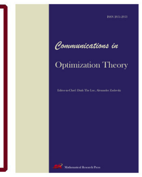
Commun. Optim. Theory 2022 (2022) 9 https://doi.org/10.23952/cot.2022.9

MATHRES Communications in Optimization Theory
Available online at http://cot.mathres.org



THE GENERIC APPROACH TO MODERN OPTIMIZATION

KAY BARSHAD

Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel

Dedication to the memory of Professor Hoang Tuy

Abstract. In this paper, we survey some recent applications of the generic approach to selected optimization problems, such as minimization of convex functionals and stochastic feasibility problems. We also overview a few recent developments related to the residuality properties of certain classes of convex functions, the abundance of which is known to be crucial in many optimization algorithms for which the elements of a certain class can be used as good approximates of given convex functions.

Keywords. Baire category; Common fixed point problem; Convex function; Generic convergence; Local uniform convexity; Lyapunov function; Residual set; Stochastic feasibility problem.

1. Introduction

The generic approach has already been successfully applied in many areas of analysis and the theory of dynamical systems (see, for example, [9, 10, 11, 13, 14, 26, 32, 35]), as well as in the Calculus of Variations and Optimization Theory (see, for example, [6, 8, 16, 21, 28, 29, 33, 34]). By applying this approach, we can investigate certain properties for the whole space and not just for a few elements in it. In this paper, we survey such applications to selected optimization problems and review some recent results regarding the existence of generic sets of certain classes of convex functions. This work is based on [1, 2, 3, 4, 5].

The following definition is one of the cornerstones of the generic approach discussed in this paper.

Definition 1.1. A subset Z of a topological space Y is called *residual* if it contains a countable intersection of open and dense subsets of Y.

We recall a general Baire category theorem for complete pseudo-metric spaces. It is a key theorem for obtaining results on residual sets.

Theorem 1.2. *Let X be a complete pseudo-metric space. Then the intersection of a countable family of open and dense subsets of X is itself dense in X.*

E-mail address: kaybarshad@technion.ac.il.
Received: June 28, 2022; Accepted: July 30, 2022.

Thus, in the case where the space under consideration is completely pseudo-metrizable, the Baire category theorem guarantees that each one of its residual subsets is also dense. In this work we are interested in the residual sets of some classes of operators, sequences of operators and of convex functions defined on normed spaces. We call elements of such sets *generic* elements. In Section 3 we also recall the notion of porosity which provides somewhat more refined results concerning the existence of residual sets of desired generic elements.

Obviously, while considering residual sets which, by definition, contain a countable intersection of open and dense sets, the topology with respect to which the sets are open should be as weak as possible, while the topology with respect to which the sets are dense should be as strong as possible. All the main results reviewed in the current paper should be read while keeping in mind this observation.

The rest of the paper is organized as follows. In Section 2 we overview the recent results on the properties of normal mappings and normal sequences of mappings, as well as recall the development of theory of weakly normal sequences of mappings. Two different definitions of normality and weak normality which depend on a given convex function are considered with applications to minimizing convex functionals. In Section 3 various methods for solving stochastic feasibility problems are surveyed. These methods are valid for certain residual subsets of the spaces under consideration. Finally, in Section 4 few recent results on the existence of residual sets of some classes of convex functions are reviewed.

2. Normality and Weak Normality

The notion of a normal mapping with respect to a continuous convex function was first introduced by Gabour, Reich and Zaslavski in 2000 for bounded, closed and convex sets (see [18]). This notion has turned out to be useful in the study of minimization problems. In this section we consider further properties of normal mappings and normal sequences of mappings under various assumptions for arbitrary nonempty, closed and convex sets, recall more general notions of weakly normal sequences of mappings and survey results with applications to the minimization of convex functions.

In Subsection 2.1 we consider the case of normal mappings and normal sequences of mappings with respect to an everywhere uniformly continuous convex function.

In Subsection 2.2 we recall the notions and study the analogous properties of operator-dependent normality and operator-dependent weak normality with respect to a given operator and a given convex function, which is uniformly continuous on bounded sets.

In this section we assume that $(X, \|\cdot\|)$ is a normed space with norm $\|\cdot\|$, $K \subset X$ is a nonempty, closed and convex subset of X and $f : K \to \mathbb{R}$ is a convex function which is bounded from below. Set

$$\inf f := \inf \{f(x) : x \in K\}.$$

The results and corollaries which we overview in this section are of interest when X is a Banach space: thus the residual sets under consideration are dense by Theorem 1.2.

2.1. **Operator-independent case.**

In this subsection we assume that f is uniformly continuous on the whole of K.

2.1.1. *Background.* Denote by $\mathfrak{A}$ the set of all bounded self-mappings $A : K \to K$ such that

$$f(Ax) \le f(x) \ \text{for each}\, x \in K \tag{2.1}$$

and by $\mathfrak{A}_c$ the set of all continuous mappings $A \in \mathfrak{A}$. For the set $\mathfrak{A}$ define a metric $\rho : \mathfrak{A} \times \mathfrak{A} \to \mathbb{R}$ by

$$\rho(A,B) := \sup\{\|Ax - Bx\| : x \in K\},\, A,B \in \mathfrak{A}.$$

Clearly, the metric space $\mathfrak{A}$ is complete if $(X, \|\cdot\|)$ is a Banach space, and the metric space $\mathfrak{A}_c$ is a closed subset of $\mathfrak{A}$. Denote by $\mathfrak{M}$ the set of all sequences of elements in $\mathfrak{A}$ and by $\mathfrak{M}_c$ the set of all sequences of elements in $\mathfrak{A}_c$. For the set $\mathfrak{M}$ we consider the following two uniformities and the topologies induced by them. The first uniformity is determined by the following basis:

$$E_1(N,\varepsilon) = \{(\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}) \in \mathfrak{M} \times \mathfrak{M} : \rho(A_n, B_n) < \varepsilon,\, n = 1, \dots, N\},$$

where $N = 1,2\dots$ and $\varepsilon > 0$. This uniformity induces a uniform topology on $\mathfrak{M}$, which we denote by τ_1 and call the *weak topology*.

The second uniformity is determined by the following basis:

$$E_2(\varepsilon) = \{(\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}) \in \mathfrak{M} \times \mathfrak{M} : \rho(A_n, B_n) < \varepsilon,\, n = 1, 2, \dots\},$$

where $\varepsilon > 0$. This uniformity induces a uniform topology on $\mathfrak{M}$, which we denote by τ_2 and call the *strong topology*. It is clear that τ_2 is indeed stronger than τ_1.

Clearly, the uniform spaces $(\mathfrak{M}, \tau_1)$ and $(\mathfrak{M}, \tau_2)$ are metrizable (by metrics ρ_1 and ρ_2, respectively) and it is not difficult to see that these metrics are complete if $(X, \|\cdot\|)$ is a Banach space.

Evidently, $\mathfrak{M}_c$ is a closed subset of $\mathfrak{M}$ with respect to the weak topology (and therefore with respect to the strong topology) and hence complete with respect to both the strong and weak topologies. Denote by $\mathfrak{M}_b$ the set of all bounded sequences of elements in $\mathfrak{A}$ and by $\mathfrak{M}_{bc}$ the set of all bounded sequences of elements in $\mathfrak{A}_c$. It can easily be verified that $\mathfrak{M}_b$ and $\mathfrak{M}_{bc}$ are closed subsets of $\mathfrak{M}$ with respect to the strong topology. Evidently, the relative strong topology on $\mathfrak{M}_b$ is determined by the metric $d : \mathfrak{M}_b \times \mathfrak{M}_b \to \mathbb{R}$ defined by

$$d(\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}) := \sup\{\rho(A_n, B_n)\}_{n=1}^{\infty} \ \{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \in \mathfrak{M}_b.$$

Definition 2.1. A mapping $A : K \to K$ is called *normal* with respect to f if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \ge \inf(f) + \varepsilon$, the inequality

$$f(Ax) < f(x) - \delta(\varepsilon)$$

is true. A sequence $\{A_n\}_{n=1}^{\infty}$ of operators $A_n : K \to K$ is called *normal* with respect to f if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in K$ satisfying $f(x) \ge \inf(f) + \varepsilon$ and each integer $n = 1, 2, \dots$, the inequality

$$f(A_n x) < f(x) - \delta(\varepsilon)$$

holds.

Example 2.2. Let $X = \mathbb{R}$ and $K = [0, \infty)$. Define $A : K \to K$ by $Ax := 2^{-1}|\sin x|$ for each $x \in K$. Let $f : K \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x^2, & x \le 1, \\ 2x - 1, & x > 1 \end{cases}$$

for each $x \in K$. Clearly, $A \in \mathfrak{A}_c$, that is, $\mathfrak{A}_c \subset \mathfrak{A} \neq \emptyset$ and therefore $\mathfrak{M}_c \subset \mathfrak{M} \neq \emptyset$. Let $\varepsilon > 0$ be given and assume $x \in K$ satisfies $f(x) \geq \varepsilon$. Choose $\delta(\varepsilon) := 3 \cdot 8^{-1}\varepsilon$. Then

$$f(Ax) = 4^{-1}\sin^2 x < f(x) - \delta(\varepsilon).$$

We conclude that A is normal with respect to f.

It was shown in [18] that if K is a bounded, closed and convex set in $(X, \|\cdot\|)$, where $(X, \|\cdot\|)$ is a Banach space, then a generic element taken from the spaces $\mathfrak{A}$, $\mathfrak{A}_c$, $\mathfrak{M}$ and $\mathfrak{M}_c$ is normal with respect to f, and that the sequence of values of the function f along any trajectory of such an element tends to the infimum of f on K. These results demonstrate the importance of normal mappings for convex minimization problems. We survey analogous results for the case where the set K is a general nonempty, closed and convex set, which is not necessarily bounded. To this end, we recall the following weaker notion of normality, introduced by Barshad, Reich and Zaslavski in [2].

Definition 2.3. A sequence $\{A_n\}_{n=1}^{\infty}$ of operators $A_n : K \to K$ is called *weakly normal* with respect to f if given $\varepsilon > 0$, there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$ of positive numbers such that $\limsup_{n\to\infty} n\delta_n = \infty$, and for each positive integer n, each $x \in K$ satisfying $f(x) \geq \inf(f) + \varepsilon$ and each integer $k = 1, 2, \ldots, n$, the inequality

$$f(A_k x) < f(x) - \delta_n$$

holds.

