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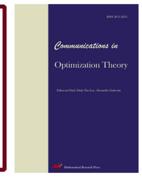

A DYNAMIC MULTILEVEL MODEL OF THE EUROPEAN GAS MARKET

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Dedicated to the memory of Professor Roland Glowinski

Abstract. The European gas market is governed by rules that are agreed on by the European Union. We present a mathematical market model that takes into account this structure, where the technical system operator (TSO) offers certain transportation capacities that can be booked and later nominated within the previously chosen bookings. The TSO also fixes booking fees and defines an operational control of the gas pipeline system in order to deliver the gas according to the nominations. Since the gas flow is governed by a system of partial differential equations, to realize this control structure partial differential equations (PDEs) should be involved in the model. While the four level gas market model has been discussed previously, in this paper we take into account the flow model by PDEs in the discussion of the model and in the reduction to a single level problem, where we also state the corresponding necessary optimality conditions. We also present some examples for the optimal control problem of the TSO.
Keywords. Coupling conditions; Flow model; Isothermal Euler equations; Nash equilibrium; Partial differential equation, Potential games.
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1. Introduction

The structure of the European gas market is governed by a set of rules that have been defined by the European Union. For a deeper understanding of the market mechanisms a representation as a mathematical market model, where the involved players take their decisions according to optimization problems that model their preferences and feasible decisions is essential.

In this paper, we follow [23] where the European gas market is modeled as a multilevel game theoretic model. This model describes the market situation, where gas traders (buyers and sellers) compete on a joint resource of a gas network. The interaction of the market participants and their competition is due to the limitations of the network, that are given in particular by the

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technical capacities of the network. Moreover, based on the assumptions of complete information and a noncooperative setting between the market participants, the traders are assumed to act strategically such the general Nash game framework is applied.

The starting point is a four-level model where the players and the decision variables of the levels are given as follows:

Level 1: The TSO (technical system operator) decides about the technical capacities q^{TC} and the booking price floors $\underline{\pi}^{book}$.

Level 2: The traders (the buyers and sellers v) decide about their individual bookings (i.e. capacity rights) of q^{book}

Level 3: The traders determine the individual gas nominations q_k^{nom}, which are only bounded from above by the associated previously chosen bookings q^{book} and linked to each other by a clearing condition.

Level 4: The TSO decides about the pressure level p_v and the flow of the transported gas at node v and the gas flow q_e on the arc e of the network, according to the nominated gas given by the vector q^{nom}.

Furthermore, in [23] the total time period $(0,T)$ is discretized and the gas flows and pressures in each time interval $[t_j,t_{j+1}]$ with $t_j \in \mathscr{T} = \{t_0 = 0, t_1, t_2, .., t_N = T\}$ are assumed to be stationary. Note that in the practical implementation of the gas market, the transport capacity products are nominated on a daily basis. Hence for the intervals $[t_j, t_{j+1}]$ a natural choice is one day. The products are nominated according to the demand that is expected for the day. This expectation is subject to a substantial uncertainty about the actual demand, which depends for example on the temperature. Of course in the physical realization of the gas transport, this transport capacity product is realized by a function of continuous time. In our model, we require that this physical realization is compatible with the nominated capacities.

Therefore in contrast to the approach in [23] in this paper we include the flow model that is governed by partial differential equations (PDEs) for the system dynamics in the multilevel model. This allows us to take into account the physics of the gas flow in the market model. We provide a method that allows to couple the market models that are based upon a finite number of variables for a discretized system (where the time interval is replaced by a finite number of subintervals) with the PDE model that models an evolution in continuous time. As an example, consider an objective functional for the optimal control model for the PDE system that is given by an integral in continuous time over the time horizon. We discuss how such on objective functional can be coupled in a consistent way with demand values where only the accumulated values on a finite number of subintervals are prescribed as input data for the optimization problem.

In [23] it has been shown, that under suitable assumptions, due to the specific structure of the optimization problems and their coupling in the presented model, the four-level model can be reduced to an aggregated bilevel problem, where on the upper level, the TSO optimizes social welfare and the efficient allocation of the gas and on the lower-level the traders determine their bookings and nominations in a single optimization problem. This reduction of the complexity of the market model is done in terms of the associated stationarity or KKT conditions [23]. In this paper we will follow a different direction concerning the simplification of the model in level two and three that is based on the description of the Nash games of the market participants as so-called potential games. Such games can be replaced by a single optimization problem due to the inherent structure of the players' objectives and constraints. Potential games have first be introduced by Monderer and Shapley in 1996 for standard noncooperative Nash games [22]. Furthermore, in 2011 Facchinei et al extend the idea of potential games to generalized Nash games, i.e. games where not only the objectives, but also the constraint sets depend on the players' strategies [8]. Moreover, applications of potential games for hierarchical games have been discussed e.g. in [18, 25]. There is also a close connection of potential games to Nash games with shared constraints [20]. Taking into account the structure of the traders maximization problems in our model, it becomes clear that both of them form a generalized Nash game, respectively, which is of the particular structure as presented in [8].

In the presented market model, although we consider the coupling of the continuous flow model of the TSO with the finite dimensional game model of the traders, there is no direct influence of the PDE dynamics on the decisions of the traders concerning their nominations and bookings. The solution of the PDE dynamics influences the booking fees $\underline{\pi}_i^{book}$ and the technical capacities q_i^{TC}. Both of which are then taken as exogenous parameters by the traders. Therefore the coupling between the PDE model and the finite Nash games is realized through $\underline{\pi}_i^{book}$ and q_i^{TC} only.

This paper has the following structure. We start our discussion in Section 2 with a dynamic version of the TSO's optimal control problem. Moreover, since we want to focus on the gas dynamics on this level, we first concentrate on the first-best benchmark model described in [12], which is a simplified version of the full four level model which allows to compare different market outcomes. After the dicussion of the gas dynamics and the problem definitions Section 2 also contains numerical examples for the optimal control problems both for the cases of one and multiple market participants.

In Section 3 we will then discuss the levels two and three. We will prove that the games on these levels are in fact so-called potential GNEPs (generalized Nash games). Thus by introducing the correct potential functions and feasible sets, we can replace the games by their associated single optimization problem, which provides an alternative to [12] concerning the aggregation of the game model.

2. Dynamics of the Cost-Optimal Gas Transport

On the lowest level, the pressure and flow of the gas in each pipe/vertex are determined by the TSO according to the traders' nominated gas load q_t^{nom}, which is (for this optimization problem) taken as external parameter.

2.1. **Optimal Control Problem of the TSO.**

For the sake of simplicity of the presentation consider a connected, directed, star-shaped graph $G = (\mathscr{V}, \mathscr{E})$ with the vertex set $\mathscr{V} = \{v_{in}, v_0, \cdots, v_n\}$ and the set of edges $\mathscr{E} = \{e_0, \cdots, e_n\} \subseteq \mathscr{V} \times \mathscr{V}$. The star-shaped graph is the simplest possible graph where we can present the coupling conditions that govern the flow through the central node where several pipes are linked. In the case of general finite graphs, exactly the same node conditions are used as a model of the flow through an arbitrary vertex where several edges (pipes, respectively) are coupled. Let us name the central node in the star-shaped graph v_C. We assume, that the graph has only one inflow node v_{in}, all other nodes are outflow nodes. This is shown in Figure 1.

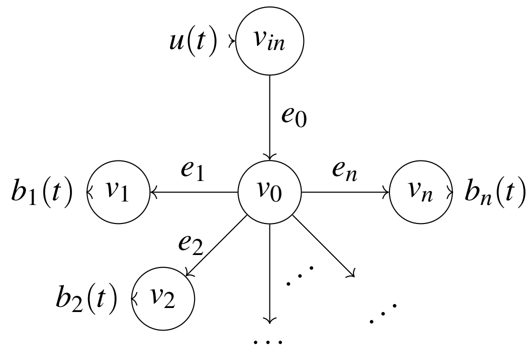

FIGURE 1. Scheme of a star-shaped graph with one inflow node

The gas dynamics for ideal gas on every edge $e \in \mathscr{E}$ corresponding to a space interval $[0, L^e]$ is given by the semilinear isothermal Euler equations (see for example [15], [6])

$$\begin{cases} \rho_t^e + q_x^e = 0 \\ q_t^e + c^2 \rho_x^e = -\dfrac{\lambda_F^e}{2D^e} \dfrac{q^e|q^e|}{\rho^e} \end{cases} \tag{2.1}$$

where ρ^e is the gas density, q^e is the gas flow, λ_F^e is the pipe friction coefficient and D^e is the pipe diameter of pipe $e \in \mathscr{E}$ respectively. The constant $c > 0$ is the speed of sound in the gas. The model is a simplification of the quasilinear isothermal Euler system, see for example [2].

Let initial conditions $\rho^e(0,x) = \rho_{\text{ini}}^e(x)$ and $q^e(0,x) = q_{\text{ini}}^e(x)$ be given by functions with L^2 regularity, that is in $L^2(0, L^e)$. We assume that the gas density $\rho_0(t)$ is given at the inflow node v_{in}, i.e., we have

$$\rho^{e_0}(t,0) = u(t)$$

with a control u in $L^2(0,T)$. Further we assume that gas demands $b_1, \cdots, b_n$ are given at the outflow nodes $v_1, \cdots, v_n$ by L^2 functions, i.e., we have

$$q^{e_i}(t, L^{e_i}) = b_i(t) \quad \text{for all } i = 1, \cdots, n.$$

At node v_0 we assume conservation of mass

$$\sum_{i=1}^{n} q^{e_i}(t,0)\,(D^{e_i})^2 \;=\; q^{e_0}(t, L^{e_0})\,(D^{e_0})^2,$$

and continuity of pressure resp. density

$$\rho^{e_i}(t,0) \;=\; \rho^{e_0}(t, L^{e_0}) \quad \text{for all } i = 1, \cdots, n.$$

We write the isothermal Euler equations (2.1) in terms of Riemann invariants. The eigenvalues of the system (2.1) are given by $\lambda_{1/2} = \pm c$ and the left eigenvectors are given by $\ell^1 = [c,\ 1]$ and $\ell^2 = [-c,\ 1]$. Thus the Riemann invariants are given by

$$R_1^e(\rho^e, q^e) = q^e + c\,\rho^e \qquad \text{and} \qquad R_2^e(\rho^e, q^e) = q^e - c\,\rho^e.$$

Vice versa this implies

$$\rho^e(R_1^e, R_2^e) = \frac{1}{2c}\,(R_1^e - R_2^e) \qquad \text{and} \qquad q(R_1^e, R_2^e) = \frac{1}{2}\,(R_1^e + R_2^e)\,.$$

Thus the isothermal Euler equations (2.1) stated in terms of Riemann invariants are

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}_t + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}_x = -\frac{c\lambda^F}{4D} \frac{(R_1+R_2)|R_1+R_2|}{R_1-R_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{2.2}$$

