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A NOTION OF CONJUGACY FOR NONCONVEX SET-VALUED MAPPINGS OF THE REAL-LINE AND RELATED PROPERTIES

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Dedication to the memory of Professor Hoang Tuy

Abstract. Given a nonconvex set-valued mappings $F:\mathbb{R}^N \rightrightarrows \mathbb{R}$, a notion of conjugate F^* is introduced with the goal that $(F^*)^* = F$. This is given by using the usual (bilinear) duality pairing. Several examples showing its geometric interpretation are presented, as well as a notion of subdifferential for such set-valued maps is also outlined.

Keywords. Convex conjugate function; Contingent cone; Contingent derivative; Duality theory; Set-valued map.

1. Introduction

It is known that the classical convex duality theory as well as some nonconvex duality schemes as that of Ekeland or Toland [6, 12, 13] are based on the notion of convex conjugate function. Given a locally convex topological vector real space X, with dual X^*, and a function $f : X \to \mathbb{R}\cup\{+\infty\}$ such that $f \not\equiv +\infty$, the convex conjugate of f is $f^* : X^* \to \mathbb{R}\cup\{+\infty\}$, given by

$$f^*(x^*) = \sup_{x\in X}\Big\{ \ \langle x^*, x\rangle - f(x) \ \Big\} \ (x^* \in X^*). \tag{1.1}$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. This notion was introduced by Fenchel [7] for the finite dimensional case and further developed mainly by Rockafellar [11]. Some other nonconvex duality theories but based on an extension due to Moreau [10] (where the usual duality pairing is substituted by a more general coupling function), of the above notion of conjugacy, can be found in [3, 5], and with respect to the continuous real-valued functions in [8, 9] and the references therein. We quote [4, 10] for more recent results in this direction. However, we attempt to develop a conjugacy notion based on the usual duality pairing. The purpose of

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this paper is to contribute to the development of the conjugation theory. More precisely, we define the *conjugate* of a set-valued map $F:\mathbb{R}^N \rightrightarrows \mathbb{R}$ and derive its main properties useful in optimization theory. This notion of conjugacy is related to the Legendre transform more than to the conjugate of single-valued maps. Such a notion is introduced in such a way that $(F^*)^* = F$ for some instances.

The paper is organized as follows. Section 2 provides the basic definitions related to set-valued mappings. In Section 3, the proposal of (convex) conjugate for set-valued mappings of the real-line is introduced, along with its main properties. Also several instances satisfying the desired property are exhibited. Section 4 discusses the important class $F(x) = [f(x), +\infty[$.

2. Basic Definitions and Preliminaries

We recall some basic definitions useful in set-valued analysis; see [1, 2] for general references. Given any two sets X, Y, a set-valued map F from X to Y, denoted by $F : X \rightrightarrows Y$, is a map that associates to any $x \in X$ a subset $F(x)$ of Y. The subsets $F(x)$ are called the *images* or the *values* of F. As usual in convex analysis, we use the following notations:

$$\text{Dom } F \doteq \Big\{ x \in X : \ F(x) \neq \emptyset \Big\},$$

is the *domain* of F. If $\text{Dom } F = X$, we say that map F is strict. It is said to be proper if $\text{Dom } F \neq \emptyset$. Sometimes, given a set-valued map $F : C \to Y$, we extend it to the whole set X by setting $F_C(x) = F(x)$ if $x \in C$ and $F_C(x) = \emptyset$ if $x \notin C$. The set

$$\text{Gr } F \doteq \Big\{ (x,y) \in X \times Y : \ y \in F(x) \Big\}$$

is called the *graph* of F and it characterizes the set-valued map F. In fact, let $\mathscr{F}$ be a nonempty set of $X \times Y$. Then it defines a set-valued map as follows:

$$F(x) = \Big\{ y \in Y : (x,y) \in \mathscr{F} \Big\}.$$

Here $\text{Gr } F = \mathscr{F}$. The *range* $\mathscr{R}(F)$ is defined by $\mathscr{R}(F) \doteq \bigcup_{x\in X} F(x)$. The inverse F^{-1} of a set-valued map F from X to Y is the set-valued map from $\mathscr{R}(F)$ to X defined by

$$F^{-1}(y) \doteq \Big\{ x \in X : y \in F(x) \Big\} = \Big\{ x \in X : (x,y) \in \text{Gr } F \Big\}.$$

In the remaining of this section X, Y are normed spaces. We say that the subset of the dual X^* of X defined by

$$K^- \doteq \Big\{ x^* \in X^* : \langle x^*, x\rangle \leq 0 \ \forall x \in K \Big\}$$

is the (negative) *polar cone* of $K \subseteq X$. By K' we mean the set of accumulation points of K.

3. The Conjugate of a Set-Valued Map

Given a convex differentiable function $f : \mathbb{R}^N \to \mathbb{R}$ such that the supremum on the right hand side of (1.1) is attained at $\bar{x}$, we obtain the classical formula of the Legendre transform:

$$f^*(x^*) = \langle x^*, \bar{x}\rangle - f(\bar{x}), \ x^* = f'(\bar{x}). \tag{3.1}$$

Thus $\bar{x}$ is a critical point to the map $x \mapsto \langle x^*, x\rangle - f(x)$. This motivates the following definition.

Definition 3.1. Let $F : \mathbb{R}^N \rightrightarrows \mathbb{R}$ be a proper set-valued map.

