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OBSERVES FOR LINEAR TIME-VARYING SYSTEMS WITH QUASIDERIVATIVE COEFFICIENTS

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Dedicated to the memory of Professor Rafail Gabasov

Abstract. In this paper, on the basis of quasiderivatives, we consider the state observation and estimation problems for linear time-varying systems of ordinary differential equations. The quasiderivatives are defined for some lower triangular matrix, and the simplest rules of the quasidifferentiation are described. The conditions for linear independence of continuous quasidifferentiable functions are established in terms of the Wronski matrix. The method for constructing state estimators for linear time-varying systems based on the quasidifferentiability of the coefficients is proposed. For uniformly observable systems with quasidifferentiable coefficients, we obtain conditions for the existence of an exponential observer and describe a constructive method for designing such observers.

Keywords. Frobenius form; Hessenberg form; Linear time-varying system; Observability matrix; Quasidifferentiation.

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1. INTRODUCTION

The state estimation for deterministic linear finite-dimensional observation systems is based on the assumption that the system has a certain type of observability. In the time-invariant case, the complete observability of the system guarantees the existence of an asymptotic estimator [1]. It is clear that the problem is significantly more complicated for systems with variable coefficients. For time-varying linear systems, there exist several concepts of observability, including complete observability [2], differential observability [3]–[8], uniform observability [4]–[10], uniformly complete observability [11], observability via resolving operations [2], approximate observability [12], uniform pointwise observability[13], observability in the class of Chebyshev function systems [14], Hessenberg observability[15], etc. The notion of uniformly complete observability reflects the specific properties necessary for the existence of asymptotic estimators most adequately [11]. But it is very difficult to verify it in terms of the coefficients of the original observation system, and it is therefore inefficient from the constructive standpoint. We also

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note that the form of information about the output function is important when constructing state estimation procedures; for example, it is important whether we know only the function itself, or its successive derivatives, or its values at a given sequence of points, etc.

The problem of estimating the states of observation systems from available information has been intensively studied because of its importance in various plant positioning systems [1, 2], [9]–[22]. When constructing feedback controls, one usually needs to know the system state vector. However, the direct measurement of the state vector in real-world situations may be hindered by technical restrictions, the impossibility to organize the observation process, or extremely high costs. The possible errors in the observation channel distorting the exact value of the state vector must be taken into account as well.

The notion of uniform observability apparently plays the most important role in the state estimation theory from the constructive standpoint; this notion was first introduced in [10] in terms of the observability matrix as a purely technical tool for constructing canonical forms. Uniform observability was systematically studied in [4]–[9] based on its definition in terms of output functions.

The construction of asymptotic state estimators implies the construction of a dynamical system whose input is fed with the output function of the original system, while the state of the estimator must asymptotically approximate the state of the original system in some sense. The approach proposed in this paper is based on the quasidifferentiation technique, which permits significantly weakening the well-known smoothness requirements imposed on the coefficients when constructing exponential state estimators.

Currently, known conditions of the observability and controllability for linear time-varying systems of ordinary differential equations are formulated either implicitly by using the fundamental matrix or explicitly by the observability and controllability matrices [2, 3, 8] and [23]–[26]. However, a fundamental matrix is usually unknown, and its calculation is quite a challenge; on the other hand, the existence of the observability matrices requires a high smoothness of the coefficients [2, 26]. Therefore, an important objective is to obtain new efficient conditions of the controllability and observability with minimal assumptions on the differentiability of system parameters.

The functions that define the coefficients of the dynamical systems may violate the classical differentiability assumptions, and that may lead to improper use of the obtained mathematical results. However, there is a need to explore the time-varying systems that do not satisfy differentiability assumptions, for example, in studies of nonlinear systems by the linearization method. As the research has shown, the quasiderivative concept allows us to extend the class of the systems for which we can set effective (in terms of the systems' coefficients) necessary and sufficient conditions for the observability, controllability, stability, and existence of canonical forms. The quasidifferentiation methodology makes possible determination [6] and exploration of the classical property of the uniform observability for a broader class of systems than in previous studies.

2. Asymptotic State Observes

Consider a linear time-varying system of ordinary differential equations

$$\frac{dx(t)}{dt} = A(t)x(t), \tag{2.1}$$

where $x(t)$ is the column state n-vector at time t. The independent variable t varies on the interval $T=[t_0,t_1]$, and the entries of the $n\times n$ matrix $A(t)$ are continuous on T. Assume that system (2.1) permits measuring a scalar output function $y(t)$ related to the state $x(t)$ by

$$y(t)=c(t)x(t), \quad t\in T, \tag{2.2}$$

where $c(t)$ is a given row n-vector with continuous components on T.

We identify each such system (2.1), (2.2) with the corresponding pair (A,c) and denote the set of all such pairs by Σ_n; i.e., $\Sigma_n=C(T,\mathbb{R}^{n\times n})\times C(T,\mathbb{R}^n)$. Here $C(T,\mathbb{R}^{n\times n})$ is the set of continuous $n\times n$ matrix functions, and $C(T,\mathbb{R}^n)$ is the set of row n-vector functions with continuous components on T.

Let X be a compact set in $\mathbb{R}^n$. If some initial state $x_0=x(t_0)\in X$ is realized in system (2.1) at time t_0, then it generates a solution $x(t)=x(t,x_0)$ and hence an output signal $y(t)=y(t,x_0)$, $t\in T$, according to (2.2).

In what follows, we assume that the independent variable t ranges in the set of nonnegative real numbers, i.e., $T=\mathbb{R}_+$, and the set X of initial states coincides with $\mathbb{R}^n$.

Consider the approach to the problem of estimating the state vector of an observation system based on the construction of a dynamical system, called an observer (estimator) of the solution $w(t)$, which converges to the state $x(t)$ of the original system for any initial states in the following sense: $\lim\limits_{t\to\infty}\|x(t)-w(t)\|=0$.

