtain the following condition ensuring the non-occurrence of the Lavrentiev phenomenon.

Corollary 3.3 (**Non-occurrence of the Lavrentiev phenomenon**)**.** *In addition to the Basic Assumptions, Condition* (S)*, the Radial Convexity Assumption* (RC) *on* Λ *suppose, moreover, that:*

(B^w_Λ) *For all $K>0$ there is $r_K>0$ such that Λ is bounded on*

$$\big(I\times(B^n_K\cap\mathscr{S})\times(B^m_{r_K}\cap\mathscr{U})\big)\cap\mathrm{Dom}(\Lambda).$$

Then the Lavrentiev phenomenon does not occur for ($\mathscr{P}$).

Proof. Let $(y_j,u_j)_j$ be a minimizing sequence for ($\mathscr{P}$) such that

$$\forall j\in\mathbb{N}\qquad F(y_j,u_j)\le\inf(\mathscr{P})+\frac{1}{j+1}.$$

Fix $j\in\mathbb{N}$. The assumptions imply the validity of the conditions of Theorem 3.1. Indeed, Hypothesis (B^w_Λ) implies ($\mathrm{B}^w_{y_j,\Lambda}$) of Theorem 3.1. The application of Theorem 3.1 yields $(\overline{y}_j,\overline{u}_j)\in W^{1,1}(I;\mathbb{R}^n)\times L^\infty(I;\mathbb{R}^m)$ satisfying the desired boundary conditions and constraints and, moreover,

$$F(\overline{y}_j,\overline{u}_j)\le F(y_j,u_j)+\frac{1}{j+1}\le\inf(\mathscr{P})+\frac{2}{j+1}.$$

Thus $(\overline{y}_j,\overline{u}_j)$ is a minimizing sequence of pairs with the desired properties. □

4. Proof of Theorem 3.1

4.1. **A fundamental Lemma.**

The next result is an important tool in the proof of Theorem 3.1.

Lemma 4.1. *Suppose that Λ satisfies the Basic and Structure assumptions. Let $\mathscr{K}$ be a bounded subset of $\mathscr{S}$. Let, for any $(s,z,v)\in\mathrm{Dom}(\Lambda)\cap(I\times\mathscr{S}\times\mathscr{U})$,*

$$P(s,z,v)\in\partial_\mu\left(\Lambda\left(s,z,\frac{v}{\mu}\right)\mu\right)_{\mu=1}.$$

Assume that there is $r>0$ such that Λ is bounded on $(I\times\mathscr{K}\times(B^m_r\cap\mathscr{U}))\cap\mathrm{Dom}(\Lambda)$. Then

$$\sup_{\substack{s\in I,z\in\mathscr{K},|v|\ge r\\ v\in\mathscr{U}}}P(s,z,v)<+\infty.\tag{4.1}$$

The proof of Lemma 4.1 follows narrowly the arguments given in [19, Lemma 4.18, Proposition 4.24] and the new arguments involved in [7, Proposition 3.15] in a different framework. For the convenience of the reader, we give the full details with a simpler argument.

Proof. If the set

$$\{(s,z,v)\in\mathrm{Dom}(\Lambda):\,s\in I,z\in\mathscr{K},|v|\ge r,v\in\mathscr{U}\}$$

is empty there is nothing to prove. Otherwise, let $(s,z,v)\in\mathrm{Dom}(\Lambda)$ with $s\in I,z\in\mathscr{K}$ and $|v|\ge r$, $v\in\mathscr{U}$. Since the (s,y)-sections of $\mathrm{Dom}(\Lambda)$ are star-shaped, then

$$\left(s,z,\frac{r}{2}\frac{v}{|v|}\right)\in\mathrm{Dom}(\Lambda)$$

and thus

$$\Lambda\left(s,z,\frac{r}{2}\frac{v}{|v|}\right)\frac{2|v|}{r}-\Lambda(s,z,v)\ge P(s,z,v)\left(\frac{2|v|}{r}-1\right),\tag{4.2}$$

from which we deduce that

$$P(s,z,v)\le\Lambda\left(s,z,\frac{r}{2}\frac{v}{|v|}\right)\frac{\frac{2|v|}{r}}{\frac{2|v|}{r}-1}.\tag{4.3}$$