(i) The *convex conjugate* of the map F is the set-valued map $F^* : \mathbb{R}^N \rightrightarrows \mathbb{R}$, possibly with empty values, defined by

$$F^*(p) = \bigcup_{(x,y)\in S^*_F(p)} \Big\{ \langle p,x\rangle - y \Big\},$$

where

$$S^*_F(p) \doteq \Big\{ (x,y) \in \text{Gr}\, F :\ \forall\, x_n \to x, x_n \neq x,\ \exists\, y_n \in F(x_n), y_n \to y \text{ such that}$$
$$\limsup_{n\to+\infty} \frac{\langle p, x_n - x\rangle - y_n + y}{\|x_n - x\|} \leq 0 \Big\}. \tag{3.2}$$

We set $\cup_\emptyset = \emptyset$.

(ii) The *concave conjugate* of the map F is the set-valued map $F_* : \mathbb{R}^N \rightrightarrows \mathbb{R}$, possibly with empty values, defined by

$$F_*(p) = \bigcup_{(x,y)\in S^F_*(p)} \Big\{ \langle p,x\rangle - y \Big\},$$

where

$$S^F_*(p) \doteq \Big\{ (x,y) \in \text{Gr}\, F :\ \forall\, x_n \to x, x_n \neq x,\ \exists\, y_n \in F(x_n), y_n \to y \text{ such that}$$
$$\liminf_{n\to+\infty} \frac{\langle p, x_n - x\rangle - y_n + y}{\|x_n - x\|} \geq 0 \Big\}. \tag{3.3}$$

Before going on, some remarks are in order.

Remark 3.2. If $x \in \text{Dom}\, F \setminus (\text{Dom}\, F)'$, then $(x,y) \in S^F_*(p) \cap S^*_F(p)$ for all $y \in F(x)$ and all $p \in \mathbb{R}^N$. Thus $\langle p,x\rangle - y \in F^*(p) \cap F_*(p)$ for all $y \in F(x)$. Hence

$$(\text{Dom}\, F)' = \emptyset \Longrightarrow S^F_*(p) = S^*_F(p) = \text{Gr}\, F.$$

The importance of S^*_F lies in the following proposition. We recall that K' denotes the set of accumulations points of K.

Proposition 3.3. *Let $C \subset \mathbb{R}^N$ be a nonempty convex set with $C' \neq \emptyset$ and $f : C \to \mathbb{R}$.*

(a) *If f is a continuously differentiable real single-valued map in an open neighborhood of C, then*

$$S^*_f(p) = \Big\{ (x, f(x)) :\ x \in C,\ \langle \nabla f(x) - p, x' - x\rangle \geq 0\ \forall\, x' \in C \Big\}; \tag{3.4}$$
$$S^f_*(p) = \Big\{ (x, f(x)) :\ x \in C,\ \langle \nabla f(x) - p, x' - x\rangle \leq 0\ \forall\, x' \in C \Big\}.$$

Hence, if C is open, then $S^*_f(p) = S^f_*(p) = \Big\{(x, f(x)) :\ x \in C,\ \nabla f(x) = p \Big\}$.

(b) *If f is a convex real single-valued map in C, then $S^*_f(p) = \Big\{(x, f(x)) :\ x \in C,\ p \in \partial f(x) \Big\}$. Here, we assumed the function f is extended to the whole space $\mathbb{R}^N$ by taking $f(x) = +\infty$ if $x \notin C$.*

Similar results also hold for S^f_*.

Proof. (a) Let $x \in C$ such that, $\forall\, x_n \to x$, $x_n \neq x$, $x_n \in C$ (which implies $f(x_n) \to f(x)$):

$$\limsup_{n\to+\infty} \frac{\langle p, x_n - x\rangle - f(x_n) + f(x)}{\|x_n - x\|} \le 0. \tag{3.5}$$

Then, given any $x' \neq x$ in C, we consider in particular, $x_n := x + t_n(x' - x)$ for $t_n \downarrow 0$ as $n \to +\infty$. Substituting in (3.5), it reduces

$$\frac{1}{\|x' - x\|} \lim_{n\to+\infty} \left\{ \langle p, x' - x\rangle - \frac{f(x + t_n(x' - x)) - f(x)}{t_n} \right\} \le 0.$$

This implies $\langle \nabla f(x) - p, x' - x\rangle \ge 0$, $\forall\, x' \in C$, proving that x is in the set of the right hand side of (3.4). Conversely, if $x \in C$ is such that $\langle \nabla f(x) - p, x' - x\rangle \ge 0\ \forall\, x' \in C$, then, for every sequence $x_n \to x$, $x_n \in C$, we

$$\frac{\langle p, x_n - x\rangle - f(x_n) + f(x)}{\|x_n - x\|} \le \frac{\langle \nabla f(x), x_n - x\rangle - f(x_n) + f(x)}{\|x_n - x\|}.$$

Consequently,

$$\limsup_{n\to+\infty} \frac{\langle p, x_n - x\rangle - f(x_n) + f(x)}{\|x_n - x\|} \le 0.$$

Thus $(x, f(x)) \in S_f^*(p)$, which completes the proof of Part (a) of the proposition.

(b) Let $x \in C$ such that (3.5) is satisfied. Then, given any $x' \neq x$ in C, by taking $x_n = x + t_n(x' - x)$ for $t_n \downarrow 0$ as $n \to +\infty$ in (3.5), it implies, by convexity

$$\frac{\langle p, x' - x\rangle}{\|x' - x\|} + \frac{f(x) - f(x')}{\|x' - x\|} \le \limsup_{n\to+\infty} \frac{\langle p, x_n - x\rangle - f(x_n) + f(x)}{\|x_n - x\|} \le 0.$$

Thus, $p \in \partial f(x)$. The remaining inclusion is trivially obtained.

