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GLOBAL OPTIMALITY CONDITIONS FOR DC-CONSTRAINED OPTIMAL CONTROL PROBLEMS

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Dedicated to the memory of Professor Hoang Tuy

Abstract. The paper addresses the Optimal Control (OC) Problem along the state-linear Control System where the objective and inequality constraints are given by Bolza functionals with state-DC functions. With the help of the Exact Penalization Theory, the original problem is reduced to an auxiliary penalized problem without constraints, the cost functional of which is represented as a difference of two state-convex functionals. Such structure allows us to develop the Global Optimality Conditions with the constructive (algorithmic) property, providing a possibility to escape any local pitfall with an improving the objective of the original problem, if possible. Besides, the relations with the modern OC Theory (say, the PMP) are established. The effectiveness of the developed approach is demonstrated by examples.

Keywords. Bolza functional; DC function; Global Optimality Conditions; Exact Penalization Theory; Optimal Control Problem.

1. INTRODUCTION

Nowadays, the situation in the numerical optimization can be viewed, apparently, as under the crucial impact, strong pression and the increasing demand from diverse applied areas for new ideas and the design of non-standard numerical methods capable to find a global solution to different kind of applied problems.

Actually, it is not difficult to point out that the development in the field of optimization methods is not so impressive, as it was in 50th till 70th [4, 5, 6, 7, 15, 18, 19, 32, 38, 40, 41], and moreover as the technical progress in computational sciences and software.

To partially explain the difference, it is worth to note that the optimization problems (and not only!) must be separated into two parts: convex and nonconvex since the set of convex problems can be viewed as "solvable case", i.e., under minimal computability assumptions a

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convex problem is computationally tractable [15, 16, 38, 41]. It means that the computational efforts required to solve the problem grow moderately with the problem's dimension.

In contrast to this even the simplest-type of non-convex problems [8, 13, 16, 21, 34, 35, 36, 37, 38, 40, 41] often turn out to be difficult for numerical solution, because real-life problems usually have a lot (often a huge!) of local extrema and stationary points situated rather far from a global solution. Therefore here, we have to express our gratitude and profound respect to the pioneers in the field of the Global Optimization, among whom was professor H. Tuy. The fundamental works of the pre-eminent scientist overtake the frontiers of the time in various directions of nonconvex optimization [32, 33, 34, 35, 36, 37].

It is worth noting that H. Tuy was the first in the whole world to make turn our attention to the special properties of concave minimization problem [32]. On the other hand, he was a driver in publishing the famous monograph [13] that marked the boundary, the results and outcomes of 30-years-long working on the field. These results showed us that in non-convex problems the direct applications of classical optimization methods may have unpredictable consequences, and sometimes may even distract one from the desired solution. Moreover, it was demonstrated that one may apply direct selection methods (B&B and cut's ideas), but carefully, because of "the curse of dimension". Besides, one can, without doubts, find in [32, 33, 34, 35, 36, 37] and other works of H.Tuy, a number of suggestions for seeking other ways when it comes to finding to global solution to an applied problem.

Furthermore, at present it is readily to point out that the new applied areas and new fields appeared for applications the Global Optimization Theory and Methods. These streams are viewered by numerous confirmed specialists in optimization as the most attractive and promising fields of investigation, as challenges, and even as modeling paradigms in optimization in XXI century. In particular, one gives the following examples [13, 16]

- the search for equilibriums in competitions (conflict situations or games);
- hierarchical optimization problems;
- dynamical optimization problems.

All these problems possess explicit or implicit nonconvexity, generated by the structures [29]. Unexpectedly for us, we turned out to be on this main stream, but rather prepared, i.e. be endowed with a relevant mathematical tools.

It means that in the recent three decades, we have managed to construct the Global Search Theory, which is related to the modern optimization theory and which unexpectedly has turned out to be rather efficient in the aspect of computational solving, especially for problems of high dimensions [29, 30]. The approach is based on necessary and sufficient Global Optimality Conditions (GOSs) for problems with DC-data. On the other hand, the GOCs (even for general optimization problems with equality and inequality constraints) [21, 22, 23, 26, 27] can be viewed as the kernel of the approach. Furthermore, we have developed a family of local search methods (LSMs), which employ the special structure of a problem in question, and, on the other hand, represents a joint ensemble of methods [29, 30, 31].

The most important and beneficial feature of the developed approach is the fact, that the procedures of escape stationary and local pitfalls, based on the GOCs, are unique, original and quite efficient even in the case of any simple implementation [22, 23, 26, 27, 28, 29].

The paper is keeping on the investigations above and address the optimal control (OC) problem with inequality constraints and objective given by the Boltza functionals with state-DC data. In Section 2, after statement of the OC problem, we apply the Exact Penalization Theory (EPT) aiming at getting a problem without constraints. In Section 3, one performs the transformation of the problem's statement into DC form, when every functional has DC representation, and therefore the objective of the penalized problem is the same. The operation allows us to develop in Section 4 the new necessary Global Optimality Conditions (GOCs) (Theorem 4.1), while the sufficient GOC's are presented in Section 6.

In Section 5, we prove Theorem 5.1 which substantiates "the constructive property" of the GOC's to improve a current iteration, if it is not an ε-solution to the original OC problem. The effectiveness of the GOC's demonstrated by examples.

2. Problem's Statement and Exact Penalization

Let us address the state-linear control system

$$\left.\begin{array}{c} \dot{x}(t) = A(t)x(t) + B(u(t),t) \quad \overset{\circ}{\forall} t \in T :=]t_0, t_1[, \\ x(t_0) = x_0, \quad -\infty < t_0 < t_1 < \infty; \end{array}\right\} \tag{2.1}$$

$$u(\cdot) \in \mathscr{U} := \{u(\cdot) \in L^r_\infty(T) \mid u(t) \in U \quad \overset{\circ}{\forall} t \in T\}; \tag{2.2}$$

(where the sign $\overset{\circ}{\forall}$ denotes "almost everywhere" in the sense of the Lebesque measure), under standard assumptions [3, 9, 17, 26, 38, 39] when the $(n\times n)$-matrix $A(t) = [a_{ij}(t)]_1^n$ and the mapping $(u,t) \to B(u,t) : I\!R^r \times I\!R \to I\!R^n$ are continuous in the variables $(u,t) \in I\!R^r \times I\!R$ on a rather large open set from $I\!R^r \times I\!R$ including the set $U \times T$ with the compact $U \subset I\!R^r$.

Then [3, 4, 17, 38, 39] for any control $u(\cdot) \in \mathscr{U}$ and any $x_0 \in I\!R^n$ the system of ODE's (2.1) has a unique absolutely continuous solution $x(\cdot,u) \in AC_n(T)$, $x(t) = x(t,u)$, $t \in \overline{T}$.

Furthermore, let us consider the functionals

$$J_i(x,u) := J_i(x(\cdot),u(\cdot)) := \varphi_{1i}(x(t_1)) + \int_T \varphi_i(x(t),u(t),t)dt, \quad i \in \{0\}\cup I,\ I := \{1,...,m\}, \tag{2.3}$$

where the functions $\varphi_{1i} : I\!R^n \to I\!R$ can be represented in the DC-form

$$\varphi_{1i}(x) := g_{1i}(x) - h_{1i}(x) \quad \forall x \in \Omega_1 \subset I\!R^n, \tag{2.4}$$

with an open convex subset Ω_1 of $I\!R^n$ such that $\mathscr{R}(t_1) \subset \Omega_1$, where $\mathscr{R}(t_1)$ is the reachable set of the control system (CS) (2.1)–(2.2) at the terminal moment t_1. Besides, the functions $g_{1i}(\cdot)$ and $h_{1i}(\cdot)$ are convex on Ω_1, which implies that $\varphi_{1i}(x)$ turns out to be DC functions, $i \in \{0\}\cup I$.