In other words, the state vector of the estimator $w(t)$ serves as an approximate estimate of the state $x(t)$ of the original observation system, and the vector $w(t)$ can be used to construct feedback controls. In the literature, estimators were constructed for observation systems with sufficiently smooth coefficients [1, 10], [18]–[22] or output functions [20]. In this paper, instead of the differentiability of the outputs, we use their quasidifferentiability with respect to some lower triangular matrix $P(t)$. This permits significantly strengthening the already known results. The matrix $P(t)$ can be obtained constructively for systems written in upper Hessenberg form and hence for all systems reducible to the Hessenberg form by linear nonsingular changes of variables. In this connection, we present a criterion for the reducibility of the original system to the Hessenberg form and propose an algorithm for constructing the Hessenberg form. Our approach is based on the notion of P-uniform observability for systems with quasidifferentiable output functions, which permits constructing the canonical Frobenius form under certain conditions.

Following [1, 10], the system of differential equations

$$\frac{dw(t)}{dt}=A(t)w(t)+k(t)\big(y(t)-c(t)w(t)\big), \quad t\in T=\mathbb{R}_+, \tag{2.3}$$

with an arbitrary column n-vector $k(t)$ (the vector of gain factors) will be called an estimator of states of system (2.1), (2.2), because, for any $\tau\in\mathbb{R}_+$ and $x_0\in\mathbb{R}^n$ defining the initial conditions $x(\tau)=w(\tau)=x_0$, the solutions $x(t)=x(t,\tau,x_0)$ and $w(t)=w(t,\tau,x_0)$ of respective systems (2.1) and (2.3) coincide for all $t\geq\tau$. Therefore, if relation $x(\tau)=w(\tau)=x_0$ is satisfied, then the state of the observer (2.3) is exactly described by the solutions of system (2.1).

Since the initial state x_0 of system (2.1) is unknown by the statement of the observation problem, we see that one cannot construct the exact solution $x(t)$ using the estimator (2.3). Therefore, it is natural to consider the problem of determining a gain vector $k(t)$ for which the solution $w(t)$ of the observer system (2.3) asymptotically approximates (estimates) the solution

$x(t) = x(t,\tau,x_0)$ for an arbitrary $x_0 \in \mathbb{R}^n$ that generates the output function $y(t)$ given by system (2.1), (2.2). In this case, the initial state of the observer system can be taken arbitrarily.

Let $\varepsilon(t)$ be the estimation error, i.e., the difference $\varepsilon(t) = x(t) - w(t)$. From systems (2.1) and (2.3), we find that the estimation error $\varepsilon(t)$ satisfies the time-varying linear system of differential equations

$$\frac{d\varepsilon(t)}{dt} = \big(A(t) - k(t)c(t)\big)\varepsilon(t), \quad t \in T = \mathbb{R}_+. \tag{2.4}$$

Then the problem of asymptotic estimation of states of system (2.1) is equivalent to the problem of determining the gain vector $k(t)$ for which $\varepsilon(t) \to 0$ as $t \to \infty$. If the rate of approximation of the vector $x(t)$ by the vector $w(t)$ is important when the state $x(t)$ is estimated (i.e., the estimation error $\varepsilon(t)$ must tend to zero with a prescribed rate, for example, exponentially), then the choice of the gain coefficient $k(t)$ must ensure the desired rate of approximation.

Let us give a definition of so-called exponential estimators. Let ρ be a positive number. We say that the observer (2.3) exponentially estimates the state of system (2.1) with a rate ρ if the estimation error satisfies the inequality $\|\varepsilon(t)\| \leq C_{\rho,\tau}\exp(-\rho(t-\tau))$, $t \geq \tau$, where $C_{\rho,\tau}$ is a positive constant. In what follows, we describe a constructive method for determining exponential state estimators for P-uniformly observable systems based on the method of Frobenius forms for systems with quasidifferentiable coefficients.

3. Notion of Quasiderivatives

Here the quasiderivative is defined by following [27] – [31].

Let $T = [t_0, t_1]$ be a segment of the real line $\mathbb{R}$, and let p be a given nonnegative integer. We denote by $\mathscr{U}_p(T)$ the set of all lower triangular $(p+1)\times(p+1)$ matrices $P(t)$ with continuous on T elements $p_{ki}(t)$, $k,i \in \{0,1,\ldots,p\}$ such that $p_{kk}(t) \neq 0$, $t \in T$, $k \in \{0,1,\ldots,p\}$.

Let us take a matrix $P(t)$ from the set $\mathscr{U}_p(T)$. Then for a continuous function $w : T \to \mathbb{R}$ the quasiderivatives

$$\overset{0}{_P}w(t),\ \overset{1}{_P}w(t),\ \ldots,\ \overset{p}{_P}w(t)$$

of the order $0,1,\ldots,p$ with respect to the matrix $P(t)$ are defined by the following recurrence rules:

$$\overset{0}{_P}w(t) = p_{00}(t)w(t), \quad \overset{1}{_P}w(t) = p_{11}(t)\frac{d\big(\overset{0}{_P}w(t)\big)}{dt} + p_{10}(t)\big(\overset{0}{_P}w(t)\big),\ \ldots,$$

$$\overset{k}{_P}w(t) = p_{kk}(t)\frac{d\big(\overset{k-1}{_P}w(t)\big)}{dt} + \sum_{i=0}^{k-1} p_{ki}(t)\big(\overset{i}{_P}w(t)\big), \quad k \in \{2,3,\ldots,p\}. \tag{3.1}$$

It is assumed that the operation of the differentiation in formulas (3.1) can be performed and result in continuous functions. In the case $p = 0$, quasidifferentiable means that the product $p_{00}(t)w(t)$ is continuously differentiable.