Since $\mathscr{U}$ is a cone, the assumptions imply that $\Lambda\left(s,z,\frac{r}{2}\frac{v}{|v|}\right)\le C$ for some constant C depending only on $\mathscr{K}$. Moreover, $\frac{2|v|}{r}\ge 2$ and the function $\frac{\alpha}{\alpha-1}$ is bounded in $[2,+\infty[$; the conclusion follows. □

Remark 4.2. Taking into account the geometric interpretation of the subdifferentials given in Remark 2.6, Condition (4.1) in Lemma 4.1 means that the intersection with the ordinate axis of the tangent lines to $r\mapsto\Lambda(s,z,rv)$ at $r=1$ are bounded below by a constant if $|v|\ge r$. In the smooth case Condition (4.1) may be rewritten as

$$\sup_{\substack{s\in I,z\in\mathscr{K},|v|\ge r\\ v\in\mathscr{U}}}\Lambda(s,z,v)-v\cdot\nabla_v\Lambda(s,z,v)<+\infty. \tag{4.4}$$

4.2. **Change of variables and approximations.** We shall often make use of the following change of variables formula for Lebesgue integrals.

Proposition 4.3 (**Change of variables for Lebesgue integrals**). [3, Corollary 3.16] *Let $f\ge 0$ be measurable and $\gamma:I\to I$ be bijective, absolutely continuous with $\gamma'>0$ on I. Then, for every measurable $A\subset I$, $f\in L^1(A)\Leftrightarrow(f\circ\gamma)\gamma'\in L^1(\gamma^{-1}(A))$ and*

$$\int_A f(s)\,ds=\int_{\gamma^{-1}(A)}f(\gamma(\tau))\gamma'(\tau)\,d\tau.$$

The following approximation argument will be used in the sequel; we refer to [11] for its proof.

Lemma 4.4. *Let $f\in L^1(I)$ and $(\varphi_k)_k$ be a sequence of bijective, absolutely continuous functions $\varphi_k:I\to I_k\supset I$ with a Lipschitz inverse ψ_k, such that:*

- *For all k, $\varphi_k'>0$ on I;*
- *The sequence of Lipschitz constants $(\|\psi_k\|_\infty)_k$ is bounded;*
- *For each $t\in I$, $\varphi_k(t)\to t$ as $k\to+\infty$.*

Then $\displaystyle\int_{\psi_k(I)}|f\circ\varphi_k-f|\,ds\to 0$ as $k\to+\infty$.

4.3. **Proof of Theorem 3.1.**

Proof. Fix $\eta>0$.

i) Definition of $\Xi(k)$.

Let, for $z\in\mathscr{S},v\in\mathscr{U}$,

$$P(s,z,v)\in\partial_\mu\left[\Lambda\left(s,z,\frac{v}{\mu}\right)\mu\right]_{\mu=1}.$$

For $k>0$ we define

$$\Xi(k):=\sup_{\substack{s\in I,z\in y(I),|v|\ge k\\ (s,z,v)\in\mathrm{Dom}(\Lambda)\\ v\in\mathscr{U}}}P(s,z,v).$$

We may assume that $\Xi(k)>-\infty$ for all $k>0$, otherwise there is $k>0$ such that $|u(s)|\le k$ a.e. on I and the conclusion of Theorem 3.1 follows trivially. Hypothesis $(\mathrm{B}^w_{y,\Lambda})$ ensures the validity of (4.1) of Lemma 4.1, with $\mathscr{K}=y(I),r=r_y$. Its application shows that

$$\forall k\ge r_y\qquad \Xi(k)\in\mathbb{R}. \tag{4.5}$$

ii) For every $k>0$ define

$$S_k := \{s \in I : |u(s)| > k\}, \quad \varepsilon_k := \int_{S_k} \left(\frac{|u(s)|}{k} - 1 \right) ds.$$

