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# HOMOCLINICS OF SUPERQUADRATIC OR ASYMPTOTICALLY QUADRATIC FOURTH ORDER DIFFERENTIAL EQUATIONS

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**Abstract.** In this paper, we study the existence of homoclinic and ground state homoclinic solutions for the fourth order differential equation $u^{(4)}(x)+2q(x)u^{(3)}(x)+\left(q^2(x)+q'(x)+\omega\right)u''(x)+\omega q(x)u'(x)+a(x)u(x)=f(x,u(x))$ when the potential $F(x,u)=\int_0^u f(x,v)dv$ is superquadratic or asymptotically quadratic in the second variable. We apply the critical point theory and variational methods. To the best of our knowledge, the existence of homoclinic solutions of this type of equations was not previously studied.

**Keywords.** Concentration compactness principle; Fourth order differential equation; Homoclinic solutions; Monotonicity tric; variational methods.

## 1. INTRODUCTION

Consider the problem of finding homoclinic solutions for the following fourth-order differential equation

$$u^{(4)}(x)+2q(x)u^{(3)}(x)+\left(q^2(x)+q'(x)+\omega\right)u''(x)+\omega q(x)u'(x)+a(x)u(x)=f(x,u(x)), \qquad (\mathscr{F})$$

where $\omega$ is a constant, $a,q\in C(\mathbb{R},\mathbb{R})$, and $f\in C(\mathbb{R}^2,\mathbb{R})$ are three real functions periodic in the first variable. Here as usual, we say that a solution $u$ of $(\mathscr{F})$ is homoclinic (to 0) if $u\in C^4(\mathbb{R},\mathbb{R})$, $u\neq 0$, and $u(x)\longrightarrow 0$ as $|x|\longrightarrow\infty$. A ground state homoclinic solution is a homoclinic solution that minimizes the energy associated to equation $(\mathscr{F})$ among all homoclinic solutions. Many problems arising in science and engineering call for the solving of partial or ordinary differential equations and systems. These equations are difficult to solve, and there are very few general techniques that can be applied to solve them. In the three last decades, critical point theory and variational methods have been highly successful in solving nonlinear problems in partial and ordinary differential equations. If $q=0$, equation $(\mathscr{F})$ takes the following form $u^{(4)}(x)+\omega u''(x)+a(x)u(x)=f(x,u(x))$ and the existence and multiplicity of homoclinic solutions for this equation have been investigated recently by many mathematicians via critical point theory and variational methods; see, e.g., [1]-[17] and the references therein. However, to the best of our knowledge, there is no research regarding the existence of homoclinic solutions for equation $(\mathscr{F})$. In the present paper, we are interested in the existence of homoclinic and ground state homoclinic solutions for $(\mathscr{F})$ when the nonlinearity $F(x,u)$ is superquadratic or asymptotically quadratic at infinity in the second variable by using the monotonicity trick of Jeanjean and the concentration compactness principle.

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The remaining of this paper is organized as follows. Section 2 is devoted to some preliminary results. In Section 3, we study the existence of ground state homoclinic solution for $(\mathscr{F})$ under superquadratic growth. In Section 4, the last section, we prove the existence of homoclinic solution for $(\mathscr{F})$ under asymptotically quadratic growth.

## 2. Preliminaries

To prove our main result via the critical point theory, we need to establish the variational setting for $(\mathscr{F})$. In the following, since $Q$ is continuous and periodic, then, for all $1 \leq s < \infty$, we can endow $L^s(\mathbb{R})$ with the following equivalent norm

$$\|u\|_{L^s} = \Big( \int_{\mathbb{R}} e^{Q(x)} |u(x)|^s dx \Big)^{\frac{1}{s}}.$$

Let $H^2(\mathbb{R})$ be the Sobolev space with inner product and norm given respectively by

$$< u,v >_{H^2} = \int_{\mathbb{R}} e^{Q(x)} \left[ u''(x)v''(x) + u'(x)v'(x) + u(x)v(x) \right] dx$$

and

$$\|u\|_{H^2} = < u,u >_{H^2}^{\frac{1}{2}}$$

for all $u,v \in H^2(\mathbb{R})$. In this paper, we assume that $a$ satisfies the following condition
$(\mathscr{A})$ $a : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and there exists a constant $a_0$ such that

$$0 < a_0 \leq a(x) \ and \ \omega \leq 2\sqrt{a_0}.$$

**Lemma 2.1.** [18, Lemma 8] *Assume that $a$ satisfies $(\mathscr{A})$. Then there exists a constant $c_0 > 0$ such that*

$$\int_{\mathbb{R}} e^{Q(x)} \left[ u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2 \right] dx \geq c_0 \|u\|_{H^2}^2, \ \forall u \in H^2(\mathbb{R}).$$

By Lemma 2.1, we define

$$E = \left\{ u \in H^2(\mathbb{R}) / \int_{\mathbb{R}} e^{Q(x)} \left[ u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2 \right] dx < \infty \right\}$$

with inner product

$$< u,v > = \int_{\mathbb{R}} e^{Q(x)} \left[ u''(x)v''(x) - \omega u'(x)v'(x) + a(x)u(x)v(x) \right] dx$$

and corresponding norm

$$\|u\| = \Big( \int_{\mathbb{R}} e^{Q(x)} \left[ u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2 \right] dx \Big)^{\frac{1}{2}}.$$

It is easy to verify that $E$ is a Hilbert space. Since $E = H^2(\mathbb{R})$, then it is known that $E$ is continuously embedded into $L^s_Q(\mathbb{R})$ for $2 \leq s \leq \infty$, and $E \hookrightarrow L^s_{loc}(\mathbb{R})$ is compact for all $2 \leq s < \infty$. Hence, there exists a constant $\eta_s > 0$ such that

$$\|u\|_{L^s} \leq \eta_s \|u\|, \quad \forall u \in E. \tag{2.1}$$

Now, we prove that equation $(\mathscr{F})$ possesses a mountain pass type solution. For this purpose, we apply a monotonicity trick due to Jeanjean [5] together with the concentration compactness principle [13].

**Lemma 2.2.** [19] *Let $E$ be a Banach space and $I \subset \mathbb{R}^+$ be an interval. Consider a family $(\Phi_\lambda)_{\lambda \in I}$ of continuously differentiable functionals on $E$ of the form $\Phi_\lambda(u) = A(u) - \lambda B(u)$, $\forall \lambda \in I$, where $B(u) \geq 0$ for all $u \in E$, $A(u) \longrightarrow +\infty$ or $B(u) \longrightarrow +\infty$ as $\|u\| \longrightarrow \infty$. Assume that there exist two points $v_1, v_2 \in E$ such that*

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{x \in [0,1]} \Phi_\lambda(\gamma(x)) > \max\left\{ \Phi_\lambda(v_1), \Phi_\lambda(v_2) \right\}, \ \forall \lambda \in I,$$

*where*

$$\Gamma = \{\gamma \in C([0,1],E)/\gamma(0) = v_1, \gamma(1) = v_2\}.$$

*Then, for almost* $\lambda \in I$*, there is a sequence* $(v_n) \subset E$ *such that*
*(i)* $(v_n)$ *is bounded in* $E$*,*
*(ii)* $\Phi_\lambda(v_n) \longrightarrow c_\lambda$*,*
*(iii)* $\Phi'_\lambda(v_n) \longrightarrow 0$ *on* $E'$*.*
*Moreover, the map* $\lambda \longrightarrow c_\lambda$ *is continuous from the left.*

**Definition 2.3.** Let $(u_n)$ be a bounded sequence in a Banach space. We say that $(u_n)$ is vanishing if, for each $R > 0$,

$$\lim_{n\longrightarrow\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} e^{Q(x)} |u_n|^2 dx = 0,$$

and $(u_n)$ is nonvanishing if there exist $\sigma > 0$, $R > 0$, and $(y_n) \subset \mathbb{R}$ such that

$$\liminf_{n\longrightarrow\infty} \int_{y_n-R}^{y_n+R} e^{Q(x)} |u_n|^2 dx \geq \sigma.$$

In the vanishing case, we have the following result, which is a special case of Lions [20].