To explain the notion of quasiderivatives, let us consider the system of equations

$$\frac{dx_1(t)}{dt} = \alpha_0(t)x_3(t), \quad \frac{dx_2(t)}{dt} = x_1(t) + \alpha_1(t)x_3(t), \quad \frac{dx_3(t)}{dt} = x_2(t) + \alpha_2(t)x_3(t),$$

with continuous functions $\alpha_0(t)$, $\alpha_1(t)$, $\alpha_2(t)$. We have $x_2(t) = \frac{dx_3(t)}{dt} - \alpha_2(t)x_3(t)$. Obviously, $x_2(t)$ is continuously differentiable function, but functions $\frac{dx_3(t)}{dt}$ and $\alpha_2(t)x_3(t)$ are not. So we may call expression $\frac{dx_3(t)}{dt} - \alpha_2(t)x_3(t)$ as the quasiderivative ${}^{1}_{P}x_3(t)$ of function $x_3(t)$ with respect to the matrix

$$P(t) = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_2(t) & 1 & 0 \\ -\alpha_1(t) & 0 & 1 \end{pmatrix}.$$

We denote by $C_P(T)$ the family of all continuous functions that have continuous quasiderivatives (3.1) with respect to a given matrix $P \in \mathscr{U}_p(T)$. Clearly, every p times continuously differentiable function is quasidifferentiable with respect to the identity $(p+1) \times (p+1)$ matrix. However, it is easy to demonstrate that some nondifferentiable in the usual sense function could be quasidifferentiable with respect to some matrix $P \in \mathscr{U}_p(T)$. Let f_1 and f_2 be two functions from the set $C_P(T)$. From formulas (3.1), it follows that

$${}^{i}_{P}\big(\lambda_1 f_1(t) + \lambda_2 f_2(t)\big) = \lambda_1\big({}^{i}_{P}f_1(t)\big) + \lambda_2\big({}^{i}_{P}f_2(t)\big), \quad i \in \{0, 1, \dots, p\},$$

for any real values λ_1, λ_2. So $C_P(T)$ is a vector space over the real numbers.

For the sake of simplicity, we skip rules on how to derive quasiderivatives for the product of the functions. However, we note that

$${}^{0}_{P}\big(f_1(t)f_2(t)\big) = \frac{{}^{0}_{P}f_1(t)\,{}^{0}_{P}f_2(t)}{p_{00}(t)},$$

$${}^{1}_{P}\big(f_1(t)f_2(t)\big) = \frac{1}{p_{00}(t)}\Big({}^{1}_{P}f_1(t)\,{}^{0}_{P}f_2(t) + {}^{0}_{P}f_1(t)\,{}^{1}_{P}f_2(t)\Big)$$
$$- \frac{p_{11}(t)\dfrac{dp_{00}(t)}{dt} + p_{10}(t)p_{00}(t)}{p_{00}^2(t)}\,{}^{0}_{P}f_1(t)\,{}^{0}_{P}f_2(t)$$

if the derivatives exist. Let us assume that $P(t), R(t) \in \mathscr{U}_p(T)$ are matrices such that their sum $Q(t) = P(t) + R(t)$ also belongs to the set $\mathscr{U}_p(T)$. Then ${}^{i}_{Q}f(t) = {}^{i}_{P}f(t) + {}^{i}_{R}f(t)$, and for any nonzero $\lambda \in \mathbb{R}$ the equality ${}^{i}_{\lambda P}f(t) = \lambda\big({}^{i}_{P}f(t)\big)$ is fulfilled, $i \in \{0, 1, \dots, p\}$. Unlike the usual derivatives, a quasiderivative ${}^{1}_{P}C$ of a constant C is not equal to zero, generally speaking. Let us consider n real scalar functions $f_i \in C_P(T)$, $i \in \{1, 2, \dots, n\}$, where $p \geq n-1$, and let f be a vector function with entries $f_1, f_2, \dots, f_n$. The matrix ${}_{P}W(f)(t) = {}_{P}W(f_1, f_2, \dots, f_n)(t)$ of the rows $\Big({}^{0}_{P}f_i(t),\ {}^{1}_{P}f_i(t),\ \dots,\ {}^{n-1}_{P}f_i(t)\Big)$, $i \in \{1, 2, \dots, n\}$ is called the Wronski matrix relatively to matrix $P(t)$. The proofs of the following theorems are given in [6].

Theorem 3.1. *If there is $t^* \in T$ such that $\det {}_{P}W(f_1, f_2, \dots, f_n)(t^*) \neq 0$, then system of functions $f_1(t), f_2(t), \dots, f_n(t)$ are linearly independent on T.*

Theorem 3.2. *System of functions $f_1(t), f_2(t), \dots, f_n(t)$ are linearly independent on each interval $(\tau_0, \tau_1) \subset T$ if and only if*

$$\det {}_{P}W(f_1, f_2, \dots, f_n)(t) \neq 0 \tag{3.2}$$

on the set of points t that is everywhere dense in T.

Let $\mathscr{P}_n(f) = \mathscr{P}_n(f_1, f_2, \dots, f_n)$ be a subset of the set $\mathscr{U}_n(T)$ such that $f_i \in C_P(T)$, $i \in \{1, 2, \dots, n\}$ for any matrix $P \in \mathscr{P}_n(f)$. Analysis of (3.1) and (3.2) raise the following question. Let $P(t)$ and $R(t)$ be the elements of the set $\mathscr{P}_n(f)$. Is there some $t^* \in T$ such that the matrix ${}_{P}W(f)(t^*)$ is nonsingular and the matrix ${}_{R}W(f)(t^*)$ is degenerate?