□

Remark 3.4. (i) It is easily seen that

$$\text{int (Gr F)} \subset S_F^*(p) \cap S_*^F(p).$$

In fact, for every (x, y) in int (Gr F), there exists a neighborhood U of (x, y) such that $U \subset$ Gr F. Then, for every $x_n \in$ Dom F, $x_n \to x$, $x_n \neq x$, we set $y_n \doteq y + \langle p, x_n - x\rangle$. Thus, $y_n \to y$ and therefore $(x_n, y_n) \in U$ for all $n \in \mathbb{N}$ sufficiently large. This implies $y_n \in F(x_n)$ and

$$\frac{\langle p, x_n - x\rangle - y_n + y}{\|x_n - x\|} = 0,$$

proving the desired assertion.

(ii) Given $(x, y) \in$ Gr F with F, taking convex and closed values, the sets

$$D_F^*(x,y) = \left\{ p \in \mathbb{R}^N :\ (x,y) \in S_F^*(p) \right\},\ D_*^F(x,y) = \left\{ p \in \mathbb{R}^N :\ (x,y) \in S_*^F(p) \right\},$$

are convex and closed. Actually the first set can be seen as the subdifferential of F at $(x, y) \in$ Gr F, and the second as the superdifferential.

(iii) Let $C \subset \mathbb{R}^N$ be a nonempty set and $f : C \to \mathbb{R}$ be any function, and consider the set-valued map $F : \mathbb{R}^N \rightrightarrows \mathbb{R}$, $F(x) = [f(x), +\infty[$ if $x \in C$ and $F(x) = \emptyset$ if $x \notin C$. Then, for every p, $(x, y) \in S_F^*(p)$ implies $(x, y + \lambda) \in S_F^*(p)$ for all $\lambda > 0$. Set-valued maps of this kind will be studied in more detail in Section 4.

In what follows $d_C(x)$ stands for the distance from x to C, i.e.,

$$d_C(x) \doteq \inf\{ \|x-y\| : \; y \in C \},$$

where $\|\cdot\|$ denotes the Euclidean norm.

Definition 3.5. Let C be a nonempty subset of $\mathbb{R}^N$, $x \in \overline{C}$.

(i) The subset

$$T_C(x) \doteq \left\{ v : \; \liminf_{h\to 0^+} \frac{d_C(x+hv)}{h} = 0 \right\}$$

is called the *contingent cone* to C at x.

(ii) The subset

$$T_C^\flat(x) \doteq \left\{ v : \; \lim_{h\to 0^+} \frac{d_C(x+hv)}{h} = 0 \right\}$$

is called the *intermediate or adjacent cone* to C at x.

For the main properties of these cones we refer to Chapter IV in the book [2]. We simply recall the following characterization, which is easily obtained from the definition.

Proposition 3.6. *Let C be a nonempty set in $\mathbb{R}^N$. Then,*

(a) $v \in T_C(x)$ if and only if $\exists\, t_n \downarrow 0$, $\exists\, v_n \to v$ and $\forall\, n \in \mathbb{N}$, $x + t_n v_n \in C$.
(b) $v \in T_C^\flat(x)$ if and only if $\forall\, t_n \downarrow 0$, $\exists\, v_n \to v$ and $\forall\, n \in \mathbb{N}$, $x + t_n v_n \in C$.

In order to establish some relationships between S_F^*, S_*^F and sets involving contingent or adjacent cones, we introduce some notations:

$$S_F^0(p) := \left\{ (x,y) \in \mathrm{Gr}\, F : \; \forall\, x_n \to x, x_n \neq x, \; \exists\, y_n \in F(x_n), y_n \to y \text{ such that } \lim_{n\to+\infty} \frac{\langle p, x_n - x\rangle - y_n + y}{\|x_n - x\|} = 0 \right\}; \tag{3.6}$$

$$S_F(p) := \left\{ (x,y) \in \mathrm{Gr}\, F : \; \exists\, x_n \to x, x_n \neq x, \; \exists\, y_n \in F(x_n), y_n \to y \text{ such that } \lim_{n\to+\infty} \frac{\langle p, x_n - x\rangle - y_n + y}{\|x_n - x\|} = 0 \right\};$$

$$G_F^\flat(p) := \left\{ (x,y) \in \mathrm{Gr}\, F : \; \forall\, u \in T^\flat_{\mathrm{Dom}\, F}(x), \; (u, \langle p, u\rangle) \in T^\flat_{\mathrm{Gr}\, F}(x,y) \right\};$$

$$G_F(p) := \left\{ (x,y) \in \mathrm{Gr}\, F : \; \forall\, u \in T_{\mathrm{Dom}\, F}(x), \; (u, \langle p, u\rangle) \in T_{\mathrm{Gr}\, F}(x,y) \right\}.$$

Proposition 3.7. *Let $F : \mathbb{R}^N \rightrightarrows \mathbb{R}$ be a proper set-valued mapping. Then*

(a) $S_F^0(p) \subset G_F^\flat(p) \cap G_F(p)$;
(b)

$$S_F(p) = \left\{ (x,y) \in \mathrm{Gr}\, F : \exists\, u \in T_{\mathrm{Dom}\, F}(x), \|u\| = 1, \; (u, \langle p, u\rangle) \in T_{\mathrm{Graph}\, F}(x,y) \right\} \tag{3.7}$$