**Lemma 2.4.** [20] *Let* $(u_n)$ *be a bounded sequence, if, for any* $R > 0$ $\lim_{n\longrightarrow\infty} \sup_{y\in\mathbb{R}} \int_{y-R}^{y+R} e^{Q(x)} |u_n|^2 dx = 0$*, then* $u_n \longrightarrow 0$ *in* $L^s(\mathbb{R})$ *for* $2 < s < \infty$*.*

## 3. Superquadratic Growth

In this Section, we are concerned with the existence of ground state homoclinic solutions to the fourth order differential equation ($\mathscr{F}$), where $a, q \in C(\mathbb{R},\mathbb{R})$ and $f \in C(\mathbb{R}^2,\mathbb{R})$ are such that $a(x), Q(x) = \int_0^x q(s)ds$ and $f(x,u)$ are $T-$periodic in $x$ and $F(x,u) = \int_0^u f(x,v)dv$ is superquadratic in the second variable not satisfying the global ($\mathscr{AR}$) superquadratic condition. More precisely, we take the following conditions
$(F_1)$ $f(x,u) = o(|u|)$ as $u \longrightarrow 0$ uniformly in $x$, $F(x,0) = 0$, and $F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^2$,
$(F_2)$ there exist constants $p > 2$ and $C_0 > 0$ such that

$$|f(x,u)| \leq C_0\left(1 + |u|^{p-1}\right),\ \forall (x,u) \in \mathbb{R}^2,$$

$$(F_3) \qquad \lim_{|u|\longrightarrow\infty} \frac{F(x,u)}{|u|^2} = +\infty,\ for\ a.e.\ x \in \mathbb{R},$$

$(F_4)$ there exists a constant $\sigma \geq 1$ such that

$$\widetilde{F}(x,su) \leq \sigma \widetilde{F}(x,u),\ \forall (s,x,u) \in [0,1] \times \mathbb{R}^2,$$

where $\widetilde{F}(x,u) = \frac{1}{2} f(x,u)u - F(x,u)$.

Our main result in this section reads as follows.

**Theorem 3.1.** *Assume that* ($\mathscr{A}$) *and* $(F_1) - (F_4)$ *are satisfied. Then equation* ($\mathscr{F}$) *possesses at least one ground state homoclinic solution.*

It is important to notice that conditions $(F_2)$ and $(F_4)$ imply that $F(x,u)$ is superquadratic both at the origin and at infinity, which is different from the ($\mathscr{AR}$)- condition.

Let us state the following example to illustrate our Theorem 3.1.

**Example 3.2.** Let $a(x) = \frac{3}{2} + cos\left(\frac{2\pi}{T}x\right)$ and $F(x,u) = \left(1 + sin\left(\frac{2\pi}{T}x\right)\right)|u|^2 ln\left(1 + |u|^2\right)$ for all $(x,u) \in \mathbb{R}^2$. It is easy to check that $a$ and $F$ satisfy all the conditions of Theorem 3.1. However, since $F\left(\frac{3T}{4}, u\right) = 0$ for all $u \in \mathbb{R}$, it does not satisfy the ($\mathscr{AR}$)-condition.

3.1. **Proof of Theorem 3.1.** Now we are in a position to establish the corresponding variational framework to obtain the existence of ground state homoclinic solution for ($\mathscr{F}$). For this end, define the energy functional $\Phi$ associated to equation ($\mathscr{F}$)

$$\Phi(u)=\frac{1}{2}\int_{\mathbb{R}}e^{Q(x)}\left[u^{''}(x)^2-\omega u^{'}(x)^2+a(x)u(x)^2\right]dx-\int_{\mathbb{R}}e^{Q(x)}F(x,u(x))dx$$

on the Hilbert space $E$ introduced in Section 2. By $(F_1)$ and $(F_2)$, for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$|f(x,u)|\leq\varepsilon|u|+C_\varepsilon|u|^{p-1} \tag{3.1}$$

and

$$0\leq F(x,u)\leq\frac{\varepsilon}{2}|u|^2+\frac{C_\varepsilon}{p}|u|^p \tag{3.2}$$

for all $(x,u)\in\mathbb{R}^2$. Hence, it is known that $\Phi\in C^1(E,\mathbb{R})$ and

$$\begin{aligned}\Phi'(u)v&=\int_{\mathbb{R}}e^{Q(x)}\left[u''(x)v''(x)-\omega u'(x)v'(x)+a(x)u(x)v(x)\right]dx-\int_{\mathbb{R}}e^{Q(x)}f(x,u(x))v(x)dx\\&=<u,v>-\int_{\mathbb{R}}e^{Q(x)}f(x,u(x))v(x)dx\end{aligned}$$

for all $u,v\in E$. Moreover, the nontrivial critical points of $\Phi$ on $E$ are homoclinic solutions of ($\mathscr{F}$). Now, we define on $E$ the family of functionals

$$\Phi_\lambda(u)=A(u)-\lambda B(u),\ \lambda\in[1,2],$$

where

$$A(u)=\frac{1}{2}\int_{\mathbb{R}}e^{Q(x)}\left[u^{''}(x)^2-\omega u^{'}(x)^2+a(x)u(x)^2\right]dx$$

and

$$B(u)=\int_{\mathbb{R}}e^{Q(x)}F(x,u(x))dx.$$

We also present some lemmas which are used in the subsequent discussion.

**Lemma 3.3.** *Assume that* ($\mathscr{A}$) *and* $(F_1)-(F_3)$ *are satisfied. Then*
*(i) There exists* $u_0\in E\setminus\{0\}$ *such that* $\Phi_\lambda(u_0)<0$*, for all* $\lambda\in[1,2]$*,*
*(ii)*
$$c_\lambda=\inf_{\gamma\in\Gamma}\max_{x\in[0,1]}\Phi_\lambda(\gamma(x))>\max\{\Phi_\lambda(v_1),\Phi_\lambda(v_2)\},\ \forall\lambda\in[1,2],$$

*where*

$$\Gamma=\{\gamma\in C([0,1],E)/\gamma(0)=0,\gamma(1)=u_0\}.$$

*Proof.* (i) Let $e_0\in C_0^\infty(\mathbb{R})\setminus\{0\}$. By $(F_3)$, the fact $F(x,u)\geq0$ and Fatou's lemma, we have

$$\lim_{s\longrightarrow\infty}\frac{\Phi_\lambda(se_0)}{s^2}\leq\lim_{s\longrightarrow\infty}\frac{\Phi_1(se_0)}{s^2}=\frac{1}{2}\|e_0\|^2-\lim_{s\longrightarrow\infty}\int_{e_0\neq0}e^{Q(x)}\frac{F(x,se_0)}{|se_0|^2}|e_0|^2dx\leq-\infty,$$

for all $\lambda\in[1,2]$. Hence, there is $s_0>0$ large enough such that $\Phi_1(s_0e_0)<0$. Then, setting $u_0=s_0e_0$, we obtain $\Phi_\lambda(u_0)\leq\Phi_1(u_0)<0$ and (i) holds.