Theorem 3.3. [6] *For each $t \in T$ all matrices ${}_PW(f)(t)$, $P \in \mathscr{P}_n(f)$ are either degenerate or not degenerate simultaneously.*

Application of the quasidifferentiability produces the nontrivial problem of finding at least one element $P(t)$ of set $\mathscr{U}_p(T)$ with respect to which the output functions of a system of observation are quasidifferentiable. Here we will describe one class of observation systems with a scalar output for which this problem is easily solved. Then we will show how to construct matrix $P(t)$ for a broader class of equations by using the invariant of the set of the outputs under linear nonsingular transformations on the space of the states.

The linear time-varying system of observation

$$\frac{dx(t)}{dt} = H(t)x(t), \quad y(t) = g(t)x(t) \tag{3.3}$$

is said to have upper Hessenberg form if continuous on the interval T $n \times n$ matrix $H(t)$ and row vector $g(t)$ are defined as follows:

$$H(t) = \begin{pmatrix} r_{11}(t) & r_{12}(t) & r_{13}(t) & \dots & r_{1,\,n-1}(t) & r_{1n}(t) \\ r_{21}(t) & r_{22}(t) & r_{23}(t) & \dots & r_{2,\,n-1}(t) & r_{2n}(t) \\ 0 & r_{32}(t) & r_{33}(t) & \dots & r_{3,\,n-1}(t) & r_{3n}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & r_{n-1,\,n-1}(t) & r_{n-1,\,n}(t) \\ 0 & 0 & 0 & \dots & r_{n,\,n-1}(t) & r_{nn}(t) \end{pmatrix}, \tag{3.4}$$

$$g(t) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & r_{10}(t) \end{pmatrix}.$$

Assume, that

$$r_{k,k-1}(t) \neq 0, \quad t \in T, \quad k \in \{1, 2, \dots, n\}. \tag{3.5}$$

Let us construct $(n+1) \times (n+1)$ matrix

$$P(t) = \begin{pmatrix} \frac{1}{r_{10}(t)} & 0 & 0 & \dots & 0 & 0 \\ -\frac{r_{nn}(t)}{r_{n,n-1}(t)} & \frac{1}{r_{n,n-1}(t)} & 0 & \dots & 0 & 0 \\ -\frac{r_{n-1,n}(t)}{r_{n-1,n-2}(t)} & -\frac{r_{n-1,n-1}(t)}{r_{n-1,n-2}(t)} & \frac{1}{r_{n-1,n-2}(t)} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{r_{2n}(t)}{r_{21}(t)} & -\frac{r_{2,n-1}(t)}{r_{21}(t)} & -\frac{r_{2,n-2}(t)}{r_{21}(t)} & \dots & \frac{1}{r_{21}(t)} & 0 \\ -r_{1n}(t) & -r_{1,n-1}(t) & -r_{1,n-2}(t) & \dots & -r_{11}(t) & 1 \end{pmatrix}, \tag{3.6}$$

which, obviously, belongs to the set $\mathscr{U}_n(T)$.

Simple arguments validate the following lemma.

Lemma 3.4. *If conditions (3.5) are met, then each output function $y(t)$ of system (3.3) in upper Hessenberg form has n quasiderivative ${}^k_P y(t)$, $t \in T$, $k \in \{0,1,\dots,n\}$ with respect to matrix $P(t)$ in form (3.6).*

4. Quasidifferentiability and Uniform Observability

In the classical statement [2], the complete observability problem and the closely related differential observability problem [3] are stated as existence problems requiring a one-to-one correspondence between the output signals $y(t,x_0)$, $t \in T$ and the initial conditions $x_0 = x(t_0)$ generating them. Coefficient conditions for various types of observability of system (2.1), (2.2) are, as a rule, expressed in terms of the classical observability matrix $S(t)$ [2, 26] which, under certain conditions, is formed from the rows $s_i(t)$ constructed by the recursion scheme

$$s_0(t) = c(t), \quad s_i(t) = s_{i-1}(t)A(t) + \frac{ds_{i-1}(t)}{dt}, \quad i \in \{1,2,\dots,n-1\}.$$

It is well known [2] that if the classical observability matrix $S(t)$ of system (2.1), (2.2) is nonsingular at some point $t^* \in T$ (and hence system (2.1), (2.2) is completely observable on T), then one can use the value of the output signal and its successive derivatives at time t^* to obtain the vector $x(t^*)$ in an algebraic way. But determining the vector $x(t)$ at points t other than t^* leads to a rather difficult problem of integration of a time-varying system of ordinary differential equations. Obviously, the problems of integration can be avoided if one requires the matrix $S(t)$ to be nonsingular for all $t \in T$. We also note that, by the statement of the observability problem, only the output function but not its derivatives are known, and determining its successive derivatives is a rather complicated computational problem [22]. In what follows, we give the definition [4] - [9] of uniform observability in terms of the output function via its successive quasiderivatives.

Let us consider system $(A,c) \in \Sigma_n$. Also by $\mathscr{Y}_T(A,c)$, let us denote the set of all its output functions

$$\mathscr{Y}_T(A,c) = \left\{ y \in C(T,\mathbb{R}) : \ y(t) = c(t)F(t)x_0, \ t \in T, \ x_0 \in \mathbb{R}^n \right\},$$

where $F(t)$ is a fundamental matrix of system (2.1), normalized at the point t_0.

Let $P(t)$ be a given matrix in the set $U_m(T)$. We say that system (2.1), (2.2) is of the class $\{P,m\}$ and write $(A,c) \in \{P,m\}$ if each of its output functions $y(t,x_0)$, $t \in T$, $y \in \mathscr{Y}_T(A,c)$ has continuous quasiderivatives of order m with respect to the matrix $P(t)$. If $P(t) = E_n$ (where E_n is the $n \times n$ identity matrix), then we say that system (2.1), (2.2) is a system of the class $n-1$.