Proof. (a): Let (x,y) be in the set given by (3.6) $u \in T^\flat_{\mathrm{Dom}\, F}(x)$. By the characterization in terms of sequences of the intermediate or adjacent cone, we have $\forall\, t_n \downarrow 0$, $\exists\, u_n \to u$ such that $x + t_n u_n \in \mathrm{Dom}\, F$ for all $n \in \mathbb{N}$. Set $x_n \doteq x + t_n u_n$, which converges to x, with $x_n \neq x$ if $u \neq 0$. Thus, there exists $y_n \in F(x_n)$ with $y_n \to y$ such that

$$\lim_{n\to\infty} \frac{\langle p, x_n - x\rangle - y_n + y}{|x_n - x|} = 0.$$

We set $v_n \doteq \frac{1}{t_n}(y_n - y)$ and obtain $(x,y)+t_n(u_n,v_n) \in \text{Gr } F$. We claim that $v_n \to \langle p,u\rangle$. In fact,

$$\begin{aligned}\langle p,u\rangle - v_n &= \langle p,u\rangle - \frac{y_n-y}{t_n} = \langle p,u\rangle + \|u_n\|\frac{-y_n+y}{\|x_n-x\|}\\ &= \langle p,u\rangle + |u_n|\left\{\frac{\langle p,x_n-x\rangle - y_n+y}{\|x_n-x\|} - \frac{\langle p,x_n-x\rangle}{\|x_n-x\|}\right\}\\ &= \langle p,u\rangle + \|u_n|\frac{\langle p,x_n-x\rangle - y_n+y}{\|x_n-x\|} - \langle p,u_n\rangle.\end{aligned}$$

Hence, $v_n \to \langle p,u\rangle$ as desired. This proves that $(u,\langle p,u\rangle) \in T^{\flat}_{\text{Graph } F}(x,y)$ and so $(x,y) \in G^{\flat}_F(p)$. Similarly, one shows that $(x,y) \in G_F(p)$.

(b) Let us prove the inclusion "$\supset$". Let (x,y) be in the set of the right hand side of (3.7). Then, because of the characterization in terms of sequences of contingent cones, $\exists\, t_n \downarrow 0$, $\exists\, u_n \to u$, $\exists\, v_n \to \langle p,u\rangle$ such that $y+t_nv_n \in F(x+t_nu_n)$ for some $u \in T_{\text{Dom } F}(x), u \neq 0$. Setting $x_n \doteq x+t_nu_n$, $y_n \doteq y+t_nv_n$, we obtain that $y_n \in F(x_n)$ and

$$\lim_{n\to+\infty}\frac{\langle p,x_n-x\rangle - y_n+y}{\|x_n-x\|} = \lim_{n\to+\infty}\frac{t_n\langle p,u_n\rangle - t_nv_n}{t_n\|u_n\|} = 0.$$

Therefore, $(x,y) \in S_F(p)$. It remains to prove the inclusion "$\subset$". Let (x,y) be in $G_F(p)$. Then $\exists\, x_n \to x$, $x_n \neq x$, $\exists\, y_n \to y$ with $y_n \in F(x_n)$ such that

$$\lim_{n\to+\infty}\frac{\langle p,x_n-x\rangle - y_n+y}{\|x_n-x\|} = 0.$$

It is our purpose to show that (x,y) is in the set in (3.7), that is, we show that there exists $u \in T_{\text{Dom } F}(x)$, $u \neq 0$ such that $(u,\langle p,u\rangle) \in T_{\text{Graph } F}(x,y)$. The later will be proved through its characterization in terms of sequences. We set $t_n \doteq |x_n - x|$, $u_n \doteq \frac{1}{t_n}(x_n-x)$ and $v_n \doteq \frac{1}{t_n}(y_n-y)$. Thus, $\|u_n\| = 1$ and therefore, up to a subsequence, $u_n \to u$, $\|u\| = 1$. In addition, $\forall\, n \in \mathbb{N}$, $x_n = x+t_nu_n \in \text{Dom } F$. We claim that $v_n \to \langle p,u\rangle$. In fact, this follows in a similar way to that of Part (a). □

Some of the basic properties of the above conjugacy notion are listed in the following proposition.

Proposition 3.8. *Let $F,G : \mathbb{R}^N \rightrightarrows \mathbb{R}$ be two proper set-valued maps. Then*

(a) If $F \subseteq G$, in the sense that $F(x) \subseteq G(x)\ \forall\, x$, then $F^* \subseteq G^*$;
(b) $(F+c)^*(p) = F^*(p) - c$, whenever $c \in \mathbb{R}$;
(c) $(F+\langle x_0,\cdot\rangle)^*(p) = F^*(p-x_0)$, $\forall\, x_0 \in \mathbb{R}^N$;
(d) $(\lambda F)^*(p) = \lambda F^*(\frac{p}{\lambda})$ whenever $\lambda > 0$.

Similar results also hold for F_*.

Proposition 3.9. *Let $C \subset \text{Dom } F \subset \mathbb{R}^N$ be a nonempty open set. If $F(x) = \mathbb{R}$ whenever $x \in C$, then $F_*(p) = F^*(p) = \mathbb{R}$ for every $p \in \mathbb{R}^N$. Hence, $F^{**} = F_{**} = (F_*)^* = (F^*)_* = \mathbb{R}$ in $\mathbb{R}^N$.*

Proof. For a fixed $p \in \mathbb{R}^N$, given any $z \in \mathbb{R}$, let us take any $x \in C$ and any sequence x_n converging to x. Since C is open, we can assume that $x_n \in C$ and $x_n \neq x$ for all $n \in \mathbb{N}$. We write $z = \langle p,x\rangle - y$ for some $y \in \mathbb{R}$. By setting $y_n \doteq \langle p,x_n\rangle - z + \|x_n-x\|^2$, we have

$$\frac{\langle p,x_n-x\rangle - y_n+y}{\|x_n-x\|} = -\|x_n-x\|.$$

Taking the limit as $n \to +\infty$, the latter proves that $F_*(p) = F^*(p) = \mathbb{R}$ for all $p \in \mathbb{R}^N$. □

The proof of next proposition is straightforward.

Proposition 3.10. *Let $C \subset \mathbb{R}^N$ and $K \subset \mathbb{R}$ be nonempty sets with $C' \neq \emptyset$. Let us consider the set-valued map $F(x) = K$ if $x \in C$ and $F(x) = \emptyset$ if $x \notin C$. Then $F^*(0) = -K$.*

What follows is a consequence of Remark 3.2.