For the initial conditions we have

$$R^e_{1,\mathrm{ini}}(t) = q^e_{\mathrm{ini}} + c\rho^e_{\mathrm{ini}} \qquad \text{and} \qquad R^e_{2,\mathrm{ini}}(t) = q^e_{\mathrm{ini}} - c\rho^e_{\mathrm{ini}}.$$

Due to the structure of the Riemann invariants for the boundary conditions we have the feedback laws

$$R_1^{e_0}(t,0) \;-\; R_2^{e_0}(t,0) \;=\; 2\,c\,u(t)$$

and

$$R_1^{e_i}(t,L^{e_i}) \;+\; R_2^{e_i}(t,L^{e_i}) \;=\; 2\,b_i(t) \quad \text{for all } i=1,\cdots,n.$$

For the coupling conditions at node v_0 we have

$$\sum_{i=1}^{n} \Big(R_1^{e_i}(t,0)+R_2^{e_i}(t,0)\Big)(D^{e_i})^2 \;=\; \Big(R_1^{e_0}(t,L^{e_0})+R_2^{e_0}(t,L^{e_0})\Big)(D^{e_0})^2,$$

and

$$\Big(R_1^{e_i}(t,0)-R_2^{e_i}(t,0)\Big) \;=\; \Big(R_1^{e_0}(t,L^{e_0})-R_2^{e_0}(t,L^{e_0})\Big) \quad \text{for all } i=1,\cdots,n.$$

Due to regular gas nominations and gas renominations we consider a time interval decomposition in N equally distributed subintervals:

$$[0,T] \;=\; \bigcup_{j=0}^{N-1} [t_j,t_{j+1}],$$

where the $[t_j,t_{j+1}]$ all have the same length. We set $\Delta t := T/N$. Let q^{nom}_{i,t_j} be the gas demand of node v_i in the time interval $[t_j,t_{j+1}]$. This means that q^{nom}_{i,t_j} is the amount of gas that should be transported to node v_i in the time interval $[t_j,t_{j+1}]$. This yields the following constraint for the continuous flow as a function of time:

$$\int_{t_j}^{t_{j+1}} b_i(\tau)\,d\tau \;=\; q^{\mathrm{nom}}_{i,t_j}.$$

We introduce a time dependent function $\hat{q}^{\mathrm{nom}}_i(t)$ that is defined by

$$\int_{t_j}^{t_{j+1}} \hat{q}^{\mathrm{nom}}_i(\tau)\,d\tau \;=\; q^{\mathrm{nom}}_{i,t_j}.$$

The given values of q^{nom}_{i,t_j} can be used to construct the functions $\hat{q}^{\mathrm{nom}}_i(t)$ depending on the desired regularity. A piecewise constant function $\hat{q}^{\mathrm{nom}}_i(t)$ can be constructed to satisfy the equations

$$\hat{q}^{\mathrm{nom}}_i(t) = \begin{cases} \frac{q^{\mathrm{nom}}_{i,t_0}}{\Delta t}, & t=0 \\ \frac{q^{\mathrm{nom}}_{i,t_j}}{\Delta t}, & t \in \big(t_j,t_{j+1}\big] \end{cases} \qquad j=0,\cdots,N-1.$$

For a piecewise linear function with possible kinks at the grid points $t_1,..t_{N-1}$, we set

$$\hat{q}^{\mathrm{nom}}_i(t_0) = \frac{q^{\mathrm{nom}}_{i,t_0}}{\Delta t} \qquad \text{and} \qquad \hat{q}^{\mathrm{nom}}_i(t_N) = \frac{q^{\mathrm{nom}}_{i,t_{N-1}}}{\Delta t},$$

and for $j = 1, \cdots, N-1$ we define

$$\hat{q}_i^{\text{nom}}(t_j) = \frac{1}{2}\left(\frac{q_{i,t_j}^{\text{nom}}}{\Delta t} + \frac{q_{i,t_{j-1}}^{\text{nom}}}{\Delta t}\right).$$

Now for $j = 0, \cdots, N-1$ we can choose values for $\hat{q}_i^{\text{nom}}(t_j + \frac{\Delta t}{2})$, s.t. we have

$$\int_{t_j}^{t_j+\Delta t} \hat{q}_i^{\text{nom}}(\tau)\, d\tau \;+\; \int_{t_j+\Delta t}^{t_{j+1}} \hat{q}_i^{\text{nom}}(\tau)\, d\tau \;=\; q_{i,t_j}^{\text{nom}},$$

for the piecewise linear function $\hat{q}_i^{\text{nom}}(t)$. An explicit representation is given by

$$\hat{q}_i^{\text{nom}}\left(t_j + \frac{\Delta t}{2}\right) \;=\; \frac{2\, q_{i,t_j}^{\text{nom}}}{\Delta t} - \frac{\hat{q}_i^{\text{nom}}(t_{j+1})}{2} - \frac{\hat{q}_i^{\text{nom}}(t_j)}{2}.$$

An example on a graph with a single outflow node (i.e., $n = 1$) for $T = 6h$, $\Delta t = 1h$ and

$$q^{\text{nom}} = \begin{bmatrix} 2.7\cdot 10^5 & 2.8\cdot 10^5 & 3\cdot 10^5 & 2.5\cdot 10^5 & 2.4\cdot 10^5 & 2.8\cdot 10^5 \end{bmatrix} \quad \left[kg/m^2\right],$$

is shown in Figure 2. With the piecewise linear approximation it could happen that some values of $\hat{q}_i^{\text{nom}}(t)$ are negative, however, if the values $q_{t_j}^{nom}$ are chosen sufficiently large relative to the values of the differences $|q_{t_j}^{nom} - q_{t_{j+1}}^{nom}|$ this situation can be avoided. Surely one can use other piecewise linear approximations to avoid negative values at all but for our case this approximation works perfectly.

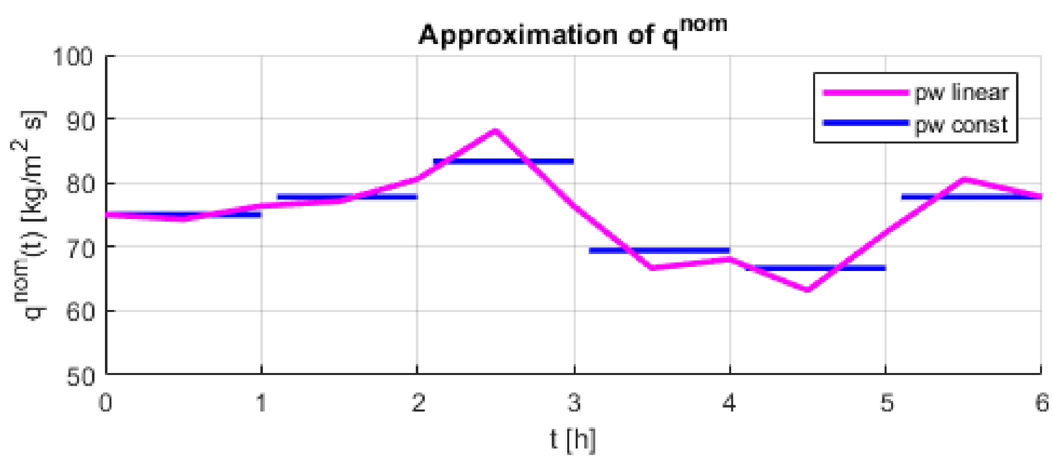

FIGURE 2. $\hat{q}^{\text{nom}}(t)$ in $\frac{kg}{m^2\, s}$ as piecewise constant approximation (blue) and piecewise linear approximation (pink) of q^{nom}

In the sequel for a real number x we use the notation

$$(x)_+ = \frac{1}{2}(x + |x|) = \max\{x, 0\}.$$

Let $\rho_{\min}^{e_i} > 0$ denote a given lower bound for the density at the end of edge e_i ($i = 1, \cdots, n$).

2.2. **The capacity of the network.** One of the tasks of the TSO is to determine the set of possible bookings that can be transported through the networks. In other words, the TSO has to compute the maximal demand that can be satisfied by the network. For this purpose, the TSO has to find capacity vectors in such a way that it is guaranteed that for all possible bookings (and thus nominations), the demand for gas can the be satisfied. A worst case scenario is often too conservative in this context. Therefore it makes sense to consider a probabilistic constraint that guarantees that the probability that the nominated gas can be delivered is greater than or equal to a prescribed parameter value. Probabilistic constraints in the operation of gas pipeline networks have been studied in [11, 24]. Up to now, only the static case has been studied in detail in the literature. In the model with probabilistic constraints, the demand is considered as a random variable where information on the probability distribution is available. In the sequel we assume that the TSO knows the technical capacities q_i^{TC} at the nodes v_i.

2.2.1. *A Dynamic Optimization Problem on One Edge with a Single Player.* While the study of PDE constrained optimal control problems is a very active field, the investigation of market models with PDE dynamics is less established. Therefore, in order to provide a convenient access to the problem, we first look at the optimal control problem for the TSO for the simplest possible case where there is only one single edge in the graph of the network and there is only a single player at the boundary node before we proceed to the more general case with several players at the boundary nodes of a star-shaped graph in Section 2.2.2.

We use simplified dynamics that are obtained from (2.2) by linearization around a steady state $(\bar{R}_1, \bar{R}_2)$. For the linearized source term we use the notation

$$M \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$
$$= -\frac{c\lambda^F}{4D}\left(\frac{(\bar{R}_1+\bar{R}_2)|\bar{R}_1+\bar{R}_2|}{\bar{R}_1-\bar{R}_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{2|\bar{R}_1+\bar{R}_2|}{(\bar{R}_1-\bar{R}_2)^2} \begin{bmatrix} -\bar{R}_2 & \bar{R}_1 \\ -\bar{R}_2 & \bar{R}_1 \end{bmatrix} \begin{bmatrix} R_1-\bar{R}_1 \\ R_2-\bar{R}_2 \end{bmatrix} \right).$$

We consider the optimal control problem of the TSO on a single edge with a single player and fixed demands $q_{t_j}^{\text{nom}}$, i.e.,

$$\mathbf{OCP}(q^{\text{nom}}) \begin{cases} \max\limits_{u,b} \quad \gamma_1 \sum_{j=0}^{N-1} \int_0^{q_{t_j}^{\text{nom}}} P_{t_j}(s)\, ds - \gamma_2 \|u\|^2_{L^2(0,T)} - \gamma_3 \|b-\hat{q}^{\text{nom}}\|^2_{L^2(0,T)} \\ \qquad - \gamma_4 \|\left(\rho_{\min} - \rho(\cdot, L)\right)_+\|^2_{L^2(0,T)} \\ \text{s.t.} \quad \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}_t + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}_x = M \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \\ R_1(0,x) = R_{1,\text{ini}}(x), \\ R_2(0,x) = R_{2,\text{ini}}(x), \\ R_1(t,0) - R_2(t,0) = 2\,c\,u(t), \\ R_1(t,L) + R_2(t,L) = 2\,b(t), \\ \int_{t_j}^{t_{j+1}} b(\tau)\, d\tau = \int_{t_j}^{t_{j+1}} \hat{q}^{\text{nom}}(\tau)\, d\tau = q_{t_j}^{\text{nom}}, \quad j \in \{0,..,N-1\}. \end{cases}$$

where the objective functional is given in terms of an inverse demand function $P(s,t)$ together with a quadratic control cost and two tracking terms. Note that the inverse demand function does not depend on u and b. In the market model it plays a role for the decision of the buyers and sellers about their nominations, see Section 3. The inverse demand function gives the price as a function of the quantity that can be sold for this price. In our case the quantities are given by the physical data that describe the nominations. In order to simplify notation, we consider the inverse demand function as a function of the dimensionless quantity s.