It was proved in [6] that system (2.1), (2.2) is of the class $\{P,n-1\}$ if and only if for each $k \in \{0,1,\dots,n-1\}$ the row vector functions

$$s_0(t) = p_{00}(t)c(t), \quad s_k(t) = p_{kk}(t)\left(s_{k-1}(t)A(t) + \frac{ds_{k-1}(t)}{dt}\right) + \sum_{i=0}^{k-1} p_{ki}(t)s_i(t) \tag{4.1}$$

exist and are continuous.

One can readily verify the relations

$$s_k(t)x(t) = {}^k_P y(t), \ k \in \{0,1,\dots,n-1\}, \ t \in T,$$

which, with regard to the notation

$$S_P(t) = \begin{pmatrix} s_0(t) \\ s_1(t) \\ \dots \\ s_{n-1}(t) \end{pmatrix}, \quad Y(t,x_0) = \begin{pmatrix} {}^0_P y(t,x_0) \\ {}^1_P y(t,x_0) \\ \dots \\ {}^{n-1}_P y(t,x_0) \end{pmatrix},$$

leads to the system of equations

$$S_P(t)x(t) = Y(t,x_0),\ t \in T$$

for the state vector $x(t)$.

Let $\mathscr{P}_{n-1}(A,c) \subset U_{n-1}(T)$ be the set of matrices P with respect to which system (2.1), (2.2) is of the class $\{P, n-1\}$. Clearly, for each matrix $P \in \mathscr{P}_{n-1}(A,c)$ one can define the matrix $S_P(t)$ by formulas (4.1). It was proved in [6] that for each $t \in T$ all matrices $S_P(t)$, $P \in \mathscr{P}_{n-1}(A,c)$ are simultaneously singular or simultaneously nonsingular. The matrix $S_P(t)$ will be called the observability matrix for system (2.1), (2.2) of the class $\{P, n-1\}$.

Definition 4.1. System (2.1), (2.2) of the class $\{P, n-1\}$ is said to be P-uniformly observable on T if for each $x_0 \in X$ the output functions $y(t) = y(t,x_0)$, $y \in \mathscr{Y}_T(A,c)$ have continuous quasiderivatives with respect to the matrix P and the mapping

$$x(t) \to \left({}^0_P y(t), {}^1_P y(t), \dots, {}^{n-1}_P y(t)\right) \tag{4.2}$$

is injective for each $t \in T$.

By the above cited property of the observability matrix $S_P(t)$ thus constructed, the injectivity of the mapping (4.2) is independent of the choice of the matrix $P \in \mathscr{P}_{n-1}(A,c)$.

Note that the property of P-uniform observability will be substantially used in what follows when constructing asymptotic estimators of observation systems. Therefore, let us present a criterion for P-uniform observability [6].

Theorem 4.2. *System (2.1), (2.2) of the class $\{P, n-1\}$ is P-uniformly observable on T if and only if $\operatorname{rank} S_P(t) = n$ for each $t \in T$.*

5. Construction of the Frobenius Form

Several problems of mathematical control theory can be studied rather easily if the original linear time-varying observation system (A,c) can be transformed to a Frobenius form (A^0, c^0)

$$A^0(t) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \alpha_0(t) \\ 1 & 0 & 0 & \dots & 0 & \alpha_1(t) \\ 0 & 1 & 0 & \dots & 0 & \alpha_2(t) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & \alpha_{n-1}(t) \end{pmatrix}, \quad c^0 = \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

by an appropriate transformation group $\mathscr{G}$. Here the functions $\alpha_i(t)$ are of the class $C(T,\mathbb{R})$, $i \in \{0, 1, \dots, n-1\}$.

We point out that methods known in the literature for constructing Frobenius forms are based on the use of the classical observability matrix $S(t)$, which only exists for sufficiently smooth systems [23]. Assume that $\mathscr{G}$ is the group of all $n \times n$ matrices $G(t)$ of the class $C^1(T, \mathbb{R}^{n\times n})$

that are nonsingular for each $t \in T$. The action of the group $\mathscr{G}$ on the pair (A,c) from Σ_n is defined in a standard way as

$$G*(A,c) = \big(G^{-1}AG - G^{-1}\frac{dG}{dt},\ cG\big), \quad G \in \mathscr{G}. \tag{5.1}$$

By $\mathscr{O}(A,c)$ we denote the orbit of the system $(A,c) \in \Sigma_n$ with respect to the action of the group $\mathscr{G}$ explained above. It is easily seen that if in the orbit $\mathscr{O}(A,c)$ of a system $(A,c) \in \Sigma_n$ there exists a pair (A^0,c^0) in Frobenius form, then it is unique. Therefore, the transformation $G \in \mathscr{G}$ for which $G*(A,c) = (A^0,c^0)$ is unique as well.

If the matrix $P_0 \in U_n(T)$ is defined as

$$P_0(t) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -\alpha_{n-1}(t) & 1 & 0 & \dots & 0 & 0 \\ -\alpha_{n-2}(t) & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_1(t) & 0 & 0 & \dots & 1 & 0 \\ -\alpha_0(t) & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

then one can readily verify that the observability matrix $S^0_{P_0}(t)$ for the pair (A^0,c^0) is a backward identity matrix (i.e., a matrix with units on the secondary diagonal and zeros elsewhere). Since the backward identity matrix is nonsingular, we see that the system (A^0,c^0) is P_0-uniformly observable on T.