Remark 2.4. It is not difficult to see that for each $\alpha \in (0,1)$ and each $\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \in \mathfrak{M}$, their convex combination, $\alpha \{A_n\}_{n=1}^{\infty} + (1-\alpha)\{B_n\}_{n=1}^{\infty}$, is also an element of $\mathfrak{M}$ and if one of them is normal, then the sequence $\alpha \{A_n\}_{n=1}^{\infty} + (1-\alpha)\{B_n\}_{n=1}^{\infty}$ is also normal. Evidently, each normal sequence of mappings is, in particular, weakly normal, but not *vice versa*, as is shown in the following example.

Example 2.5. Let $X = \mathbb{R}$ and $K = (-\infty, 1]$. Let $g : K \to \mathbb{R}$ be defined by

$$g(x) := \begin{cases} x & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

for each $x \in K$. For each positive integer n, define $A_n : K \to K$ by

$$A_n x := \left(1 - n^{-2^{-1}}\right)^{2^{-1}} g(x)$$

for each $x \in K$. Let $f : K \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & x > 0, \\ 0, & \text{otherwise} \end{cases}$$

for each $x \in K$. Clearly, f is convex. Let $\varepsilon > 0$ be arbitrary. For each positive integer n, set $\delta_n := n^{-2^{-1}}\varepsilon$. Then $\inf(f) = 0$ and for each $x \in K$ such that $f(x) \geq \varepsilon$ and each $k = 1, 2, \ldots, n$, we have

$$f(A_k x) = \left(1 - k^{-2^{-1}}\right) f(x) \leq f(x) - k^{-2^{-1}}\varepsilon = f(x) - \delta_k \leq f(x) - \delta_n.$$

Clearly, $\lim_{n\to\infty} n\delta_n = \infty$. Therefore the sequence $\{A_n\}_{n=1}^{\infty}$ is weakly normal with respect to f, but it is not normal with respect to f because $\lim_{n\to\infty} f(A_n x) = f(x)$ for each $x \in K$ such that $f(x) \geq \varepsilon$. As a matter of fact, we also have $\{A_n\}_{n=1}^{\infty} \in \mathfrak{M}_c$, that is, $\mathfrak{M}_c \subset \mathfrak{M} \neq \emptyset$.

In the sequel we assume that the function f is clearly understood and therefore use the notions of normality and weak normality without referring explicitly to f.

2.1.2. *Statements of the main results.* The following results have been obtained by Barshad, Reich and Zaslavski in [2]. They generalize the corresponding results in [18] and [27].

Theorem 2.6. *There exist sets $\mathscr{F} \subset \mathfrak{M}$, $\mathscr{F}_b \subset \mathscr{F} \cap \mathfrak{M}_b$, $\mathscr{F}_c \subset \mathscr{F} \cap \mathfrak{M}_c$ and $\mathscr{F}_{bc} \subset \mathscr{F}_b \cap \mathfrak{M}_c$ of weakly normal sequences of mappings which are countable intersections of open (in the relative weak topology) and dense (respectively, in the weak topology, in the relative strong topology, in the relative weak topology and in the relative strong topology) sets in, respectively, $\mathfrak{M}$, $\mathfrak{M}_b$, $\mathfrak{M}_c$ and $\mathfrak{M}_{bc}$ such that for each $\{A_n\}_{n=1}^{\infty} \in \mathscr{F}$, the following assertion holds:*

For each $\varepsilon > 0$ and each $B_0 \in \mathfrak{A}$, there exists a neighborhood U (in the weak topology) of $\{A_n\}_{n=1}^{\infty}$ and a positive integer N satisfying

$$f(B_N \dots B_1 B_0 x) < \inf(f) + \varepsilon$$

for each $\{B_n\}_{n=1}^{\infty} \in U$ and each $x \in K$.

Theorem 2.7. *There exist a set $\mathscr{F} \subset \mathfrak{A}$ of normal mappings, which is a countable intersection of open and dense sets in $\mathfrak{A}$, and a set $\mathscr{F}_c \subset \mathscr{F} \cap \mathfrak{A}_c$ of normal mappings, which is a countable intersection of open and dense sets in $\mathfrak{A}_c$, such that for each $A \in \mathscr{F}$, the following assertion holds:*

For each $\varepsilon > 0$, there exists a neighborhood U of A in $\mathfrak{A}$ such that for each $B_0 \in \mathfrak{A}$, there is a positive integer N satisfying

$$f\left(B^N B_0 x\right) < \inf(f) + \varepsilon$$

for each $B \in U$ and each $x \in K$. In particular, for each $B \in U$, there is a positive integer N such that we have

$$f\left(B^N x\right) < \inf(f) + \varepsilon$$

for each $x \in K$.

Theorem 2.8. *There exist sets $\mathscr{F}_b \subset \mathfrak{M}_b$ and $\mathscr{F}_{bc} \subset \mathscr{F}_b \cap \mathfrak{M}_c$ of normal sequences of mappings, which are countable intersections of open (in the relative strong topology) and dense (in the relative strong topology) sets in, respectively, $\mathfrak{M}_b$ and $\mathfrak{M}_{bc}$, such that for each $\{A_n\}_{n=1}^{\infty} \in \mathscr{F}$, the following assertion holds:*

For each $\varepsilon > 0$, there exists a neighborhood U (in the strong topology) of $\{A_n\}_{n=1}^{\infty}$ such that for each $B_0 \in \mathfrak{A}$ there is a positive integer N satisfying

$$f\left(B_{r(N)} \dots B_{r(1)} B_0 x\right) < \inf(f) + \varepsilon$$

for each $\{B_n\}_{n=1}^{\infty} \in U$, each mapping $r : \{1,2,\dots\} \to \{1,2,\dots\}$ and each $x \in K$. In particular, for each $\{B_n\}_{n=1}^{\infty} \in U$ and each mapping $r : \{1,2,\dots\} \to \{1,2,\dots\}$, there is a positive integer N such that

$$f\left(B_{r(N)} \dots B_{r(1)} x\right) < \inf(f) + \varepsilon$$

for each $x \in K$.

2.2. **Operator-dependent case.** The importance of concepts related to normality and weak normality for convex minimization problems was plainly demonstrated in [2, 18, 27] and Subsection 2.1 above, where the function f is uniformly continuous on the set K. In this subsection we survey analogous results under different assumptions. Denote by $\mathscr{B}$ the set of all bounded mappings $B : K \to K$. In contrast with previous studies, here the concepts of normality and weak normality are considered with respect to a given operator $B_0 \in \mathscr{B}$ and not globally, and the residual sets depend on the operator B_0. This operator-dependent approach turns out to be useful for removing the somewhat restrictive requirement on the function f to be uniformly continuous on all of K. In this subsection we assume that f is uniformly continuous on bounded subsets of K. In the case where K is an unbounded set we also assume that

$$\lim_{x\to\infty} f(x) = \infty. \tag{2.2}$$

We also review numerous applications to generic minimization problems on arbitrary open balls.

2.2.1. *Background.* Denote by $\mathscr{A}$ the set of all mappings $A \in \mathfrak{B}$ which satisfy

$$f(Ax) \le f(x) \text{ for each } x \in K \tag{2.3}$$

and by $\mathscr{A}_c$ the set of all continuous mappings $A \in \mathscr{A}$. For the set $\mathscr{A}$ we define a metric $\rho : \mathfrak{A} \times \mathfrak{A} \to \mathbb{R}$ by

$$\rho(A,B) := \sup\{\|Ax - Bx\| : x \in K\},\, A,B \in \mathscr{A}.$$

Clearly, the metric space $\mathscr{A}$ is complete if $(X, \|\cdot\|)$ is a Banach space, and the metric space $\mathscr{A}_c$ is a closed subset of $\mathscr{A}$. Denote by $\mathscr{M}$ the set of all sequences of elements in $\mathscr{A}$ and by $\mathscr{M}_c$ the set of all sequences of elements in $\mathscr{A}_c$. For the set $\mathscr{M}$ we consider the following two uniformities and the topologies induced by them. The first uniformity is determined by the following basis:

$$E_1(N,\varepsilon) = \{(\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}) \in \mathscr{M} \times \mathscr{M} : \rho(A_n,B_n) < \varepsilon, n = 1,\dots,N\},$$

where $N = 1,2\dots$ and $\varepsilon > 0$. This uniformity induces a uniform topology on $\mathscr{M}$, which we denote by τ_1 and call the *weak topology*.

The second uniformity is determined by the following basis:

$$E_2(\varepsilon) = \{(\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}) \in \mathscr{M} \times \mathscr{M} : \rho(A_n,B_n) < \varepsilon, n = 1,2,\dots\},$$

where $\varepsilon > 0$. This uniformity induces a uniform topology on $\mathscr{M}$, which we denote by τ_2 and call the *strong topology*. Clearly, τ_2 is indeed stronger than τ_1.

Evidently, the uniform spaces $(\mathscr{M}, \tau_1)$ and $(\mathscr{M}, \tau_2)$ are metrizable (by metrics ρ_1 and ρ_2, respectively) and it is not difficult to see that these metrics are complete if $(X, \|\cdot\|)$ is a Banach space.

Clearly, $\mathscr{M}_c$ is a closed subset of $\mathscr{M}$ with respect to the weak topology (and therefore with respect to the strong topology) and hence complete with respect to both the strong and weak topologies. Denote by $\mathscr{M}_b$ the set of all bounded sequences of elements in $\mathscr{A}$ and by $\mathscr{M}_{bc}$ the set of all bounded sequences of elements in $\mathscr{A}_c$. It can easily be verified that $\mathscr{M}_b$ and $\mathscr{M}_{bc}$ are closed subsets of $\mathscr{M}$ with respect to the strong topology. Evidently, the relative strong topology on $\mathscr{M}_b$ is determined by the metric $d : \mathscr{M}_b \times \mathscr{M}_b \to \mathbb{R}$ defined by

$$d(\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}) := \sup\{\rho(A_n,B_n)\}_{n=1}^{\infty}\ \{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \in \mathscr{M}_b.$$

For each $B_0 \in \mathfrak{B}$, we set

$$d_{B_0} := \sup\{|f(B_0 x)| : x \in K\} \text{ and } S_{B_0} := \{x \in K : f(x) \le d_{B_0}\}.$$

Definitely, the assumption that f is convex and uniformly continuous on bounded sets implies that d_{B_0} is finite. Note also that the set S_{B_0} is always bounded (in the case where K is unbounded this follows from (2.2)) and hence f is uniformly continuous on it.

Definition 2.9. Let $B_0 \in \mathfrak{B}$. A mapping $A : K \to K$ is called B_0-*normal* with respect to f if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in S_{B_0}$ satisfying $f(x) \ge \inf(f) + \varepsilon$, the inequality

$$f(Ax) < f(x) - \delta(\varepsilon)$$

is true. A sequence $\{A_n\}_{n=1}^{\infty}$ of operators $A_n : K \to K$ is called B_0-*normal* with respect to f if given $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for each $x \in S_{B_0}$ satisfying $f(x) \ge \inf(f) + \varepsilon$ and each integer $n = 1, 2, \ldots$, the inequality

$$f(A_n x) < f(x) - \delta(\varepsilon)$$

holds.