Similar as in [14], the L^2-regularity assumptions for the initial data and the boundary data, in particular for the control lead to states with L^2-regular boundary traces, such that the existence of a solution of the optimal control problem $\mathbf{OCP}(q^{\text{nom}})$ can be shown. Since the function $x \mapsto ((x)_+)^2$ is differentiable, all terms of the objective function are differentiable with respect to the decision variables u and b. For the numerical solution, an upwind/downwind discretization of the corresponding PDE system is used. For the approximation of the integral in the last constraint the trapezoidal rule is used.

In the following part we present two examples. The constants and weights are given in Table 1. According to the real gas data all values in our examples are close to real world values. For the implementation we used MATLAB® 2019a and the *fmincon.m* routine with default settings.

Variable	Letter	Value	Unit
lower density bound	$\rho_{\min}$	38	kg/m^3
speed of sound in the gas	c	343	m/s
pipe friction coefficient	λ^F	0.05	
pipe diameter	D	0.5	m
pipe length	L	30	km
final time	T	6	h
weights	$[\gamma_1, \gamma_2, \gamma_3, \gamma_4]$	$[1, 1, 1, 40]$	

TABLE 1. Values for the example with one edge.

Example 1 In the first example we consider a single time interval

$$[0,T] = [t_0, t_1].$$

We set $q^{\text{nom}} = 1.62 \cdot 10^6 \; kg/m^2$ and choose the piecewise constant approximation

$$\hat{q}^{\text{nom}}(t) \;=\; \frac{q^{\text{nom}}}{T} \;=\; 75 \,\frac{kg}{m^2 s}.$$

A dimensionless and time independent inverse demand function is given by

$$P(x) = \frac{400}{x+25}. \tag{2.3}$$

Thus we have

$$\int_0^{q^{\text{nom}}_{t_0}} P(s)\, ds \;=\; 400\,\left(\ln(q^{\text{nom}}_{t_0} + 25) - \ln(25)\right).$$

https://gaslib.zib.de/

For the linearization of the isothermal Euler equations we use the stationary state with boundary conditions $u^S = 34\ kg/m^3$ and $b^S = 75\ kg/m^2$. As initial condition we use the stationary state with boundary conditions $u^S = 38.5\ kg/m^3$ and $b^S = 77\ km/m^2$. The solutions $u(t)$ and $b(t)$ of **OCP** are shown in Figure 3.

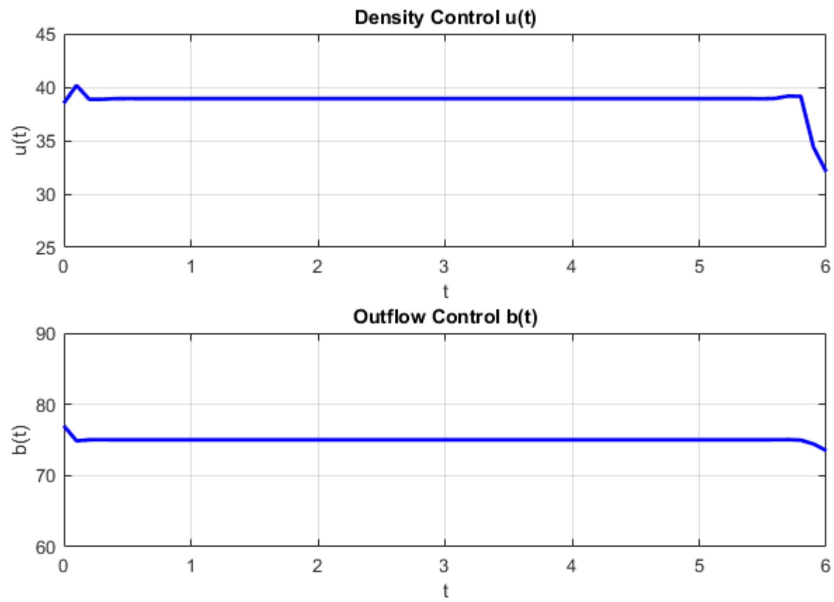

FIGURE 3. Density control $u(t)$ and outflow control $b(t)$ of **OCP**

The plus-term (the $\max\{0,\cdot\}$) in the objective function guarantees that in the interior of the time interval the gas density is not too far below 38 kg/m^3 as one can see in Figure 4. Note that at the end of the time interval the control is almost switched off, since then it does no longer pay off to invest in the control cost in order to decrease the tracking term. The numerical results indicate that the optimal state and the optimal control have a turnpike property that is in the interior of the time interval the optimal state and the optimal control are almost constant. Turnpike results for similar optimal control problems have been stated for example in [13].

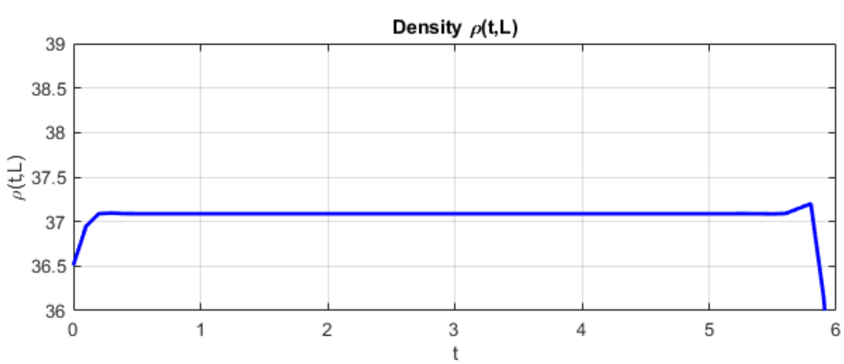

FIGURE 4. Gas density at the end of the pipe

Example 2 Now we consider an equidistant decomposition of the time interval

$$[0,T] \;=\; \bigcup_{j=0}^{5} [t_j, t_{j+1}], \tag{2.4}$$

with $t_j = j$ and $\Delta T = t_{j+1} - t_j$. We set

$$\begin{aligned} q^{\text{nom}} &= \begin{bmatrix} q^{\text{nom}}_{t_0} & q^{\text{nom}}_{t_1} & q^{\text{nom}}_{t_2} & q^{\text{nom}}_{t_3} & q^{\text{nom}}_{t_4} & q^{\text{nom}}_{t_5} \end{bmatrix} \\ &= \begin{bmatrix} 2.7\cdot 10^5 & 2.8\cdot 10^5 & 3\cdot 10^5 & 2.5\cdot 10^5 & 2.4\cdot 10^5 & 2.8\cdot 10^5 \end{bmatrix} \quad \left[kg/m^2\right]. \end{aligned}$$

The values in q^{nom} are chosen s.t. we have

$$\sum_{j=0}^{5} q^{\text{nom}}_{t_j} = 1.62\cdot 10^6 \; kg/m^2,$$

thus we use the same linearization of the nonlinear source term of (2.2) as in the previous example: We use the stationary state with boundary conditions $u^S = 34\; kg/m^3$ and $b^S = 75\; kg/m^2$. We use the inverse demand function given in (2.3). With a piecewise constant approximation $\hat{q}^{\text{nom}}(t)$ we have

$$\sum_{j=0}^{5} \int_0^{q^{\text{nom}}_{t_j}} P(s)\, ds \;=\; \sum_{j=0}^{5} 400 \left(\ln\left(q^{\text{nom}}_{t_j} + 25\right) - \ln(25) \right).$$

This term also holds for the piecewise linear approximation $\hat{q}^{\text{nom}}(t)$. We also use the same initial condition as in the previous example: We use the stationary state with boundary conditions $u^S = 38.5\; kg/m^3$ and $b^S = 77\; km/m^2$. The solutions $u(t)$ and $b(t)$ of **OCP** are shown in Figure 5. The density at the end of the gas pipeline (cf. the non differentiable term in the objective function) is shown in Figure 6.

Since now the demand curve is given by a piecewise constant function with different values, the outflow control shows a similar structure whereas the density control remains closer to a constant control, except for a terminal arc similar as in the first example. The corresponding solutions for the piecewise linear approximation of q^{nom} are shown in Figure 7. Note that while the density control $u(t)$ does not change much, in the outflow control $b(t)$ the piecewise linear structure is clearly visible. The density at the end of the pipe $\rho(t,L)$ is shown in Figure 8. It shows a turnpike structure similar to Figure 6.

In real world applications the gas nomination q^{nom} can slightly change e.g., due to short-term renominations. In order to guarantee that the computed density control is also valid in scenarios with uncertain q^{nom} one can compute the probabilistic robustness of the controls, i.e., the probability that the density bound is satisfied even if q^{nom} is considered to be uncertain. This has been done in [16] for stationary gas networks and in [24] for both stationary and dynamic flow networks.