(ii) By (2.1) and (3.2), we have

$$\int_{\mathbb{R}}e^{Q(x)}F(x,u)dx\leq\frac{\varepsilon}{2}\|u\|_{L^2}^2+\frac{C_\varepsilon}{p}\|u\|_{L^p}^p\leq\frac{\varepsilon\eta_2^2}{2}\|u\|^2+\frac{\eta_p^pC_\varepsilon}{p}\|u\|^p.$$

Hence

$$\Phi_\lambda(u)\geq\Phi_2(u)\geq\left(\frac{1}{2}-\varepsilon\eta_2^2\right)\|u\|^2-\frac{2\eta_p^pC_\varepsilon}{p}\|u\|^p.$$

By taking $\varepsilon$ small enough, we deduce that there exist constants $\alpha>0$ and $0<\rho<\|u_0\|$ such that

$$\Phi_{\lambda_{|\partial B_\rho(0)}}\geq\alpha\ \textit{for all}\ \lambda\in[1,2],\ \textit{where}\ B_\rho=\{u\in E/\|u\|<\rho\}.$$

Let $\Gamma = \{\gamma \in C([0,1],E)/\gamma(0) = 0, \gamma(1) = u_0\}$. Since, for any $\gamma \in \Gamma$, we have $\gamma(0) = 0 < \rho < \gamma(1) = \|u_0\|$, then there exists $x_\gamma \in ]0,1[$ such that $\rho = \gamma(x_\gamma)$, and then

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{x\in[0,1]} \Phi_\lambda(\gamma(x)) \geq \alpha > \max\{\Phi_\lambda(0), \Phi_\lambda(u_0)\}.$$

The proof of Lemma 3.3 is completed. □

Combining Lemma 2.1 and Lemma 3.3, we obtain the following lemma.

**Lemma 3.4.** *Assume that $(\mathscr{A})$ and $(F_1)-(F_3)$ are satisfied. Then, for any $\lambda \in [1,2]$, there exists a bounded sequence $(u_n) \subset E$ such that $\Phi_\lambda(u_n) \longrightarrow c_\lambda$ and $\Phi'_\lambda(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$.*

**Lemma 3.5.** *Assume that $(\mathscr{A})$, $(F_1)$, and $(F_2)$ are satisfied. Then, for any bounded vanishing sequence $(u_n) \subset E$, we have $\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} e^{Q(x)} \widetilde{F}(x,u_n)dx = 0$.*

*Proof.* Using (3.1) and (3.2), we obtain

$$\int_{\mathbb{R}} e^{Q(x)} F(x,u_n)dx \leq \frac{\varepsilon}{2} \|u_n\|_{L^2}^2 + \frac{C_\varepsilon}{p} \|u_n\|_{L^p}^p$$

and

$$\left| \int_{\mathbb{R}} e^{Q(x)} f(x,u_n)u_n dx \right| \leq \varepsilon \|u_n\|_{L^2}^2 + C_\varepsilon \|u_n\|_{L^p}^p.$$

Since $(u_n)$ is vanishing, Lemma 2.4 implies that

$$\int_{\mathbb{R}} e^{Q(x)} F(x,u_n)dx \longrightarrow 0 \text{ and } \int_{\mathbb{R}} e^{Q(x)} f(x,u_n)u_n dx \longrightarrow 0$$

as $n \longrightarrow \infty$, and the proof of Lemma 3.5 is completed. □

**Lemma 3.6.** *Assume that $(\mathscr{A})$, $(F_1)$, $(F_2)$, and $(F_4)$ are satisfied. Then, for all bounded sequence $(u_n) \subset E$ satisfying*

$$0 < \lim_{n\longrightarrow\infty} \Phi_\lambda(u_n) \leq c_\lambda \text{ and } \lim_{n\longrightarrow\infty} \Phi'_\lambda(u_n) = 0,$$

*there is $(y_n) \subset \mathbb{Z}$ such that, up to a subsequence, $\widetilde{u}_n(x) = u_n(x + y_n T)$ satisfies*

$$\widetilde{u}_n \rightharpoonup u_\lambda \neq 0,\ \Phi_\lambda(u_\lambda) \leq c_\lambda \text{ and } \Phi'_\lambda(u_\lambda) = 0.$$

*Proof.* Since $\Phi'_\lambda(u_n)u_n \longrightarrow 0$, one has

$$\lim_{n\longrightarrow\infty} \lambda \int_{\mathbb{R}} e^{Q(x)} \widetilde{F}(x,u_n)dx = \lim_{n\longrightarrow\infty} \left( \Phi_\lambda(u_n) - \frac{1}{2}\Phi'_\lambda(u_n)u_n \right) = \lim_{n\longrightarrow\infty} \Phi_\lambda(u_n) > 0$$

which together with Lemma 3.5 implies that $(u_n)$ is nonvanishing. Hence, there exist constants $\sigma > 0$, $R > 0$, and a subsequence $(\widetilde{y}_n) \subset \mathbb{R}$ such that

$$\liminf_{n\longrightarrow\infty} \int_{\widetilde{y}_n - R}^{\widetilde{y}_n + R} e^{Q(x)} |u_n|^2 dx \geq \sigma > 0.$$

Letting $\widetilde{u}_n(x) = u_n(x + y_n T)$, we choose $(y_n) \subset \mathbb{Z}$ such that

$$\liminf_{n\longrightarrow\infty} \int_{-2R}^{2R} e^{Q(x)} |\widetilde{u}_n|^2 dx \geq \frac{\sigma}{2} > 0. \tag{3.3}$$

Since $a(x)$ and $F(x,u)$ are $T-$periodic in $x$, then $\|\widetilde{u}_n\| = \|u_n\|$, $\Phi_\lambda(\widetilde{u}_n) = \Phi_\lambda(u_n)$, and

$$\Phi'_\lambda(\widetilde{u}_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.4}$$

Indeed, for any $v\in E$, set $v_n(x)=v(x-y_nT)$. It is clear that $\|v_n\|=\|v\|$ and

$$\begin{aligned}\left|\Phi_\lambda'(\widetilde{u}_n)v\right| &= \left|\int_{\mathbb{R}} e^{Q(x)}\left[\widetilde{u}_n''v''-\omega\widetilde{u}_n'v'+a\widetilde{u}_nv-\lambda f(x,\widetilde{u}_n)v\right]dx\right|\\ &= \left|\int_{\mathbb{R}} e^{Q(x)}\left[u_n''v_n''-\omega u_n'v_n'+au_nv_n-\lambda f(x,u_n)v_n\right]dx\right|\\ &= \left|\Phi_\lambda'(u_n)v_n\right|\le \left\|\Phi_\lambda'(u_n)\right\|\|v_n\|=\left\|\Phi_\lambda'(u_n)\right\|\|v\|\longrightarrow 0,\end{aligned}$$

which implies (3.4). Since $(\widetilde{u}_n)$ is still bounded, up to a subsequence if necessary, there exists $u_\lambda\in E$ such that

$$\begin{aligned}&\widetilde{u}_n \rightharpoonup u_\lambda \ in\ E,\\ &\widetilde{u}_n \longrightarrow u_\lambda \ in\ L^s_{loc}(\mathbb{R})\ for\ s\in]2,\infty[,\\ &\widetilde{u}_n \longrightarrow u_\lambda \ a.e.\ in\ \mathbb{R},\end{aligned} \tag{3.5}$$

and $u_\lambda\neq 0$ by (3.3). We claim that for all compact $K\subset\mathbb{R}$, and $f(x,\widetilde{u}_n)\longrightarrow f(x,u_\lambda)$ in $L^2(K)$. Arguing indirectly, we may assume that there exist a constant $\varepsilon_0>0$ and a subsequence $(\widetilde{u}_{n_k})$ such that

$$\int_K e^{Q(x)}\left|f(x,\widetilde{u}_{n_k})-f(x,u_\lambda)\right|^2dx\ge\varepsilon_0,\ \forall k\in\mathbb{N}. \tag{3.6}$$