Example 2.10. Let $X = \mathbb{R}$ and $K = [0, \infty)$. Define $B_0 \in \mathfrak{B}$ by $B_0(x) := |\cos x|$ for each $x \in K$, and define $A : K \to K$ by $Ax := \begin{cases} 2^{-1}x, & x \le 1, \\ 2x - 3 \cdot 2^{-1}, & 1 < x \le 3 \cdot 2^{-1}, \\ x, & 3 \cdot 2^{-1} < x \le 2, \\ 2, & x > 2 \end{cases}$ for each $x \in K$. Let $f : K \to \mathbb{R}$ be defined by $f(x) := x^2$ for each $x \in K$. Evidently, f is convex and uniformly continuous on bounded sets. Clearly, A is B_0-normal with respect to f. We also have $A \in \mathscr{A}_c$, that is, $\mathscr{A}_c \subset \mathscr{A} \neq \emptyset$ and therefore $\mathscr{M}_c \subset \mathscr{M} \neq \emptyset$.

Definition 2.11. Let $B_0 \in \mathfrak{B}$. A sequence $\{A_n\}_{n=1}^{\infty}$ of operators $A_n : K \to K$ is called B_0-*weakly normal* with respect to f if given $\varepsilon > 0$, there exists a sequence $\{\delta_n\}_{n=1}^{\infty}$ of positive numbers such that $\limsup_{n \to \infty} n\delta_n = \infty$ and for each positive integer n, each $x \in S_{B_0}$ satisfying $f(x) \ge \inf(f) + \varepsilon$ and each integer $k = 1, 2, \ldots, n$, the inequality

$$f(A_k x) < f(x) - \delta_n$$

holds.

Remark 2.12. Assume $B_0 \in \mathfrak{B}$. It is not difficult to see that for each $\alpha \in (0,1)$ and each $\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty} \in \mathscr{M}$, the convex combination, $\alpha\{A_n\}_{n=1}^{\infty} + (1-\alpha)\{B_n\}_{n=1}^{\infty}$, is also an element of $\mathscr{M}$ and if one of them is B_0-normal with respect to f, then the sequence $\alpha\{A_n\}_{n=1}^{\infty} + (1-\alpha)\{B_n\}_{n=1}^{\infty}$ is also B_0-normal with respect to f. Each B_0-normal sequence of mappings with respect to f is, in particular, B_0-weakly normal with respect to f, but not *vice versa* as is shown in the following example.

Example 2.13. Let $X=\mathbb{R}$ and $K=[0,\infty)$. Let $B_0\in\mathfrak{B}$ be defined by $B_0(x):=\sin^2 x$ for each $x\in K$. For each positive integer n, define $A_n:K\to K$ by

$$A_n x := \begin{cases} \left(1-n^{-2^{-1}}\right)^{2^{-1}} x, & x\le 1,\\ \left(3-2\left(1-n^{-2^{-1}}\right)^{2^{-1}}\right)x+3\left(\left(1-n^{-2^{-1}}\right)^{2^{-1}}-1\right), & 1<x\le 3\cdot 2^{-1},\\ x, & 3\cdot 2^{-1}<x\le 2,\\ 2, & x>2 \end{cases}$$

for each $x\in K$. Let $f:K\to\mathbb{R}$ be defined by $f(x):=x^2$ for each $x\in K$. It is clear that f is convex and uniformly continuous on bounded sets. Let $\varepsilon>0$ be arbitrary. For each positive integer n, set $\delta_n:=n^{-2^{-1}}\varepsilon$. Then $\inf(f)=0$ and for each $x\in S_{B_0}$ such that $f(x)\ge\varepsilon$ and each $k=1,2,\dots,n$, we have

$$f(A_k x)=\left(1-k^{-2^{-1}}\right)f(x)\le f(x)-k^{-2^{-1}}\varepsilon=f(x)-\delta_k\le f(x)-\delta_n.$$

Clearly, $\lim_{n\to\infty} n\delta_n=\infty$. Therefore the sequence $\{A_n\}_{n=1}^\infty$ is B_0-weakly normal with respect to f, but it is not B_0-normal with respect to f because $\lim_{n\to\infty} f(A_n x)=f(x)$ for each $x\in S_{B_0}$ such that $f(x)\ge\varepsilon$. As a matter of fact, we also have $\{A_n\}_{n=1}^\infty\in\mathscr{M}_c$, that is, $\mathscr{M}_c\subset\mathscr{M}\neq\emptyset$.

In the sequel we assume that the function f is clearly understood and therefore we use the notions of operator-dependent normality and operator-dependent weak normality without referring explicitly to f.

Remark 2.14. Note that in the case where $K=X$ we may replace the uniform continuity of f on bounded sets by the boundedness of f on bounded sets.

2.2.2. *Statements of the main results.* Barshad, Reich and Zaslavski proved the following theorems presented in [3]. These theorems generalize the corresponding results in [18] and [27].

Theorem 2.15. *Let $B_0\in\mathfrak{B}$. Then there exist sets $\mathscr{F}\subset\mathscr{M}$, $\mathscr{F}_b\subset\mathscr{F}\cap\mathscr{M}_b$, $\mathscr{F}_c\subset\mathscr{F}\cap\mathscr{M}_c$ and $\mathscr{F}_{bc}\subset\mathscr{F}_b\cap\mathscr{M}_c$ of B_0-weakly normal sequences of mappings, which are countable intersections of open (in the relative weak topology) and dense (respectively, in the weak topology, in the relative strong topology, in the relative weak topology and in the relative strong topology) sets in, respectively, $\mathscr{M}$, $\mathscr{M}_b$, $\mathscr{M}_c$ and $\mathscr{M}_{bc}$ such that for each $\{A_n\}_{n=1}^\infty\in\mathscr{F}$, the following assertion holds:*

For each $\varepsilon>0$, there exist a neighborhood U (in the weak topology) of $\{A_n\}_{n=1}^\infty$ and a positive integer N satisfying

$$f(B_N\dots B_1B_0x)<\inf(f)+\varepsilon$$

for each $\{B_n\}_{n=1}^\infty\in U$ and each $x\in K$.

Theorem 2.16. *Let $B_0\in\mathfrak{B}$. Then there exist a set $\mathscr{F}\subset\mathscr{A}$ of B_0-normal mappings, which is a countable intersection of open and dense sets in $\mathscr{A}$, and a set $\mathscr{F}_c\subset\mathscr{F}\cap\mathscr{A}_c$ of B_0-normal mappings, which is a countable intersection of open and dense sets in $\mathscr{A}_c$, such that for each $A\in\mathscr{F}$, the following assertion holds:*

For each $\varepsilon>0$, there exist a neighborhood U of A in $\mathscr{A}$ and a positive integer N satisfying

$$f\left(B^N B_0 x\right)<\inf(f)+\varepsilon$$

for each $B \in U$ and each $x \in K$.

Theorem 2.17. *Let $B_0 \in \mathfrak{B}$. Then there exist sets $\mathscr{F}_b \subset \mathscr{M}_b$ and $\mathscr{F}_{bc} \subset \mathscr{F}_b \cap \mathscr{M}_c$ of B_0-normal sequences of mappings, which are countable intersections of open (in the relative strong topology) and dense (in the relative strong topology) sets in, respectively, $\mathscr{M}_b$ and $\mathscr{M}_{bc}$, such that for each $\{A_n\}_{n=1}^{\infty} \in \mathscr{F}$, the following assertion holds:*

For each $\varepsilon > 0$, there exist a neighborhood U (in the strong topology) of $\{A_n\}_{n=1}^{\infty}$ and a positive integer N satisfying

$$f\left(B_{r(N)} \dots B_{r(1)} B_0 x\right) < \inf(f) + \varepsilon$$

for each $\{B_n\}_{n=1}^{\infty} \in U$, each mapping $r : \{1,2,\dots\} \to \{1,2,\dots\}$ and each $x \in K$.

2.2.3. *Normality, weak normality and metric projections onto closed balls.* Let d be the metric induced on K by $\|\cdot\|$, $x_0 \in K$ and $r > 0$. Since K is convex, there exists a metric projection $P_{B_d(x_0,r)}$ of K onto $B_d(x_0,r)$. If X is a strictly convex space, then this metric projection is unique. By taking $B_0 = P_{B_d(x_0,r)}$ in the main results of this subsection the following corollaries are obtained.

Corollary 2.18. *There exist sets $\mathscr{F} \subset \mathscr{M}$, $\mathscr{F}_b \subset \mathscr{F} \cap \mathscr{M}_b$, $\mathscr{F}_c \subset \mathscr{F} \cap \mathscr{M}_c$ and $\mathscr{F}_{bc} \subset \mathscr{F}_b \cap \mathscr{M}_c$ of $P_{B_d(x_0,r)}$-weakly normal sequences of mappings, which are countable intersections of open (in the relative weak topology) and dense (respectively, in the weak topology, in the relative strong topology, in the relative weak topology and in the relative strong topology) sets in, respectively, $\mathscr{M}$, $\mathscr{M}_b$, $\mathscr{M}_c$ and $\mathscr{M}_{bc}$ such that for each $\{A_n\}_{n=1}^{\infty} \in \mathscr{F}$, the following assertion holds:*

For each $\varepsilon > 0$, there exist a neighborhood U (in the weak topology) of $\{A_n\}_{n=1}^{\infty}$ and a positive integer N satisfying

$$f(B_N \dots B_1 x) < \inf(f) + \varepsilon$$

for all $\{B_n\}_{n=1}^{\infty} \in U$ and $x \in B_d(x_0,r)$.

Corollary 2.19. *There exist a set $\mathscr{F} \subset \mathscr{A}$ of $P_{B_d(x_0,r)}$-normal mappings, which is a countable intersection of open and dense sets in $\mathscr{A}$, and a set $\mathscr{F}_c \subset \mathscr{F} \cap \mathscr{A}_c$ of $P_{B_d(x_0,r)}$-normal mappings, which is a countable intersection of open and dense sets in $\mathscr{A}_c$, such that for each $A \in \mathscr{F}$, the following assertion holds:*

For each $\varepsilon > 0$, there exist a neighborhood U of A in $\mathscr{A}$ and a positive integer N satisfying

$$f\left(B^N x\right) < \inf(f) + \varepsilon$$

for all $B \in U$ and $x \in B_d(x_0,r)$.