Proposition 3.11. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a set-valued map. If $(\text{Dom } F)' = \emptyset$ and $F(x)' = \emptyset$ for all $x \in \text{Dom } F$, then*

$$F_*(p) = F^*(p) = \bigcup_{(x,y) \in \text{ Graph } F} \Big\{ \langle p, x\rangle - y \Big\}.$$

Taking into account Proposition 3.3, we obtain the following.

Proposition 3.12. *Let $C \subset \mathbb{R}^N$ be a nonempty convex set with $C' \neq \emptyset$. Let $f_i : C \to \mathbb{R}$, $i \in I$, be real single-valued maps such that, for every $x \in C$, every $x_n \to x$, $x_n \in C$, every $i_n \in I$, the following holds*

$f_{i_n}(x_n) \to f_i(x)$ as $n \to \infty \Longrightarrow \exists\, n_0 \in \mathbb{N}, \exists\, i_0 \in I : i_n = i_0\ \forall n \geq n_0$ and $f_{i_0}(x) = f_i(x)$. Let us consider the set-valued map

$$F(x) \quad = \quad \begin{cases} \bigcup_{i \in I}\{f_i(x)\} & if \quad x \in C; \\ \emptyset & if \quad x \notin C. \end{cases}$$

(a) If each f_i, $i \in I$, is continuously differentiable in a neighborhood of C, then

$$F^*(p) = \bigcup_{i \in I} \bigcup_{\bar{x} \in C} \Big\{ \langle p, \bar{x}\rangle - f_i(\bar{x}) : \langle \nabla f_i(\bar{x}) - p, x' - \bar{x}\rangle \geq 0\ \forall\, x' \in C \Big\}; \tag{3.8}$$

$$F_*(p) = \bigcup_{i \in I} \bigcup_{\bar{x} \in C} \Big\{ \langle p, \bar{x}\rangle - f_i(\bar{x}) : \langle \nabla f_i(\bar{x}) - p, x' - \bar{x}\rangle \leq 0\ \forall\, x' \in C \Big\}.$$

Hence, if C is open, then

$$F_*(p) = F^*(p) = \bigcup_{i \in I} \bigcup_{\bar{x} \in C} \Big\{ \langle p, \bar{x}\rangle - f_i(\bar{x}) :\ p = \nabla f_i(\bar{x}) \Big\}.$$

(b) If each f_i, $i \in I$, is convex in C, then

$$F^*(p) = \bigcup_{i \in I} \bigcup_{\bar{x} \in C} \Big\{ \langle p, \bar{x}\rangle - f_i(\bar{x}) :\ p \in \partial f_i(\bar{x}) \Big\}. \tag{3.9}$$

Remark 3.13. The previous proposition gives the possibility, in case $F(x) = \{f(x)\}$, that $f^*(p)$ (in the usual sense) being in $\mathbb{R}$, could not be in $F^*(p)$. In fact, consider $f(x) = x + e^x$, $p = 1$, which satisfies $f^*(1) = \sup_{\mathbb{R}}\{-e^x\} = 0$. However, $F^*(1) = \emptyset$ as one can directly check it. This fact says, in some sense, that the notion of conjugacy recently introduced is related to the Legendre transform more than to the conjugate of single-valued maps.

In the following examples, we prefer to compute the conjugate of set-valued maps in a direct way instead of obtaining them as the applications of previous results.

Example 3.14. Let us consider, for a given $x_0 \in \mathbb{R}^N$, the set-valued map $F : \mathbb{R}^N \rightrightarrows \mathbb{R}$, defined by $F(x) = \{\langle x_0, x\rangle\}$. Then,

$$F_*(p) = F^*(p) \quad = \quad \begin{cases} \{0\} & if \quad p = x_0; \\ \emptyset & if \quad p \neq x_0. \end{cases}$$

Thus, by Proposition 3.11, we obtain $(F^*)_* = (F_*)^* = F_{**} = F^{**} = F$ in $\mathbb{R}^N$. In fact, in case $p = x_0$, we simply take $x = 0$ in the definition of $F^*(x_0)$ or $F_*(x_0)$ and show that $(0,0)$ is in $S^*_F(x_0) \cap S^F_*(x_0)$. If $p \neq x_0$, we proceed as follows. Assume that $z \in \mathbb{R}$ can be written as $z = \langle p, x\rangle - \langle x_0, x\rangle$ for some $x \in \mathbb{R}^N$. Then, by taking $x_n \doteq x + t_n u$ with $t_n \downarrow 0$ and suitable $u \neq 0$, the quotient in (3.2) reduces $\langle p - x_0, u\rangle / \|u\|$. This is positive (negative) if we choose $u = p - x_0$ ($u = -p + x_0$). Therefore, $F^*(p) = F_*(p) = \emptyset$ if $p \neq x_0$.

Example 3.15. Let $F : \mathbb{R}^N \rightrightarrows \mathbb{R}$ be the set-valued map defined by $F(x) = \{\|x\|\}$. Here $\|\cdot\|$ stands for the euclidean norm in $\mathbb{R}^N$. Then,

$$F^*(p) = \begin{cases} \{0\} & if \quad |p| \leq 1; \\ \emptyset & if \quad \|p\| > 1, \end{cases} \qquad F_*(p) = \begin{cases} \{0\} & if \quad \|p\| = 1; \\ \emptyset & if \quad \|p\| \neq 1, \end{cases}$$

$F^{**}(x) = F(x), x \in \mathbb{R}^N$; $F_{**}(0) = (F_*)^*(0) = (F^*)_*(0) = \{0\}$ and $F_{**}(x) = (F_*)^*(x) = (F^*)_*(x) = \emptyset$ if $x \neq 0$.