By (3.5), we can assume that $\sum_{k=1}^\infty\|\widetilde{u}_{n_k}-u_\lambda\|_{L^2(K)}<\infty$ and $\sum_{k=1}^\infty\|\widetilde{u}_{n_k}-u_\lambda\|_{L^{2(p-1)}(K)}<\infty$. Let $w(x)=\sum_{k=1}^\infty|\widetilde{u}_{n_k}(x)-u_\lambda(x)|$ for all $x\in K$. Then $w\in L^2(K)\bigcap L^{2(p-1)}(K)$. By (3.1), there holds for all $k\in\mathbb{N}$ and $x\in\mathbb{R}$

$$\begin{aligned}\left|f(x,\widetilde{u}_{n_k})-f(x,u_\lambda)\right|^2 &\le 2\Big(\left|f(x,\widetilde{u}_{n_k})\right|^2+\left|f(x,u_\lambda)\right|^2\Big)\\ &\le 2\Big[\Big(\varepsilon|\widetilde{u}_{n_k}|+C_\varepsilon|\widetilde{u}_{n_k}|^{p-1}\Big)^2+[\Big(\varepsilon|u_\lambda|+C_\varepsilon|u_\lambda|^{p-1}\Big)^2\Big]\\ &\le C_1\Big[|\widetilde{u}_{n_k}|^2+|\widetilde{u}_{n_k}|^{2(p-1)}+|u_\lambda|^2+|u_\lambda|^{2(p-1)}\Big]\\ &\le C_1\Big[\Big(|\widetilde{u}_{n_k}-u_\lambda|+|u_\lambda|\Big)^2+\Big(|\widetilde{u}_{n_k}-u_\lambda|\\ &\quad+|u_\lambda|\Big)^{2(p-1)}+|u_\lambda|^2+|u_\lambda|^{2(p-1)}\Big]\\ &\le C_2\Big[|w|^2+|w|^{2(p-1)}+|u_\lambda|^2+|u_\lambda|^{2(p-1)}\Big]\end{aligned}$$

where $C_1,C_2$ are positive constants. Combining this with (3.5), Lebesgue's Dominated Convergence Theorem implies $\lim_{k\longrightarrow\infty}\int_K e^{Q(x)}\left|f(x,\widetilde{u}_{n_k})-f(x,u_\lambda)\right|^2dx=0$, which contradicts (3.6). Hence the claim above is true. It follows that

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}}e^{Q(x)}\Big(f(x,\widetilde{u}_n)-f(x,u_\lambda)\Big)\psi dx=0,\ \forall\psi\in C_0^\infty(\mathbb{R},\mathbb{R}^N),$$

which implies that $\Phi_\lambda'$ is weakly sequentially continuous. Hence, by (3.4), we deduce

$$\Phi_\lambda'(u_\lambda)=0. \tag{3.7}$$

Now, by $(F_4)$ and Fatou's lemma, one obtains

$$\begin{aligned} c_\lambda &\geq \lim_{n\longrightarrow\infty}\left(\Phi_\lambda(\widetilde{u}_n)-\frac{1}{2}\Phi_\lambda'(\widetilde{u}_n)\widetilde{u}_n\right)\\ &=\lim_{n\longrightarrow\infty}\lambda\int_{\mathbb{R}}e^{Q(x)}\widetilde{F}(x,\widetilde{u}_n)dx\\ &\geq\lambda\int_{\mathbb{R}}e^{Q(x)}\widetilde{F}(x,u_\lambda)dx\\ &=\Phi_\lambda(u_\lambda)-\frac{1}{2}\Phi_\lambda'(u_\lambda)u_\lambda\\ &=\Phi_\lambda(u_\lambda). \end{aligned}$$

The proof of Lemma 3.6 is completed. □

As a consequence of Lemma 2.4 and Lemma 3.6, we have the following lemma.

**Lemma 3.7.** *Assume that* $(\mathscr{A})$, $(F_1)$, $(F_2)$, *and* $(F_4)$ *are satisfied. Then there exist* $(\lambda_n)\subset[1,2]$ *and* $(u_n)\subset E\setminus\{0\}$ *such that*

$$\lambda_n\longrightarrow 1,\ \Phi_{\lambda_n}(u_n)\leq c_{\lambda_n}\ \textit{and}\ \Phi_{\lambda_n}'(u_n)=0.$$

**Lemma 3.8.** *The sequence* $(u_n)$ *obtained in Lemma 3.7 is bounded.*

*Proof.* Suppose by contradiction that $\|u_n\|\longrightarrow\infty$ as $n\longrightarrow\infty$. Set $w_n=\frac{u_n}{\|u_n\|}$. Then $\|w_n\|=1$, and by Lion's concentration compactness principle [13], either $(w_n)$ is vanishing or it is nonvanishing. Hence the proof of the lemma will be completed if we show that $(w_n)$ is neither vanishing nor nonvanishing. Assume that $(w_n)$ is vanishing. Let $(s_n)\subset[0,1]$ be a sequence such that $\Phi_{\lambda_n}(s_nu_n)=\max_{s\in[0,1]}\Phi_{\lambda_n}(su_n)$. For any $M>0$, let $v_n=\left(\frac{2\sqrt{M}}{\|u_n\|}\right)u_n=2\sqrt{M}w_n$. Since $(v_n)$ is vanishing and bounded, by Lemma 3.5 and (3.1), one has

$$\int_{\mathbb{R}}e^{Q(x)}\widetilde{F}(x,v_n)dx\longrightarrow 0\ as\ n\longrightarrow\infty.$$

Now, for $n$ large enough, $\frac{2\sqrt{M}}{\|u_n\|}\in]0,1[$, and by the definition of $s_n$, we deduce that

$$\Phi_{\lambda_n}(s_nu_n)\geq\Phi_{\lambda_n}(v_n)=2M-\lambda_n\int_{\mathbb{R}}e^{Q(x)}F(x,v_n)dx\geq M,$$

which implies that

$$\Phi_{\lambda_n}(s_nu_n)\longrightarrow+\infty\ as\ n\longrightarrow\infty. \tag{3.8}$$

Since $\Phi_{\lambda_n}(0)=0$ and $\Phi_{\lambda_n}(u_n)\leq c_{\lambda_n}\leq c_1$, then $s_n\in]0,1[$ and

$$\Phi_{\lambda_n}'(s_nu_n)s_nu_n=s_n\frac{d}{ds}\Big(\Phi_{\lambda_n}(su_n)\Big)_{|s=s_n}=0. \tag{3.9}$$

Therefore, using (3.8) and (3.9), we deduce that

$$\begin{aligned} \int_{\mathbb{R}}e^{Q(x)}\widetilde{F}(x,s_nu_n)dx &=\frac{1}{\lambda_n}\left(\Phi_{\lambda_n}(s_nu_n)-\frac{1}{2}\Phi_{\lambda_n}'(s_nu_n)s_nu_n\right)\\ &=\frac{1}{\lambda_n}\Phi_{\lambda_n}(s_nu_n)\longrightarrow+\infty\ as\ n\longrightarrow\infty. \end{aligned}$$

However, it follows from $(F_4)$ and Lemma 3.7 that

$$\begin{aligned} \int_{\mathbb{R}}e^{Q(x)}\widetilde{F}(x,s_nu_n)dx &\leq\sigma\int_{\mathbb{R}}e^{Q(x)}\widetilde{F}(x,u_n)dx\\ &\leq\frac{\sigma}{\lambda_n}\left[\Phi_{\lambda_n}(u_n)-\frac{1}{2}\Phi_{\lambda_n}'(u_n)u_n\right]\\ &=\frac{\sigma}{\lambda_n}\Phi_{\lambda_n}(u_n)\leq\frac{\sigma}{\lambda_n}c_{\lambda_n}\leq\sigma c_1,\ \forall n\in\mathbb{N}, \end{aligned}$$

yielding a contradiction.