Corollary 2.20. *There exist sets $\mathscr{F}_b \subset \mathscr{M}_b$ and $\mathscr{F}_{bc} \subset \mathscr{F}_b \cap \mathscr{M}_c$ of $P_{B_d(x_0,r)}$-normal sequences of mappings, which are countable intersections of open (in the relative strong topology) and dense (in the relative strong topology) sets in, respectively, $\mathscr{M}_b$ and $\mathscr{M}_{bc}$, such that for each $\{A_n\}_{n=1}^{\infty} \in \mathscr{F}$, the following assertion holds:*

For each $\varepsilon > 0$, there exist a neighborhood U (in the strong topology) of $\{A_n\}_{n=1}^{\infty}$ and a positive integer N satisfying

$$f\left(B_{r(N)} \dots B_{r(1)} x\right) < \inf(f) + \varepsilon$$

for all $\{B_n\}_{n=1}^{\infty} \in U$, all mappings $r : \{1,2,\dots\} \to \{1,2,\dots\}$ and all points $x \in B_d(x_0,r)$.

3. Stochastic Feasibility Problems

In this section we consider (generalized) stochastic feasibility problems. These are the problems of finding almost common fixed points of measurable (with respect to a probability measure) families of mappings. We survey an implementation of the generic approach based on both residuality and porosity in order to solve them. Such an implementation based on residuality has been presented by Gabour, Reich and Zaslavski in 2001 in the case where the set K is bounded. The strong convergence results recalled here provide iterative methods (in the case where the set K is not necessarily bounded) for finding an almost common fixed point of a generic measurable family of mappings . Some of these results involve the case where a subset of the almost common fixed point set is a nonexpansive retract of K. These results are applicable to both the consistent case (that is, the case where the aforesaid almost common fixed points exist) and the inconsistent case (that is, the case where there are no common fixed points at all).

We assume that $(X, \|\cdot\|)$ is a normed vector space with norm $\|\cdot\|$, $K \subset X$ is a subset of X, and $(\Omega, \mathscr{A}, \mu)$ is a probability measure space (more information on measure spaces and measurable mappings can be found, for example, in [17]). As previously, the main results reviewed in this section are meaningful if we assume that X is a Banach space so that all residual sets, the existence of which is obtained, are dense by Theorem 1.2.

Recall that a topological vector space V with the topology T is said to be a *locally convex space* if there exists a family $\mathscr{P}$ of pseudo-norms on V such that the family of open balls $\left\{B_\rho(x_0, \varepsilon) : x_0 \in V, \varepsilon > 0, \rho \in \mathscr{P}\right\}$ is a subbasis for T and $\cap_{\rho \in \mathscr{P}} Z_\rho = \{0\}$, where $Z_\rho = \{x \in V : \rho(x) = 0\}$ for each $\rho \in \mathscr{P}$. Clearly, every normed space (as a topological vector space with respect to its norm) is a locally convex space. In the sequel we use the following result (see Theorem 3.9 in [12]).

Theorem 3.1. *Let V be a real locally convex topological vector space, and let A and B be two disjoint closed and convex subsets of V. If either A or B is compact, then A and B are strictly separated, that is, there is $\alpha \in \mathbb{R}$ and a continuous linear functional $\phi : V \to \mathbb{R}$ such that $\phi(a) > \alpha$ for each $a \in A$ and $\phi(b) < \alpha$ for each $b \in B$.*

3.1. **Residuality-based methods.**

In this subsection we recall generic methods for finding almost common fixed points by using the notion of residuality.

3.1.1. *Background.* Assume that $K \subset X$ is a nonempty, closed and convex subset of X. Denote by $\mathscr{N}$ the set of all bounded and nonexpansive mappings $A : K \to K$, that is, all bounded mappings $A : K \to K$ such that $\|Ax - Ay\| \leq \|x - y\|$ for each $x, y \in K$. For the set $\mathscr{N}$, define a metric $\rho_{\mathscr{N}} : \mathscr{N} \times \mathscr{N} \to \mathbb{R}$ by

$$\rho_{\mathscr{N}}(A, B) := \sup\{\|Ax - Bx\| : x \in K\}, A, B \in \mathscr{N}.$$

Clearly, the metric space $(\mathscr{N}, \rho_{\mathscr{N}})$ is complete if $(X, \|\cdot\|)$ is a Banach space. Denote by $\mathscr{M}$ the set of all sequences $\{A_n\}_{n=1}^\infty \subset \mathscr{N}$. For the set $\mathscr{M}$, we consider two uniformities determined by the bases

$$E_1'(N, \varepsilon) = \{(\{A_n\}_{n=1}^\infty, \{B_n\}_{n=1}^\infty) \in \mathscr{M} \times \mathscr{M} : \rho_{\mathscr{N}}(\{A_n\}_{n=1}^\infty, \{B_n\}_{n=1}^\infty) < \varepsilon, n = 1, 2, \ldots N\}$$

and

$$E_2'(\varepsilon) = \{(\{A_n\}_{n=1}^\infty, \{B_n\}_{n=1}^\infty) \in \mathscr{M} \times \mathscr{M} : \rho_{\mathscr{N}}(\{A_n\}_{n=1}^\infty, \{B_n\}_{n=1}^\infty) < \varepsilon, n = 1, 2, \ldots\},$$

where $\varepsilon > 0$ and $N = 1, 2, \ldots$. These two uniformities induce two uniform topologies, respectively, τ_1' and τ_2' on $\mathscr{M}$. These topologies on $\mathscr{M}$ will be called the weak and strong topologies, respectively. Clearly, τ_1' is indeed weaker than τ_2' and the uniform space $\mathscr{M}$ with each of these topologies is metrizable (by metrics $\rho_{\tau_1'}$ and $\rho_{\tau_2'}$, respectively) and it is not difficult to show that these metrics are complete if $(X, \|\cdot\|)$ is a Banach space.

Denote by $\mathscr{N}_\Omega$ the set of all bounded mappings $T : \Omega \to \mathscr{N}$ such that for each $x \in K$, the mapping $T_x' : \Omega \to K$, defined, for each $\omega \in \Omega$, by $T_x'(\omega) := T(\omega)(x)$ for each $x \in K$, is measurable. It is not difficult to see that if $T \in \mathscr{N}_\Omega$, then T_x' is integrable on Ω. For each $T \in \mathscr{N}_\Omega$, define an operator $\widetilde{T} : K \to K$ by $\widetilde{T}x = \int_\Omega T_x'(\omega)\, d\mu(\omega)$. By Theorem 3.1, this is indeed a self-mapping of K. Note that the mapping defined on $\mathscr{N}_\Omega$ by $T \mapsto \widetilde{T}$ is onto $\mathscr{N}$. Clearly, for each $T \in \mathscr{N}_\Omega$, we have $\widetilde{T} \in \mathscr{N}$. Thus we consider the topology defined by the following pseudo-metric on $\mathscr{N}_\Omega$:

$$\rho_{\mathscr{N}_\Omega}(T, S) := \rho_{\mathscr{N}}\left(\widetilde{T}, \widetilde{S}\right), \; T, S \in \mathscr{N}_\Omega.$$

It is not difficult to see that the pseudo-metric space $\left(\mathscr{N}_\Omega, \rho_{\mathscr{N}_\Omega}\right)$ is complete if $(X, \|\cdot\|)$ is a Banach space. The topology defined by the pseudo-metric $\rho_{\mathscr{N}_\Omega}$ on $\mathscr{N}_\Omega$ will be called the weak topology. For the set $\mathscr{N}_\Omega$ we define a metric $d_{\mathscr{N}_\Omega} : \mathscr{N} \times \mathscr{N} \to \mathbb{R}$ by

$$d_{\mathscr{N}_\Omega}(T, S) := \sup\left\{\rho_{\mathscr{N}}(T(\omega), S(\omega)) : \omega \in \Omega\right\}, \; T, S \in \mathscr{N}_\Omega.$$

The topology defined by the metric $d_{\mathscr{N}_\Omega}$ on $\mathscr{N}_\Omega$ will be called the strong topology. It is not difficult to see that this topology is indeed stronger than the topology defined on $\mathscr{N}_\Omega$ by the pseudo-metric $\rho_{\mathscr{N}_\Omega}$. Clearly, the metric space $\left(\mathscr{N}_\Omega, d_{\mathscr{N}_\Omega}\right)$ is complete if $(X, \|\cdot\|)$ is a Banach space.

Denote by $\mathscr{M}_\Omega$ the set of all sequences $\{T_n\}_{n=1}^\infty \subset \mathscr{N}_\Omega$. For the set $\mathscr{M}_\Omega$, we consider two uniformities determined by the bases

$$E_1(N, \varepsilon) = \left\{(\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty) \in \mathscr{M}_\Omega \times \mathscr{M}_\Omega : \rho_{\mathscr{N}_\Omega}(T_n, S_n) < \varepsilon, n = 1, 2, \ldots N\right\}$$

and

$$E_2(\varepsilon) = \left\{(\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty) \in \mathscr{M}_\Omega \times \mathscr{M}_\Omega : \rho_{\mathscr{N}_\Omega}(T_n, S_n) < \varepsilon, n = 1, 2, \ldots\right\},$$

where $\varepsilon > 0$ and $N = 1, 2, \ldots$. These two uniformities induce two uniform topologies, τ_1 and τ_2, respectively, on $\mathscr{M}_\Omega$. These topologies on $\mathscr{M}_\Omega$ will be called the weak and strong topologies, respectively. Clearly, τ_1 is indeed weaker than τ_2 and the uniform space $\mathscr{M}_\Omega$ with each of these topologies is pseudo-metrizable (by pseudo-metrics ρ_{τ_1} and ρ_{τ_2}, respectively) and and it is not difficult to see that these pseudo-metrics are complete if $(X, \|\cdot\|)$ is a Banach space.

We denote by $\mathscr{M}_\Omega^b$ the set of all sequences of mappings $\{T_n\}_{n=1}^\infty \subset \mathscr{N}_\Omega$ which are bounded in $\left(\mathscr{N}_\Omega, d_{\mathscr{N}_\Omega}\right)$ and by $\mathscr{M}_\Omega^B$ the set of all sequences of mappings $\{T_n\}_{n=1}^\infty \subset \mathscr{N}_\Omega$ which are bounded in $\left(\mathscr{N}_\Omega, \rho_{\mathscr{N}_\Omega}\right)$. It is not difficult to see that $\mathscr{M}_\Omega^b \subset \mathscr{M}_\Omega^B$. For the set $\mathscr{M}_\Omega^b$, define a metric $d_{\mathscr{M}_\Omega^b} : \mathscr{M}_\Omega^b \times \mathscr{M}_\Omega^b \to \mathbb{R}$ by

$$d_{\mathscr{M}_\Omega^b}(\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty) := \sup\left\{d_{\mathscr{N}_\Omega}(T_n, S_n) : n = 1, 2 \ldots\right\}, \{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty \in \mathscr{N}_\Omega.$$

The topology defined by the metric $d_{\mathscr{M}_\Omega^b}$ on $\mathscr{M}_\Omega^b$ will be denoted by τ_3. It is not difficult to see that this topology is stronger than the relative strong topology (and therefore stronger than the relative weak topology) on $\mathscr{M}_\Omega^b$. Clearly, the metric space $\left(\mathscr{M}_\Omega^b, d_{\mathscr{M}_\Omega^b}\right)$ is complete if $(X, \|\cdot\|)$ is a Banach space, and $\mathscr{M}_\Omega^B$ is a closed subset of $\mathscr{M}_\Omega$ with respect to the strong topology.