2.2.2. *A Dynamic Optimization Problem on a Star-Shaped Graph with Multiple Players.* We consider a finite number of players $k = 1, \cdots, M$, where we sort the list of players k as follows: $k = 1, .., m_1, m_1+1, .., m_1+m_2, .., \sum_{i=1}^{n-1} m_i + m_n$. Hence, the players at node i are given by the list $(k = \sum_{j=1}^{i-1} m_j + 1, ..., \sum_{j=1}^{i} m_j)$ and the number $M = \sum_{i=1}^{n} m_i$ corresponds to the total number of

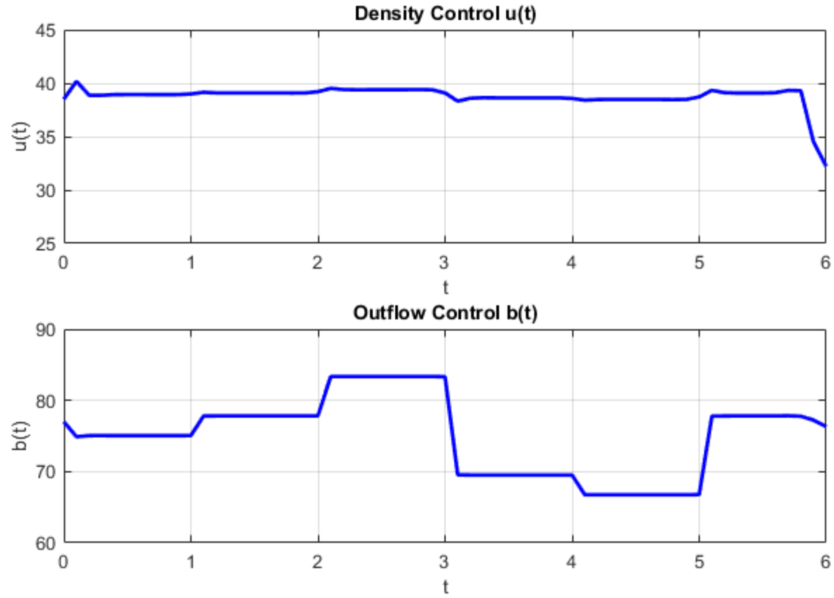

FIGURE 5. Density control $u(t)$ and outflow control $b(t)$ for the piecewise constant approximation of q^{nom}

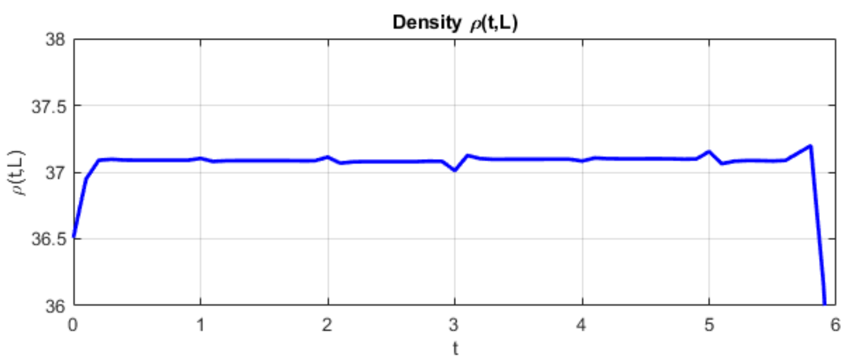

FIGURE 6. Gas density at the end of the pipe for the piecewise constant approximation of q^{nom}

players. Moreover, let $\mathscr{P}_i$ be the set of players at node v_i, i.e. $\mathscr{P}_i = \{\sum_{j=1}^{i-1} m_j + 1, ..., \sum_{j=1}^{i-1} m_j + m_i\}$.

On every node v_i $(i = 1, \cdots, n)$ on the star-shaped graph Figure 1. Consider the time interval decomposition

$$[0,T] = \bigcup_{j=0}^{N-1} [t_j, t_{j+1}],$$

where the $[t_j, t_{j+1}]$ all have the same length. Let q_{i,t_j}^{nom} be the gas demand of node v_i in the time interval $[t_j, t_{j+1}]$. With q_{k,t_j}^{nom} we denote the gas demand of player k $(k = 1, \cdots, M)$ in the time

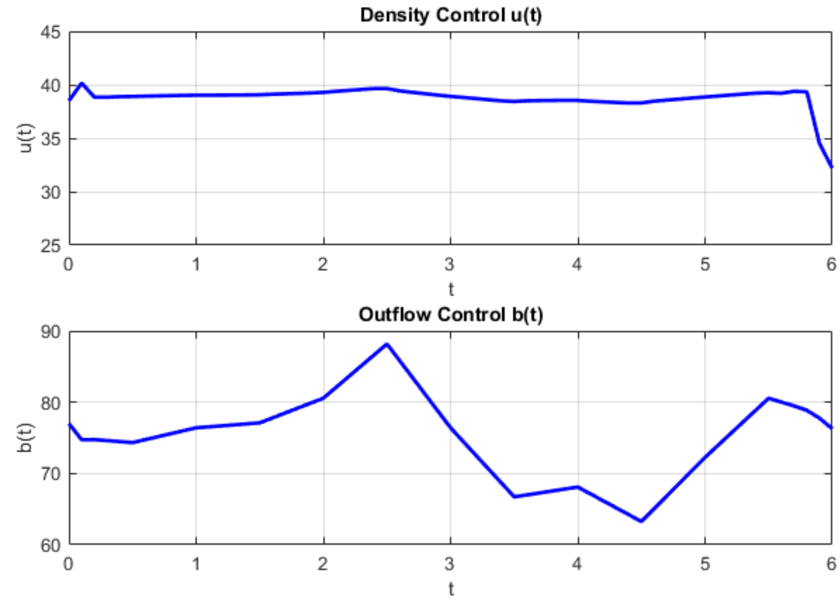

FIGURE 7. Density control $u(t)$ and outflow control $b(t)$ for the piecewise linear approximation of q^{nom}

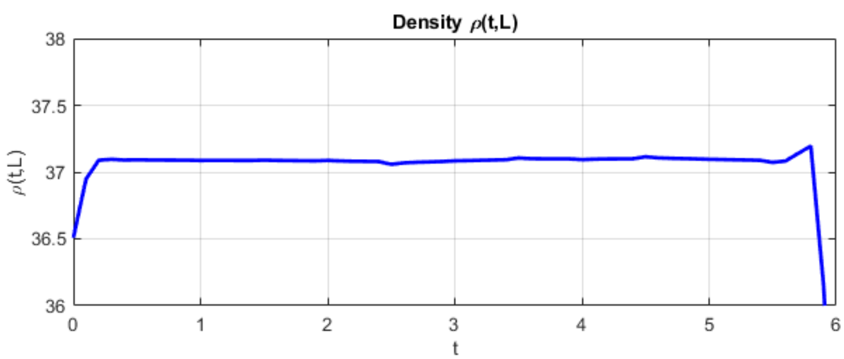

FIGURE 8. Gas density at the end of the pipe for the piecewise linear approximation of q^{nom}

interval $[t_j, t_{j+1}]$. Thus we have that

$$q_{i,t_j}^{\text{nom}} = \sum_{k \in \mathscr{P}_i} q_{k,t_j}^{\text{nom}} \qquad \text{for each node } i = 1,..,n. \tag{2.5}$$

Thus note that in a slight abuse of notation we distinguish the nominated gas demand using the variable q^{nom} in connection with two different indices k and i: we use $q_{k,t}^{\text{nom}}$ to indicate the nomination of an individual player k, whereas $q_{i,t}^{\text{nom}}$ is used to denote the overall nominated amount of gas at node i (i.e. nominations summed up over all players k at node i). In the case of a single player at each node (see Section 2.2.1) these two variables coincide. However, in the case of multiple players at a node the variables differ and fulfill the condition (2.5). Furthermore, q_{i,t_j}^{nom} is the i-th row and j-th column of the matrix $q^{\text{nom}} \in \mathbb{R}^{n \times N}$.

Given the overall demand q^{nom} the system operator has to solve the optimal control problem, accounting for social welfare.

$$
\mathbf{OCP}(q^{\text{nom}}) \begin{cases}
\max_{u,b} \gamma_1 \sum_{j=0}^{N-1} \sum_{i=1}^{n} \sum_{k\in\mathscr{P}_i} \int_0^{q^{\text{nom}}_{k,t_j}} P_k(s)\, ds \\
\quad - \gamma_2 \, \|u\|^2_{L^2(0,T)} - \gamma_3 \sum_{i=1}^{n} \|b_i - \hat{q}_i^{\text{nom}}\|^2_{L^2(0,T)} \\
\quad - \gamma_4 \sum_{i=1}^{n} \|\left(\rho^{e_i}_{\min} - \rho^{e_i}(\cdot, L_i)\right)_+\|^2_{L^2(0,T)} \\
\text{s.t.} \quad \text{for all } e \in \mathscr{E} \text{ we have the linear hyperbolic PDE} \\
\quad \begin{bmatrix} R_1^e \\ R_2^e \end{bmatrix}_t + \begin{bmatrix} \lambda_1^e & 0 \\ 0 & \lambda_2^e \end{bmatrix} \begin{bmatrix} R_1^e \\ R_2^e \end{bmatrix}_x = M^e \begin{bmatrix} R_1^e \\ R_2^e \end{bmatrix} + \begin{bmatrix} d_1^e \\ d_2^e \end{bmatrix}, \\
\quad \text{with initial conditions} \\
\quad R_1^e(0,x) = R^e_{1,\text{ini}}(x), \\
\quad R_2^e(0,x) = R^e_{2,\text{ini}}(x), \\
\quad \text{boundary condition for the density at the inflow node} \\
\quad R_1^{e_0}(t,0) - R_2^{e_0}(t,0) = 2\,c\,u(t), \\
\quad \text{boundary condition for the flow at the outflow nodes} \\
\quad R_1^{e_i}(t,L_i) + R_2^{e_i}(t,L_i) = 2\,b_i(t), \\
\quad \text{coupling conditions at the inner node} \\
\quad \sum_{j=1}^{n} \left(R_1^{e_j}(t,0) + R_2^{e_j}(t,0)\right)(D^{e_j})^2 = \left(R_1^{e_0}(t,L^{e_0}) + R_2^{e_0}(t,L^{e_0})\right)(D^{e_0})^2, \\
\quad \left(R_1^{e_i}(t,0) - R_2^{e_i}(t,0)\right) = \left(R_1^{e_0}(t,L^{e_0}) - R_2^{e_0}(t,L^{e_0})\right) \text{ for all } i = 1,\cdots,n, \\
\quad \text{and the gas demand condition } (i = 1,\cdots,n;\ j = 0,\cdots,N-1) \\
\quad \int_{t_j}^{t_{j+1}} b_i(\tau)\, d\tau = \int_{t_j}^{t_{j+1}} \hat{q}_i^{\text{nom}}(\tau)\, d\tau = q^{\text{nom}}_{i,t_j}.
\end{cases}
$$

As in the previous section the objective function of the optimal control problem is differentiable. For the numerical solution, an upwind/downwind discretization of the corresponding PDE system is used. For the approximation of the integrals in the last constraint the trapezoidal rule is used.

In the following part we present two examples on the star-shaped graph with three edges shown in Figure 9.