Suppose, that system (A,c) transformed to a Frobenius form (A^0,c^0) with respect to the group $\mathscr{G}$. Then the matrix $G(t)$ of the transformation $G*(A,c) = (A^0,c^0)$ connects the observability matrices of the systems (A,c) and (A^0,c^0) by the relation $G(t) = S_P^{-1}(t)S_P^0(t)$ [6]. Using the matrix $P_0(t)$ and the properties of the backward identity matrix, we see that the inverse matrix of $G(t)$ has the form

$$G^{-1}(t) = \begin{pmatrix} s'_{n-1}(t) & s'_{n-2}(t) & \dots & s'_0(t) \end{pmatrix},$$

where the rows $s_i(t)$ are constructed by formulas (4.1) with the matrix $P_0(t)$.

Since the action (5.1) of the group $\mathscr{G}$ must preserve the asymptotic properties of the observer, it is natural to study when the matrix $G(t)$ of the transformation $G*(A,c) = (A^0,c^0)$ belongs to the Lyapunov group $\mathscr{L}_n$ [5], i.e., to the group of invertible continuously differentiable $n \times n$ matrices bounded on $\mathbb{R}_+$ together with their inverses.

The above considerations imply the following assertion.

Lemma 5.1. *Assume that the system (A,c) has a Frobenius form (A^0,c^0) with respect to the actions of a transformation group $\mathscr{G}$. For the matrix of the transformation $G*(A,c) = (A^0,c^0)$ belong to the Lyapunov group, it is necessary and sufficient that the observability matrix $S_{P_0}(t)$ of the system (A,c) belongs to the Lyapunov group.*

An analysis of the relation

$$\Big(G^{-1}AG - G^{-1}\frac{dG}{dt}, cG\Big) = (A^0,c^0)$$

shows that, to reduce the pair (A,c) to the system (A^0,c^0) by using the group $\mathscr{G}$, it is necessary to satisfy the conditions $c \in C^1(T,\mathbb{R}^n)$ and $c(t) \neq 0$, $t \in T$. We assume in what follows that they are satisfied.

In what follows, a key role is played by the representation of the nonsingular continuously differentiable $n\times n$ matrix $G\in\mathscr{G}$ in the form of a product of a continuously differentiable orthogonal matrix $G_o(t)$ and an upper triangular matrix $G_\Delta(t)$, $G(t)=G_o(t)G_\Delta(t)$, $t\in T$, where $G_o\in\mathscr{O}\mathscr{L}_n$ and $G_\Delta\in\mathscr{G}_\Delta$, and $\mathscr{G}_\Delta$, $\mathscr{O}\mathscr{L}_n$ are subgroups of the group $\mathscr{G}$, respectively, consisting of all upper triangular and orthogonal matrices for each $t\in T$. The existence of such a representation is a straightforward consequence of the Perron triangularization theorem [32].

If the matrix $G(t)$ belongs to the Lyapunov group, then each of the matrices $G_o(t)$ and $G_\Delta(t)$ obviously belongs to this group.

Using the expansion $G(t)=G_o(t)G_\Delta(t)$, we can rewrite the relation $G*(A,c)=(A^0,c^0)$ as follows:

$$\Big(G_o'(t)A(t)+\frac{dG_o'(t)}{dt}\Big)G_o(t)=G_\Delta(t)A^0(t)G_\Delta^{-1}(t)+\frac{dG_\Delta(t)}{dt}G_\Delta^{-1}(t)$$

and $c(t)G_o(t)=c^0G_\Delta^{-1}(t)$ (where the prime stands for transposition).

Simple matrix calculations show that the mapping $(A^0,c^0)\to G^{-1}*(A^0,c^0)$ leads to the observation systems (H,c^*) with n-vector functions $c^*(t)=\big(0\ 0\ \dots\ 0\ r_{10}(t)\big)$ and matrices $H(t)=\big(r_{ij}(t)\big)_{i,j=1}^n$ in the upper Hessenberg form (3.3), (3.4); i.e., $r_{ij}(t)\equiv 0$, $t\in T$, for $i>j+1$ $i,j\in\{1,2,\dots,n\}$, and the functions $r_{ij}(t)$ have the following additional properties:

$$r_{i+1,j}\in C^1(T,\mathbb{R}),\quad r_{i+1,i}(t)\neq 0,\quad t\in T,\quad i\in\{0,1,\dots,n-1\}.$$

Assume that $G*(A,c)=\big(G_oG_\Delta*(A,c)\big)=(A^0,c^0)$ for some matrix $G\in\mathscr{G}$. By $p_n(t)$, $p_{n-1}(t)$, …, $p_1(t)$ we denote the first, second, …, n-th row, respectively, of the matrix $G_o'(t)$, and by $g_{ij}(t)$ we denote the entries of the matrix $G_\Delta(t)$. It is well known [6] that the functions $p_i(t)$, $i\in\{1,2,\dots,n\}$ can be determined by the recursion formulas

$$p_1(t)=c(t)\|c(t)\|^{-1},\quad b_{10}(t)=\|c(t)\|,\quad \|c(t)\|\neq 0, \tag{5.2}$$

$$b_{n+1-i,n+1-j}(t)=\Big(p_i(t)A(t)+\frac{dp_i(t)}{dt}\Big)p_j'(t),\ j\in\{1,2,\dots,i\},\ i\in\{1,2,\dots,n-1\}, \tag{5.3}$$

$$b_{n+1-i,n-i}(t)=\|p_i(t)A(t)+\frac{dp_i(t)}{dt}-\sum_{k=1}^{i}b_{n+1-i,n+1-k}(t)p_k(t)\|, \tag{5.4}$$

$$p_{i+1}(t)=b_{n+1-i,n-i}^{-1}(t)\Big(p_i(t)A(t)+\frac{dp_i(t)}{dt}-\sum_{k=1}^{i}b_{n+1-i,n+1-k}(t)p_k(t)\Big), \tag{5.5}$$

and the functions $g_{ij}(t)$, i.e., the entries of the matrix $G_\Delta(t)$, by the recursion rules

$$g_{nn}(t)=\frac{1}{\|c(t)\|},\quad g_{i-1,i-1}(t)=\frac{g_{ii}(t)}{b_{i,i-1}(t)},\quad g_{i,j+1}(t)=\sum_{k=1}^{j}b_{ik}(t)g_{kj}(t)-\frac{dg_{ij}(t)}{dt}, \tag{5.6}$$

where the calculations are successively carried out for the indexing sets

$$i=1,\ j\in\{2,3,\dots,n-1\};\ i=2,\ j\in\{3,4,\dots,n-1\};\ \dots;\ i=n-1,\ j=n-1.$$