The following two propositions follow easily from the definitions.

Proposition 3.2. *The mapping $T \to \widetilde{T}$ from $\mathcal{N}_\Omega$ with the weak topology onto $(\mathcal{N}, \rho_{\mathcal{N}})$ is continuous.*

Proposition 3.3. *The mapping $\{T_n\}_{n=1}^\infty \to \left\{\widetilde{T}_n\right\}_{n=1}^\infty$ from $\mathcal{M}_\Omega$ with the strong (respectively, weak) topology onto $\mathcal{M}$ with the strong (respectively, weak) topology is continuous.*

We denote by $\mathcal{N}_\Omega^{reg}$ the set of all operators $T \in \mathcal{N}_\Omega$ for which there exists $x \in K$ such that $\widetilde{T}x = x$, and by $\mathcal{M}_\Omega^{reg}$ the set of all sequences $\{T_n\}_{n=1}^\infty \in \mathcal{M}_\Omega$ for which there exists $x \in K$ such that $\widetilde{T}_n x = x$ for all $n = 1, 2, \ldots$. We denote by $\overline{\mathcal{N}_\Omega^{reg}}$ the closure of $\mathcal{N}_\Omega^{reg}$ in $\mathcal{N}_\Omega$ with respect to the strong topology, by $\overline{\mathcal{M}_\Omega^{reg} \cap \mathcal{M}_\Omega^B}$ the closure of $\mathcal{M}_\Omega^{reg} \cap \mathcal{M}_\Omega^B$ with respect to the strong topology and by $\overline{\mathcal{M}_\Omega^{reg} \cap \mathcal{M}_\Omega^b}$ the closure of $\mathcal{M}_\Omega^{reg} \cap \mathcal{M}_\Omega^b$ with respect to the τ_3 topology.

Finally, we consider $F \subset K$ which is a nonempty, bounded, closed and convex subset of K. Denote by $\mathcal{N}_\Omega^{(F)}$ the set of all operators $T \in \mathcal{N}_\Omega$ such that for almost all $\omega \in \Omega$, we have $T(\omega)x = x$ for each $x \in F$. It is not difficult to see that $\mathcal{N}_\Omega^{(F)}$ is a closed subset of $\mathcal{N}_\Omega$ with respect to the strong topology. Denote by $\mathcal{M}_\Omega^{(F)}$ the set of all sequences $\{T_n\}_{n=1}^\infty \in \mathcal{N}_\Omega^{(F)}$ and by $\overline{\mathcal{M}_\Omega^{(F)}}$ the closure of $\mathcal{M}_\Omega^{(F)}$ in $\mathcal{M}_\Omega$ with the weak topology. It is not difficult to see that $\mathcal{M}_\Omega^{(F)} \cap \mathcal{M}_\Omega^b$ is a closed subset of $\mathcal{M}_\Omega^b$ with respect to the τ_3 topology. Denote by $\overline{\mathcal{M}_\Omega^{(F)} \cap \mathcal{M}_\Omega^B}$ the closure of $\mathcal{M}_\Omega^{(F)} \cap \mathcal{M}_\Omega^B$ with respect to the strong topology.

Recall that a mapping $P : K \to F$ is a *nonexpansive retraction* of K onto F if $P \in \mathcal{N}$ and $Px = x$ for all $x \in F$, and F is a *nonexpansive retract* of K if there exists a nonexpansive retraction P of K onto F. More information on nonexpansive retractions and nonexpansive retracts can be found, for example, in [20] and [22], and in references therein.

3.1.2. *Statements of the main results.* The theorems below have been presented by Barshad, Reich and Zaslavski in [1]. Some of these theorems include, *inter alia*, the extensions of all the results which were obtained in [19] to an unbounded set K.

Recall that for each $S \in \mathcal{N}_\Omega$, a point $x \in K$ is an *almost common fixed point* of the family $\{S(\omega)\}_{\omega\in\Omega}$ if $S(\omega)x = x$ for almost all $\omega \in \Omega$. Similarly, for each $\{S_n\}_{n=1}^\infty \in \mathcal{M}_\Omega$, a point $x \in K$ is an *almost common fixed point* of the family $\{S_n(\omega)\}_{\omega\in\Omega, n=1,2\ldots}$ if $S_n(\omega)x = x$ for all $n = 1, 2, \ldots$ and almost all $\omega \in \Omega$.

Theorem 3.4. *There exist sets $\mathcal{F} \subset \mathcal{N}_\Omega$ and $\mathcal{F}' \subset \mathcal{F} \cap \overline{\mathcal{N}_\Omega^{reg}}$, which are countable intersections of open (in the relative weak topology) and dense (respectively, in the strong topology and in the relative strong topology) subsets of, respectively, $\mathcal{N}_\Omega$ and $\overline{\mathcal{N}_\Omega^{reg}}$ so that for each $S \in \mathcal{F}$, the following assertion holds:*

There exists $x_S \in K$, which is the unique fixed point of the operator $\widetilde{S}$, such that for each $x \in K$, the sequence $\left\{\widetilde{S}^n x\right\}_{n=1}^\infty$ converges to x_S, uniformly on K, and the set of almost common fixed points of the family $\{S(\omega)\}_{\omega\in\Omega}$ is contained in $\{x_S\}$. Moreover, for each $\varepsilon > 0$, there exist a positive integer N and a neighborhood U of S in $\mathcal{N}_\Omega$ with the weak topology such that for each integer $n \geq N$ and each $R \in U$, we have

$$\left\|\widetilde{R}^n x - x_S\right\| < \varepsilon$$

for each $x \in K$.

Theorem 3.5. *There exist sets $\mathscr{F} \subset \overline{\mathscr{M}_\Omega^{reg} \cap \mathscr{M}_\Omega^B}$ and $\mathscr{F}' \subset \mathscr{F} \cap \mathscr{M}_\Omega^b$, which are countable intersections of open (in the relative strong topology) and dense (respectively, in the relative strong topology and in the relative τ_3 topology) subsets of, respectively, $\overline{\mathscr{M}_\Omega^{reg} \cap \mathscr{M}_\Omega^B}$ and $\overline{\mathscr{M}_\Omega^{reg} \cap \mathscr{M}_\Omega^b}$, so that for each $\{S_n\}_{n=1}^\infty \in \mathscr{F}$, there exists $x_S \in K$, which is the unique common fixed point of the operators $\widetilde{S_n}$, $n = 1,2,\ldots$, such that for each $x \in K$, the sequence $\left\{\widetilde{S_n}^k x\right\}_{k=1}^\infty$ converges to x_S, uniformly on K, for each $n = 1,2,\ldots$, and the set of almost common fixed points of the family $\{S_n(\omega)\}_{\omega\in\Omega, n=1,2\ldots}$ is contained in $\{x_S\}$. Moreover, the following assertion holds:*

For each $\varepsilon > 0$, there exist a positive integer N and a neighborhood U of $\{S_n\}_{n=1}^\infty$ in $\mathscr{M}_\Omega$ with the strong topology such that for each $\{R_n\}_{n=1}^\infty \in U$, each integer $n \geq N$ and each mapping $r : \{1,\ldots,n\} \to \{1,2,\ldots\}$, we have

$$\left\|\widetilde{R_{r(n)}}\ldots\widetilde{R_{r(1)}}x - x_S\right\| < \varepsilon$$

for each $x \in K$.

Theorem 3.6. *There exist sets $\mathscr{F} \subset \mathscr{M}_\Omega$, $\mathscr{F}' \subset \mathscr{F} \cap \mathscr{M}_\Omega^B$ and $\mathscr{F}'' \subset \mathscr{F}' \cap \mathscr{M}_\Omega^b$, which are countable intersections of open (in the relative weak topology) and dense (respectively, in the relative weak topology, in the relative strong topology and in the τ_3 topology) subsets of, respectively, $\mathscr{M}_\Omega$, $\mathscr{M}_\Omega^B$, and $\mathscr{M}_\Omega^b$, so that for each $\{S_n\}_{n=1}^\infty \in \mathscr{F}$, the following assertion holds:*

For each $\varepsilon > 0$, there exist a positive integer N and a neighborhood U of $\{S_n\}_{n=1}^\infty$ in $\mathscr{M}_\Omega$ with the weak topology such that for each $\{R_n\}_{n=1}^\infty \in U$ and each integer $n \geq N$, we have

$$\left\|\widetilde{R_n}\ldots\widetilde{R_1}x - \widetilde{R_n}\ldots\widetilde{R_1}y\right\| < \varepsilon$$

for all $x,y \in K$.

Theorem 3.7. *There exist sets $\mathscr{F} \subset \mathscr{M}_\Omega^B$ and $\mathscr{F}' \subset \mathscr{F} \cap \mathscr{M}_\Omega^b$, which are countable intersections of open (in the relative strong topology) and dense (respectively, in the relative strong topology and in the τ_3 topology) subsets of, respectively, $\mathscr{M}_\Omega^B$ and $\mathscr{M}_\Omega^b$, so that for each $\{S_n\}_{n=1}^\infty \in \mathscr{F}$, the following assertion holds:*

For each $\varepsilon > 0$, there exist a positive integer N and a neighborhood U of $\{S_n\}_{n=1}^\infty$ in $\mathscr{M}_\Omega$ with the strong topology such that for each $\{R_n\}_{n=1}^\infty \in U$, each integer $n \geq N$ and each mapping $r : \{1,\ldots,n\} \to \{1,2,\ldots\}$, we have

$$\left\|\widetilde{R_{r(n)}}\ldots\widetilde{R_{r(1)}}x - \widetilde{R_{r(n)}}\ldots\widetilde{R_{r(1)}}y\right\| < \varepsilon$$

for each $x,y \in K$.