Then

$$|S_k| \to 0, \quad 0 \le \varepsilon_k \le \frac{\|u\|_1}{k} \to 0 \text{ as } k \to +\infty. \tag{4.6}$$

Indeed,

$$k|S_k| \le \int_{S_k} |u(s)|\, ds \le \|u\|_1.$$

iii) Choice of k and definition of Ξ.
We choose $k \ge r_y$ in such a way that

$$\varepsilon_k \le \frac{\|u\|_1}{k} \le \varepsilon_*. \tag{4.7}$$

Set

$$\Theta := 2\left(\|\Lambda(s,y,u)\|_1 + \beta\|u\|_1 + \|\gamma\|_1\right),$$

with β, γ as in Condition (S) corresponding to $K = \|y\|_\infty$. We choose k large enough in such a way that

$$\frac{\|u\|_1}{k}(\Theta + \Xi(r_y)^+) \le \eta, \tag{4.8}$$

so that, from (4.6) and the fact that $\Xi(k) \le \Xi(r_y)$ (whence $\Xi(k)^+ \le \Xi(r_y)^+$), we have

$$\varepsilon_k(\Theta + \Xi(k)^+) \le \frac{\|u\|_1}{k}(\Theta + \Xi(r_y)^+) \le \eta, \tag{4.9}$$

From now on we set $\Xi := \Xi(k)$.

iv) The change of variable φ_k. We introduce the following absolutely continuous change of variable $\varphi_k : I \to \mathbb{R}$ defined by

$$\varphi_k(t) := t, \quad \text{for a.e. } \tau \in I \quad \varphi_k'(\tau) := \begin{cases} \dfrac{|u(\tau)|}{k} & \text{if } \tau \in S_k, \\ 1 & \text{otherwise.} \end{cases} \tag{4.10}$$

Clearly φ_k is strictly increasing with $\varphi_k' \ge 1$ a.e. on I. Therefore the image of φ_k is $[t, T_h]$, for some $T_h \ge 1$ and thus $\varphi_k : [t,T] \to [t,T_h]$ is bijective; let us denote by ψ_k its inverse, restricted to $[t,T]$. Then ψ_k is absolutely continuous and even Lipschitz, since $\|\psi_k'\|_\infty \le 1$.

v) Set

$$u_k(s) := \frac{u(\psi_k(s))}{\varphi_k'(\psi_k(s))} \quad y_k := y \circ \psi_k. \tag{4.11}$$

Then (y_k, u_k) is admissible.
It follows from [24, Corollary 5] and [23, Chapter IX, Theorem 5] that y_k is absolutely continuous and that, for a.e. $s \in [t,T]$,

$$\begin{aligned} y_k'(s) &= y'(\psi_k(s))\frac{1}{\varphi_k'(\psi_k(s))} \\ &= b(y(\psi_k(s)))\frac{u(\psi_k(s))}{\varphi_k'(\psi_k(s))} \\ &= b(y_k(s))u_k(s). \end{aligned}$$

Since y_k is defined via a reparametrization of y, we still have that $y_k(s) \in \mathscr{S}$ for all s. Moreover, $y_k(t) = y(\psi_k(t)) = y(t)$. However, it may happen that $y_k(T) = y(\psi_k(T)) \neq y(T)$, since in general $\psi_k(T) < T$. Notice also that

$$u_k(s) = \frac{1}{\varphi_k'(\psi_k(s))}u(\psi_k(s)) \in \mathscr{U} \text{ a.e. } s \in [t,T], \tag{4.12}$$

the set $\mathscr{U}$ being a cone.