The components of the n-vector function $\alpha(t)=\big(\alpha_0(t),\alpha_1(t),\dots,\alpha_{n-1}(t)\big)'$ determining the Frobenius form (A^0,c^0) are calculated by the formulas

$$\alpha_{n-1}(t)=\frac{\sum_{j=1}^{n}b_{nj}(t)g_{jn}(t)-\dfrac{dg_{nn}(t)}{dt}}{g_{nn}(t)},$$

$$\alpha_{n-k}(t) = \frac{\sum\limits_{j=1}^{n} b_{n+1-k,j}(t)g_{jn}(t) - \sum\limits_{j=1}^{k-1} g_{n+1-k,n+1-j}(t)\alpha_{n-j}(t) - \dfrac{dg_{n+1-k,n}(t)}{dt}}{g_{n+1-k,n+1-k}(t)}, \tag{5.7}$$

where $k \in \{2,3,\ldots,n\}$. We point out that the existence of these functions imposes a number of restrictions on the elements of the pair (A,c). These restrictions are the continuous differentiability of the rows $p_1(t)$, $p_2(t)$, $\ldots$, $p_n(t)$ and the entries $g_{ij}(t)$ and the inequalities $b_{i,i-1}(t) \neq 0$, which must be satisfied for all indices $i \in \{1,2,\ldots,n\}$.

Note that, under the condition of existence of the orthogonal matrix $G_o(t)$, the coefficients of the system $(H,c^*) = G_o * (A,c)$ in the upper Hessenberg form for the pair (A,c) are determined as follows:

$$r_{ij}(t) = b_{n-i+1,n-j+1}(t), \quad i \in \{1,2,\ldots,n\};\ j \in \{i-1,\ldots,n\}.$$

Theorem 5.2. *A pair (A,c) has a Frobenius form (A^0,c^0) with respect to the action of the Lyapunov group if and only if the functions $g_{ij}(t)$, $p_i(t)$, $i \in \{1,2,\ldots,n\}; j \in \{i,i+1,\ldots,n\}$ are continuously differentiable and bounded on the set T and the inequalities*

$$b_{i,i-1}(t) \neq 0, \quad \rho_1 \leq |g_{ii}(t)| \leq \rho_2 \quad i \in \{1,2,\ldots,n\}, \quad t \in T$$

are satisfied for some positive numbers ρ_1, ρ_2.

Proof. Necessary and sufficient conditions for the existence such matrix G that

$$G * (A,c) = (A^0,c^0)$$

follow from the theorem 3.5 in [6], and the additional condition

$$\rho_1 \leq g_{ii}(t) \leq \rho_2,\ i \in \{1,2,\ldots,n\}$$

guarantees that the upper triangular matrix G_Δ constructed from the functions $g_{ij}(t)$ belongs to the Lyapunov group. The latter ensures that the transformation $G(t) = G_o(t)G_\Delta(t)$ belongs to this group. The proof of the theorem is complete.

The above reasoning justifies the following method for determining the canonical form. We use formulas (5.2)–(5.6) to obtain the functions $b_{ij}(t)$, the row vectors $p_1(t),\ p_2(t),\ \ldots,\ p_n(t)$, and the entries $g_{ij}(t)$. Based on relations (5.7), we determine the coefficients

$$\alpha_k(t),\ k \in \{0,1,\ldots,n-1\}$$

of the Frobenius form. If the conditions for the calculations by these formulas are violated, then there is no Frobenius form (A^0,c^0) with respect to the group $\mathscr{G}$ for the system (A,c).

As was already noted, the lower triangular matrices $P(t)$ with respect to which the output functions of system (2.1), (2.2) are at least n times continuously quasidifferentiable play an important role in observability problems. The construction used in the preceding section allows us to indicate one such matrix. Assume that the functions $b_{ij}(t)$ are defined by formulas (5.2) –

(5.4) and $b_{i,i-1}(t)\neq 0,\ i\in\{1,2,\dots,n\},\ t\in T$. Set

$$P(t)=\begin{pmatrix} b_{10}^{-1}(t) & 0 & \dots & 0 & 0\\ -b_{nn}(t)b_{n,n-1}^{-1}(t) & b_{n,n-1}^{-1}(t) & \dots & 0 & 0\\ -b_{n-1,n}(t)b_{n-1,n-2}^{-1}(t) & -b_{n-1,n-1}(t)b_{n-1,n-2}^{-1}(t) & \dots & 0 & 0\\ \cdot & \cdot & \dots & \cdot & \cdot\\ -b_{2n}(t)b_{21}^{-1}(t) & -b_{2,n-1}(t)b_{21}^{-1}(t) & \dots & b_{21}^{-1}(t) & 0\\ -b_{1n}(t) & -b_{1,n-1}(t) & \dots & -b_{11}(t) & 1 \end{pmatrix}. \tag{5.8}$$

Theorem 5.3. *If the matrix $P(t)$ of the form (5.8) is constructed for the pair (A,c), then each output function $y(t)$, $t\in T$ of system (2.1), (2.2) is n times continuously quasidifferentiable with respect to the matrix $P(t)$.*

The proof of this theorem was given in [6].