On the other hand, the functions $\varphi_i(x,u,t)$, $\varphi_i : \Omega(t) \times U \times T \to I\!R$ have the next decompositions:

$$\varphi_i(x,u,t) := g_i(x,u,t) - h_i(x,t), \ \ i \in \{0\}\cup I \ \ \forall x \in \Omega(t) \ \ \forall (u,t) \in U \times T, \tag{2.5}$$

where $\Omega(t)$ is a sufficiently large open convex subset of $I\!R^n$, such that $\mathscr{R}(t) \subset \Omega(t)$, $t \in T$, where $\mathscr{R}(t)$ is the reachable set of the CS (2.1)–(2.2) at the moment $t \in T$. Here the functions $g_i(x,u,t)$ are continuous in the variables $(x,u,t) \in I\!R^{n+r+1}$, and the mappings $x \to g_i(x,u,t) : \Omega(t) \to I\!R$ are convex $\forall (u,t) \in U \times T$[10, 11, 12, 18, 19], $i \in \{0\}\cup I$. Besides, the functions $h_i(x,t)$ are continuous on $(x,t) \in \Omega(t) \times T$, and mappings $x \to h_i(x,t)$ are convex on $\Omega(t)$ $\forall t \in T$, $i \in \{0\}\cup I$.

In what follows, we will call the convexity property of the functions $g_{1i}(x)$, $g_i(x,u,t)$, $h_{1i}(x)$, $h_i(x,t)$ relatively to the variable x as state-convex, meanwhile the properties of the functions $\varphi_{1i}(x)$, $\varphi_i(x,u,t)$ to be represented as in (2.4) and (2.5), will be said to be state-DC, $i \in \{0\}\cup I$.

Further,since $x(\cdot)=x(\cdot,u),\ u\in\mathcal{U}$, is the solution to the CS (2.1) corresponding to $u\in\mathcal{U}$, the next denotations looks to be natural: $J_i(u):=J_i(x(\cdot,u),u),\ i\in I\cup\{0\}$.

In addition, we assume that data from above is differentiable with respect to the state. Therefore, due to the state-convexity of the functions above, in particular, the "convexity inequalities" hold true ($\nabla:=\nabla_x$) [10, 11, 12, 18, 19, 38]:

$$\left.\begin{array}{ll}(a): & \langle\nabla h_{1i}(y),x-y\rangle\le h_{1i}(x)-h_{1i}(y)\quad\forall x,y\in\Omega_1,\\ (b): & \langle\nabla h_i(y(t),t),x(t)-y(t)\rangle\le h_i(x(t),t)-h_i(y(t),t)\\ & \forall x(t),y(t)\in\Omega(t),\ \forall t\in T,\quad i\in\{0\}\cup I.\end{array}\right\}\tag{2.6}$$

Let us now address the following optimal control (OC) problem:

$$(\mathcal{P}):\qquad \left.\begin{array}{c}J_0(u)\overset{\triangle}{=}J_0(x(\cdot,u),u(\cdot))\downarrow\min\limits_{u(\cdot)},\quad u(\cdot)\in\mathcal{U},\\ J_i(u)\overset{\triangle}{=}J_i(x(\cdot,u),u(\cdot))\le 0,\quad i\in I\overset{\triangle}{=}\{1,...,m\},\end{array}\right\}\tag{2.7}$$

along the control system (2.1)–(2.2) [3, 4, 9, 17, 20].

It is clear that, in virtue of nonconvexity (relatively to the state $x(\cdot,u),\ x(t,u)=x(t),\ t\in T,u\in\mathcal{U}$) of the terminal parts $\varphi_{1i}(\cdot)$ and the integrand $\varphi_i(x,u,t)$, every functional $J_i(x,u),\ i\in\{0\}\cup I$, the feasible region of Problem $(\mathcal{P})$, and Problem $(\mathcal{P})$ itself, as a whole, turn out to be nonconvex. It means, that Problem $(\mathcal{P})$ may possess a big number of locally optimal and stationary (say, in the sense of the PMP) processes, which may be rather far from a set Sol$(\mathcal{P})$ of global solutions (globally optimal controls, processes, if one exists), even with respect to the value of the cost functional $J_0(\cdot,u)$ [2, 3, 4, 8, 9, 10, 11, 12, 13, 15, 16, 21, 24, 25, 26, 27, 29].

In order to deal with Problem $(\mathcal{P})$–(2.7), let us now apply the very popular approach of the Exact Penalization Theory. To this end introduce the penalty functional $\pi(x,u)$ for Problem $(\mathcal{P})$ in the next form [1, 2, 3, 4, 5, 6, 7, 15, 19, 22, 40, 41, 42, 43]:

$$\pi(x,u):=\pi(u):=\max\{0,J_1(u),...,J_m(u)\}\tag{2.8}$$

and address the auxiliary (penalized) problem

$$(\mathcal{P}_\sigma):\qquad J_\sigma(u):=J_\sigma(x(\cdot,u),u(\cdot))\downarrow\min_u,\quad u(\cdot)\in\mathcal{U},\tag{2.9}$$

along the control system (2.1)–(2.2) with the cost functional defined by

$$J_\sigma(u):=J_0(x(\cdot,u),u(\cdot))+\sigma\pi(x(\cdot,u),u(\cdot)),\tag{2.10}$$

where $\sigma>0$ is a penalty parameter.

Let us recall now that the key feature of the Exact Penalization Theory [1, 2, 4, 5, 6, 7, 22, 31, 40, 41, 42, 43] consists in the existence of the threshold value $\sigma_*>0$ of the penalty parameter for which Problems $(\mathcal{P})$ and $(\mathcal{P}_\sigma)$ are equivalent in the sense that

$$\mathcal{V}(\mathcal{P})=\mathcal{V}(\mathcal{P}_\sigma)\quad\text{and}\quad \mathrm{Sol}(\mathcal{P})=\mathrm{Sol}(\mathcal{P}_\sigma)\quad\forall\sigma>\sigma_*\tag{2.11}$$

(see [12], Chapter VII, Lemma 1.21 and [1, 4, 5, 7, 14, 40, 41, 42, 43]).

In other words, the existence of the exact (threshold) value $\sigma_*>0$ of the penalty parameter implies that instead of solving a sequence $\{(\mathcal{P}_{\sigma_k})\}$ of unconstrained problem with $\sigma_k\uparrow\infty$ we need to consider only a single problem $(\mathcal{P}_\sigma)$ with $\sigma\ge\sigma_*$. On the other hand, it is well-known that if a process $(z(\cdot),w(\cdot))$ (a control $w(\cdot)\in\mathcal{U}$) is a global solution to Problem $(\mathcal{P}_\sigma)$: $(z(\cdot),w(\cdot))\in\mathrm{Sol}(\mathcal{P}_\sigma)$, $z(t)=x(t,w),\ t\in T,\ w(\cdot)\in\mathcal{U}$, and, besides $(z(\cdot),w(\cdot))$ is feasible in Problem $(\mathcal{P})$, i.e. $J_i(z,w)\le 0,\ i\in I$, then $(z(\cdot),w(\cdot))$ is a global solution to Problem $(\mathcal{P})$. It is worth noting that the inverse assertion, in general, does not hold. Moreover, one can prove the existence of the exact penalty (threshold) value $\sigma_* > 0$ for local and global solutions under various Constraint Qualification (CQ) conditions (e.g. MFCQ, Slater, etc.), the error bound properties, the calmness of the constraint system, etc. [1, 4, 5, 6, 7, 40, 41, 42, 43].