The constants and weights are given in Table 2. According to the real gas data all values in our examples are close to real world values. Note that the pipe friction coefficient, the pipe length and the pipe diameter are equal on every edge. For the implementation we also used MATLAB® 2019a and the *fmincon.m* routine with default settings.

https://gaslib.zib.de/

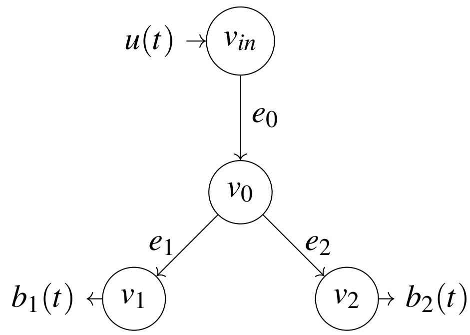

FIGURE 9. Scheme of a star-shaped graph with three edges

Variable	Letter	Value	Unit
lower density bound	$\rho_{\min}^{e_1}$	38	kg/m^3
	$\rho_{\min}^{e_2}$	40	kg/m^3
speed of sound in the gas	c	343	m/s
pipe friction coefficient	λ^F	0.05	
pipe diameter	D	0.5	m
pipe length	L	30	km
final time	T	6	h
weights	$[\gamma_1, \gamma_2, \gamma_3, \gamma_4]$	$[1,1,1,5]$	

TABLE 2. Values for the example with one edge.

Consider two players $k = 1,2$ at node v_1 and a single player $k = 3$ at node v_2.

Example 3 As in Example 1 we consider again a single time interval $[0,T] = [t_0,t_1]$ with a single integral nomination q^{nom} that is approximated with one constant function on the whole time interval. We set $q_{1,t_0}^{\text{nom}} = 6.48 \cdot 10^5\ kg/m^2$, $q_{2,t_0}^{\text{nom}} = 2.16 \cdot 10^5\ kg/m^2$ and $q_{3,t_0}^{\text{nom}} = 7.56 \cdot 10^5\ kg/m^2$. Thus we have $q^{\text{nom}} = [8.64 \cdot 10^5\ kg/m^2,\ 7.56 \cdot 10^5\ kg/m^2]^\top$ and we choose the piecewise constant approximation

$$\hat{q}^{\text{nom}}(t) = \frac{q^{\text{nom}}}{T} = \begin{bmatrix} 40\ \frac{kg}{m^2 s} \\ 35\ \frac{kg}{m^2 s} \end{bmatrix}.$$

In this case the piecewise linear approximation is identical. For convenience we consider the same dimensionless inverse demand function for every player, i.e.,

$$P_1(x) = P_2(x) = P_3(x) = \frac{400}{x+25}.$$

Thus we have

$$\sum_{i=1}^{2} \sum_{k \in \mathscr{P}_i} \int_0^{q_{k,t_0}^{\text{nom}}} P_{k,t_0}(s)\, ds = 400 \left(\ln\left(q_{1,t_0}^{\text{nom}} + 25\right) - \ln(25) \right)$$
$$+ 400 \left(\ln\left(q_{2,t_0}^{\text{nom}} + 25\right) - \ln(25) \right) + 400 \left(\ln\left(q_{3,t_0}^{\text{nom}} + 25\right) - \ln(25) \right).$$

For the linearization of the isothermal Euler equations we use the stationary state on three edges with boundary conditions $u^S = 47\ kg/m^3$, $b_1^S = 40\ kg/m^2$ and $b_2^S = 35\ kg/m^2$. As initial condition we use the stationary state on three edges with boundary conditions $u^S = 38\ kg/m^3$, $b_1^S = 40\ kg/m^2$ and $b_2^S = 35\ kg/m^2$. The solutions $u(t)$, $b_1(t)$ and $b_2(t)$ of **OCP** are shown in *Figure 10* and the densities at the end of the pipes $\rho_1(t,L_1)$ and $\rho_2(t,L_2)$ are shown in Figure 11.

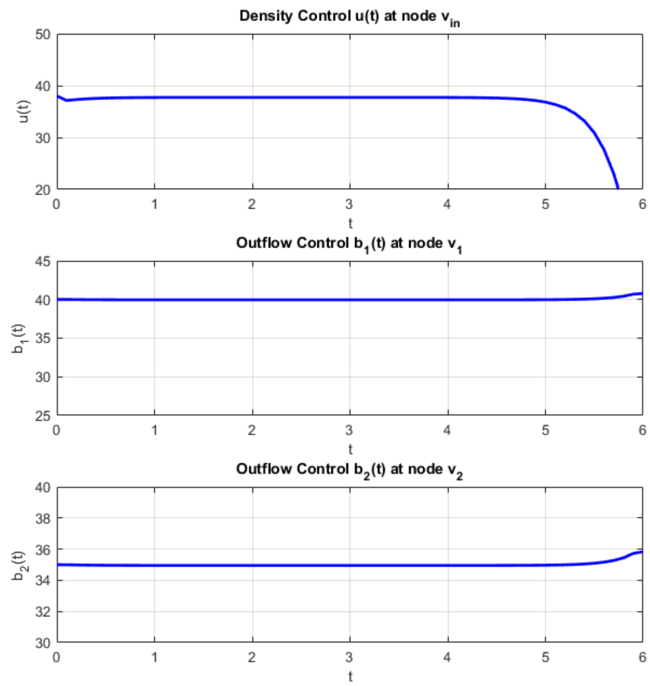

FIGURE 10. Density control and outflow controls for a single time interval

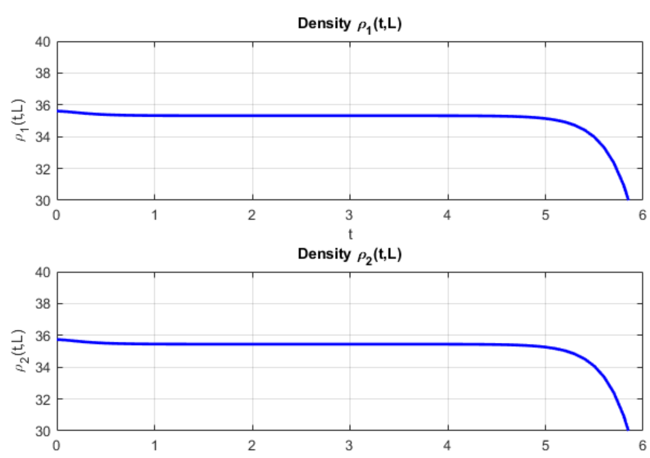

FIGURE 11. Densities at the end of the pipes for a single time interval

Since here the demand is constant, similar as in the first example in Section 2.2.1 the objective function enforces a turnpike structure of the optimal state and the optimal control. Both are almost constant in the interior of the time interval except for a terminal arc and an initial arc.

Example 4 In the next example we consider the equidistant time interval decomposition (2.4) from Section 2.2.1. For the players $k = 1,2$ at note v_1 and $k = 3$ at node v_2 we set

$$\begin{aligned} q_1^{\text{nom}} &= \begin{bmatrix} q_{1,t_0}^{\text{nom}} & q_{1,t_1}^{\text{nom}} & q_{1,t_2}^{\text{nom}} & q_{1,t_3}^{\text{nom}} & q_{1,t_4}^{\text{nom}} & q_{1,t_5}^{\text{nom}} \end{bmatrix} \\ &= \begin{bmatrix} 1.10\cdot 10^5 & 1.04\cdot 10^5 & 1.07\cdot 10^5 & 1.12\cdot 10^5 & 1.10\cdot 10^5 & 1.05\cdot 10^5 \end{bmatrix} \quad \left[kg/m^2\right], \end{aligned}$$

$$\begin{aligned} q_2^{\text{nom}} &= \begin{bmatrix} q_{2,t_0}^{\text{nom}} & q_{2,t_1}^{\text{nom}} & q_{2,t_2}^{\text{nom}} & q_{2,t_3}^{\text{nom}} & q_{2,t_4}^{\text{nom}} & q_{2,t_5}^{\text{nom}} \end{bmatrix} \\ &= \begin{bmatrix} 0.34\cdot 10^5 & 0.36\cdot 10^5 & 0.35\cdot 10^5 & 0.38\cdot 10^5 & 0.36\cdot 10^5 & 0.37\cdot 10^5 \end{bmatrix} \quad \left[kg/m^2\right], \end{aligned}$$

and

$$\begin{aligned} q_3^{\text{nom}} &= \begin{bmatrix} q_{3,t_0}^{\text{nom}} & q_{3,t_1}^{\text{nom}} & q_{3,t_2}^{\text{nom}} & q_{3,t_3}^{\text{nom}} & q_{3,t_4}^{\text{nom}} & q_{3,t_5}^{\text{nom}} \end{bmatrix} \\ &= \begin{bmatrix} 1.26\cdot 10^5 & 1.30\cdot 10^5 & 1.25\cdot 10^5 & 1.29\cdot 10^5 & 1.24\cdot 10^5 & 1.22\cdot 10^5 \end{bmatrix} \quad \left[kg/m^2\right]. \end{aligned}$$

The values in q^{nom} are chosen s.t. we have

$$\sum_{j=0}^{5} q_{1,t_j}^{\text{nom}} = 6.48\cdot 10^5 \, \frac{kg}{m^2} \quad \text{and} \quad \frac{1}{T}\sum_{j=0}^{5} q_{1,t_j}^{\text{nom}} = 30 \, \frac{kg}{m^2\, s},$$

$$\sum_{j=0}^{5} q_{2,t_j}^{\text{nom}} = 2.16\cdot 10^5 \, \frac{kg}{m^2} \quad \text{and} \quad \frac{1}{T}\sum_{j=0}^{5} q_{2,t_j}^{\text{nom}} = 10 \frac{kg}{m^2\, s},$$

and

$$\sum_{j=0}^{5} q_{3,t_j}^{\text{nom}} = 7.56\cdot 10^5 \, \frac{kg}{m^2} \quad \text{and} \quad \frac{1}{T}\sum_{j=0}^{5} q_{3,t_j}^{\text{nom}} = 35 \frac{kg}{m^2\, s}.$$

Thus we use the same linearization of the isothermal Euler equations as in the previous example. The piecewise constant approximation and the piecewise linear approximation at the nodes v_1 and v_2 are shown in Figure 12. The objective function is given by

$$\gamma_1 \sum_{j=0}^{N-1} \sum_{i=1}^{2} \sum_{k\in P_i} 400 \left(\ln\left(q_{k,t_j}^{\text{nom}} + 25\right) - \ln(25) \right).$$

We also use the same initial data as in the previous example: We use the stationary state with boundary conditions $u^S = 38kg/m^2$, $b_1^S = 40kg/m^2$ and $b_2^S = 35kg/m^2$.

The solutions $u(t)$, $b_1(t)$ and $b_2(t)$ of OCP for the piecewise linear approximation of q^{nom} are shown in Figure 13 and the densities at the end of the pipes $\rho_1(t,L_1)$ and $\rho_2(t,L_2)$ are shown in Figure 14. We only show the results for the piecewise linear approximation of q^{nom} here since the piecewise constant approximation yields less regular controls and the optimization with the piecewise linear approximation performs better.