vi) *u_k is bounded, y_k is Lipschitz.*
It is convenient to write explicitly the function $u_k(s)$, which is given by

$$u_k(s) = \begin{cases} k\dfrac{u(\psi_k(s))}{|u(\psi_k(s))|} & \text{if } \psi_k(s) \in S_k, \\ u(\psi_k(s)) & \text{otherwise.} \end{cases}$$

Since $|u(s)| \le k$ a.e. out of S_k then

$$\|u_k\|_\infty \le k. \tag{4.13}$$

Notice that, since b is bounded on bounded sets, then

$$y_k' = b(y_k)u_k$$

is bounded too.

vii) *The following estimate holds:*

$$\forall \tau \in [t,T] \quad \|\varphi_k(\tau) - \tau\|_\infty \le \int_t^T \left|\varphi_k'(s) - 1\right| ds \le \varepsilon_k \le \varepsilon_*. \tag{4.14}$$

Indeed, for all $\tau \in [t,T]$ we have

$$\begin{aligned} |\varphi_k(\tau) - \tau| &\le \int_t^\tau \left|\varphi_k'(s) - 1\right| ds \\ &\le \int_{S_k} \left(\frac{|u(s)|}{k} - 1\right) ds \le \varepsilon_k \le \varepsilon_*, \end{aligned}$$

in virtue of (4.7).
In the next steps viii – xi we compare $F(y_k)$ with $F(y)$. We will use a consequence of the radial convex subgradient inequality (2.4) that follows from the radial convexity assumption: For every $(\tilde{s}, z, v) \in \mathrm{Dom}(\Lambda) \cap (I \times \mathscr{S} \times \mathscr{U})$ and $\mu > 0$ such that $\left(\tilde{s}, z, \dfrac{v}{\mu}\right) \in \mathrm{Dom}(\Lambda)$ we have

$$\forall \mu > 0 \quad \Lambda\left(\tilde{s}, z, \frac{v}{\mu}\right)\mu \le \Lambda(\tilde{s}, z, v) + P\left(\tilde{s}, z, \frac{v}{\mu}\right)(\mu - 1). \tag{4.15}$$

viii) For a.e. in $S_k \cap \psi_k(I)$ we have

$$\Lambda\left(\varphi_k, y, k\frac{u}{|u|}\right)\frac{|u|}{k} \le \Lambda\big(\varphi_k, y, u\big) + \left(\frac{|u|}{k} - 1\right)\Xi. \tag{4.16}$$

Since $|u| > k$ a.e. on S_k, the Structure Assumptions on $\mathrm{Dom}(\Lambda)$ imply

$$\Lambda\left(\varphi_k, y, k\frac{u}{|u|}\right) < +\infty \quad \text{a. e. in } S_k \cap \psi_k(I).$$

By applying (4.15) with $\tau \in S_k \cap \psi_k(I)$, $\tilde{s} = \varphi_k(\tau)$, $\mu = \dfrac{|u(\tau)|}{k} > 1$, $z = y(\tau)$ and $v = u(\tau)$ we obtain

$$\Lambda\left(\varphi_k, y, k\frac{u}{|u|}\right)\frac{|u|}{k} \le \Lambda(\varphi_k, y, u) + P\left(\varphi_k, y, k\frac{u}{|u|}\right)\left(\frac{|u|}{k} - 1\right). \tag{4.17}$$

Since $|u| > k$ a.e. in S_k, we deduce that

$$P\left(\varphi_k, y, k\frac{u}{|u|}\right)\left(\frac{|u|}{k} - 1\right) \le \left(\frac{|u|}{k} - 1\right)\Xi \quad \text{a.e. in } S_k \cap \psi_k(I),$$

so that (4.16) follows directly from (4.17).

ix) The following estimate of $F(y_k)$ holds:

$$F(y_k) \le \int_{\psi_k(I)} \Lambda(\varphi_k, y, u)\, d\tau + \varepsilon_k\, \Xi^+. \tag{4.18}$$