Assume that $(w_n)$ is nonvanishing. Then, as in the proof of (3.6), by the translation invariance of equation $(\mathscr{F})$, one has $w_n \rightharpoonup w$ in $E$ and $w_n(x) \longrightarrow w(x)$ a.e. in $\mathbb{R}$ for some $w \in E \setminus \{0\}$. On the set $\Omega = \{x \in \mathbb{R}/w(x) \neq 0\}$, one has $|u_n(x)| \longrightarrow +\infty$. It follows from $(F_3)$ that

$$\frac{F(x,u_n)}{|u_n|^2}|w_n|^2 \longrightarrow +\infty \text{ as } n \longrightarrow \infty.$$

Therefore, taking into account $meas(\Omega) > 0$ and using Fatou's lemma, we obtain

$$\int_{\mathbb{R}} e^{Q(x)}\frac{F(x,u_n)}{\|u_n\|^2}dx \geq \int_{\Omega} e^{Q(x)}\frac{F(x,u_n)}{|u_n|^2}|w_n|^2 dx \longrightarrow +\infty \text{ as } n \longrightarrow \infty.$$

On the other hand, since $0 \leq \Phi_{\lambda_n}(u_n) \leq c_{\lambda_n} \leq c_1$, we deduce that

$$\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} e^{Q(x)}\frac{F(x,u_n)}{\|u_n\|^2}dx = \frac{1}{2},$$

a contradiction. The proof of Lemma 3.8 is completed. □

Now, we are in the position to complete the proof of Theorem 3.1. by Lemma 3.5, Lemma 3.6, and (3.1), we have, for any $v \in E$,

$$\Phi'(u_n)v = \Phi'_{\lambda_n}(u_n)v + (\lambda_n - 1)\int_{\mathbb{R}} e^{Q(x)}f(x,u_n)vdx \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence $\Phi'(u_n) \longrightarrow 0$. Combining (3.1) and Lemma 3.8 yields

$$\begin{aligned}\|u_n\|^2 &= \lambda_n \int_{\mathbb{R}} e^{Q(x)}f(x,u_n)u_n dx \\ &\leq \varepsilon\|u_n\|_{L^2}^2 + C_\varepsilon\|u_n\|_{L^p}^p \\ &\leq \varepsilon\eta_2^2\|u_n\|^2 + C_\varepsilon\eta_p^p\|u_n\|^p\end{aligned} \tag{3.10}$$

which implies that $\|u_n\| \geq d, \forall n \in \mathbb{N}$ for some constant $d > 0$. If $(u_n)$ is vanishing, Lemma 2.4 and (3.10) imply that $u_n \longrightarrow 0$, a contradiction. Hence $(u_n)$ is nonvanishing. Proceeding as in the proof of Lemma 3.7, we can obtain a sequence $(y_n) \subset \mathbb{Z}$ such that if $\widetilde{u}_n(x) = u_n(x + y_n T)$, then $\widetilde{u}_n \longrightarrow \widetilde{u}$ and $\Phi'(\widetilde{u}) = 0$. Therefore equation $(\mathscr{F})$ possesses a nontrivial homoclinic solution.

Finally, we prove the existence of ground state homoclinic solution of $(\mathscr{F})$. Set

$$K = \{u \in E \setminus \{0\}/\Phi'(u) = 0\}$$

and $m = \inf_K \Phi(u)$. Using $(F_4)$ and the fact $\widetilde{u} \in K$, we obtain $0 \leq m \leq f(\widetilde{u})$. By the definition of $m$, there exists a sequence $(v_n) \subset K$ such that $f(v_n) \longrightarrow m$ as $n \longrightarrow \infty$. Following the same procedures as in the proof of Lemma 3.8, we have that $(v_n)$ is bounded. Since $(v_n) \subset K$ and $\Phi'(v_n) = 0$, similar to (3.10), we obtain that $\|v_n\| \geq d_1 > 0$ for all $n$ and $(v_n)$ is nonvanishing. Hence, arguing as in (3.5)-(3.7), there exists $\widetilde{v} \in E \setminus \{0\}$ such that $\Phi'(\widetilde{v}) = 0$ and $\Phi(\widetilde{v}) \leq m$. Noting that $\widetilde{v} \in K$, one has $\Phi(\widetilde{v}) \geq m$. Thus $\Phi(\widetilde{v}) = m$. This ends the proof.

## 4. ASYMPTOTICALLY QUADRATIC CASE

In this Section, we are concerned with the existence of homoclinic solution for the fourth order differential equation $(\mathscr{F})$, where $a, q \in C(\mathbb{R},\mathbb{R})$ and $f \in C(\mathbb{R}^2,\mathbb{R})$ are such that $a(x), Q(x) = \int_0^x q(s)ds$ and $f(x,u)$ are $T-$periodic in $x$, and $F(x,u) = \int_0^u f(x,v)dv$ is asymtotically quadratic with respect to the second variable. More precisely, we assume that $F(x,u)$ is of the form $F(x,u) = \frac{1}{2}S|u|^2 + V(x,u)$, where $S$ is a positive constant and we take the following assumptions:

$(F_5)$ $$F(x,u) \geq 0,\ \forall (x,u) \in \mathbb{R}^2,$$

$(F_6)$ there exist positive constants $c, r$ and $p > 2$ such that

$$|f(x,u)| \leq c\,|u|^{p-1},\ \forall x \in \mathbb{R},\ |u| \leq r,$$

$(F_7)$
$$V(x,u) = o(|u|)\ as\ |u| \longrightarrow \infty,$$

$(F_8)$ $\widetilde{F}(x,u) = \frac{1}{2}f(x,u)u - F(x,u) \geq 0$ for all $(x,u) \in \mathbb{R}^2$, and there are positive constants $a_1, b_1 > 0$, $R > r$ and $\alpha \in ]1,2[$ such that

$$\widetilde{F}(x,u) \geq \begin{cases} a_1\,|u|^p,\ \forall x \in \mathbb{R},\ |u| \leq r, \\ b_1\,|u|^\alpha,\ \forall x \in \mathbb{R},\ |u| \geq R. \end{cases}$$

Before stating the main result of this section, we consider the eigenvalue problem

$$\begin{cases} \frac{d^4u}{dx^4} + 2q\frac{d^3u}{dx^3} + (q^2 + q' + \omega)\omega\frac{d^2u}{dx^2} + \omega\frac{du}{dx} + au = \lambda u, \\ u \in E, \end{cases}$$

which has a non-decreasing sequence of positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$.

**Theorem 4.1.** *Assume that* $(\mathscr{A})$, $(F_5) - (F_8)$ *and the following condition*

$$S > \lambda_1 \tag{C}$$

*are satisfied. Then equation* $(\mathscr{F})$ *possesses at least one homoclinic solution.*

**Remark 4.2.** Let

$$F(x,u) = \begin{cases} \left(\frac{1}{2}S - \theta(x)\right)|x|^p\ if\ |x| \leq 1, \\ \frac{1}{2}S\,|x|^2 - \theta(x)\,|x|^\alpha\ if\ |x| \geq 1, \end{cases}$$

where $\theta \in C(\mathbb{R},\mathbb{R})$ is periodic in $t$, $0 < \inf_{x\in\mathbb{R}} \theta(x) \leq \sup_{x\in\mathbb{R}} \theta(x) < \frac{S}{2}$, and $1 < \alpha < 2 < p$. It is easy to check that the above function $F$ satisfies conditions $(F_5) - (F_8)$.