Let us assume, in what follows, that some regularity conditions, ensuring the existence of the threshold value $\sigma_* > 0$ of the penalty parameters in Problem $(\mathscr{P})$, when it is needed, are satisfied.

3. DC Decomposition of the Data

Let us show, first of all, that every functional $J_i(u) = J_i(x,u)$ defined in (2.3) can be represented in the DC form, i.e.

$$J_i(u) := G_i(x,u) - F_i(x), \quad i \in \{0\} \cup I, \tag{3.1}$$

where the functionals $G_i(\cdot)$ and $F_i(\cdot)$ are state-convex. Indeed, employing (2.3)–(2.5), we have

$$\left.\begin{array}{ll} (a): & G_i(x,u) := g_{1i}(x(t_1)) + \int\limits_T g_i(x(t),u(t),t)dt, \\ (b): & F_i(x) := h_{1i}(x(t_1)) + \int\limits_T h_i(x(t),t)dt, \quad i \in \{0\} \cup I, \end{array}\right\} \tag{3.1$'$}$$

which yields the desirable state-convexity property to $G_i(x,u)$ and $F_i(x)$. In particular, for the functionals $F_i(x)$ one can get the feature similar to the convexity inequalities (2.6) [10, 12, 18, 19]. Actually, under the above assumptions one can introduce a differential of the functional $F_i(\cdot)$ by the next way:

$$\langle\langle \nabla F_i(y(\cdot)), x(\cdot)\rangle\rangle := \langle \nabla h_{1i}(y(t_1)), x(t_1)\rangle + \int\limits_T \langle \nabla h_i(y(t),t), x(t)\rangle dt, \tag{3.2}$$

where $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_n$ is the inner product in $I\!R^n$, and, say, $x(\cdot), y(\cdot) \in AC_n(\overline{T})$. Besides , one can addresses the case when $y(\cdot) \in PC(\overline{T})$ with the linear space $PC(T)$ of piecewise continuous functions. Therefore, one can consider the pair $(\nabla h_{1i}(y(t_1)); \nabla h_i(y(\cdot),\cdot))$ as a gradient of $F_i(\cdot)$ at a function $y(\cdot) \in AC_n(T)$ or $y(\cdot) \in PC(\overline{T})$:

$$\nabla F_i(y(\cdot)) := (\nabla h_{1i}(y(t_1)), \nabla h_i(y(t),t), t \in T).$$

As a consequence, due to (2.6) and (3.2), the following inequality holds true:

$$\left.\begin{array}{c} \langle\langle \nabla F_i(y(\cdot)), x(\cdot) - y(\cdot)\rangle\rangle \le F_i(x(\cdot)) - F_i(y(\cdot)) \\ \forall x(\cdot), y(\cdot) \in AC_n(T) \;\; (i \in \{0\} \cup I). \end{array}\right\} \tag{2.6$'$}$$

Furthermore, it can be readily seen that, thanks to the presentations (2.4), (2.5), (2.8)–(2.10), (3.1)-(3.1′), the cost functional $J_\sigma(x,u)$ of the penalized problem $(\mathscr{P}_\sigma)$–(2.9)–(2.10) has the next form:

$$\begin{array}{c} J_\sigma(x,u) := G_0(x,u) - F_0(x) + \sigma\pi(x,u) \stackrel{\triangle}{=} \\ = G_0(x,u) - F_0(x) + \sigma \max\{0; [G_i(x,u) - F_i(x)],\ i \in I\}. \end{array} \tag{2.10$'$}$$

Let us now show that the penalty function $\pi(x,u)$ defined in (2.8) can also be represented as a state-DC functional, i.e. $\pi(x,u) = G_\pi(x,u) - F_\pi(x)$, where $G_\pi(\cdot)$ and $F_\pi(\cdot)$ also have the state-convexity property. In this case, $J_\sigma(x,u)$ turns out to be also state-DC.

Indeed, one can easily derive from (2.8) that

$$\left.\begin{aligned} \pi(x,u) \overset{\triangle}{=} \max\{0; [G_i(x,u)-F_i(x)],\ i\in I\} \pm \sum_{j\in I} F_j(x) = \\ = \max\{\sum_{j\in I} F_j(x); [G_i(x,u)+\sum_{k\in I}^{k\neq i} F_k(x)],\ i\in I\} - \sum_{j\in I} F_j(x). \end{aligned}\right\} \tag{3.3}$$

Therefore, with the help of the denotations

$$\left.\begin{aligned} &(a): G_\pi(x,u) := \max\{\sum_{j\in I} F_j(x); [G_i(x,u)+\sum_{k\in I}^{k\neq i} F_k(x)],\ i\in I\} \\ &(b): F_\pi(x) := \sum_{j\in I} F_j(x), \end{aligned}\right\} \tag{3.4}$$

one gets the following DC decomposition of the penalty functional:

$$\pi(x,u) = G_\pi(x,u) - F_\pi(x), \tag{3.5}$$

where the functionals $G_\pi(\cdot)$ and $F_\pi(\cdot)$ clearly preserve the state-convexity property in virtue of (3.1), (3.1′), and (3.4).

Hence, as claimed above, the cost functional $J_\sigma(x,u)$ defined in (2.10), because of (2.10′), (3.3)–(3.5), has the next state DC decomposition:

$$\left.\begin{aligned} J_\sigma(x,u) \overset{\triangle}{=} G_0(x,u) - F_0(x) + \sigma[G_\pi(x,u) - F_\pi(x)] = \\ = [G_0(x,u)+\sigma G_\pi(x,u)] - [F_0(x)+\sigma F_\pi(x)] = G_\sigma(x,u) - F_\sigma(x). \end{aligned}\right\} \tag{3.6}$$

Here, the functionals $G_\sigma(x,u)$ and $F_\sigma(x)$ thanks to (3.1′), have the following forms:

$$\begin{gathered} G_\sigma(x,u) := G_0(x,u) + \sigma G_\pi(x,u) = g_{10}(x(t_1)) + \int_T g_0(x(t),u(t),t)dt + \\ +\sigma \max\{\sum_{j\in I}[h_{1j}(x(t_1)) + \int_T h_j(x(t),t)dt]; \\ [g_{1i}(x(t_1)) + \int_T g_i(x(t),u(t),t)dt + \sum_{k\in I}^{k\neq i}\Big(h_{1k}(x(t_1)) + \int_T h_k(x(t),t)dt\Big)], i\in I\}; \end{gathered} \tag{3.7}$$

$$\begin{gathered} F_\sigma(x) := F_0(x) + \sigma F_\pi(x) \overset{\triangle}{=} h_{10}(x(t_1)) + \int_T h_0(x(t),t)dt + \\ +\sigma \sum_{i\in I}[h_{1i}(x(t_1)) + \int_T h_i(x(t),t)dt] = \\ = h_{10}(x(t_1)) + \sigma\sum_{i\in I} h_{1i}(x(t_1)) + \int_T [h_0(x(t),t) + \sigma\sum_{i\in I} h_i(x(t),t)]dt. \end{gathered} \tag{3.8}$$