(a): The computation of F^* results from Proposition 3.12. We now prove that $F^{**}(x) = F(x)$ for all $x \in \mathbb{R}^N$. To that end, we can use part (a) of Proposition 3.3. However, we will prove that directly. For a fixed $x \in \mathbb{R}^N$, we start by proving that any $z < \|x\|$ is not in $F^{**}(x)$ for $x \neq 0$. If $z = \langle x, p\rangle$ for $\|p\| < 1$, then, we take $p_n = p + t_n x$ with $t_n \downarrow 0$ in the definition of $S^*_{F^*}(x)$. Thus

$$\frac{\langle x, p_n\rangle - \langle x, p\rangle}{\|p_n - p\|} = \frac{t_n\|x\|^2}{t_n\|x\|} = \|x\| > 0,$$

proving that $z \notin F^{**}(x)$ for $z = \langle x, p\rangle < \|x\|$ with $\|p\| < 1$. If, on the contrary, $z = \langle x, p\rangle < \|x\|$ with $\|p\| = 1$, we choose $\bar{p}$ such that $\|x\| = \langle x, \bar{p}\rangle$ with $\|\bar{p}\| = 1$, and we take $p_n = p + t_n(\bar{p} - p)$ with $t_n \downarrow 0$. Notice that $\bar{p} \neq p$ and $\|p_n\| \leq 1$. Moreover

$$\frac{\langle x, p_n - p\rangle}{\|p_n - p\|} = \frac{\langle x, \bar{p} - p\rangle}{\|\bar{p} - p\|} > 0.$$

This proves that $z \notin F^{**}(x)$ for $z = \langle x, p\rangle < \|x\|$ with $\|p\| = 1$ and therefore, the proof that any $z < \|x\|$ is not in $F^{**}(x)$ if $x \neq 0$ is concluded. That $\|x\| \in F^{**}(x)$, follows directly by computing the quotient involved in (3.2). This completes the proof that $F^{**} = F$.

(b): Let us compute F_*. Let us fix any z in $F_*(p)$. Then, there exists $x \in \mathbb{R}^N$ such that $z = \langle p, x\rangle - \|x\|$, and $(x, \|x\|)$ is in $S^F_*(p)$. However, if $z < 0$ (which implies $x \neq 0$), then, by choosing $x_n \doteq x + t_n x = (1 + t_n)x$ with $t_n \downarrow 0$, the quotient appearing in the definition of $S^F_*(p)$ reduces $(\langle p, x\rangle - \|x\|)/\|x\|$. Hence, $z \notin F_*(p)$ for all $z < 0$ and for all $p \in \mathbb{R}^N$. In a similar way, one can prove, by taking $x_n \doteq x - t_n x = (1 - t_n)x$ in the definition $S^F_*(p)$, that $z \notin F_*(p)$ for all $z > 0$ for all $p \neq 0$. On the other hand, it is not difficult to show that $0 \notin F_*(p)$ if $\|p\| > 1$. Therefore, $F_*(p) = \emptyset$ if $\|p\| > 1$. That $F_*(p) = \emptyset$ if $\|p\| < 1$ follows directly from the definition. It only remains to deal with the case $\|p\| = 1$. If this is the case, we recall the function $x \mapsto \|x\|$ is differentiable at every $x \neq 0$ with derivative $x/\|x\|$. Then, we simply take $x = p$ and prove that $(x, \|x\|) \in S^F_*(p)$, i.e. $0 = \langle p, x\rangle - \|x\| \in F_*(p)$ if $\|p\| = 1$. The proof of the remaining assertions are left as simple exercises.

In what follows, we present instances where $F = F^{**}$ in the entire Dom F, and another, they coincide in Dom F^{**}.

Example 3.16. Let us consider $F(x) = \{\frac{1}{3}x^3\}$, $x \in \mathbb{R}$. Then by Part (a) of Proposition 3.12, we have

$$F_*(p) = F^*(p) \quad = \quad \begin{cases} \{\pm\frac{2}{3}p^{3/2}\} & \text{if} \quad p \geq 0; \\ \emptyset & \text{if} \quad p < 0. \end{cases}$$

Applying again the same proposition, we obtain $F^{**}(x) = \{\frac{1}{3}x^3\} = F(x)$, for all $x \in \mathbb{R}$, and $F^{**} = F_{**} = (F^*)_* = (F_*)^*$ in $\mathbb{R}$.

Example 3.17. Let us consider $F(x) = \{\pm\sqrt{1-x^2}\}$, $\|x\| \leq 1$; $F(x) = \emptyset$. Then by Part (a) of Proposition 3.12, we have

$$F^*(p) = \{\pm\sqrt{1+p^2}\}, p \in \mathbb{R}.$$

Applying again the same proposition, we obtain $F^{**}(x) = F(x)$, for all $x \in \mathbb{R}$.

Example 3.18. Let us consider the set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) \quad = \quad \begin{cases} \{\pm\sqrt{x}\} & \text{if} \quad x \geq 0; \\ \emptyset & \text{if} \quad x < 0. \end{cases}$$

Then, it is not difficult to show that

$$F^*(p) \quad = \quad \begin{cases} \{-\frac{1}{4p}\} & \text{if} \quad p \neq 0; \\ \{0\} & \text{if} \quad p = 0. \end{cases}$$

Thus, we obtain

$$F^{**}(x) \quad = \quad \begin{cases} \{\pm\sqrt{x}\} & \text{if} \quad x > 0; \\ \emptyset & \text{if} \quad x \leq 0. \end{cases}$$

Notice that Dom $F^{**} \subseteq$ Dom F and $F^{**}(x) = F(x)$ for all $x \in$ Dom F^{**}.