4.1. **Proof of Theorem 4.1.** Consider the functional $\Phi$ defined on the space $E$ introduced in Section 2 by

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(x)}\left[u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2\right]dx - \int_{\mathbb{R}} e^{Q(x)}F(x,u(x))dx \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} e^{Q(x)}F(x,u(x))dx. \end{aligned}$$

It is known that $\Phi$ is continuously differentiable on $E$. For all $u, v \in E$, we have

$$\begin{aligned} \Phi'(u)v &= \int_{\mathbb{R}} e^{Q(x)}\left[u''(x)v''(x) - \omega u'(x)v'(x) + a(x)u(x)v(x)\right]dx - \int_{\mathbb{R}} e^{Q(x)}f(x,u(x))v(x)dx \\ &= < u,v > - \int_{\mathbb{R}} e^{Q(x)}f(x,u(x))v(x)dx \end{aligned}$$

Moreover, the critical points of $\Phi$ are classical solutions of $(\mathscr{F})$ satisfying $\dot{u}(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.

In the following, we will reason by successive lemmas.

**Lemma 4.3.** *Assume that* $(\mathscr{A})$, $(F_6)$, *and* $(F_7)$ *are satisfied. Then, for any bounded vanishing sequence* $(u_n) \in E$, *we have* $\lim_{n\longrightarrow\infty}\int_{\mathbb{R}} e^{Q(x)}\widetilde{F}(x,u_n(x))dx = 0$.

*Proof.* By $(F_6)$ and $(F_7)$, for every $\varepsilon > 0$, there exists a positive constant $C_\varepsilon$ such that

$$|f(x,u)| \leq \varepsilon\,|u| + C_\varepsilon\,|u|^{p-1},\ \forall(x,u) \in \mathbb{R}^2. \tag{4.1}$$

Since $F(x,0) = 0$, we deduce

$$|F(x,u)| \leq \frac{\varepsilon}{2}\,|u|^2 + \frac{C_\varepsilon}{p}\,|u|^p,\ \forall(x,u) \in \mathbb{R}^2. \tag{4.2}$$

Let $(u_n) \in E$ be a bounded vanishing sequence. Then Lemma 2.4 implies that $u_n \longrightarrow 0$ in $L^s(\mathbb{R})$ for all $s \in ]2,\infty[$. Combining this with (4.1), (4.2), and Hölder's inequality yields

$$\left|\int_{\mathbb{R}} e^{Q(x)} f(x,u_n(x))u_n(x)dx\right| \leq \varepsilon \|u_n\|_{L^2}^2 + C_\varepsilon \|u_n\|_{L^p}^p \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and

$$\int_{\mathbb{R}} e^{Q(x)} F(x,u_n(x)) \leq \frac{\varepsilon}{2} \|u_n\|_{L^2}^2 + \frac{C_\varepsilon}{p} \|u_n\|_{L^p}^p \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence $\int_{\mathbb{R}} e^{Q(x)} \widetilde{F}(x,u_n(x))dx \longrightarrow 0$ as $n \longrightarrow \infty$ and the proof of the lemma is completed. □

In the following, we define on $E$ the family of functionals

$$\Phi_\lambda(u) = A(u) - \lambda B(u),\ \lambda \in [1,2]$$

where

$$A(u) = \frac{1}{2}\int_{\mathbb{R}} e^{Q(x)} \left[u''(x)^2 - \omega u'(x)^2 + a(x)u(x)^2\right]dx$$

and

$$B(u) = \int_{\mathbb{R}} e^{Q(x)} F(x,u(x))dx.$$

**Lemma 4.4.** *Assume that $(\mathscr{A})$, $(C)$ and $(F_7)$ are satisfied, then there exists $v_0 \in E \setminus \{0\}$ such that $\Phi_1(v_0) = \Phi(v_0) < 0$.*

*Proof.* Since $S > \lambda_1$, we can choose a nonnegative function $\varphi \in E$ such that

$$\int_{\mathbb{R}} e^{Q(x)} |\varphi(x)|^2 dx = 1 \text{ and } \int_{\mathbb{R}} e^{Q(x)} \left[\varphi''(x)^2 - \omega\varphi'(x)^2 + a(x)\varphi(x)^2\right]dx < S.$$

Assumption $(W_7)$ implies that, for all $x \in \mathbb{R}$ with $\varphi(x) \neq 0$,

$$\lim_{s\longrightarrow\infty} \frac{F(x,s\varphi(x)}{s^2} = \lim_{s\longrightarrow\infty} \frac{F(x,s\varphi(x)}{|s\varphi(x)|^2} |s\varphi(x)|^2 = \frac{1}{2}S|\varphi(x)|^2,$$

which together with $(C)$ and Fatou's lemma implies

$$\begin{aligned}
\lim_{s\longrightarrow\infty} \frac{\Phi(s\varphi)}{s^2} &= \frac{1}{2}\int_{\mathbb{R}} e^{Q(x)} \left[\varphi''(x)^2 - \omega\varphi'(x)^2 + a(x)\varphi(x)^2\right]dx - \lim_{s\longrightarrow\infty}\int_{\mathbb{R}} e^{Q(x)} \frac{F(x,s\varphi(x))}{s^2}dx \\
&< \frac{S}{2} - \int_{\mathbb{R}} e^{Q(x)} \lim_{s\longrightarrow\infty} \frac{F(x,s\varphi(x))}{s^2}dx \\
&< \frac{S}{2} - \int_{\mathbb{R}} e^{Q(x)} \frac{S}{2}|\varphi(x)|^2 dx = 0.
\end{aligned}$$

Consequently, there exists a positive constant $s_0$ large enough such that the element $v_0 = s_0\varphi$ satisfies $v_0 \neq 0$ and $\Phi(v_0) < 0$. The proof of Lemma 4.4 is completed. □

Now, let

$$c_\lambda = \inf_{\gamma\in\Gamma} \sup_{x\in[0,1]} \Phi_\lambda(\gamma(x))$$

where

$$\Gamma = \{\gamma \in C([0,1],E)/\gamma(0) = 0, \gamma(1) = v_0\}.$$

**Lemma 4.5.** *Assume that $(\mathscr{A})$ and $(F_6)-(F_8)$ are satisfied. Then for any sequence $(u_n) \subset E$ satisfying*

$$0 < \lim_{n\longrightarrow\infty} \Phi_\lambda(u_n) \leq c_\lambda \text{ and } \Phi'_\lambda(u_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty,$$

*there exists a subsequence $(u_{n_k})$ such that*

$$u_{n_k} \rightharpoonup u_\lambda \neq 0 \text{ with } \Phi_\lambda(u_\lambda) \leq c_\lambda \text{ and } \Phi'_\lambda(u_\lambda) = 0.$$

*Proof.* Note that

$$\int_{\mathbb{R}} e^{Q(x)}\widetilde{F}(x,u_n(x))dx = \frac{1}{\lambda}\left[\Phi_\lambda(u_n) - \frac{1}{2}\Phi'_\lambda(u_n)u_n\right] \longrightarrow \frac{1}{\lambda}\lim_{n\longrightarrow\infty}\Phi_\lambda(u_n) > 0.$$

Since $(u_n)$ is bounded, then Lemma 4.3 implies that $(u_n)$ does not vanish, i.e., there exist positive constants $r,\delta > 0$ and a sequence $(s_n) \subset \mathbb{R}$ such that

$$\lim_{n\longrightarrow\infty}\int_{I_r(s_n)} e^{Q(x)}|u_n|^2 dx \geq \delta, \tag{4.3}$$

where $I_r(s_n) = [s_n - r, s_n + r]$. From the boundedness of $(u_n)$, we can assume, after passing to a subsequence, that $u_n \rightharpoonup u_\lambda$ in $E$ and $u_n \longrightarrow u_\lambda$ in $L^2_{loc}(\mathbb{R})$, which together with (4.3) implies that $u_\lambda \neq 0$. By the weakly sequentially continuity of $\Phi_\lambda$ and the fact $\Phi'_\lambda(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$, we obtain

$$\Phi'_\lambda(u_\lambda)v = \lim_{n\longrightarrow\infty}\Phi'_\lambda(u_n)v = 0, \ \forall v \in E.$$