It is not difficult to point out from (3.6)–(3.8) that the functionals $G_\sigma(x,u)$ and $F_\sigma(x)$ are also endowed with the state-convexity property [10, 11, 12, 18, 19]. Moreover, with the help of (2.6), (3.2), (3.8), one can see that the functional $F_\sigma(x(\cdot))$ is differentiable in the sense that $\forall y(\cdot) \in PC(T)$ we have

$$\begin{gathered} \langle\langle \nabla F_\sigma(y(\cdot)), x(\cdot)\rangle\rangle \overset{\triangle}{=} \langle\langle \nabla F_0(y(\cdot)), x(\cdot)\rangle\rangle + \sigma\sum_{i\in I}\langle\langle \nabla F_i(y(\cdot)), x(\cdot)\rangle\rangle \overset{\triangle}{=} \\ = \langle x(t_1), \nabla h_{10}(y(t_1)) + \sigma\sum_{i\in I}\nabla h_{1i}(y(t_1))\rangle + \int_T \langle x(t), \nabla h_0(y(t),t) + \sigma\sum_{i\in I}\nabla h_i(y(t),t)\rangle dt. \end{gathered} \tag{3.9}$$

In addition, due to the state-convexity of $F_\sigma(\cdot)$ the following ”convexity inequality” holds true for every $u(\cdot)\in\mathcal{U}$:

$$\langle\langle \nabla F_\sigma(y(\cdot)), x(\cdot,u) - y(\cdot)\rangle\rangle \le F_\sigma(x(\cdot,u)) - F_\sigma(y(\cdot)). \tag{3.10}$$

4. Global Optimality Conditions (GOCs)

First, let us assume now that in the original problem $(\mathscr{P})$–(2.7) the feasible set $\mathscr{F} = \mathscr{F}(\mathscr{P})$ is not empty, i.e.

$$\mathscr{F} := \{(x(\cdot),u(\cdot)) \mid x(t) = x(t,u), t \in T,\ u(\cdot) \in \mathscr{U};\ J_i(u) \le 0, i \in I\} \neq \emptyset \tag{4.1}$$

and the optimal value $\mathscr{V}(\mathscr{P})$ of Problem $(\mathscr{P})$ is finite, i.e.

$$(\mathscr{A}_0): \qquad \mathscr{V}(\mathscr{P}) := \inf_{u(\cdot)} \{J_0(u) \mid u(\cdot) \in \mathscr{U}, (x(\cdot,u),u(\cdot)) \in \mathscr{F}\} > -\infty. \tag{4.2}$$

In addition, let us suppose that the set of globally optimal processes (controls) is not empty, as well, i.e.

$$\mathrm{Sol}(\mathscr{P}) := \{(x(\cdot),u(\cdot)) \in \mathscr{F} \mid J_0(u) = \mathscr{V}(\mathscr{P})\} \neq \emptyset\} \tag{4.3}$$

(or equivalently one can say that (4.3) amounts to

$$\mathrm{Sol}(\mathscr{P}) := \{u(\cdot) \in \mathscr{U} \mid (x(\cdot),u(\cdot)) \in \mathscr{F}, J_0(u) = \mathscr{V}(\mathscr{P})\} \neq \emptyset). \tag{4.3$'$}$$

In other words, a process $(z(\cdot),w(\cdot)) \in \mathscr{F}$, i.e. $z(t) = x(t,w)$, $t \in T$, $w(\cdot) \in \mathscr{U}$, $\pi(w) \overset{\triangle}{=} \pi(z,w) = 0$ (or a control $w(\cdot) \in \mathscr{U}$) is said to be globally optimal to Problem $(\mathscr{P})$–(2.7) ($(z(\cdot),w(\cdot)) \in \mathrm{Sol}(\mathscr{P})$ or $w(\cdot) \in \mathrm{Sol}(\mathscr{P})$), if the next inequality holds true:

$$\left.\begin{array}{c} J_0(z,w) \overset{\triangle}{=} J_0(w) \le J_0(u) \overset{\triangle}{=} J_0(x(\cdot,u),u) \\ \forall (x(\cdot,u),u) \in \mathscr{F} \text{ (or } \forall u(\cdot) \in \mathscr{U} : (x(\cdot,u),u(\cdot)) \in \mathscr{F}). \end{array}\right\} \tag{4.4}$$

Since $(z(\cdot),w(\cdot)) \in \mathscr{F}$, which amounts to $\pi(z(\cdot),w(\cdot)) = 0$, let us use the next denotation:

$$\zeta := J_0(w) = J_\sigma(w) \overset{\triangle}{=} J_0(w) + \sigma\pi(z(\cdot),w(\cdot)). \tag{4.5}$$

Let us denote by $PC(T)$ the linear space of all piecewise continuous functions. Now we are ready to prove the following result.

Theorem 4.1. *Let a feasible in Problem $(\mathscr{P})$ process $(z(\cdot),w(\cdot))$ be globally optimal to Problem $(\mathscr{P})$–(2.7) and $\sigma > \sigma_*$ with $\sigma_* > 0$ being the threshold value of the penalty parameter σ, so that $Sol(\mathscr{P}) = Sol(\mathscr{P}_\sigma)$ $\forall \sigma > \sigma_*$. Then, for each pair $(y(\cdot),\beta) \in PC(T) \times I\!R$, $y(\cdot) : T \to I\!R^n$, satisfying the equation (see (3.8))*

$$F_\sigma(y(\cdot)) \overset{\triangle}{=} F_0(y(\cdot)) + \sigma F_\pi(y(\cdot)) = \beta - \zeta, \tag{4.6}$$

the following principal inequality holds true:

$$G_\sigma(x(\cdot,u),u) - \beta \ge \langle\langle \nabla F_\sigma(y(\cdot)), x(\cdot,u) - y(\cdot) \rangle\rangle \quad \forall u(\cdot) \in \mathscr{U}. \tag{4.7}$$

Proof. Since $w(\cdot) \in \mathrm{Sol}(\mathscr{P}) = \mathrm{Sol}(\mathscr{P}_\sigma)$, then with the help of (4.4) one has $\forall u(\cdot) \in \mathscr{U}$

$$\zeta := J_\sigma(w) \overset{\triangle}{=} G_\sigma(z,w) - F_\sigma(z) \le G_\sigma(x(\cdot,u),u) - F_\sigma(x(\cdot,u)).$$

Whence, thanks to the equation (4.6), we have

$$G_\sigma(x(\cdot,u),u) \ge \beta - F_\sigma(y(\cdot)) + F_\sigma(x(\cdot,u)) \quad \forall u(\cdot) \in \mathscr{U}.$$

Using now the state-convexity of $F_\sigma(\cdot)$ and the inequality (3.10), we finally obtain (4.7). #

Remark 4.2. It can be readily seen that Theorem 4.1 reduces the solution of the nonconvex OC problem $(\mathscr{P})$–(2.7), first, to the nonconvex OC problem $(\mathscr{P}_\sigma)$–(2.9)–(2.10) without inequality constraints $J_i(u) \le 0,\ i \in I$. Besides, the latter problem $(\mathscr{P}_\sigma)$ is reduced to a study of the family of the state-convex (partially) linearized problems of the next form (see (4.7)):

$$(\mathscr{P}_\sigma L(y)):\ \Phi_{\sigma y}(x(\cdot,u),u) := G_\sigma(x(\cdot,u),u) - \langle\langle \nabla F_\sigma(y(\cdot)), x(\cdot)\rangle\rangle \downarrow \min_{u(\cdot)},\ \ u(\cdot) \in \mathscr{U}, \tag{4.8}$$

depending on the pair $(y(\cdot),\beta) \in PC(T) \times I\!R$, which fulfills the equation (4.6) with the finctionals given by (3.8), (4.5).