Indeed we have

$$F(y_k) = \int_t^T \Lambda(s, y_k, u_k)\, ds. \tag{4.19}$$

Taking into account (4.11), the change of variables $s = \varphi_k(\tau), \tau \in I$ yields (in what follows, for the sake of brevity, we omit the independent variables τ, s):

$$\begin{aligned} F(y_k) &= \int_{\psi_k(I)} \Lambda\left(\varphi_k, y, \frac{u}{\varphi_k'}\right)\varphi_k'\, d\tau \\ &= \int_{S_k\cap\psi_k(I)} \Lambda\left(\varphi_k, y, k\frac{u}{|u|}\right)\frac{|u|}{k}\, d\tau + \int_{\psi_k(I)\setminus S_k} \Lambda(\varphi_k, y, u)\, d\tau, \end{aligned} \tag{4.20}$$

We deduce from (4.16) that, a.e. in $S_k \cap \psi_k(I)$,

$$\Lambda\left(\varphi_k, y, k\frac{u}{|u|}\right)\frac{|u|}{k} \le \Lambda\big(\varphi_k, y, u\big) + \left(\frac{|u|}{k} - 1\right)\Xi^+, \tag{4.21}$$

whence

$$\int_{S_k\cap\psi_k(I)} \Lambda\left(\varphi_k, y, k\frac{u}{|u|}\right)\frac{|u|}{k}\, d\tau \le \int_{S_k\cap\psi_k(I)} \Lambda\big(\varphi_k, y, u\big)\, d\tau + \Xi^+\varepsilon_k. \tag{4.22}$$

Therefore, (4.18) follows now immediately from (4.20) and (4.22).

x) The following estimate holds:

$$\int_{\psi_k(I)} \Lambda\big(\varphi_k, y, u\big)\, d\tau \le F(y) + \varepsilon_k \Theta. \tag{4.23}$$

Indeed, a.e. on I we have

$$\Lambda\big(\varphi_k, y, u\big) = Q_k(\tau) + \Lambda\big(\tau, y, u\big),$$

where

$$Q_k(\tau) := \Lambda\big(\varphi_k, y, u\big) - \Lambda\big(\tau, y, u\big).$$

Condition (S) (with $K := \|y\|_\infty$) and Step *vii* imply that

$$\begin{aligned}\int_{\psi_k(I)} |Q_k(\tau)|\, d\tau &\le \varepsilon_k \int_t^T \kappa\Lambda(\tau, y(\tau), u(\tau)) + \beta|u(\tau)| + \gamma(\tau)\, d\tau \\ &\le \varepsilon_k \Theta,\end{aligned}$$

from which follows (4.23).

xi) Final estimate of $F(y_k)$. From (4.18) of Step *ix* and (4.23) of Step *x*, we obtain

$$F(y_k) \le F(y) + \varepsilon_k \left(\Theta + \Xi^+\right). \tag{4.24}$$

The choice of k in (4.9) gives

$$F(y_k) \le F(y) + \eta.$$

xii) Convergence of u_k to u in $L^1(I;\mathbb{R}^m)$. The change of variable $s = \varphi_k(\tau)$ gives

$$\begin{aligned}\int_t^T |u_k - u|\, ds &= \int_t^T \left| \frac{u(\psi_k)}{\varphi_k'(\psi_k)} - u \right| ds \\ &= \int_{\psi_k(I)} \left| \frac{u}{\varphi_k'} - u(\varphi_k) \right| \varphi_k'\, d\tau \\ &\le \int_{\psi_k(I)\setminus S_k} *\, d\tau + \int_{\psi_k(I)\cap S_k} *\, d\tau\end{aligned} \tag{4.25}$$

where in the above $*$ stands for $\left| \dfrac{u}{\varphi_k'} - u(\varphi_k) \right| \varphi_k'$. It follows from the definition of φ_k in Step *iv* that:

- Since $\varphi_k' = 1$ on $I \setminus S_k$ then

$$\begin{aligned}\int_{\psi_k(I)\setminus S_k} *\, d\tau &= \int_{\psi_k(I)\setminus S_k} |u - u(\varphi_k)|\, d\tau \\ &\le \int_{\psi_k(I)} |u - u(\varphi_k)|\, d\tau \to 0 \quad k \to +\infty,\end{aligned} \tag{4.26}$$

  as a consequence of Lemma 4.4 since, from Step *vii*, $\|\varphi_k(\tau) - \tau\|_\infty \to 0$.
- Since $\varphi_k' = \dfrac{|u|}{k}$ on S_k then:

$$\begin{aligned}\int_{\psi_k(I)\cap S_k} *\, d\tau &:= \int_{\psi_k(I)\cap S_k} \left| \frac{u}{\varphi_k'} - u(\varphi_k) \right| \varphi_k'\, d\tau \\ &\le \int_{\psi_k(I)\cap S_k} |u|\, d\tau + \int_{\psi_k(I)\cap S_k} |u(\varphi_k)| \varphi_k'\, d\tau \\ &= \int_{\psi_k(I)\cap S_k} |u|\, d\tau + \int_{I\cap\varphi_k(S_k)} |u|\, ds.\end{aligned} \tag{4.27}$$

  Since $u \in L^1([t,T];\mathbb{R}^m)$ and, from Step *ii*, $|S_k| \to 0$ as $k \to +\infty$ we obtain

$$\int_{\psi_k(I)\cap S_k} |u|\, d\tau \to 0 \quad k \to +\infty.$$

Moreover,

$$\begin{aligned}|\varphi_k(S_k)| &= \int_{\varphi_k(S_k)} 1\,ds = \int_{S_k} \varphi_k'\,d\tau \\ &= \int_{S_k} \frac{|y'|}{k}\,d\tau \le \frac{\|y'\|_1}{k} \to 0 \quad k \to +\infty,\end{aligned}$$

so that

$$\int_{I\cap\varphi_k(S_k)} |u|\,ds \to 0 \quad k \to +\infty.$$

It follows from (4.27) that $\int_{\psi_k(I)\cap S_k} * \, d\tau \to 0$ as $k \to +\infty$.

Therefore, we deduce from (4.25), together with (4.26) and (4.27) that

$$\|u_k - u\|_1 = \int_t^T |u_k - u|\,ds \to 0 \quad k \to +\infty.$$

xiii) Convergence of y_k to y in $L^\infty(I;\mathbb{R}^n)$.
Accordingly to the definition of y_k, for all $s \in I$ we have

$$\begin{aligned}|y_k(s) - y(s)| &= |y(\psi_k(s)) - y(s)| \\ &\le \sup\{|y(\tau) - y(\varphi_k(\tau))| : \tau \in \psi_k(I)\}.\end{aligned}$$

Now, from (4.14), $\varphi_k(\tau)$ converges uniformly to τ and y is uniformly continuous on I, thus y_k converges uniformly to y.

xiv) Proof of Claim 2. Assume that $b : \mathbb{R}^n \to L(\mathbb{R}^m;\mathbb{R}^n)$ is Lipschitz with constant K. Then

$$\begin{aligned}\|y_k' - y'\|_1 &= \|b(y_k)u_k - b(y)u\|_1 \\ &\le \|b(y_k)u_k - b(y)u_k\|_1 + \|b(y)u_k - b(y)u\|_1 \\ &\le \|b(y_k) - b(y)\|_{L(\mathbb{R}^m;\mathbb{R}^n)}\|u_k\|_1 + \|b(y)\|_{L(\mathbb{R}^m;\mathbb{R}^n)}\|u - u_k\|_1 \\ &\le K\|y_k - y\|_\infty \|u_k\|_1 + \|b(y)\|_{L(\mathbb{R}^m;\mathbb{R}^n)}\|u - u_k\|_1.\end{aligned}$$