6. Construction of a ρ-Exponential State Estimator

Assume that for system (2.1), (2.2) there exists a Frobenius form (A^0,c^0) with respect to the actions of the Lyapunov group $\mathscr{L}_n$ and $G(t)$ is a transformation for which the relation

$$G*(A,c)=(A^0,c^0)$$

is satisfied. The coefficients of the Frobenius form (A^0,c^0) form a column n-vector $\alpha(t)$. Take distinct real numbers λ_1, λ_2, …, λ_n satisfying the inequality $\lambda_i<\rho$ for a given positive number ρ and construct the polynomial

$$(\xi-\lambda_1)(\xi-\lambda_2)\cdots(\xi-\lambda_n)=\xi^n-\beta_{n-1}\xi^{n-1}-\dots-\beta_1\xi-\beta_0.$$

Assume that β is the column n-vector with components $(\beta_0,\beta_1,\dots,\beta_{n-1})$.

In the time-varying system of differential equations (2.4) satisfied by the estimation error $\varepsilon(t)=x(t)-w(t)$, the gain factor $k(t)$ can be determined by the formula

$$k^*(t)=G(t)\big(\alpha(t)-\beta\big). \tag{6.1}$$

Theorem 6.1. *Assume that the functions $g_{ij}(t),\ p_i(t),\ i\in\{1,2,\dots,n\};\ j\in\{i,i+1,\dots,n\}$ constructed by formulas (5.2) – (5.6) are continuously differentiable and bounded on the set T and inequalities (5.8) are satisfied for some positive numbers ρ_1, ρ_2 and for any $t\in T$. Then system (2.3) with the gain factor $k(t)$ of the form (6.1) is a ρ-exponential estimator for system (2.1) - (2.2).*

Proof. Under the assumptions of the theorem, we use the row vectors $p_1(t), p_2(t),\dots,p_n(t)$ and the functions $g_{ij}(t)$ $(i=1,2,\dots,n; j=i,i+1,\dots,n)$ to construct the orthogonal matrix $G_o(t)$ and the upper triangular matrix $G_\Delta(t)$. By theorem 5.2, the matrix $G(t)=G_o(t)*G_\Delta(t)$ is the matrix of a Lyapunov transformation that takes the pair (A,c) to the Frobenius form (A^0,c^0).

From system $\dot\varepsilon(t)=\big(A(t)-k^*(t)c(t)\big)\varepsilon(t)$, after the change of variables $\varepsilon(t)=G(t)z(t)$, we obtain the following system of differential equations for the vector function $z(t)$:

$$\dot z(t)=G^{-1}(t)\Big(A(t)-G(t)\big(\alpha(t)-\beta\big)c(t)\Big)G(t)z(t)-G^{-1}(t)\dot G(t)z(t).$$

Since $G^{-1}(t)A(t)G(t) - G^{-1}(t)\dot{G}(t) = A^0(t)$, and $c(t)G(t) = c^0$, we have a time-invariant linear system of ordinary differential equations

$$\dot{z}(t) = \big(A^0(t) - (\alpha(t) - \beta)c^0\big)z(t) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & \beta_0 \\ 1 & 0 & 0 & \dots & 0 & \beta_1 \\ . & . & . & \dots & \dots & . \\ 0 & 0 & 0 & \dots & 0 & \beta_{n-2} \\ 0 & 0 & 0 & \dots & 1 & \beta_{n-1} \end{pmatrix} z(t)$$

with given characteristic numbers $\lambda_1, \lambda_2, \dots, \lambda_n,\ \lambda_i < -\rho, i \in \{1, 2, \dots, n\}$. Since the Lyapunov transformation preserves the exponential stability of the system, we see that the estimator (2.3) with the gain factor (6.1) is a ρ-exponential estimator for system (2.1), (2.2). The proof of the theorem is complete. An essential point in the construction of the gain factor $k(t)$ of the exponential estimator (2.3) is the construction of the Frobenius form and the corresponding transformation in the Lyapunov group. Since this Frobenius form exists [5, 6] only for P-uniformly observable systems (2.1), (2.2), the first natural question is the problem of quasidifferentiability of the output functions of the pair (A, c) with respect to some matrix $P \in U_n$. By theorem 5.3, the assumptions of theorem 6.1 guarantee the quasidifferentiability of all output functions of system (2.1), (2.2) with respect to the matrix P of the form (6.1).

7. Conclusions

The current states of a system are crucial to know in many problems in the theory of controlled motions, for instance, when control actions are based on a feedback. In many cases, the coordinates of the objects cannot be directly observed (measured); however, there is an information about their states, given by some output function (signal). The essence of the observability problem is to determine whether it is possible to unambiguously infer current (or initial) states of a system from the observations.

The existent coefficient conditions for the observability are based on the high degree of smoothness of either the coefficients [2, 3, 8, 26] or the output functions [5]. In this paper we use the quasidifferentiability (rather than the differentiability) of the output functions with respect to some lower triangular matrix $P(t)$. This allows us to establish explicit conditions for various types of observability that significantly strengthen known ones. Matrix $P(t)$ is easily obtained for systems in upper Hessenberg form and thus for all systems that can be transformed into Hessenberg form by linear variables substitutions. Thereby we also give the criteria of reducibility to Hessenberg form and the method of its construction.

On the basis of quasiderivatives we consider the state estimation problems for linear time-varying systems of ordinary differential equations. Conditions for linear independence of continuous quasidifferentiable functions are established in terms of the Wronski matrix. The method for constructing state estimators for linear time-varying systems based on the quasidifferentiability of the coefficients is proposed. For uniformly observable systems with quasidifferentiable coefficients, we obtain conditions for the existence of an exponential observer and describe a constructive method for designing such observers.

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