It is worth noting that the linearization was carried out to the "united" nonconvexity of Problem $(\mathscr{P})$–(2.7) generated only by the state-nonconvex functionals $J_0(u),\ldots,J_m(u)$, but not by state-linear control system (CS) (2.1). This nonconvexity is accumulated by the functional $F_\sigma(\cdot)$ (see $(\mathscr{P})$–(2.7), $(\mathscr{P}_\sigma)$–(2.9)–(2.10),(3.8)). Thus, it is clear that Problem $(\mathscr{P}_\sigma L(y))$–(4.8) turns out to be state-convex, and therefore more opportune and easier for globally solving, say, with the help of the Pontryagin's Principle (the PMP) based methods than the original Problem $(\mathscr{P})$–(2.7). More precisely, the verification of the principal inequality (4.7) of the GOCs can be implemented by the solution of the linearized state-convex problems $(\mathscr{P}_\sigma L(y))$–(4.8), combining it with the simultaneous varying the parameters $(y(\cdot),\beta)$: $y(\cdot) \in PC(T)$, $\beta \in I\!R$, satisfying the equation (4.6).

On the other hand, using the consecutive solution of the linearized problem $(\mathscr{P}_\sigma L(y))$ (say, of the kind $(\mathscr{P}_{\sigma_s} L(x^s(\cdot))$, i.e. linearized at the current iterate $x^s(t) = x(t,u^s(\cdot)), t \in T, u^s(\cdot) \in \mathscr{U}$, $x^{s+1} \in \mathrm{Sol}(\mathscr{P}_{\sigma_s} L(x^s(\cdot)))$ combined with a suitable change of the penalty parameter, provides a first simplest scheme of local search with satisfactory convergence properties (see [2, 28, 29, 30, 31]).

Hence, the idea of linearization with respect to basic nonconvexities of an applied problem under study, has shown itself to be effective and beneficial from the viewpoint of numerical finding global solutions to nonconvex optimization problems.

Remark 4.3. Suppose, we found a triple $(y(\cdot),\beta,u(\cdot))$: $y(\cdot) \in PC(T)$, $\beta \in I\!R$, $u(\cdot) \in \mathscr{U}$, such that (see (3.8), (4.5)) $F_\sigma(y(\cdot)) = \beta - \zeta$, and the inequality (4.7) is violated

$$G_\sigma(x(\cdot,u),u) - \beta < \langle\langle \nabla F_\sigma(y(\cdot)), x(\cdot,u) - y(\cdot)\rangle\rangle.$$

Then, thanks to the state-convexity of $F_\sigma(\cdot)$ (see (3.10)), we have

$$0 > G_\sigma(x(\cdot,u),u) - \beta - F_\sigma(x(\cdot,u)) + F_\sigma(y(\cdot))$$

and, further, due to the equation (4.6), it yields

$$0 > G_\sigma(x(\cdot,u),u) - F_\sigma(x(\cdot,u)) - \zeta \stackrel{\triangle}{=} J_\sigma(u) - J_\sigma(w),$$

$$\text{or } J_\sigma(w) > J_\sigma(u),\ (z(\cdot),w(\cdot)) \in \mathscr{F} = \mathscr{F}(\mathscr{P}),\ u(\cdot) \in \mathscr{U}.$$

Hence, the process $(z(\cdot),w(\cdot))$ can not be a solution to the penalized problem $(\mathscr{P}_\sigma)$–(2.9)–(2.10) (see (4.4)).

Moreover, if the process $(x(\cdot,u),u(\cdot))$ is feasible in the original problem $(\mathscr{P})$–(2.7), i.e. $\pi(w) = 0 = \pi(u)$, we obtain $(x(\cdot,u),u(\cdot)) \in \mathscr{F}$, $J_0(w) = J_\sigma(w) > J_\sigma(u) = J_0(u)$, and the control $u(\cdot) \in \mathscr{U}$ turns out to be better than $w(\cdot) \in \mathscr{U}$ for Problem $(\mathscr{P})$–(2.7).

It means that the conditions (4.6)–(4.7) of Theorem 4.1 possess the classical constructive (algorithmic) property: once the conditions are violated, then one can find a feasible control which is better than this one under scrutiny [5, 15, 21, 22, 23, 24, 38, 39].

Let us consider now an example to demonstrate the effectiveness of this property.

Example 4.4. Consider the next "terminal" optimal control problem

$$\left.\begin{array}{r} \dot{x}(t) = u(t) - 1, \quad \overset{\circ}{\forall} t \in T :=]0,2[, \quad x(0) = 1, \\ \mathscr{U} := \{u(\cdot) \in L_\infty(T) \mid u(t) \in [-2,2] \ \overset{\circ}{\forall} t \in T\}, \end{array}\right\} \tag{4.9}$$

$$\left.\begin{array}{r} J_0(u) := \varphi_{10}(x(2,u)) = [x(2)]^2 \downarrow \min_u, \\ J_1(u) := \varphi_{11}(x(2,u)) = 4 - [x(2)-1]^2 \leq 0. \end{array}\right\} \tag{4.10}$$

Let us investigate the control $u_*(t) \equiv 2$.It is not difficult to see that the corresponding state is $x_*(t) := x(t,u_*) \equiv 1+t, \ \ x_*(2) = 3$, and

$$\begin{aligned} J_1(u_*) &= \varphi_{11}(x_*(2)) = 4 - [3-1]^2 = 0, \\ J_0(u_*) &= \varphi_{10}(x_*(2)) = [3]^2 = 9 =: \zeta. \end{aligned} \tag{4.11}$$

Let us verify whether the control $u_*(\cdot)$ satisfies the Pontryagin's Principle. Clearly, the adjoint system at the process $(x_*(\cdot), u_*(\cdot))$ has the form

$$\dot{\psi}(t) = -A(t)^T \psi(t) \equiv 0, \quad \psi(t) \equiv const,$$

$$\psi(t) \equiv \psi(t_1) = \nabla_x \mathscr{L}(x_*(t_1), \lambda_0, \lambda_1) = 2\lambda_0 x_*(2) - 2\lambda_1[x_*(2) - 1]$$

with $\lambda_0 = \dfrac{1}{2}$, $\lambda_1 = 1$ which yields $\psi(t) \equiv \psi(t_1) = -1$.

On the other hand, we see that the Hamiltonian is as follows

$$\mathscr{H}(x,u,\psi,t) = \psi(t)[u(t)-1] \ \text{ and}$$

$$\psi(t)[u_*(t)-1] = \min_v \{\psi(t)[v-1] \mid v \in [-2,2]\} =$$

$$= \min_v \{(-1)[v-1] \mid v \in [-2,2]\} = (-1)[2-1] = -1 \ \ \overset{\circ}{\forall} t \in T.$$

Thus, the Pontryaguin's Principle (the PMP) holds true for $u_*(t) \equiv 2$, or $(x_*(\cdot), u_*(\cdot))$ is the Pontryaguin's extremal.

Let us verify now whether the control $u_*(t) \equiv 2$ is a global solution to the problem (4.9)–(4.10). In order to apply the GOCs (4.6)–(4.7) of Theorem 4.1, let us produce a DC decomposition of the cost functional

$$J_\sigma(u) \overset{\triangle}{=} J_0(u) + \sigma \max\{0, J_1(u)\} \overset{\triangle}{=} \varphi_{10}(x(t_1)) + \sigma \max\{0; \varphi_{11}(u)\}$$

to the penalized problem $(\mathscr{P}_\sigma)$–(2.9)–(2.10).