Hence $\Phi'_\lambda(u_\lambda) = 0$. Combining $(F_8)$ with Fatou's lemma yields

$$\begin{aligned} c_\lambda &\geq \lim_{n\longrightarrow\infty}\Phi_\lambda(u_n) = \lim_{n\longrightarrow\infty}\left[\Phi_\lambda(u_n) - \frac{1}{2}\Phi'_\lambda(u_n)u_n\right] \\ &= \lim_{n\longrightarrow\infty}\lambda\int_{\mathbb{R}} e^{Q(x)}\widetilde{F}(x,u_n(x))dx \geq \lambda\int_{\mathbb{R}} e^{Q(x)}\widetilde{F}(x,u_\lambda(x))dx \\ &= \Phi_\lambda(u_\lambda) - \frac{1}{2}\Phi'_\lambda(u_\lambda)u_\lambda = \Phi_\lambda(u_\lambda). \end{aligned}$$

The proof of Lemma 4.5 is completed. □

**Lemma 4.6.** *Assume that $(\mathscr{A})$ and $(F_5)-(F_7)$ are satisfied. Then for any $\lambda \in [1,2]$, there exists a sequence $(v_n) \subset E$ such that*

$$(v_n) \text{ is bounded}, \ \Phi_\lambda(v_n) \longrightarrow c_\lambda \text{ and } \Phi'_\lambda(v_n) \longrightarrow 0. \tag{4.4}$$

*Proof.* For the $v_0 \in E$ obtained in Lemma 4.4, we have $f(v_0) < 0$. It follows from $(F_5)$ that $\Phi_\lambda(v_0) \leq f(v_0) < 0$, for all $\lambda \in [1,2]$. By (2.1) and (4.2), we have

$$\int_{\mathbb{R}} e^{Q(x)}F(x,u(x))dx \leq \frac{\varepsilon}{2}\eta_2^2\|u\|^2 + \frac{C_\varepsilon}{p}\eta_p^p\|u\|^p, \ \forall u \in E.$$

Since $\varepsilon$ is arbitrary, then $\int_{\mathbb{R}} e^{Q(x)}F(x,u(x))dx = o(\|u\|^2)$ as $u \longrightarrow 0$. Hence, there exists a constant $0 < r_0 < \|v_0\|$ such that

$$\int_{\mathbb{R}} e^{Q(x)}F(x,u(x))dx \leq \frac{1}{4}\|u\|^2, \ \forall \|u\| \leq r_0.$$

For all $\gamma \in \Gamma$, there is $s_\gamma \in [0,1]$ such that $\|\gamma(s_\gamma)\| = r_0$ and

$$\begin{aligned} \max_{s\in[0,1]}\Phi_\lambda(\gamma(s)) &= \Phi_\lambda(\gamma(s_\gamma)) \\ &= \frac{1}{2}\|\gamma(s_\gamma)\|^2 - \int_{\mathbb{R}} e^{Q(x)}F(x,\gamma(s_\gamma))dx \\ &\geq \frac{1}{4}\|\gamma(s_\gamma)\|^2 = \frac{r_0^2}{4} \end{aligned}$$

which implies that

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]}\Phi_\lambda(\gamma(s)) \geq \frac{r_0^2}{4} > 0, \ \forall\lambda \in [1,2] \tag{4.5}$$

and

$$c_\lambda > \max\{\Phi_\lambda(0), \Phi_\lambda(1)\}.$$

Hence, the family $(\Phi_\lambda)_{\lambda\in[1,2]}$ satisfies the hypotheses of Lemma 2.2, which completes the proof of Lemma 4.6. □

Combining Lemma 4.5 and Lemma 4.6, we deduce that there exist a sequence $(\lambda_n) \subset [1,2]$ converging to 1 and a sequence $(u_n) \subset E$ satisfying

$$u_n \neq 0,\ \Phi_{\lambda_n}(u_n) \leq c_{\lambda_n} \text{ and } \Phi'_{\lambda_n}(u_n) = 0. \tag{4.6}$$

Since

$$\frac{1}{2}\|u_n\|^2 - \lambda_n \int_{\mathbb{R}} e^{Q(x)} F(x, u_n(x))dx \leq c_{\lambda_n}$$

and

$$\|u_n\|^2 = \lambda_n \int_{\mathbb{R}} e^{Q(x)} f(x, u_n(x))u_n dx,$$

we deduce that

$$\int_{\mathbb{R}} e^{Q(x)} \widetilde{F}(x, u_n(x))dx \leq \frac{c_{\lambda_n}}{\lambda_n},\ \forall n \in \mathbb{N}.$$

It is clear that $(\frac{c_{\lambda_n}}{\lambda_n})$ is decreasing and bounded by $c_1$, which implies that $\int_{\mathbb{R}} e^{Q(x)} \widetilde{F}(x, u_n(x))dx \leq c_1$, $\forall n \in \mathbb{N}$.

**Lemma 4.7.** *Assume that $(\mathscr{A})$ and $(F_6) - (F_8)$ are satisfied. Then the sequence obtained in (4.5) is bounded.*

*Proof.* Using $(F_7)$ and $(F_8)$ respectively, we can find a positive constant $C_1$ such that

$$\int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} |f(x, u_n(x))| |u_n| dx \leq C_1 \int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} |u_n(x)|^2 dx \tag{4.7}$$

and

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} \widetilde{F}(x, u_n(x))dx \\
&= \int_{\{x\in\mathbb{R}/r\leq|u_n(x)|\leq R\}} e^{Q(x)} \widetilde{F}(x, u_n(x))dx + \int_{\{x\in\mathbb{R}/|u_n(x)|\geq R\}} e^{Q(x)} \widetilde{F}(x, u_n(x))dx \\
&\geq \frac{1}{R^\alpha} \inf_{\{x\in\mathbb{R}, r\leq|u|\leq R\}} \widetilde{F}(x,u) \int_{\{x\in\mathbb{R}/r\leq|u_n(x)|\leq R\}} e^{Q(x)} |u_n(x)|^\alpha dx \\
&\quad + b_1 \int_{\{x\in\mathbb{R}/|u_n(x)|\geq R\}} e^{Q(x)} |u_n(x)|^\alpha dx \\
&\geq C_2 \int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} |u_n(x)|^\alpha dx,
\end{aligned} \tag{4.8}$$

where $C_2 = \inf\left\{\frac{1}{R^\alpha} \inf_{\{x\in\mathbb{R}, r\leq|u|\leq R\}} \widetilde{F}(x,u), b_1\right\}$. By (4.4), we have for a positive constant $C_3$

$$\frac{\Phi_{\lambda_n}(u_n) - \frac{1}{2}\Phi'_{\lambda_n}(u_n)u_n}{\lambda_n} \leq C_3,$$

which together with $(F_8)$ and (4.8) implies

$$\begin{aligned}
C_3 &\geq \frac{\Phi_{\lambda_n}(u_n) - \frac{1}{2}\Phi'_{\lambda_n}(u_n)u_n}{\lambda_n} = \int_{\mathbb{R}} e^{Q(x)} \widetilde{F}(x, u_n(x))dx \\
&= \int_{\{x\in\mathbb{R}/|u_n(x)|\leq r\}} e^{Q(x)} \widetilde{F}(x, u_n(x))dx + \int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} \widetilde{F}(x, u_n(x))dx \\
&\geq a_1 \int_{\{x\in\mathbb{R}/|u_n(x)|\leq r\}} e^{Q(x)} |u_n|^p dx + C_2 \int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} |u_n|^\alpha dx.
\end{aligned} \tag{4.9}$$