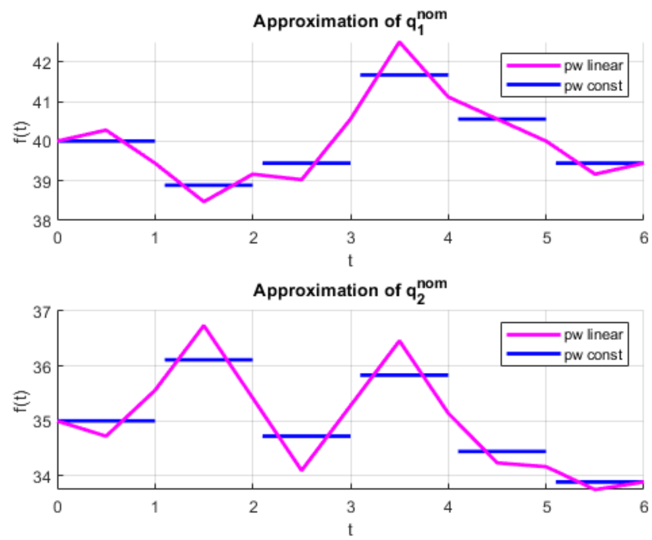

FIGURE 12. Piecewise constant and piecewise linear approximation of q^{nom} at node v_1 (above) and node v_2 (below)

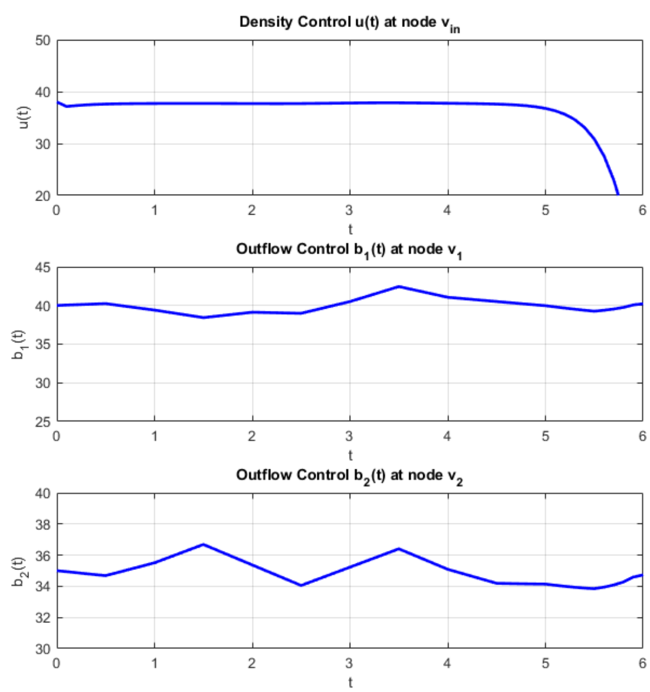

FIGURE 13. Density control and outflow controls for the piecewise linear approximation of q^{nom}

3. THE MULTILEVEL OPTIMAL CONTROL MODEL

We will now consider the Nash games of the market participants. Here, we follow the model presented in [12]. Each of the traders (sellers or buyers) maximize their own objective function

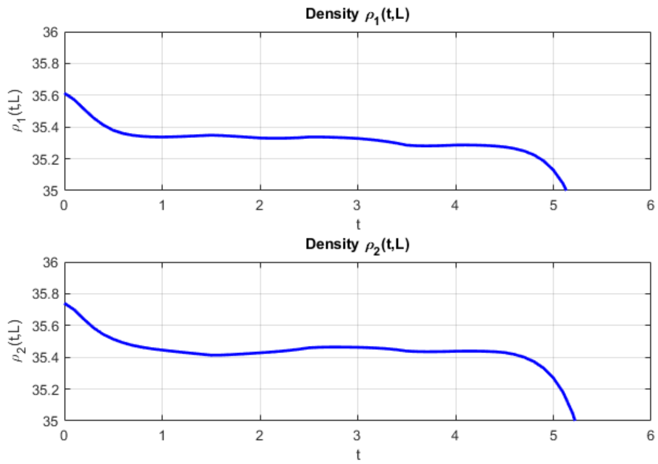

FIGURE 14. Densities at the end of the pipe for the piecewise linear approximation of q^{nom}

subject to their individual constraint set and joint constraints, that are equal to all players. As we will see, due to the structure of the game, i.e. objective functions and constraints, the generalized Nash games (GNEP) can be replaced by a single optimization problem by the use of so-called Potential functions. We will describe how these problems can be aggregated into one level, such that we end up with a bilevel discrete-continuous optimal control problem. Hence, the resulting problem links the optimal control of the gas flow with the objective of a welfare maximization (the targets of the TSO) and contains the traders' single optimization problem on a lower level (i.e. as constraint). Moreover, if the traders' maximization problem on the lower level is concave, we can replace it by its KKT conditions in order to end up with a single level optimal control problem with a joint PDE and DAE system in the constraints. For further insights into the field of bilevel optimization and optimal control of DAE systems, we refer the reader to the monographs [5] and [10], respectively.

However, let us first have a closer look on the traders' optimization problems that determine the nominations q^{nom} and the bookings q^{book}. Assume as in [12] that we can split the nodes into demand nodes $v_i \in V_-$ and supply nodes $v_i \in V_+$. (In the star shaped network, we have e.g. $V_+ = (v_0)$ and $v_i \in V_- = (v_1, .., v_n)$.)

3.1. **The Nominations.** Taking the bookings q^{book} as given external parameters, the players k i.e. the buyers and sellers, are solving a maximization problem which yield their individual gas load nominations $q^{nom}_{k,t}$ according to their bookings that appear in the constraints, where k denotes the player and $t \in T$ denotes an arbitrary time period t_j.

A gas seller $k \in \mathscr{P}_+$ (i.e. players at supply nodes $v_i \in V_+$) is solving the problem

$$\max_{q^{nom}_{k,t}} \quad \pi^{nom}_t q^{nom}_{k,t} - c^{var}_k q^{nom}_{k,t} \tag{3.1}$$
$$s.t. \qquad 0 \leq q^{nom}_{k,t} \leq q^{book}_k ,$$

where π_t^{nom} denotes the gas price and $c_k^{var} q_{k,t}^{nom}$ represents the production costs for player k. A gas buyer $k \in \mathscr{P}_-$ (i.e. players at demand nodes $v_i \in V_-$) is searching for a solution of

$$\begin{aligned} \max_{q_{k,t}^{nom}} \quad & \int_0^{q_{k,t}^{nom}} P_{k,t}(s)\, ds - \pi_t^{nom} q_{k,t}^{nom} \\ s.t. \quad & 0 \le q_{k,t}^{nom} \le q_k^{book} \end{aligned} \tag{3.2}$$

Here, we assume (as in Section 2) that the corresponding inverse demand functions for each player $k \in \mathscr{P}_-$ are constant in time on each time interval $[t_j, t_{j+1}]$. Therefore, we may write $P_{k,t}(x)$ instead of $P_k(x,t)$ for all players k and time periods $j = 1,..,N-1$, i.e. in each time interval the price setting is fixed and does not further vary in time, but it might vary for different time intervals. (This assumption holds true for the inverse demand function e.g. chosen in the examples of Section 2.)

In fact, so far these problems are not coupled, since the objectives and the constraints of (3.1) and (3.2) only depend on the kth player's decision variable $q_{k,t}^{nom}$. However, the clearing condition

$$\sum_{k \in \mathscr{P}_-} q_{k,t}^{nom} - \sum_{k \in \mathscr{P}_+} q_{k,t}^{nom} = 0 \tag{3.3}$$

is still missing which is in fact the coupling condition by which we then obtain a generalized Nash game (GNEP) with a joint constraint (3.3) as the feasible sets depend on the other players' decisions. However, since the joint constraint (3.3) is the only coupling which is equal for all players, the (GNEP) given by (3.1), (3.2) and (3.3) can be proven to be a *Potential GNEP*.

Consider a standard generalized Nash game [7] given by

$$\min_{y_k} \quad \theta_k(y_k, y_{-k}) \qquad s.t. \quad y_k \in \mathscr{Y}_k(y_{-k}) \qquad \text{for all } k = 1,..,M \tag{3.4}$$

Then, $\vartheta(y)$ is said to be an exact potential [22] for (3.4), iff the following condition holds for all $k = 1,..,M$

$$\vartheta(y, y_{-k}) - \vartheta(z, y_{-k}) = \theta_k(y, y_{-k}) - \theta_k(z, y_{-k}) \qquad \forall\, y, z \in \mathscr{Y}_k(y_{-k}).$$

In particular, if $\theta_k(y_k, y_{-k})$ is given in the form of a sum of an individual term $\theta_k^{indiv}(y_k)$ and a common term $\theta^{com}(y)$ for all $k = 1,..,M$, then it is straight forward to construct a potential (see also [8])

$$\vartheta(y) = \sum_{k=1}^{M} \theta_k^{indiv}(y_k) \, + \, \theta^{com}(y).$$

Note, that the objectives of both type of players (buyers and sellers) consist only of individual costs, i.e. we can directly define the following potential for the GNEP.

Lemma 3.1.

$$\Phi(q_t^{nom}) = \sum_{k \in \mathscr{P}_-} \int_0^{q_{k,t}^{nom}} P_{k,t}(s)\, ds - \sum_{k \in \mathscr{P}_+} c_k^{var} q_{k,t}^{nom}$$

is a potential for the GNEP given by (3.1), (3.2) and (3.3)

Proof. Summing up the individual costs of all k players we obtain the right hand side of $\Phi(q_t^{nom})$, where by condition (3.3) the terms $\pi_t^{nom} q_{k,t}^{nom}$ canceled out. □

In order to be able to replace the GNEP by a single optimization problem, we need however an additional assumption concerning the feasible sets of the players' problems.

Definition 3.2. (cf. [8])
The GNEP (3.4) is a generalized potential game, if :

(1) There exists a potential for the GNEP in the sense of [22] for (3.4).
(2) There exists a nonempty, closed set $\mathscr{Y} \subseteq \mathbb{R}^n$ such that for all $k = 1,..,M$

$$\mathscr{Y}_k(y_{-k}) \equiv \{ y_k \in Y_k \,|\, (y_k, y_{-k}) \in \mathscr{Y} \} \tag{3.5}$$

where each $Y_k \subseteq \mathbb{R}^{n_k}$ denotes an individual nonempty, closed set that satisfies $\prod_{k=1}^M Y_k \cap \mathscr{Y} \neq \emptyset$.

Now, consider the feasible sets of each player and observe that these are exactly in the form of (3.5), where the individual sets Y_k are given through the nonempty box constraints that are parameterized by the associated bookings $q_k^{book} \geq 0$ and the set $\mathscr{Y}$ is given by the clearing condition (3.3) (the shared constraint for all players). Note moreover, that the joint feasible set $\mathscr{Y}$ and the individual sets Y_k are convex sets by definition, i.e. we have the nice situation of a *jointly convex* problem.

Hence, according to [8] the individual maximization problems of each player can be aggregated into one single optimization problem in the sense that any *global solution* yields a Nash equilibrium of the GNEP.