To simplify the presentation, let us use the denotation x instead of $x(t_1,u) \in \mathscr{R}(t_1)$, where $\mathscr{R}(t_1)$ is the reachable set of the control system (4.9). Then we are getting from (4.9), (4.10):

$$\begin{aligned} \varphi_{10}(x) + \sigma \max\{0; \varphi_{11}(x)\} &= x^2 + \sigma \max\{0; 4 - (x-1)^2\} \pm \sigma (x-1)^2 = \\ &= x^2 + \sigma \max\{(x-1)^2; 4\} - \sigma (x-1)^2 =: G_\sigma(x) - F_\sigma(x), \end{aligned} \tag{4.12}$$

where the elements of DC decomposition are

$$\begin{aligned} &(a): G_\sigma(x) = x^2 + \sigma \max\{(x-1)^2; 4\}, \\ &(b): F_\sigma(x) = \sigma (x-1)^2. \end{aligned} \tag{4.13}$$

Now let us set $y := -1$, $\sigma := 1$, which yields $F_\sigma(y) = (-1-1)^2 = 4 = \beta - \zeta$, whence, thanks to (4.11), we get $\beta = 4+9 = 13$.

Further, let us address the control $\check{u}(\cdot) \in \mathscr{U}$: $\check{u}(t) = \begin{cases} -2, & t \in [0,5/3], \\ 2, & t \in [5/3,2]. \end{cases}$

Clearly, $\check{x}(t) := x(t,\check{u})$, $t \in T$ has the next presentation:

$$(a): t \in [0,5/3],\ \dot{\check{x}}(t) = -3,\ \ \check{x}(t) = 1-3t,\ \check{x}(5/3) = -4;$$
$$(b): t \in [5/3,2],\ \dot{\check{x}}(t) = 1,\ \ \check{x}(t) = -4+t,\ \check{x}(2) = -2.$$

Let us compute now the parts of the principal inequality (4.7) of Theorem 4.1 (see (4.13)):

$$G_\sigma(\check{x}(2)) = (-2)^2 + \max\{(-2-1)^2; 4\} = 4 + \max\{9;4\} = 13;$$

$$\langle \nabla F_\sigma(y), \check{x}(2) - y\rangle = \langle 2(y-1), -2+1\rangle = \langle -4, -1\rangle = 4.$$

Hence, we finally obtain

$$G_\sigma(\check{x}(2)) = 13 < 17 = 13+4 = \beta + \langle \nabla F_\sigma(y), \check{x}(2) - y\rangle,$$

so that (4.7) is broken down, and therefore the process $(x_*(\cdot), u_*(\cdot))$ is not a global solution to the penalized problem $(\mathscr{P}_\sigma)$–(2.9), since the control $\check{u}(\cdot) \in \mathscr{U}$ turns out to be better than $u_*(\cdot) \in \mathscr{U}$.

Moreover, because $\varphi_{11}(\check{x}(2)) \overset{\triangle}{=} 4 - [\check{x}(2) - 1]^2 = 4 - 9 < 0$, i.e. the process $(\check{x}(\cdot), \check{u}(\cdot))$ is feasible in the problem (4.9)–(4.10) and hence the control $u_*(\cdot)$, satisfying the PMP for (4.9)–(4.10) is not a global solution to (4.9)–(4.10).

In addition, the control $\check{u}(\cdot) \in \mathscr{U}$ is better than $u_*(\cdot)$, since $J_0(\check{u}(\cdot)) = [\check{x}(2)]^2 = 4 < 9 \overset{\triangle}{=} \zeta$. #

Remark 4.5. Suppose now, that a process $(z(\cdot), w(\cdot))$, $z(t) = x(t,w)$, $t \in T$, $w(\cdot) \in \mathscr{U}$, satisfies the GOC's (4.6)–(4.7) of Theorem 4.1, and let us set in (4.6)–(4.7) $y(t) \equiv z(t)$. It yields that, $\beta = F_\sigma(z(\cdot)) + \zeta = F_\sigma(z(\cdot)) + J_\sigma(w) = G_\sigma(z(\cdot), w(\cdot))$. Then, (4.7) provides that $\forall u(\cdot) \in \mathscr{U}$

$$G_\sigma(x(\cdot,u),\ u(\cdot)) - G_\sigma(z(\cdot), w(\cdot)) \geq \langle\langle \nabla F_\sigma(z(\cdot)),\ x(\cdot,u) - z(\cdot)\rangle\rangle.$$

It means that the control $w(\cdot)$ is a solution to the following state-convex (linearized) OC problem

$$(\mathscr{P}_\sigma L(z)): \qquad G_\sigma(x(\cdot,u),\ u) - \langle\langle \nabla F_\sigma(z(\cdot)),\ x(\cdot,u)\rangle\rangle \downarrow \min_u, \quad u(\cdot) \in \mathscr{U}, \tag{4.14}$$

along the control system (2.1)–(2.2).

Further, on account of the formulae (3.7)–(3.8), it can be readily seen that the cost functional of this state-convex problem $(\mathscr{P}_\sigma L(z))$ – (4.14) has the following form (see (3.3))

$$\begin{array}{c} J_\sigma L(z)(x(\cdot,u),u) := J_\sigma L_z(u) := G_0(x(\cdot,u),u) + \\ +\sigma \max\left\{\sum_{j\in I} F_j(x(\cdot,u)); [G_i(x(\cdot,u),u) + \sum_{k\in I}^{k\neq i} F_k(x(\cdot,u))],\ i \in I\right\} - \\ -\langle\langle \nabla F_0(z(\cdot), x(\cdot,u))\rangle\rangle - \sigma \sum_{j\in I} \langle\langle \nabla F_j(z(\cdot), x(\cdot,u))\rangle\rangle. \end{array} \tag{4.15}$$

On the other hand, one can easily show that the OC problem $(\mathscr{P}_\sigma L(z))$ – (4.14) – (4.15) (see Lemma 4.1. from [21], and also [15, 38]) amounts to the next auxiliary state-convex and smooth OC problem (with inequality constraints) as follows

$$(\mathscr{A}\mathscr{P}_\sigma L(z)): \qquad \left.\begin{array}{c} G_0(x(\cdot,u),\, u) + \sigma\gamma - \langle\langle \nabla F_0(z(\cdot)),\, x(\cdot,u)\rangle\rangle - \\ -\sigma \sum\limits_{j\in I} \langle\langle \nabla F_j(z(\cdot)), x(\cdot,u)\rangle\rangle \downarrow \min\limits_{(u,\gamma)}, \quad u(\cdot)\in\mathscr{U},\ \gamma\in I\!R, \\ \sum\limits_{j\in I} F_j(x(\cdot,u)) \le \gamma, \\ G_i(x(\cdot,u),u) + \sum\limits_{k\in I}^{k\ne i} F_k(x(\cdot,u)) \le \gamma,\ i\in I; \end{array}\right\} \tag{4.16}$$

along the control system (2.1)–(2.2).