Theorem 3.8. *Assume F is a nonexpansive retract of K. Then there exists a set $\mathscr{F} \subset \mathscr{N}_\Omega^{(F)}$, which is countable intersections of open (in the weak topology) and dense (in the strong topology) subsets of $\mathscr{N}_\Omega^{(F)}$, so that for each $S \in \mathscr{F}$, the set of almost common fixed points of the family $\{S(\omega)\}_{\omega\in\Omega}$ coincides with F and there exists a nonexpansive retraction Q of K onto F such that the following assertions hold:*

(1) The sequence of operators $\left\{\widetilde{S}^n\right\}_{n=1}^\infty$ converges to Q, uniformly on K.

(2) *For each* $\varepsilon > 0$*, there exist a positive integer* N *and a neighborhood* U *of* S *in* $\mathscr{N}_{\Omega}^{(F)}$ *with the weak topology such that for each* $R \in U$ *and each integer* $n \geq N$*, we have*

$$\left\| \widetilde{R}^{n} x - Qx \right\| < \varepsilon$$

for each $x \in K$*.*

Theorem 3.9. *Assume* F *is a nonexpansive retract of* K*. Then there exist sets* $\mathscr{F} \subset \overline{\mathscr{M}_{\Omega}^{(F)}}$*,* $\mathscr{F}' \subset \mathscr{F} \cap \mathscr{M}_{\Omega}^{B}$ *and* $\mathscr{F}'' \subset \mathscr{F}' \cap \mathscr{M}_{\Omega}^{b}$*, which are countable intersections of open (in the relative weak topology) and dense (respectively, in the relative weak topology, in the relative strong topology and in the relative* τ_3 *topology) subsets of, respectively,* $\overline{\mathscr{M}_{\Omega}^{(F)}}$*,* $\overline{\mathscr{M}_{\Omega}^{(F)} \cap \mathscr{M}_{\Omega}^{B}}$ *and* $\mathscr{M}_{\Omega}^{(F)} \cap \mathscr{M}_{\Omega}^{b}$*, so that for each* $\{S_n\}_{n=1}^{\infty} \in \mathscr{F}$*, the set of almost common fixed points of the family* $\{S_n(\omega)\}_{\omega \in \Omega, n=1,2\ldots}$ *is contained in* F *(if* $\{S_n\}_{n=1}^{\infty} \in \mathscr{F}''$*, then this set coincides with* F*) and there exists a nonexpansive retraction* Q *of* K *onto* F *such that the following assertions hold:*

(1) *The sequence of operators* $\left\{ \widetilde{S}_n \ldots \widetilde{S}_1 \right\}_{n=1}^{\infty}$ *converges to* Q*, uniformly on* K*.*

(2) *For each* $\varepsilon > 0$*, there exist a positive integer* N *and a neighborhood* U *of* $\{S_n\}_{n=1}^{\infty}$ *in* $\overline{\mathscr{M}_{\Omega}^{(F)}}$ *with the relative weak topology such that for each* $\{R_n\}_{n=1}^{\infty} \in U$ *and each integer* $n \geq N$*, we have*

$$\left\| \widetilde{R_n} \ldots \widetilde{R_1} x - Qx \right\| < \varepsilon$$

for each $x \in K$*.*

Theorem 3.10. *Assume* F *is a nonexpansive retract of* K*. Then there exist sets* $\mathscr{F} \subset \overline{\mathscr{M}_{\Omega}^{(F)} \cap \mathscr{M}_{\Omega}^{B}}$ *and* $\mathscr{F}' \subset \mathscr{F} \cap \mathscr{M}_{\Omega}^{b}$*, which are countable intersections of open (in the relative strong topology) and dense (respectively, in the relative strong topology and in the relative* τ_3 *topology) subsets of, respectively,* $\overline{\mathscr{M}_{\Omega}^{(F)} \cap \mathscr{M}_{\Omega}^{B}}$ *and* $\mathscr{M}_{\Omega}^{(F)} \cap \mathscr{M}_{\Omega}^{b}$*, so that for each* $\{S_n\}_{n=1}^{\infty} \in \mathscr{F}$*, the set of almost common fixed points of the family* $\{S_n(\omega)\}_{\omega \in \Omega, n=1,2\ldots}$ *is contained in* F *(if* $\{S_n\}_{n=1}^{\infty} \in \mathscr{F}'$*, then this set coincides with* F*) and the following assertions holds:*

For each each mapping $r : \{1,2,\ldots\} \to \{1,2,\ldots\}$*, there exists a nonexpansive retraction* Q_r *of* K *onto* F *such that:*

(1) *The sequence of operators* $\left\{ \widetilde{S_{r(n)}} \ldots \widetilde{S_{r(1)}} \right\}_{n=1}^{\infty}$ *converges to* Q_r*, uniformly on* K*.*

(2) *For each* $\varepsilon > 0$*, there exist a positive integer* N *and a neighborhood* U *of* $\{S_n\}_{n=1}^{\infty}$ *in* $\overline{\mathscr{M}_{\Omega}^{(F)}}$ *with the relative strong topology such that for each* $\{R_n\}_{n=1}^{\infty} \in U$ *and each integer* $n \geq N$*, we have*

$$\left\| \widetilde{R_{r(n)}} \ldots \widetilde{R_{r(1)}} x - Q_r x \right\| < \varepsilon$$

for each $x \in K$*.*

3.2. **Porosity-based methods.** The notion of porosity is well known in Optimization and Nonlinear Analysis. Its importance is brought out by the fact that the complement of a σ-porous subset of a complete pseudo-metric space is a residual set, while the existence of the latter is essential in many problems which apply the generic approach. Thus, under certain circumstances, some refinements of known results can be achieved by looking for porous sets. In this subsection we survey recent generic methods in which, in contrast with Subsection 3.1, we consider

σ-porous sets instead of meager ones. Namely, we review generic methods for finding almost common fixed points by using the notion of porosity.

3.2.1. *Background.* Recall that a subset E of a complete pseudo-metric space (Y,ρ) is called a *porous* subset of Y if there exist $\alpha \in (0,1)$ and $r_0 > 0$ such that for each $r \in (0,r_0]$ and each $y \in Y$, there exists a point $z \in Y$ for which

$$B_\rho(z,\alpha r) \subset B_\rho(y,r) \setminus E.$$

A subset of Y is called a *σ-porous* subset of Y if it is a countable union of porous subsets of Y. Note that since a porous set is nowhere dense, any σ-porous set is of the first category and hence its complement is residual in (Y,ρ), that is, it contains a countable intersection of open and dense subsets of (Y,ρ). For this reason, there is a considerable interest in σ-porous sets while searching for generic solutions to optimization problems. More information concerning the notion of porosity and its applications can be found, for example, in [15, 25, 29, 30].

Suppose that $F \subset K$ is a nonempty, closed, convex and bounded subset of X. Denote by $\mathfrak{N}$ the set of all nonexpansive mappings $A : K \to F$, that is, all mappings $A : K \to F$ such that $\|Ax - Ay\| \le \|x - y\|$ for each $x, y \in K$. For the set $\mathfrak{N}$, define a metric $\rho_{\mathfrak{N}} : \mathfrak{N} \times \mathfrak{N} \to \mathbb{R}$ by

$$\rho_{\mathfrak{N}}(A,B) := \sup\{\|Ax - Bx\| : x \in K\}, A, B \in \mathfrak{N}.$$

Clearly, the metric space $(\mathfrak{N}, \rho_{\mathfrak{N}})$ is complete if $(X, \|\cdot\|)$ is a Banach space.

Denote by $\mathfrak{N}_\Omega$ the set of all mappings $T : \Omega \to \mathfrak{N}$ such that for each $x \in K$, the mapping $T'_x : \Omega \to F$, defined, for each $\omega \in \Omega$, by $T'_x(\omega) := T(\omega)(x)$, is measurable. It is not difficult to see that if $T \in \mathfrak{N}_\Omega$, then T'_x is integrable on Ω. For each $T \in \mathfrak{N}_\Omega$, define an operator $\widetilde{T} : K \to F$ by $\widetilde{T}x = \int_\Omega T'_x(\omega)\, d\mu(\omega)$ for each $x \in K$. By Theorem 3.1, this is indeed a mapping the image of which is contained in F. Note that the mapping defined on $\mathfrak{N}_\Omega$ by $T \mapsto \widetilde{T}$ is onto $\mathfrak{N}$. Clearly, for each $T \in \mathfrak{N}_\Omega$, we have $\widetilde{T} \in \mathfrak{N}$. Thus we consider the topology defined by the following pseudo-metric on $\mathfrak{N}_\Omega$:

$$\rho_{\mathfrak{N}_\Omega}(T,S) := \rho_{\mathfrak{N}}\left(\widetilde{T}, \widetilde{S}\right), T, S \in \mathfrak{N}_\Omega.$$

It is not difficult to see that the pseudo-metric space $\left(\mathfrak{N}_\Omega, \rho_{\mathfrak{N}_\Omega}\right)$ is complete if $(X, \|\cdot\|)$ is a Banach space.

Denote by $\mathfrak{M}_\Omega$ the set of all sequences $\{T_n\}_{n=1}^\infty \subset \mathfrak{N}_\Omega$. We define a pseudo-metric $\rho_{\mathfrak{M}_\Omega} : \mathfrak{M}_\Omega \times \mathfrak{M}_\Omega \to \mathbb{R}$ on $\mathfrak{M}_\Omega$ in the following way:

$$\rho_{\mathfrak{M}_\Omega}\left(\{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty\right) := \sup\left\{\rho_{\mathfrak{N}_\Omega}(T_n, S_n) : n = 1, 2 \dots\right\}, \{T_n\}_{n=1}^\infty, \{S_n\}_{n=1}^\infty \in \mathfrak{M}_\Omega.$$

Obviously, this space is complete if $(X, \|\cdot\|)$ is a Banach space.

3.2.2. *Statements of the main results.* We recall the following results which have been published by Barshad, Reich and Zaslavski in [4].

Recall that for each $T \in \mathfrak{N}_\Omega$, a point $x \in K$ is an *almost common fixed point* of the family $\{T(\omega)\}_{\omega \in \Omega}$ if $T(\omega)x = x$ for almost all $\omega \in \Omega$. Similarly, for each sequence $\{T_n\}_{n=1}^\infty \in \mathfrak{M}_\Omega$, a point $x \in K$ is an *almost common fixed point* of the family $\{T_n(\omega)\}_{\omega \in \Omega, n=1,2\dots}$ if $T_n(\omega)x = x$ for all $n = 1, 2, \dots$ and almost all $\omega \in \Omega$.