Taking $s \in ]0, \frac{\alpha}{2}[$, we see that Hölder's inequality, (2.1), and (4.9) imply

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} |u_n|^2 dx \\
&= \int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} |u_n|^{2s} |u_n|^{2(1-s)} dx \\
&\leq \Big(\int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} |u_n|^{\alpha} dx\Big)^{\frac{2s}{\alpha}} \Big(\int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} |u_n|^{\frac{2\alpha(1-s)}{\alpha-2s}} )dx\Big)^{\frac{\alpha-2s}{\alpha}} \\
&\leq C_4 \|u_n\|^{2(1-s)},
\end{aligned}$$

where $C_4 = \left(\frac{C_3}{C_2}\right)^{\frac{2s}{\alpha}} \eta_{\frac{2\alpha(1-s)}{\alpha-s}}^{2(1-s)}$ and $\frac{2\alpha(1-s)}{\alpha-2s} \geq 2$. Now, since $\Phi'_{\lambda_n}(u_n)u_n = 0$, then $(F_6)$, (2.1), (4.7), and (4.9) imply

$$\begin{aligned}
\|u_n\|^2 &= \lambda_n \int_{\mathbb{R}} e^{Q(x)} f(x,u_n(x))u_n(x)dx \\
&\leq 2 \int_{\{x\in\mathbb{R}/|u_n(x)|\leq r\}} e^{Q(x)} f(x,u_n(x))u_n(x)dx \\
&+ 2 \int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} f(x,u_n(x))u_n(x)dx \\
&\leq 2c \int_{\{x\in\mathbb{R}/|u_n(x)|\leq r\}} e^{Q(x)} |u_n(x)|^{\mu} dx + 2C_1 \int_{\{x\in\mathbb{R}/|u_n(x)|\geq r\}} e^{Q(x)} |u_n(x)|^2 dx \\
&\leq 2c\frac{C_3}{a_1} + 2C_1C_4 \|u_n\|^{2(1-s)},
\end{aligned} \tag{4.10}$$

where $2(1-s) < 2$. Hence (4.10) implies that $(u_n)$ is bounded and the proof of Lemma 4.7 is completed. □

Now, we are in position to prove Theorem 4.1. Let $(u_n)$ be the bounded sequence obtained in (4.6). By taking a subsequence if necessary, we can assume that $u_n \rightharpoonup u$ and $u_n \longrightarrow u$ a.e. on $\mathbb{R}$. Using (4.6), we obtain

$$\lim_{n\longrightarrow\infty} \Phi'(u_n)v = \lim_{n\longrightarrow\infty} \left[\Phi'_{\lambda_n}(u_n)v + (\lambda_n - 1)\int_{\mathbb{R}} e^{Q(x)} f(x,u_n(x))v(x)dx\right] = 0, \ \forall v \in E.$$

We divide into two cases.

**First case:** $\limsup_{n\longrightarrow\infty} \Phi_{\lambda_n}(u_n) > 0$. In this case, the result follows from Lemma 4.5.

**Second case:** $\limsup_{n\longrightarrow\infty} \Phi_{\lambda_n}(u_n) \leq 0$. Let $(s_n) \subset [0,1]$ be such that

$$\Phi_{\lambda_n}(s_n u_n) = \max_{s\in[0,1]} \Phi_{\lambda_n}(s u_n),$$

we denote by $(v_n)$ the sequence defined by $v_n = s_n u_n$. It is clear that $(v_n)$ is bounded. Using (2.1) and (4.1), we obtain, for all $n \in \mathbb{N}$ and $u \in E$,

$$\begin{aligned}
\Phi'_{\lambda_n}(u)u &= \|u\|^2 - \lambda_n \int_{\mathbb{R}} e^{Q(x)} f(x,u(x))u(x)dx \\
&\geq \|u\|^2 - 2\int_{\mathbb{R}} e^{Q(x)} f(x,u(x))u(x)dx \\
&\geq \|u\|^2 - 2\varepsilon\eta_2^2 \|u\|^2 - 2C_\varepsilon \eta_p^p \|u\|^p.
\end{aligned}$$

Taking $\varepsilon = \frac{1}{4\eta_2^2}$, we obtain

$$\Phi'_{\lambda_n}(u)u \geq \frac{1}{2}\left[1 - 4C_\varepsilon \eta_p^p \|u\|^{p-2}\right] \|u\|^2.$$

Letting $r_1 = \left(8C_\varepsilon \eta_p^p\right)^{-\frac{1}{p-2}}$, we have

$$\Phi'_{\lambda_n}(u)u \geq \frac{1}{4}\|u\|^2, \ \forall u \in B(0,r_1). \tag{4.11}$$

Similarly, using (2.1) and (4.2), we can find a positive constant $r_2$ such that

$$\Phi_{\lambda_n}(u) \geq \frac{1}{8}\|u\|^2, \ \forall u \in B(0,r_2). \tag{4.12}$$

Combining (4.11) with the fact $\Phi'_{\lambda_n}(u_n) = 0$ yields $\|u_n\| \geq \theta$, $\forall n \in \mathbb{N}$, where $\theta = \inf(r_1,r_2)$. Let $0 < \alpha < 1$. For all $n \in \mathbb{N}$, $\bar{s}_n = \alpha \frac{\theta}{\|u_n\|} \in ]0,1[$. It follows from (4.12) that

$$\Phi_{\lambda_n}(s_n u_n) \geq \Phi_{\lambda_n}(\bar{s}_n u_n) \geq \frac{1}{8}\bar{s}_n^2\|u_n\|^2 \geq \frac{1}{8}(\alpha\theta)^2, \tag{4.13}$$

which together with $\Phi_{\lambda_n}(0) = 0$ implies that $s_n > 0$. Moreover, we have

$$\begin{aligned}\limsup_{n\longrightarrow\infty} f(u_n) &= \limsup_{n\longrightarrow\infty}\left[\Phi_{\lambda_n}(u_n) + (\lambda_n - 1)\int_{\mathbb{R}} e^{Q(x)}F(x,u_n(x))dx\right]\\ &= \limsup_{n\longrightarrow\infty}\Phi_{\lambda_n}(u_n) \leq 0,\end{aligned}$$

which together with (4.13) implies $s_n < 1$. Hence $s_n \in ]0,1[$ and then $\Phi'_{\lambda_n}(v_n)v_n = 0$ for all $n \in \mathbb{N}$ and

$$\lambda_n \int_{\mathbb{R}} e^{Q(x)}\widetilde{F}(x,v_n(x))dx = \Phi_{\lambda_n}(v_n) - \frac{1}{2}\Phi'_{\lambda_n}(v_n)v_n = \Phi_{\lambda_n}(v_n).$$

Consequently, we obtain from (4.13) that $\limsup_{n\longrightarrow\infty}\int_{\mathbb{R}} e^{Q(x)}\widetilde{F}(x,v_n(x))dx = \limsup_{n\longrightarrow\infty}\Phi_{\lambda_n}(v_n) > 0$. Since $(v_n)$ is bounded, it follows from Lemma 4.3 that $(v_n)$ does not vanish, so $(u_n)$ does not vanish. By going to a subsequence if necessary, Lemma 4.5 implies that $v_n \rightharpoonup v \neq 0$ with $\Phi'(v) = 0$ and the proof of Theorem 4.1 is finished.

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