Because this problem is state-convex, one has the rights to employ the corresponding Lagrange function $\mathscr{L}(x(\cdot,u),\gamma,\mu)$ with $\mu = (\mu_1,\ldots,\mu_{m+1})$, $\mu_i \ge 0, i \in I\cup\{m+1\}$. It is worth noting that $\mu_0 = 1$, since the Slater condition takes place for $(\mathscr{A}\mathscr{P}_\sigma L(z))$–(4.16)[21] Besides, it can be readily seen that the Lagrange function, from the view-point of the Optimal Control Theory, has the following form

$$\begin{array}{c} \mathscr{L}(x(\cdot,u),\gamma,\mu) = p(x(t_1,u)) + \sigma\gamma + \langle q(z(t_1)), x(t_1,u)\rangle + \\ + \int\limits_T [P(x(t,u),u(t),t) + \langle Q(z(t),t), x(t,u)\rangle]\, dt \end{array}$$

where the function $p(\cdot)$ and $P(\cdot)$ are state-convex. Therefore, one can address the next parametric OC problem

$$\mathscr{L}(x(\cdot,u),\gamma,\mu) \downarrow \min_{(u,\gamma)}, \quad u(\cdot)\in\mathscr{U},\ \gamma\in I\!R; \tag{4.16$'$}$$

along the CS (2.1)–(2.2). Clearly, the OC problem (4.16) can be attacked by PMP based methods [20, 39] with the real goal to find a globally optimal control. Hence, the GOC's (4.6)–(4.7) are related to the classical Optimal Control Theory [3, 4, 9, 17, 38, 39].

5. Supplementary Properties of GOC's

In this section we pay attention to a substantiation of the possibility to construct some numerical methods on the base of the GOC's of Theorem 4.1. In other words, we are looking for an answer to the natural question on whether one can really find a triple $(y(\cdot),\beta,u(\cdot))$ fulfilling the equation (4.6), and which violates the principal inequality (4.7) of Theorem 4.1.

The answer is given by the next result.

Theorem 5.1. *Let a feasible in the original Problem* $(\mathscr{P})$–(2.7) *process* $(z(\cdot),w(\cdot))$, $z(t) = x(t,w)$, $t\in T$, $w(\cdot)\in\mathscr{U}$, *is not an* ε*-solution to* $(\mathscr{P})$:

$$\inf(J_0,\mathscr{F}) + \varepsilon \overset{\triangle}{=} \mathscr{V}(\mathscr{P}) + \varepsilon < \zeta := J_0(z(\cdot),w(\cdot)). \tag{5.1}$$

Suppose, in addition, that a control $v(\cdot)\in\mathscr{U}$ *satisfies the next inequality*

$$(\mathscr{A}_0): \qquad J_0(x(\cdot,v),v(\cdot)) > \zeta - \varepsilon. \tag{5.2}$$

Then, for any penalty parameter $\sigma > 0$, *one can find a tuple* $(y(\cdot),\beta,u(\cdot))$, $y(\cdot)\in AC_n(T)$, $\beta\in I\!R$, $u(\cdot)\in\mathscr{U}$, *such that the following conditions hold true*

$$F_\sigma(y(\cdot)) = \beta - \zeta + \varepsilon; \tag{5.3}$$

$$G_\sigma(x(\cdot,u),\, u(\cdot)) < \beta + \langle\langle \nabla F_\sigma(y(\cdot)),\, x(\cdot,u) - y(\cdot)\rangle\rangle. \tag{5.4}$$

Proof. (A) If follow from (5.1) that there exists a feasible in $(\mathscr{P})$–(2.7) control $\bar{u}(\cdot) \in \mathscr{U}$, $(x(\cdot,\bar{u}),\bar{u}(\cdot)) \in \mathscr{F}$, such that

$$J_0(\bar{u}) + \varepsilon < \zeta := J_0(w). \tag{5.5}$$

Otherwise, we would have

$$J_0(w) = \zeta \le J_0(u) + \varepsilon \quad \forall u(\cdot) \in \mathscr{U} : (x(\cdot,u),u(\cdot)) \in \mathscr{F}$$

which implies that $J_0(w) = \zeta \le \mathscr{V}(\mathscr{P}) + \varepsilon$, that contradicts to (5.1).

Let us now $\sigma > 0$ be arbitrary, but fixed. Further, due to the equalities

$$\pi(z(\cdot),w(\cdot)) = 0 = \pi(x(\cdot,\bar{u}),\bar{u}),$$

it is clear that for any $\sigma > 0$ one has

$$J_\sigma(\bar{u}) + \varepsilon = J_0(\bar{u}) + \varepsilon < \zeta = J_0(w) = J_\sigma(w),$$

whence we derive

$$G_\sigma(x(\cdot,\bar{u}),\bar{u}) < F_\sigma(x(\cdot,\bar{u})) + \zeta - \varepsilon.$$

The latter inequality amounts to the relation

$$(x(\cdot,\bar{u}), G_\sigma(x(\cdot,\bar{u}),\bar{u})) \notin C, \tag{5.6}$$

where the convex set $C \subset AC_n(T) \times I\!R$ is defined as follows

$$C := epi\,[F_\sigma(\cdot) + \zeta - \varepsilon] \stackrel{\triangle}{=} \{(x(\cdot),\gamma) \in AC_n(T) \times I\!R \mid F_\sigma(x(\cdot)) + \zeta - \varepsilon \le \gamma\}. \tag{5.7}$$

On the other hand, thanks to the inequality (5.2) we have (since $\pi(\cdot) \ge 0$)

$$J_\sigma(z(\cdot),w(\cdot)) - \varepsilon \stackrel{\triangle}{=} J_0(w) - \varepsilon = \zeta - \varepsilon < J_0(x(\cdot,v),v(\cdot)) \le J_0(v) + \sigma\pi(x(\cdot,v),v) \stackrel{\triangle}{=} J_\sigma(x(\cdot,v),v(\cdot)),$$

whence, in particular, it follows $J_\sigma(x(\cdot,v),v(\cdot)) > \zeta - \varepsilon$, or, otherwise,

$$G_\sigma(x(\cdot,v),v(\cdot)) > F_\sigma(x(\cdot,v)) + \zeta - \varepsilon \tag{5.2$'$}$$

The latter inequality, due to (5.7), obviously amounts to the next inclusion

$$(x(\cdot,v), G_\sigma(x(\cdot,v),v(\cdot))) \in int\, C \stackrel{\triangle}{=} \{(x(\cdot),\gamma) \in AC_n(T) \times I\!R \mid F_\sigma(x(\cdot)) + \zeta - \varepsilon < \gamma\}. \tag{5.8}$$

(B) Now, due to the convexity of the set $C \subset AC_n(T) \times I\!R$, and thanks to (5.6) and (5.8), we obtain that there exists a pair $(y(\cdot),\beta) \in AC_n(T) \times I\!R$, which belongs to the open interval

$$](x(\cdot,\bar{u}); G_\sigma(x(\cdot,\bar{u}),\bar{u}(\cdot))); (x(\cdot,v), G_\sigma(x(\cdot,v),v(\cdot)))[\subset AC_n(T) \times I\!R,$$

and, at the same time, to the boundary of C

$$(y(\cdot),\beta) \in bd\, C \stackrel{\triangle}{=} \{(x(\cdot),\gamma) \in AC_n(T) \times I\!R \mid F_\sigma(x(\cdot)) + \zeta - \varepsilon = \gamma\}.$$

In other words, it means that there exists a number $\alpha \in]0,1[$, such that

$$(y(\cdot),\beta) = \alpha(x(\cdot,\bar{u}); G_\sigma(x(\cdot,\bar{u}),\bar{u}(\cdot))) + (1-\alpha)(x(\cdot,v); G_\sigma(x(\cdot,v),v(\cdot))) \in bd\, C.$$

Or more precisely, one has the presentation

$$\left.\begin{aligned} y(\cdot) = \alpha x(\cdot,\bar{u}) + (1-\alpha)x(\cdot,v) \in AC_n(T), \\ \beta = \alpha G_\sigma(x(\cdot,\bar{u}),\bar{u}(\cdot)) + (1-\alpha)G_\sigma(x(\cdot,v),v) = F_\sigma(y(\cdot)) + \zeta - \varepsilon, \end{aligned}\right\} \tag{5.9}$$

so that the equality (5.3) is proven.