Lemma 3.3. *The GNEP given by (3.1), (3.2) and (3.3) can be replaced by the single optimization problem*

$$\begin{aligned} \max_{q_t^{nom}} \quad & \Phi(q_t^{nom}) = \sum_{k \in \mathscr{P}_-} \int_0^{q_{k,t}^{nom}} P_{k,t}(s)\, ds - \sum_{k \in \mathscr{P}_+} c_k^{var}\, q_{k,t}^{nom} \\ s.t. \quad & 0 \leq q_{k,t}^{nom} \leq q_k^{book} \qquad k \in \mathscr{P}_+ \cup \mathscr{P}_-, \\ & \sum_{k \in \mathscr{P}_-} q_{k,t}^{nom} - \sum_{k \in \mathscr{P}_+} q_{k,t}^{nom} = 0. \end{aligned} \tag{3.6}$$

This result equals the observations that are given by Grimm et al. in [12], where it is derived by comparing the associated KKT systems.

Next, let us consider the properties of the joint optimization problem (3.6). As we have seen, the feasible set of (3.6) is nonempty and convex. Moreover, if we assume that each $P_{k,t}(s)$ is continuous and strictly decreasing, then the objective is concave, i.e. (3.6) written as a minimization problem is itself convex. Furthermore, as we have linear constraints only, the *Abadie Constraint Qualification* directly holds for (3.6), such that the associated first order stationarity (i.e. KKT-) conditions are necessary and sufficient for the global optima of (3.6). We summarize our observations and further properties of (3.6) in the following Lemma.

Lemma 3.4. *Assume that the inverse demand functions $P_{k,t}(s)$ are continuously differentiable and strictly decreasing for all players $k = 1,..,M$ and all time intervals $[t_j, t_{j+1}]$. Then*

(1) $\Phi(q_t^{nom})$ is concave with respect to all players and strictly concave with respect to players $k \in \mathscr{P}_-$.

(2) The associated minimization problem

$$\begin{aligned} \min_{q_t^{nom}} \quad & -\Phi(q_t^{nom}) = -\sum_{k\in\mathscr{P}_-}\int_0^{q_{k,t}^{nom}} P_{k,t}(s)\,ds + \sum_{k\in\mathscr{P}_+} c_k^{var}\, q_{k,t}^{nom} \\ s.t. \quad & 0 \le q_{k,t}^{nom} \le q_k^{book} \qquad k\in\mathscr{P}_+\cup\mathscr{P}_-, \\ & \sum_{k\in\mathscr{P}_-} q_{k,t}^{nom} - \sum_{k\in\mathscr{P}_+} q_{k,t}^{nom} = 0 \end{aligned} \tag{3.7}$$

is convex.

(3) Problem (3.7) satisfies the Abadie constraint qualification.

Therefore the (Karush-Kuhn-Tucker) first order necessary optimality conditions are necessary and sufficient for a global optimum $\bar{q}_t^{nom}$ of (3.6).

Proof. First, note that due to the assumptions, the entries of the Hessian of $\Phi(q_t^{nom})$ satisfy

$$\partial^2\Phi(q_t^{nom})/\partial q_{k,t}^{nom}\partial q_{\ell,t}^{nom} \le 0$$

with zeros at the offdiagonal entries, zeros for diagonal entries for $k\in\mathscr{P}_+$ and negative diagonal entries for $k\in\mathscr{P}_-$, which gives the first statement.

Next, since $\Phi(q_t^{nom})$ is concave and all constraint functions are linear, the second statement holds true.

Finally, since all constraint functions are linear, the Abadie constraint qualification is directly satisfied. This yields a convex optimization problem that satisfies a constraint qualification, thus the KKT-conditions are necessary and sufficient. □

Moreover, in [12] it has been shown that if we merge all optimization problems (3.6) of the nomination level (over all time intervals) and assume that all cost coefficients c_k^{var} are pairwise distinct and all inverse demand functions $P_{k,t}(s)$ are strictly decreasing, then the optimal solution of the joint nomination problem is unique.

Hence, by the boundedness of the feasible region and the continuity of the objective function, for any given q_k^{book} there exists a global solution of (3.7), which is unique under the previously mentioned conditions. This implies [22], moreover, that the GNEPs associated with (3.7) admit a (possibly unique) Nash equilibrium.

Furthermore, we summarize some properties of the feasible region of (3.6) seen as a parametric problem with parameter $q^{book} \ge 0$.

Lemma 3.5. *Consider (3.6) and let q_k^{book} be given, then it holds:*

(1) The objective $-\Phi(q_t^{nom})$ is jointly convex in (q^{nom}, q^{book}).
(2) The feasible set of (3.6) is nonempty, independently of the choice of q_k^{book}.
(3) The feasible region of (3.6) contains a strict interior iff $q_k^{book} > 0$ for all $k = 1,..,M$.
(4) The linear independence constraints qualification holds iff there exists at least one $q_k^{nom} \in (0, q_k^{book})$.

As the next step we consider the second level, where the decisions about the gas bookings are made.

3.2. **The Bookings.** The players of the second level are again the gas traders whose decision variables here are capacity rights, namely the bookings that determine the amount of gas which can then be nominated in the third level, i.e. they define the upper bound for the nominations q^{nom} on level three (cf. (3.6)). The traders choose their bookings in order to maximize their profit, i.e. given the booking fees $\underline{\pi}_i^{book}$ (that are determined by the TSO) they solve the problem

$$\max_{q_k^{book}} \quad \sum_{t\in T} \phi_{k,t}(q_k^{book}) - \underline{\pi}_i^{book} q_k^{book} \qquad \text{s.t.} \quad q_k^{book} \geq 0, \tag{3.8}$$

where $\phi_{k,t}(q_k^{book})$ denotes the optimal value function of (3.1) or (3.2), respectively. Hence the function value is given by the value of the individual objective of player k, i.e. of (3.1) or (3.2), respectively, for a (unique) global optimizer of (3.6) considered as a function of q_k^{book}. However, as for the nominations, here we also have a shared constraint at each node i: given the technical capacity q_i^{TC} at node v_i the bookings at this node are not allowed to exceed the capacity, i.e. it must hold

$$\sum_{k\in\mathscr{P}_i} q_k^{book} \leq q_i^{TC}. \tag{3.9}$$

Again, we can observe, that in order to determine the players bookings q_k^{book} we need to solve a potential GNEP. Furthermore, as before the objectives consist only of individual costs and individual feasible sets together with the shared constraint (3.9) that satisfy the conditions of Definition 3.2. We can thus replace the GNEP by a single optimization problem. However, in contrast to the nominations, we can first decouple the optimization problems for each node, since there is no coupling condition (shared constraint) concerning different nodes.

Lemma 3.6. *The GNEP given by (3.8) and the corresponding clearing conditions (3.9) can be replaced by the n optimization problems*

$$\begin{aligned} \max_{q_i^{book}} \quad & \sum_{k\in\mathscr{P}_i}\sum_{t\in T} \phi_{k,t}(q_k^{book}) - \underline{\pi}_i^{book} q_k^{book} \\ \text{s.t.} \quad & q_i^{book} \geq 0 \qquad k\in\mathscr{P}_i, \\ & \sum_{k\in\mathscr{P}_i} q_k^{book} \leq q_i^{TC}. \end{aligned} \tag{3.10}$$

Next, the following lemma justifies the use of suitable stationarity conditions in order to represent and compute the solution set of (3.10).

Lemma 3.7. *Given the assumptions of Lemma 3.4 the optimization problem (3.10) is convex and has a nonempty and convex feasible set satisfying the Abadie constraint qualification. Furthermore, its objective function is continuous and differentiable almost everywhere.*

Proof. First, by the structure of (3.6) and the joint convexity of $-\Phi(q^{nom})$ we can apply [9] Cor.2.2 which yields the convexity of each $\phi_{k,t}(q_k^{book})$, hence of the objective function of (3.10). Since all constraints are affine the convexity of (3.10) directly follows. Furthermore, since $q_k^{TC} \geq 0$, the vector $q_i^{book} = 0$ is always feasible (independent of q_k^{TC}). Due to the affine structure of the constraints the feasible set is therefore nonempty, convex and satisfies the Abadie constraint qualification. Applying e.g. [4] Thm 16.10 or [19], Satz 7.3.5 then gives the continuity of the convex objective function. □

Hence, we can apply first-order stationarity conditions for each problem (using an appropriate form of the subdifferential in points, where the objective is not differentiable). Now, if we merge all these conditions of the n optimization problems into one single system, the resulting system can be associated to a joint optimization problem given by

$$\begin{aligned} \max_{q^{book}} \quad & \sum_{i=1}^{n} \sum_{k \in \mathscr{P}_i} \left(\sum_{t \in T} \phi_{k,t}(q_k^{book}) - \underline{\pi}_i^{book} q_k^{book} \right) \\ s.t. \quad & q_k^{book} \geq 0, \qquad k \in \mathscr{P}_i, \quad i = 1,..,n \\ & \sum_{k \in \mathscr{P}_i} q_k^{book} \leq q_i^{TC} \qquad i = 1,..,n. \end{aligned} \tag{3.11}$$

Finally, following [12] the problems (3.6) and (3.11) of the nomination and the booking level can be merged to one single optimization problem:

$$\begin{aligned} \max_{q^{book}, q^{nom}} \quad & \sum_{i=1}^{n} \sum_{k \in \mathscr{P}_i} \left(\sum_{t \in T} \phi_{k,t}(q_k^{book}) - \underline{\pi}_i^{book} q_k^{book} \right) \\ s.t. \quad & q_k^{book} \geq 0, \qquad k \in \mathscr{P}_i, \quad i = 1,..,n \\ & \sum_{k \in \mathscr{P}_i} q_k^{book} \leq q_i^{TC} \qquad i = 1,..,n, \\ & 0 \leq q_{k,t}^{nom} \leq q_k^{book} \qquad k \in \mathscr{P}_+ \cup \mathscr{P}_-, \ t \in T \\ & \sum_{k \in \mathscr{P}_-} q_{k,t}^{nom} - \sum_{k \in \mathscr{P}_+} q_{k,t}^{nom} = 0, \qquad t \in T \end{aligned} \tag{3.12}$$

Thus, an optimization problem maximizing the overall social welfare yields an aggregated bilevel model with the optimal control problem (OCP)(q^{nom}) on the upper level and the (3.12) representing the players Nash games of the booking and the nomination on the lower level.

As before, since the feasible set of (3.12) is nonempty and compact and the objective function is continuous by Lemma 3.7, the problem admits at least one global optimum $(\bar{q}^{nom}, \bar{q}^{book})$. Therefore, we can deduce that by the theory of potential games $(\bar{q}^{nom}, \bar{q}^{book})$ defines a Nash equilibrium vector of the GNEPS for the bookings and the nominations that we started our discussion with.