Now, from the points above we know that $\|u - u_k\|_1 \to 0$ and $\|y_k - y\|_\infty \to 0$ as $k \to +\infty$; the conclusion follows. □

Remark 4.5. In the case of a final end-point constraint, instead of the initial one, Claim 1 of Theorem 3.1 may be obtained by slightly modifying Step *iv* of the proof: indeed it is enough to define

$$\varphi_k(T) := T, \quad \text{for a.e. } \tau \in I \quad \varphi_k'(\tau) = \begin{cases} \dfrac{|y'(\tau)|}{k} & \text{if } \tau \in S_k, \\ 1 & \text{otherwise.}\end{cases}$$

5. An explicit approximating sequence

The proof of Theorem 3.1 is constructive. Indeed, the approximating functions y_k are defined by $y_k := y \circ \psi_k$, where φ_k is defined in Step *iv* and depends just on u and the set $S_k := \{s \in I : |u(s)| > k\}$. One then defines the reparametrization φ_k as in Step *iv*. An explicit approximating sequence is built in the next example, provided by G. Alberti in a personal communication for a different purpose.

Example 5.1. Let $y \in W^{1,1}([0,1];\mathbb{R})$ be such that

- y is of class C^2 in $[0,1[$, $y(0) = 0, y(1) = 1$;

- $y'' > 0$ and $y' > 0$ on $[0,1[$,
- $y'(1) := \lim_{s \to 1^-} y'(s) = +\infty$.

Such a function exists, e.g., $y(s) := 1 - \sqrt{1-s}, s \in [0,1]$. For every $z \in [0,1[$ set $q(z) := y'(y^{-1}(z))$. Let

$$\Lambda(z,v) := \begin{cases} 0 & \text{if } z \in [0,1[\text{ and } v \le q(z), \\ +\infty & \text{otherwise,} \end{cases}$$

and set $F(z) := \int_0^1 \Lambda(z(s), z'(s))\, ds$ for every $z \in W^{1,1}([0,1];\mathbb{R})$. Notice that Λ is lower semi-

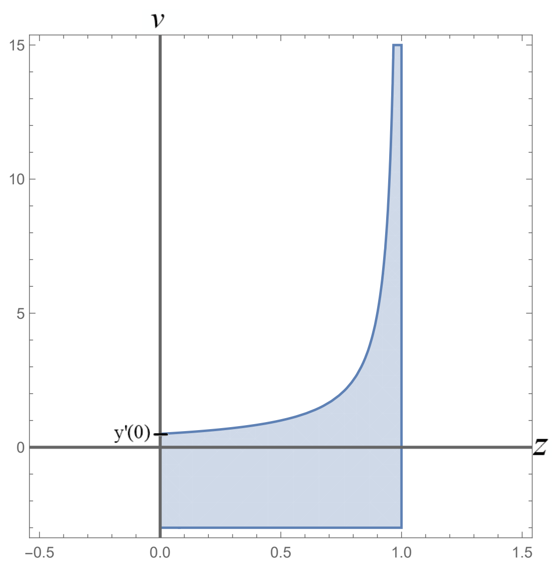

FIGURE 4. The domain of $\Lambda(\cdot,\cdot)$ in Example 5.1

continuous on $\mathbb{R}^2$ and $\Lambda(z,\cdot)$ is convex for all $z \in \mathbb{R}$. Clearly $F(y) = \min F = 0$. Then

(1) *The Lavrentiev phenomenon occurs for the problem with two end-point conditions, more precisely $F(z) = +\infty$ for every Lipschitz $z : [0,1] \to \mathbb{R}$ satisfying $z(0) = 0, z(1) = 1$.* We refer to [21, Example 3.5] for the proof.

(2) *Check of the validity of the assumptions of Theorem 3.1.*
The Lagrangian here is autonomous and satisfies the conditions for the validity of Theorem 3.1 (Λ is bounded on its effective domain). Therefore there is no gap for the one end-point problem $y(1) = 1$.