4. The Case of $F(x) = [f(x), +\infty[$

In this section, we deal with a special set-valued map, which is very important in the study of minimization problems. The graph of this special map coincides with the epigraph of a given function. More precisely, we study the case in which $F(x) = [f(x), +\infty[$ for any given function f. We start by computing F^* for F defined by $[\|x\|, +\infty[$.

Example 4.1. Let us consider the set-valued map $F(x) = [\|x\|, +\infty[$, $x \in \mathbb{R}^N$. We will prove

$$F^*(p) = \begin{cases}]-\infty, 0] & \text{if} \quad \|p\| \leq 1; \\ \mathbb{R} & \text{if} \quad \|p\| > 1. \end{cases} \qquad F_*(p) = \begin{cases}]-\infty, 0[& \text{if} \quad \|p\| < 1; \\ ]-\infty, 0] & \text{if} \quad \|p\| = 1; \\ \mathbb{R} & \text{if} \quad \|p\| > 1. \end{cases}$$

Consequently, $F^{**} = F_{**} = (F^*)_* = (F_*)^* = \mathbb{R}$ in $\mathbb{R}^N$.

(a): Let us compute F^*. Clearly $F^*(p) \subset]-\infty, 0]$ for all $\|p\| \leq 1$. Since every $z \in F^*(p)$ can be written as $z = \langle p, x\rangle - y$ for some $y \geq \|x\|$, then $z \leq (\|p\| - 1)\|x\| \leq 0$. To prove the reverse inclusion, we write any $z \leq 0$ as $z = \langle p, 0\rangle - y$ and show that $(0, -z) \in S_F^*(p)$. This follows by taking, given any $x_n \to 0$, $y_n \doteq y + \|x_n\| \geq \|x_n\|$ in (3.2). In case $\|p\| > 1$, we also show that $(0,-z) \in S_F^*(p)$ for all $z \leq 0$, that is, $]-\infty, 0] \subset F^*(p)$. To that purpose, we take $y_n \doteq -z + \|x_n\|\|p\| \geq \|x_n\|$ given any $x_n \to 0$. Let us deal with $z > 0$. We write $z = \lambda\|p\|^2 - \lambda\|p\|$ for some $\lambda > 0$ and set $x = \lambda p$, $y = \lambda\|p\| = \|x\|$, implying $z = \langle p, x\rangle - y$. Given any sequence $x_n \to x$, we take $y_n \doteq y + \|x_n - x\|\|p\| \geq \|x\| + \|x_n - x\| \geq \|x_n\|$ in (3.2) and check that $(x,y) \in S_F^*(p)$, proving that $z \in F^*(p)$. Hence $]0, +\infty[\subset F^*(p)$ if $\|p\| > 1$. This completes the proof that $F^*(p) = \mathbb{R}$ if $\|p\| > 1$.

(b) The computation of F_* is obvious.

For any given proper function $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$, the following notations will be used in the sequel:

$$f_*(p) \doteq \inf_{x \in \mathrm{dom} f} \Big\{ \langle p, x\rangle - f(x) \Big\}, \; f^*(p) \doteq \sup_{x \in \mathrm{dom} f} \Big\{ \langle p, x\rangle - f(x) \Big\}. \tag{4.1}$$

Here, $\mathrm{dom} f \doteq \{ x \in \mathbb{R}^N : \; f(x) < +\infty \}$. We set $C \doteq \mathrm{dom} f$, $m \doteq -f^*(0) = \inf_C f(x)$, $M \doteq -f_*(0) = \sup_C f(x)$. Assuming these numbers are finite, a simple estimate for the conjugate set-valued map F^* of $F(x) = [f(x), +\infty[$, is obtained from Proposition 3.8:

$$[-f_*(0), +\infty[\subset F(x) \; \subset \; [-f^*(0), +\infty[, \; x \in C. \tag{4.2}$$

Namely,

$$M^*(p) \subset F^*(p) \subset m^*(p), \; p \in \mathbb{R}^N, \tag{4.3}$$

where M^* and m^* are the conjugate of the set-valued maps $M(x) = [-f_*(0), +\infty[$, $m(x) = [-f^*(0), +\infty[$ if $x \in C$ and $M(x) = m(x) = \emptyset$ if $x \notin C$. Specific estimates for F^* can be deduced if some additional assumptions on C are imposed. For instance, if C is a subspace of $\mathbb{R}^N$. Others instances are described in the next theorems.

Theorem 4.2. *Let $C \subset \mathbb{R}^N$ be a non-empty set and $f : C \to \mathbb{R}$ be any function. Let us consider the set-valued map $F : \mathbb{R}^N \to \mathbb{R}$ defined by $F(x) = [f(x), +\infty[$, if $x \in C$, and $F(x) = \emptyset$ elsewhere. Then, for every $p \in \mathbb{R}^N$:*

$$]-\infty, f_*(p)] \; \subset \; F_*(p) \cap F^*(p) \; \subset \; F_*(p) \cup F^*(p) \; \subset]-\infty, f^*(p)]. \tag{4.4}$$

Proof. The inclusion $F_*(p) \cup F^*(p) \subset]-\infty, f^*(p)]$ is trivially verified since any z in $F_*(p) \cup F^*(p)$ can be written as $z = \langle p, x\rangle - y$ for some $y \geq f(x)$, $x \in C$, and therefore $z \leq \langle p, x\rangle - f(x) \leq f^*(p)$. Let us prove the the first inclusion of (4.4). Such an inclusion trivially holds if $f_*(p) = -\infty$. Otherwise, we proceed as follows. Given any $z \leq f_*(p) < +\infty$, we fix $x \in C$ and write $z \doteq \langle p, x\rangle - y \leq f_*(p) \leq \langle p, x\rangle - f(x)$. This implies $y \geq f(x)$. Let x_n be any sequence in C converging to x, $x_n \neq x$, and we set $y_n := y + \langle p, x_n - x\rangle$. Then $y_n = \langle p, x_n\rangle - f_*(p) + (f_*(p) - \langle p, x\rangle + y) \geq f(x_n)$. Thus

$$\frac{\langle p, x_n - x\rangle - y_n + y}{\|x_n - x\|} = 0, \text{ i.e. } z = \langle p, x\rangle - y \in F_*(p) \cap F^*(p).$$