3.3. **Aggregated Model: Technical Capacities and Booking Fees.** In this section we state the aggregated bilevel model where the PDE-constrained optimal control problem (OCP)(q^{nom}) of the TSO forms the upper level and the lower level consists of the bookings and nominations of the market participants in the form of (3.12). Note that on the upper level, the TSO determines booking fees $\underline{\pi}_i^{book}$ optimizing social welfare while ensuring that all feasible nominations can be satisfied under the condition that he does not make any profit. In contrast to [12] in the problem stated below the PDE appears as a model for the gas pipeline flow. The coupling of the boundary conditions in the PDE with the vectors q^{nom} is done via the integrals in the gas demand conditions.

$$
\begin{aligned}
\max_{u,b,\underline{\pi}_i^{book}} \ & \gamma_1 \sum_{j=0}^{N-1} \sum_{i=1}^{n} \sum_{k\in\mathscr{P}_i} \int_0^{q_{k,t_j}^{\text{nom}}} P_{k,t_j}(s)\, ds - \sum_{k\in\mathscr{P}_+} c_k^{var}\, q_{k,t_j}^{nom} \\
& - \gamma_2 \, \|u\|_{L^2(0,T)}^2 - \gamma_3 \sum_{i=1}^{n} \|b_i - \hat{q}_i^{\text{nom}}\|_{L^2(0,T)}^2 \\
& \qquad - \gamma_4 \sum_{i=1}^{n} \left\| \left(\rho_{\min}^{e_i} - \rho^{e_i}(\cdot, L_i)\right)_+ \right\|_{L^2(0,T)}^2
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad & \underline{\pi}_i^{book} \geq 0, \\
& \sum_{i\in\mathscr{P}_+} \underline{\pi}_i^{book}\, q_i^{book} = \gamma_2 \|u\|_{L^2(0,T)}^2 - \gamma_3 \sum_{i=1}^{n} \|b_i - \hat{q}_i^{\text{nom}}\|_{L^2(0,T)}^2 \\
& \qquad\qquad - \gamma_4 \sum_{i=1}^{n} \left\| \left(\rho_{\min}^{e_i} - \rho^{e_i}(\cdot, L_i)\right)_+ \right\|_{L^2(0,T)}^2, \\
& (q^{nom}, q^{book}) \qquad \text{solves} \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
& \text{for all } e \in \mathscr{E} \text{ we have the linear hyperbolic PDE} \\
& \begin{bmatrix} R_1^e \\ R_2^e \end{bmatrix}_t + \begin{bmatrix} \lambda_1^e & 0 \\ 0 & \lambda_2^e \end{bmatrix} \begin{bmatrix} R_1^e \\ R_2^e \end{bmatrix}_x = M^e \begin{bmatrix} R_1^e \\ R_2^e \end{bmatrix} + \begin{bmatrix} d_1^e \\ d_2^e \end{bmatrix}, \\
& \text{with initial conditions} \\
& R_1^e(0,x) = R_{1,\text{ini}}^e(x), \\
& R_2^e(0,x) = R_{2,\text{ini}}^e(x), \\
& \text{boundary condition for the density at the inflow node} \\
& R_1^{e_0}(t,0) - R_2^{e_0}(t,0) = 2\,c\,u(t), \\
& \text{boundary condition for the flow at the outflow nodes} \\
& R_1^{e_i}(t,L_i) + R_2^{e_i}(t,L_i) = 2\,b_i(t), \\
& \text{coupling conditions at the inner node} \\
& \sum_{i=1}^{n} \left(R_1^{e_i}(t,0) + R_2^{e_i}(t,0)\right)(D^{e_i})^2 \\
& \qquad = \left(R_1^{e_0}(t,L^{e_0}) + R_2^{e_0}(t,L^{e_0})\right)(D^{e_0})^2, \\
& \left(R_1^{e_i}(t,0) - R_2^{e_i}(t,0)\right) \\
& \qquad = \left(R_1^{e_0}(t,L^{e_0}) - R_2^{e_0}(t,L^{e_0})\right) \quad \text{for all } i = 1, \cdots, n, \\
& \text{the gas demand condition } (i = 1, \cdots, n;\, j = 0, \cdots, N-1) \\
& \int_{t_j}^{t_{j+1}} b_i(\tau)\, d\tau = \int_{t_j}^{t_{j+1}} \hat{q}_i^{\text{nom}}(\tau)\, d\tau = q_{i,t_j}^{\text{nom}}.
\end{aligned}
\tag{3.13}
$$

The discussion in Section 3.2 implies that for fees $\underline{\pi}_i^{book}$ and capacities q_i^{TC} prescribed by the TSO the constraint '(q^{nom}, q^{book}) *solves* (3.12)' is well-posed, i.e. (3.12) admits at least one global solution. Hence, the feasible set of (3.13) is nonempty. However, it cannot be assumed that the solution set of (3.12) is convex, hence the feasible set of (3.13) can also not assumed to be convex.

Note, that by Lemma 3.7, we might replace the lower level problem by its KKT conditions in order to obtain a single level optimal control problem, which can then again be solved using first order conditions for optimal control problems (see e.g. [21]). However, in general this has to be done with special care, as the nonconvex, nonsmooth structures generate further numerical and theoretical difficulties that need to be taken into account, see also for example [3].

4. Conclusions

We have stated a market model that includes a PDE-constrained optimal control problem. It is shown that the four level market model can be reduced to a single level problem based upon the potential game approach. Now we will discuss some of the many open questions that remain.

Numerical results indicate, that the optimal states and controls show a turnpike structure. Therefore it would be of interest to show rigorous turnpike results for market models. We expect that for an equidistant partition of the time horizon with a fixed number of subintervals the solutions show a turnpike structure on each subinterval of the partition for a sufficiently large time horizon.

In order to maximize the social welfare, problems with probabilistic constraints are also interesting since they allow for greater flexibility in the sense that for some customers, interruptions of the gas delivery are also possible with a given small probability, see for example [17]. To include probabilistic constraints in the optimal control problem (OCP)(q^{nom}), the boundary conditions for the flow at the outflow nodes would be stated in a probabilistic sense. This leads to probalistic constraints that would require that the boundary conditions are satisfied on the whole time interval at least with a prescribed probability parameter $p \in (0,1)$. Such constraints are called *probust constraints* (see for example [1]).

An approach for the numerical solution of (3.13) could be based upon available techniques for the optimal control of DAEs, see for example [10].

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REFERENCES

[1] D. Adelhütte, D. Aßmann, T. Gonzàlez Grandòn, M. Gugat, H. Heitsch, R. Henrion, F. Liers, S. Nitsche, R. Schultz, M. Stingl, D. Wintergerst, Joint model of probabilistic-robust (probust) constraints applied to gas network optimization, Vietnam J. Math. 49 (2021) 1097-1130.

[2] M. Banda, M. Herty, A. Klar, Coupling conditions for gas networks governed by the isothermal Euler equations, Netw. Heterog. Media 1 (2006) 295-314.

[3] F. Benita, P. Mehlitz, Bilevel optimal control with final-state-dependent finite-dimensional lower level, SIAM Journal on Optimization 26 (2016) 718-752.

[4] H. Dallmann, K.-H. Elster, Einführung in die höhere Mathematik für Naturwissenschaftler und Ingenieure, Band 2, VEB Gustav Fischer Verlag, Jena, 1981.

[5] S. Dempe, Foundations of bilevel programming, Springer New York, NY, 2002.

[6] P. Domschke, B. Hiller, J. Lang, V. Mehrmann, R. Morandin, C. Tischendorf, Gas Network Modeling: An Overview, Preprint, Technische Universität Darmstadt, 2021.

[7] F. Facchinei, C. Kanzow, Generalized Nash Equilibrium Problems, 4OR 5 (2007) 173-210.

[8] F. Facchinei, V. Piccialli, M. Sciandrone, Decomposition algorithms for generalized potential games, Computational Optimization and Applications 50 (2011) 237-262.

[9] A. V. Fiacco, J. Kyparisis, Convexity and concavity properties of the optimal value function in parametric nonlinear programming, author=Anthony V. Fiacco and Jerzy Kyparisis, Journal of Optimization Theory and Applications 48 (1983) 95-126.

[10] M. Gerdts, Optimal control of ODEs and DAEs, Walter de Gruyter & Co., Berlin, 2012.

[11] T.González Grandón, H. Heitsch, R. Henrion, A joint model of probabilistic/robust constraints for gas transport management in stationary networks, Comput. Manag. Sci. 14 (2017) 443-460.

[12] V. Grimm, L. Schewe, M. Schmidt, G. Zöttl, A multilevel model of the European entry-exit gas market, Mathematical Methods of Operations Research, 89 (2019) 223-255.

[13] M. Gugat, A turnpike result for convex hyperbolic optimal boundary control problems, Pure Appl. Funct. Anal. 4 (2019) 849-866.

[14] M. Gugat, F. Hante, On the Turnpike Phenomenon for Optimal Boundary Control Problems with Hyperbolic Systems, SIAM Journal on Control and Optimization 57 (2019) 264-289.

[15] M. Gugat, G. Leugering, A. Martin, M. Schmidt, M. Sirvent, D. Wintergerst, MIP-based instantaneous control of mixed-integer PDE-constrained gas transport problems, Comput. Optim. Appl. 70 (2018) 267-294.

[16] M. Gugat, M. Schuster, Stationary gas networks with compressor control and random loads: Optimization with probablistic constraints, Mathematical Problems in Engineering (Article ID 7984079), (2018).

[17] H. Heitsch, On probabilistic capacity maximization in a stationary gas network, Optimization, 69 (2020) 575-604.

[18] M. Herty, S. Steffensen, A. Thunen, Solving quadratic multi-leader-follower games by smoothing the follower's best response, Optimization Methods and Software 37 (2020) 772-799.

[19] F. Jarre, J. Stoer, Optimierung, Springer Verlag, Berlin Heidelberg, 2004.

[20] A. A. Kulkarni, U. V. Shanbhag, A shared-constraint approach to multi-leader multi-follower games, Set-Valued Var. Anal. 22 (2014) 691-720.

[21] D. G. Luenberger, Optimization by vector space methods, John Wiley & Sons, New York, 1997.

[22] D. Monderer, L. Shapley, *Potential games*, Games and Economic Behaviour 14 (1996) 124-143.

[23] L. Schewe, M. Schmidt, J. Thürauf, Global optimization for the multilevel European gas market system with nonlinear flow models on trees, Journal of Global Optimization 82 (2022) 627–653.

[24] M. Schuster, E. Strauch, M. Gugat, J. Lang, Probabilistic constrained optimization on flow networks, Optim. Eng. 23 (2022) 1-50.

[25] S. Steffensen, Multi-leader-follower potential games, ESAIM: ProcS. 71 (2021) 155-162.