Theorem 3.11. *There exists a set $\mathscr{F} \subset \mathfrak{M}_\Omega$ such that $\mathfrak{M}_\Omega \setminus \mathscr{F}$ is a σ-porous subset of $\mathfrak{M}_\Omega$ and for each $\{T_n\}_{n=1}^\infty \in \mathscr{F}$, the following assertion holds true:*

For each $\varepsilon > 0$, there is a positive integer N such that for each integer $n \geq N$ and each mapping $s : \{1,2,\dots\} \to \{1,2,\dots\}$, we have

$$\left\| \widetilde{T_{s(n)}} \dots \widetilde{T_{s(1)}} x - \widetilde{T_{s(n)}} \dots \widetilde{T_{s(1)}} y \right\| < \varepsilon$$

for each $x, y \in K$. Consequently, if there is an almost common fixed point of the family $\{T_n(\omega)\}_{\omega\in\Omega, n=1,2\dots}$, then it is unique and for each $x \in K$, the sequence $\left\{\widetilde{T_{s(n)}} \dots \widetilde{T_{s(1)}} x\right\}_{n=1}^{\infty}$ converges to it as $n \to \infty$, uniformly on K, for each mapping $s : \{1,2,\dots\} \to \{1,2,\dots\}$.

Theorem 3.12. *There exists a set $\mathscr{F} \subset \mathfrak{N}_\Omega$ such that the set $\mathscr{G} := \mathfrak{N}_\Omega \backslash \mathscr{F}$ a σ-porous subset of $\mathfrak{N}_\Omega$, and for each $T \in \mathscr{F}$, the following assertion holds true:*

There exists $x_T \in K$ which is the unique fixed point of the operator $\widetilde{T}$ such that for each $x \in K$, the sequence $\left\{\widetilde{T}^n x\right\}_{n=1}^{\infty}$ converges to x_T as $n \to \infty$, uniformly on K. Moreover, the set $\mathfrak{F}$ of all almost common fixed points of the family $\{T(\omega)\}_{\omega\in\Omega}$ is contained in $\{x_T\}$. As a result, if $\mathfrak{F} \neq \emptyset$, then x_T is the unique almost common fixed point of the family $\{T(\omega)\}_{\omega\in\Omega}$.

4. Residuality in Spaces of Convex Functions

In this section we survey residuality properties of locally uniformly convex and strictly convex functions. The class of totally convex functions which lies between these two classes of convex functions also possesses the same properties. See, for example, [8], where the importance of such residuality results for strictly and totally convex functions is demonstrated for several types of optimization problems.

4.1. **Spaces under consideration.** Let K be a nonempty convex subset of a normed linear space $(X, \|\cdot\|)$. We denote by $B(r)$ the open ball in $(X, \|\cdot\|)$ of center zero and radius $r > 0$. Let $\mathscr{C}$ be the set of all convex functions $f : K \to \mathbb{R}$. Denote by $\mathscr{C}_l$ the subset of all lower semicontinuous functions $f \in \mathscr{C}$, by $\mathscr{C}_c$ the subset of all continuous functions $f \in \mathscr{C}$ and by $\mathscr{C}_b$ the subset of all functions $f \in \mathscr{C}$ which are bounded on bounded subsets of K. We equip the set $\mathscr{C}$ with the topology induced by the uniformity determined by the following basis:

$$E(n) = \left\{(f,g) \in \mathscr{C} \times \mathscr{C} : |f(x) - g(x)| < n^{-1} \ \forall x \in K \cap B(n)\right\}, \tag{4.1}$$

where n is a positive integer. This topology will be denoted by τ. Clearly, this uniform space is metrizable (by metric d) and it is not difficult to see that d is complete. Definitely, $\mathscr{C}_l$, $\mathscr{C}_c$ and $\mathscr{C}_b$ are closed subsets of $\mathscr{C}$ with respect to this uniform topology. We provide these subspaces with the relative topologies inherited from $\mathscr{C}$. This topology will be called the strong topology on $\mathscr{C}_l$ and will be denoted by τ_1.

We next describe the second topology with which we equip $\mathscr{C}_l$. Recall that the epigraph of a function $f : K \to \mathbb{R}$ is the set

$$\operatorname{epi}(f) = \{(x,t) \in K \times \mathbb{R} : t \geq f(x)\}.$$

Let $\|\cdot\|_\infty$ be the norm defined on $X \times \mathbb{R}$ by $\|(x,t)\|_\infty := \max\{\|x\|, |t|\}$ for each point $(x,t) \in X \times \mathbb{R}$. Denote by $B_{\|\cdot\|_\infty}(r)$ the open ball in $(X \times \mathbb{R}, \|\cdot\|_\infty)$ of center zero and radius $r > 0$. Define the distance from $\tilde{x} = (x,t)$ to A by

$$\rho(\tilde{x}, A) := \inf_{(a,s)\in A} \{\|(x,t) - (a,s)\|_\infty\} \tag{4.2}$$

for each $\tilde{x} = (x,t) \in X \times \mathbb{R}$ and each nonempty set $A \subset X \times \mathbb{R}$. We define the Attouch-Wets metric d_{AW} on $\mathscr{C}_l$ by using distances to the epigraphs of f and g in $X \times \mathbb{R}$ as follows:

$$d_{AW}(f,g) := \sum_{n=1}^{\infty} 2^{-n} \min\left\{1, \sup_{\tilde{x}\in B_{\|\cdot\|_\infty}(n)} |\rho(\tilde{x}, \operatorname{epi}(f)) - \rho(\tilde{x}, \operatorname{epi}(g))|\right\}. \tag{4.3}$$

Since $\operatorname{epi}(f)$ is closed in $K \times \mathbb{R}$ for each $f \in \mathscr{C}_l$, we see that d_{AW} is indeed a metric on $\mathscr{C}_l$. We denote by τ_2 the topology induced by the metric d_{AW} on $\mathscr{C}_l$ and by $B_{AW}(g,r)$ the open ball in $(\mathscr{C}_l, d_{AW})$ of center $g \in \mathscr{C}_l$ and radius $r > 0$. This topology will be called the weak topology on $\mathscr{C}_l$. It was shown in [5] that τ_2 is indeed weaker than τ_1.

The τ_2 topology can be defined differently. Consider the topology induced by the uniformity determined by the basis

$$\begin{aligned} F(n) = \big\{(f,g) \in \mathscr{C}_l \times \mathscr{C}_l : |\rho(\tilde{x}, \operatorname{epi}(f)) - \rho(\tilde{x}, \operatorname{epi}(g))| < n^{-1} \text{ for each} \\ \tilde{x} \in B_{\|\cdot\|_\infty}(n)\big\}, \end{aligned} \tag{4.4}$$

where n is a positive integer. It turns out that this topology is the same as the τ_2 topology, as was proved in [5]. We refer the reader to [6] for more information concerning Attouch-Wets topologies and metrics.

We recall the following well-known result due to Alexandrov and Hausdorff (see Theorem 1.1 in [24] and references therein). It is useful in the quest for residual sets, as we show below.

Theorem 4.1. *A metrizable space X is completely metrizable if and only if it is a G_δ subset of a complete metric space.*

Since $\mathscr{C}_l$ is a closed subset of $\mathscr{C}$ with respect to the τ topology, it is a countable intersection of open subsets of $\mathscr{C}$ with respect to this topology, that is, it is a G_δ subset of $\mathscr{C}$ with respect to this topology, while $\mathscr{C}$ with this topology is a completely metrizable space. Since $\mathscr{C}_l$ with the τ_2 topology is a metrizable space, it follows from Theorem 4.1 that $\mathscr{C}_l$ with the τ_2 topology is completely metrizable. The same argument shows that $\mathscr{C}_c$ and $\mathscr{C}_b$ with their relative τ_2 topologies are completely metrizable. Nevertheless, $\mathscr{C}_l$ is not complete with respect to the metric d_{AW}, since the sequence of constant functions $\{f_n\}_{n=1}^{\infty}$, defined by $f_n(x) = -n$ for each $x \in K$ and each $n = 1,2,\dots$, is a Cauchy sequence which does not converge in $(\mathscr{C}_l, d_{AW})$.

The definitions of the sets $\mathscr{C}_c, \mathscr{C}_l$ and $\mathscr{C}$ imply that $\mathscr{C}_b \subset \mathscr{C}_c \subset \mathscr{C}_l \subset \mathscr{C}$ in the case where $K = X$, since the boundeness on bounded sets implies uniform continuity on bounded subsets of X in this case. Therefore $d_{AW}|_{\mathscr{C}_c}$ and $d_{AW}|_{\mathscr{C}_b}$ are metrics, respectively, on $\mathscr{C}_c$ and $\mathscr{C}_b$ in this case. But it is easy to see that if $K \neq X$, then $\mathscr{C}_b$ is not necessarily a subset of $\mathscr{C}_c$. Take, for example, $X = \mathbb{R}$ with the Euclidean norm, $K = [-1,1]$ and define $f : K \to \mathbb{R}$ by $f(x) := \begin{cases} x^2 & \text{if } x \in (-1,1) \\ 2 & \text{otherwise} \end{cases}$ for each $x \in K$. Clearly, f is convex and bounded on bounded subsets of K, but it is not even lower-semicontinuous.

4.2. **Strict and locally uniform convexity.** Recall that a function $f \in \mathscr{C}$ is called *strictly convex* if for each $x, y \in K$ such that $x \neq y$ and each $\lambda \in (0,1)$, we have

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y). \tag{4.5}$$

It is not difficult to see that this definition is equivalent to the requirement that f is convex and for each $x, y \in K$ such that $x \neq y$, there exists $\lambda \in (0,1)$ satisfying (4.5), as well as to the requirement that f is convex and $\lambda = 2^{-1}$ satisfies (4.5) for each $x, y \in K$ such that $x \neq y$. We denote the set of all strictly convex functions $f \in \mathscr{C}$ by $\mathscr{F}$.

Recall that a function $f \in \mathscr{C}$ is called *locally uniformly convex* if for each sequence $\{x_n\}_{n=1}^{\infty} \subset K$ and each $x \in K$,

$$\lambda f(x_n) + (1-\lambda) f(x) - f(\lambda x_n + (1-\lambda) x) \underset{n\to\infty}{\to} 0 \tag{4.6}$$

implies $\|x - x_n\| \underset{n\to\infty}{\to} 0$ for each $0 < \lambda < 1$. It is not difficult to see that this definition is equivalent to the requirement that $f \in \mathscr{C}$ and for each $x \in K$, there exists $\lambda \in (0,1)$ such that (4.6) implies that $\|x - x_n\| \underset{n\to\infty}{\to} 0$, as well as to the requirement that $f \in \mathscr{C}$ and for each $x \in K$, (4.6) with $\lambda = 2^{-1}$ implies that $\|x - x_n\| \underset{n\to\infty}{\to} 0$. It turns out (see Exercise 5.3.1 in [7]) that it is enough to consider only bounded sequences in the definition of a locally uniformly convex function.