This proves the first inclusion of (4.4) and the proof of the theorem is complete. □

Remark 4.3. It may happen that the inclusions in (4.4) can be strict. In fact, take $f(x) = e^x$ and $F(x) = [e^x, +\infty[$, $x \in \mathbb{R}$. Then $F^*(0) =]-\infty, 0[$ while $f^*(0) = 0$, $f_*(0) = -\infty$. On the other hand, by taking $f(x) = x + e^x$, $F(x) = [f(x), +\infty[$, $x \in \mathbb{R}$, we obtain $f^*(1) = 0$ while $0 \notin F^*(1)$ as one can easily check it.

Next theorem indicates that, under regularity assumptions on f, the inclusions in (4.4) can be refined.

Theorem 4.4. *Let C be a nonempty subset of $\mathbb{R}^N$ and let $f : C \to \mathbb{R}$ be a continuous real single-valued map. Let us define the set-valued map $F : \mathbb{R}^N \to \mathbb{R}$ by $F(x) = [f(x), +\infty[$ if $x \in C$ and $F(x) = \emptyset$ elsewhere. Then, for all $p \in \mathbb{R}^N$:*

$$]-\infty, f^*(p)[\ \subset\ F_*(p) \cap F^*(p)\ \subset\ F_*(p) \cup F^*(p)\ \subset\ ]-\infty, f^*(p)]. \tag{4.5}$$

In addition, $F^*(p) =]-\infty, f^*(p)]$, for every p such that the supremum involved in the definition of $f^*(p)$ is attained (this holds everywhere, for instance, if f satisfies a superlinear growth condition and C is unbounded) and $F^*(p) = F_*(p) =]-\infty, f^*(p)[$ otherwise.

Proof. The third inclusion in (4.5) proved in the previous theorem. To prove the first inclusion, we proceed as follows. If $z < f^*(p)$, then, by the definiton of $f^*(p)$, we obtain $x \in C$ such that $z < \langle p, x\rangle - f(x)$. We write $z = \langle p, x\rangle - y$, which implies $y > f(x)$. Given any sequence x_n in C converging to x, we set $y_n \doteq \langle p, x_n\rangle - z$ and we can assume, because of the continuity of f, that $\langle p, x_n\rangle - f(x_n) > z$ for all $n \in \mathbb{N}$. Thus $y_n \geq f(x_n)$, $y_n \to y$ and

$$\frac{\langle p, x_n - x\rangle - y_n + y}{\|x_n - x\|} = 0.$$

This proves that $z \in F_*(p) \cap F^*(p)$. Let us prove the second assertion of the theorem. In case the supremum involved in the definition of $f^*(p)$ is attained, we take $x \in C$ such that $f^*(p) = \langle p, x\rangle - f(x)$. Thus, if $z = f^*(p) = \langle p, x\rangle - f(x)$, then $(x, f(x)) \in S^*_F(p)$, i.e., $z = f^*(p) \in F^*(p)$. In fact, given any sequence x_n in C converging to x, we have $f(x_n) \to f(x)$ and, because of the choice of x,

$$\frac{\langle p, x_n - x\rangle - f(x_n) + f(x)}{\|x_n - x\|} \leq 0,$$

which proves the second assertion in case the supremum involved in the definition of $f^*(p)$ is attained. In case that this supremum is not attained (being possibly $+\infty$), it is clear that $f^*(p) \notin F_*(p) \cup F^*(p)$, proving the last part of the theorem. □

By Part (a) of Proposition 3.3, we have immediately the following corollary.

Corollary 4.5. *Let C be a nonempty open convex subset of $\mathbb{R}^N$ and let f be a real single-valued map, continuously differentiable in C. Consider the set-valued map $F : \mathbb{R}^N \to \mathbb{R}$ defined by $F(x) = [f(x), +\infty[$ if $x \in C$ and $F(x) = \emptyset$ if $x \notin C$. Then $F_*(p) = F^*(p) =]-\infty, f^*(p)]$, for every p such that the supremum involved in the definition of $f^*(p)$ is attained and $F_*(p) = F^*(p) =]-\infty, f^*(p)[$ otherwise.*

Proof. By the preceding theorem, we only consider the case when $f^*(p)$ is attained and we show that $f^*(p) \in F_*(p) \cap F^*(p)$. In fact, take $x \in C$ such that $f^*(p) = \langle p, x\rangle - f(x)$. Since $p = \nabla f(x)$, for every $x_n \to x$,

$$\lim_{n \to +\infty} \frac{\langle \nabla f(x), x_n - x\rangle - f(x_n) + f(x)}{\|x_n - x\|} = 0.$$

This implies that $(x, f(x)) \in S^*_F(p) \cap S^F_*(p)$. Thus $f^*(p) \in F_*(p) \cap F^*(p)$. □

Example 4.6. Take the function $f(x) = e^x$, $x \in \mathbb{R}$ which satisfies $f^*(0) = 0$, and consider $F(x) = [e^x, +\infty[$. The previous result gives $F^*(0) = F_*(0) =]-\infty, 0[$.

Conclusions A first step was given to define the conjugate of nonconvex set-valued mappings of the real-line. It was with the purpose that its biconjugate coincides with the mapping itself. The case in which the mappings are defined in infinite dimensional spaces with values in another vector normed space will be analyzed in a forthcoming paper.

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