(3) *Construction of a family of Lipschitz approximating competitors with the end-point constraint $z(1) = 1$.* We illustrate here, as an example, the construction of the "almost better" Lipschitz competitor y_k that is carried on in the proof of Theorem 3.1 for the problem with final constraint $y(1) = 1$. Let $k \in \mathbb{N}, k > \max\{y'(0), 1\}$ and let $t_k \in]0,1[$ be such that $y'(t_k) = k$; following Step *iv* of the proof of Theorem 3.1 and Remark 4.5, the change of variable $\varphi_k : [0,1] \to \mathbb{R}$ is defined by

$$\varphi_k(1) := 1, \quad \text{for a.e. } \tau \in [0,1] \quad \varphi_k'(\tau) := \begin{cases} \dfrac{y'(\tau)}{k} & \text{if } \tau \in [t_k, 1], \\ 1 & \text{otherwise.} \end{cases}$$

Therefore we have

$$\varphi_k(\tau) := \begin{cases} 1 + \dfrac{y(\tau)-1}{k} & \text{if } \tau \in [t_k, 1], \\ \varphi_k(t_k) + \tau - t_k & \text{otherwise,} \end{cases} \qquad \varphi(t_k) = 1 + \frac{y(t_k)-1}{k}.$$

Notice that $\varphi_k(0) = \varphi_k(t_k) - t_k \le 0$ since $\varphi_k' \ge 1$ on $[t_k, 1]$. Let $\tau_k \in [0,1]$ be such that

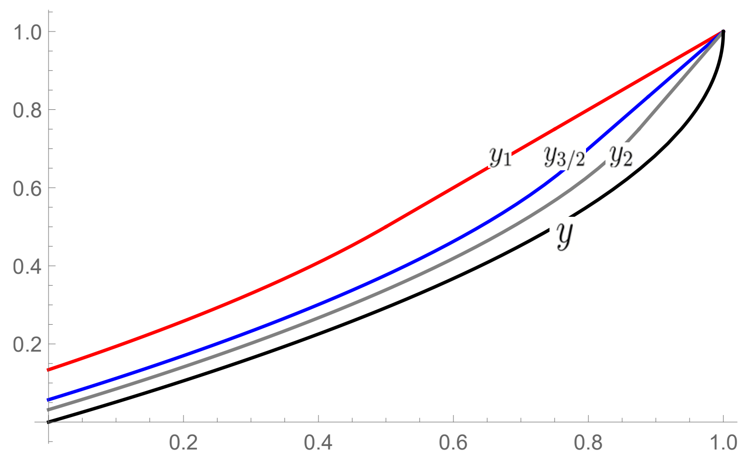

FIGURE 5. The absolutely continuous function $y(s) := 1 - \sqrt{1-s}$ (below) and some of its Lipschitz approximations (from above: $y_1, y_{3/2}, y_2$), following the recipe of the proof of Theorem 3.1.

$\varphi_k(\tau_k) = 0$, namely $\tau_k = t_k - \varphi_k(t_k)$. The inverse ψ_k of φ_k, restricted to $[0,1]$ is thus defined by

$$\psi_k(s) := \begin{cases} y^{-1}(1 + (s-1)k) & \text{if } s \in [\varphi_k(t_k), 1], \\ s + t_k - \varphi_k(t_k) & \text{if } s \in [0, \varphi_k(t_k)]. \end{cases}$$

Then $\psi_k([0,1]) = [\tau_k, 1]$. The function $y_k = y \circ \psi_k$ is thus defined as

$$y_k(s) := \begin{cases} 1 + sk - k & \text{if } s \in [\varphi_k(t_k), 1], \\ y(s + t_k - \varphi_k(t_k)) & \text{if } s \in [0, \varphi_k(t_k)]. \end{cases}$$

Figure 7 depicts the graphs of some of these approximations for some values of k and $y(s) := 1 - \sqrt{1-s}$.

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