Recall that a normed space $(X, \|\cdot\|)$ is called strictly convex if $\|x + y\| < 2$ whenever $x, y \in X$ are such that $x \neq y$ and $\|x\| = \|y\| = 1$, and locally uniformly convex if $\|x - x_n\| \underset{n\to\infty}{\to} 0$ whenever $\|x_n + x\| \underset{n\to\infty}{\to} 2$ for each sequence $\{x_n\}_{n=1}^{\infty}$ in the closed unit ball of center zero and each x in this ball. It is not difficult to see that $(X, \|\cdot\|)$ is strictly convex (respectively, locally uniformly convex) if and only if the square of its norm is a strictly convex function (respectively, locally uniformly convex function), as well as that a locally uniformly convex function is strictly convex and in the case where $K = X$ and the dimension of the vector space X is finite, the converse is also true. But in the case where the dimension of X is infinite, a strictly convex function may no longer be locally uniformly convex (see, for instance, the example on pages 229--230 in [23]). We denote the set of all locally uniformly convex functions $f \in \mathscr{C}$ by $\mathscr{G}$.

4.3. **Discussion on the results.** It was shown in [8] that under the assumption of the existence of a continuous and strictly convex function $f_* \in \mathscr{C}_b$, the set $\mathscr{F}$ of all strictly convex functions defined on K is residual in $\mathscr{C}$ with the τ topology, and that the sets $\mathscr{F} \cap \mathscr{C}_l$ and $\mathscr{F} \cap \mathscr{C}_c$ are residual in, respectively, $\mathscr{C}_l$ and $\mathscr{C}_c$ with their relative τ topologies. It can be verified that the same result is true for the set $\mathscr{F} \cap \mathfrak{M}_b$ in the space $\mathscr{C}_b$ with the relative τ topology. In the next subsection, we review a stronger result obtained in [5], which provides all these residuality properties for the set $\mathscr{G}$ (which is a subset of $\mathscr{F}$) of all locally uniformly convex functions defined on K.

In another recent work described in [31], residual sets of the space $\Gamma(X)$ are considered, where $\Gamma(X)$ is the set all lower semicontinuous functions $f : X \to \mathbb{R}$ with the τ_2 topology and $(X, \|\cdot\|)$ is a real Banach space. In particular, it is shown there that if $(X, \|\cdot\|)$ is a locally uniformly convex space (respectively, a strictly convex space), then the set of all locally uniformly convex (respectively, strictly convex) functions is residual in $\Gamma(X)$. By using similar techniques, it has been shown in [5], that in the case where $K = X$ (where $(X, \|\cdot\|)$ is not necessarily a Banach space), the relative strong topology of $\mathscr{C}_b$ is the same as its relative weak topology and hence, the sets $\mathscr{F} \cap \mathscr{C}_b$ and $\mathscr{G} \cap \mathscr{C}_b$ are residual in $\mathscr{C}_b$ with both of these topologies. For a finite dimensional vector space, X we know that each convex function $f : X \to \mathbb{R}$ is continuous, and therefore this result can be applied to all of $\mathscr{C}$, because $\mathscr{C}_b = \mathscr{C}$ in this case.

4.4. **Statements of the main results.** In this subsection we review the recent results obtained by Barshad, Reich and Zaslavski in [5].

Theorem 4.2. *Suppose that there exists a a strictly convex function $f_* \in \mathscr{C}_b$. Then the sets $\mathscr{G}$ and $\mathscr{G} \cap \mathscr{C}_b$ are residual in, respectively, $\mathscr{C}$ and $\mathscr{C}_b$ with the relative τ topology. If, in addition, f_* is lower semicontinuous (respectively, continuous), then the set $\mathscr{G} \cap \mathscr{C}_l$ (respectively, $\mathscr{G} \cap \mathscr{C}_c$) is residual in $\mathscr{C}_l$ (respectively, $\mathscr{C}_c$) with the relative strong topology.*

Remark 4.3. In particular, this result is true for a strictly convex normed linear space X, since the square of its norm is a continuous strictly convex function which is bounded on bounded sets. An important class of such spaces consists of inner product spaces and, in particular, Hilbert spaces.

Theorem 4.4. *In the case where $K = X$, the relative weak topology of $\mathscr{C}_b$ is the same as the relative strong topology of $\mathscr{C}_b$. As a result, if $\mathscr{F} \cap \mathscr{C}_b \neq \emptyset$, then the set $\mathscr{G} \cap \mathscr{C}_b$ (and therefore $\mathscr{F} \cap \mathscr{C}_b$) is residual in $\mathscr{C}_b$ with both of these topologies.*

References

[1] K. Barshad, S. Reich, A. J. Zaslavski, Generic convergence of methods for solving stochastic feasibility problems, J. Nonlinear Var. Anal. 5 (2021), 331-351.

[2] K. Barshad, S. Reich, A. Zaslavski, Generic properties of normal mappings, Linear Nonlinear Anal. 6 (2020) 205--218.

[3] K. Barshad, S. Reich, A. Zaslavski, Generic properties of operator-dependent normal mappings, Appl. Set-Valued Anal. Optim. 3 (2021) 133-147.

[4] K. Barshad, S. Reich, A. J. Zaslavski, Porosity-based methods for solving stochastic feasibility problems, Stud. Univ. Babes-Bolyai Math. 67 (2022) 11-19.

[5] K. Barshad, S. Reich, A. J. Zaslavski, Residuality properties of certain classes of convex functions on normed linear spaces, J. Convex Anal. 29 (2022) 795-806.

[6] J. Beer, R. Lucchetti, Convex optimization and epi-distance topology, Trans. Amer. Math. Soc. 327 (1991) 795-813.

[7] J. M. Borwein, J. D. Vanderwerff, Convex Functions: Constructions, Characterizations and Counterexamples, Cambridge University Press, Cambridge, 2010.

[8] D. Butnariu, S. Reich, A. J. Zaslavski, There are many totally convex functions, J. Convex Anal. 13 (2006) 623-632.

[9] D. Butnariu, S. Reich, A. J. Zaslavski: Generic power convergence of operators in Banach spaces, Numer. Funct. Anal. Optim. 20 (1999) 629-650.

[10] D. Butnariu, S. Reich, A. J. Zaslavski: Asymptotic behavior of relatively nonexpansive operators in Banach spaces, J. Appl. Anal. 7 (2001) 151-174.

[11] D. Butnariu, S. Reich, A. J. Zaslavski: Asymptotic behavior of quasi-nonexpansive mappings, in: Inherently Parallel Algorithms in Feasibility and Optimization and their Applications, D. Butnariu, Y. Censor, S. Reich (eds.), pp. 49-68, Elsevier, Amsterdam, 2001.

[12] J. B. Conway, A Course in Functional Analysis, 2nd ed., Springer-Verlag, New York, 1990.

[13] F. S. De Blasi, J. Myjak, Generic flows generated by continuous vector fields in Banach spaces, Adv. Math. 50 (1983) 266-280.

[14] F. S. De Blasi, J. Myjak, Sur la convergence des approximations successives pour les contractions non linéaries dans un espace de Banach, C. R. Acad. Sc. Paris 283 (1976) 185-187.

[15] F. S. de Blasi, J. Myjak, Sur la porosité de l'ensemble des contractions sans point fixe, Comptes Rendus de l'Académie des Sciences Paris 308 (1989) 51-54.

[16] A. L. Donchev, T. Zolezzi , Well-posed Optimization Problems, Lect. Notes in Math. 1543, Springer, Berlin-Heidelberg-New York, 1993.

[17] N. Dunford, J.T. Schwartz, Linear Operators Part I: General Theory. J. Wiley & Sons, New York, 1957.

[18] M. Gabour, S. Reich, A. J. Zaslavski, A class of dynamical systems with a convex Lyapunov function, In: Constructive, Experimental, and Nonlinear Analysis, Canadian Mathematical Society Conference Proceedings, vol. 27, pp. 83-91, 2000.

[19] M. Gabour, S. Reich, A. J. Zaslavski, Generic convergence of algorithms for solving stochastic feasibility problems, In: Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications, pp. 279-295, Elsevier, Amsterdam, 2001.

[20] G. Goebel, S. Reich, Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings, Marcel Dekker, New and Basel, 1984.

[21] A. D. Ioffe, A. J. Zaslavski, Variational principles and well-posedness in optimization and calculus of variations, SIAM J. Control Optim. 38 (2000) 566-581.

[22] E. Kopecká, S. Reich, Nonexpansive retracts in Banach spaces, Banach Center Publications 77 (2007) 161-174.

[23] A. R. Lovaglia, Locally uniformly convex Banach spaces, Trans. Amer. Math. Soc. 78 (1955) 225-238.

[24] E. Michael, A note on completely metrizable spaces, Proc. Amer. Math. Soc. 96 (1986) 513-522.

[25] S. Reich, Genericity and porosity in nonlinear analysis and optimization, ESI Preprint 1756, 2005, Proceedings of CMS'05 (Computer Methods and Systems), pp. 9-15, Kraków, 2005.

[26] S. Reich, A. J. Zaslvski, Convergence of generic infinite products of nonexpansive and uniformly continuous operators, Nonlinear Anal. 36 (1999) 1049-1065.

[27] S. Reich, A. J. Zaslavski, On the minimization of convex functionals, In: Calculus of Variations and Differential Equations, Res. Notes Math. 410, pp. 200-209, Chapman & Hall/CRC, Boca Raton, Fl, 2000.

[28] S. Reich, A. J. Zaslavski, Well-posedness and porosity in best approximation problems, Topological Meth. Nonlinear Anal. 18 (2001) 395-408.

[29] S. Reich, A. J. Zaslavski, Genericity in Nonlinear Analysis, Springer, New York, 2014.

[30] S. Reich, A. J. Zaslavski, Genericity and porosity in fixed point theory: a survey of recent results, Fixed Point Theory App. 2015 (2015) 195.

[31] J. Vanderwerff, On the residuality of certain classes of convex functions, Pure Appl. Funct. Anal. 5 (2020) 791-806.

[32] G. Vidossich: Existence, uniqueness and approximation of fixed points as a generic property, Bol. Soc. Bras. Mat. 5 (1974) 17-29.

[33] A. J. Zaslavski, Optimal programs on infinite horizon, 1 and 2, SIAM J. Control Optim. 33 (1995) 1643-1686.

[34] A. J. Zaslavski, Existence of solutions of optimal control problems for a generic integrand without convexity assumptions, Nonlinear Anal. 43 (2001) 339-361.

[35] A. J. Zaslavski, Well-posedness and porosity in optimal control without convexity assumptions, Calc. Var. Partial Differ. Equ. 13 (2001) 265-293.