Besides, the equalities (5.9) can be rewritten as follows

$$\left.\begin{aligned} x(\cdot,\bar{u}) &= \alpha^{-1}[y(t)-(1-\alpha)x(\cdot,v)],\\ G_\sigma(x(\cdot,\bar{u}),\bar{u}(\cdot)) &= \alpha^{-1}[\beta-(1-\alpha)G_\sigma(x(\cdot,v),v(\cdot))]. \end{aligned}\right\} \tag{5.9'}$$

(C) Suppose now, by contradiction, that the inequality (5.4) is not satisfied at the triple $(y(\cdot),\beta)\in bd\,C$ and $\bar{u}(\cdot)\in\mathscr{U}$, constructed above, i.e.

$$G_\sigma(x(\cdot,\bar{u}),\,\bar{u})\geq\beta+\langle\langle\nabla F_\sigma(y(\cdot)),\,x(\cdot,\bar{u})-y(\cdot)\rangle\rangle. \tag{5.10}$$

Whence, with the help of the presentation (5.9') it yields

$$\begin{gathered} 0\geq\beta-G_\sigma(x(\cdot,\bar{u}),\,\bar{u})+\langle\langle\nabla F_\sigma(y(\cdot)),\,x(\cdot,\bar{u})-y(\cdot)\rangle\rangle=\\ =\beta-\alpha^{-1}[\beta-(1-\alpha)G_\sigma(x(\cdot,v),v(\cdot))]+\langle\langle\nabla F_\sigma(y(\cdot)),\alpha^{-1}[y(\cdot)-(1-\alpha)x(\cdot,v)]\rangle\rangle=\\ =\frac{1-\alpha}{\alpha}[G_\sigma(x(\cdot,v),v(\cdot))-\beta]+\frac{1-\alpha}{\alpha}\langle\langle\nabla F_\sigma(y(\cdot)),y(\cdot)-x(\cdot,v)\rangle\rangle. \end{gathered}$$

Furthermore, thanks to the definition (3.2) of $\langle\langle\nabla F_\sigma(y(\cdot)),x(\cdot)\rangle\rangle$, (2.6′), the equation (5.3), the condition $(\mathscr{A}_0)$–(5.2), and the convexity inequality (3.9) we are getting

$$0\geq\frac{1-\alpha}{\alpha}[G_\sigma(x(\cdot,v),\,v)-\beta+F_\sigma(y(\cdot))-F_\sigma(x(\cdot,v))]=\frac{1-\alpha}{\alpha}[J_\sigma(x(\cdot,v),\,v)-\zeta+\varepsilon]>0,$$

that is impossible.

Hence, the conjecture of the part (C) of the proof led us to the absurdity, and therefore, it is incorrect. Thus, (5.4) is proved as well, as Theorem 5.1. □

Remark 5.2. It is worth noting that the penalty parameter $\sigma\geq 0$ plays different roles in Theorem 4.1 and 5.1. Indeed, in Theorem 4.1 the value σ of penalty parameter should be grater than the threshold value: $\sigma>\sigma_*\geq 0$, which provides the equivalence between $(\mathscr{P})$ and $(\mathscr{P}_\sigma)$: $Sol(\mathscr{P})=Sol(\mathscr{P}_\sigma)$. Meanwhile, in Theorem 5.1, the penalty parameter value can be arbitrary, but should remain only positive: $\sigma>0$. Regardless of that, one can find a triple $(y(\cdot),\beta,u(\cdot))$: $y(\cdot)\in AC_n(T)$, $\beta\in I\!R$, $u(\cdot)\in\mathscr{U}$, satisfying (5.3) and improving, at the same time, the current value $J_\sigma(w)=J_0(w)+\sigma\pi(z(\cdot),w(\cdot))$ of the cost functional of the penalized OC Problem $(\mathscr{P}_\sigma)$ in question. Clearly, this fact is very promising and important for successful computational implementations of the methods based on Theorems 4.1 and 5.1.

In order to demonstrate the effectiveness the GOC's (4.6)–(4.7) of Theorems 4.1 and 5.1, let us address the next example of a nonconvex OC problem with the Bolza functionals.

Example 5.3. Let us address the control system (CS) which is suggested by some applications [3, 9, 17, 20, 38, 39]

$$\left.\begin{gathered} \dot{x}_1(t)=x_2(t),\ \dot{x}_2(t)=u(t),\quad \overset{\circ}{\forall} t\in T:=]0,1[,\quad x_1(0)=1,\quad x_2(0)=0,\\ u(\cdot)\in\mathscr{U}:=\{u(\cdot)\in L_\infty(T)\mid u(t)\in[-4,4]\ \overset{\circ}{\forall} t\in T\}. \end{gathered}\right\} \tag{5.11}$$

The principal goal consists in the minimization of the next cost functional:

$$\left.\begin{gathered} J_0(u):=\frac{1}{4}[x_1(1)-3]^2-3\int_T[x_1(t)-s(t)]^2dt\downarrow\min_u,\\ u(\cdot)\in\mathscr{U}, \end{gathered}\right\} \tag{5.12}$$

along the CS (5.11). In addition, the possibilities of the control are bounded by the next inequality constraint

$$J_1(u) := \int_0^1 [x_2(t) - s'(t)]^2 dt - \frac{3}{4}[x_2(1)]^2 \leq 0, \tag{5.13}$$

where $s(t) = t^2 + 1,\ s'(t) = 2t,\ t \in [0,1],\ s(1) = 2.$

(A) Let us consider the control $u_0(t) \equiv 4,\ t \in T$, and first, verify whether $u_0(\cdot) \in \mathscr{U}$ satisfies the constraint (5.13). Actually, one can see that $u_0(\cdot)$ provides for the following state $x_0(\cdot) = (x_{01}(\cdot), x_{02}(\cdot)) \in AC_2(T)$:

$$x_{02}(t) = 4t,\ \ x_{02}(1) = 4,\ x_{01}(t) = 2t^2 + 1,\ \ x_{01}(1) = 3. \tag{5.11$'$}$$

It is not difficult to compute that

$$\left.\begin{aligned} J_1(u) \overset{\triangle}{=} \int_0^1 [x_{02}(t) - s'(t)]^2 dt - \frac{3}{4}[x_{02}(1)]^2 = \\ \int_0^1 [4t - 2t]^2 dt - \frac{3}{4}16 = \int_0^1 4t^2 dt - 12 = \frac{4}{3} - 12 < 0. \end{aligned}\right\}$$

Therefore $u_0(\cdot)$ is feasible for the OC problem (5.11)–(5.13), but the inequality constraint (5.13) is not active at $u_0(\cdot)$.

Furthermore, let us verify now whether the control $u_0(\cdot)$ satisfies the Pontryagin's Principle (the PMP). To this end, introduce the Hamiltonian $\mathscr{H}(x, y, \psi, t)$ by

$$\mathscr{H}(x,u,\psi,t) :=< \langle \psi(t), A(t)x + B(u,t) \rangle > +\lambda_0 \varphi_0(x,u,t) = \psi_1 x_2 + \psi_2 u - 3\lambda_0 [x_1 - s(t)]^2, \tag{5.14}$$

where the conjugate state $\psi(\cdot) \in AC_2(T)$, satisfies the following adjoint system of ODEs

$$\left.\begin{aligned} \dot{\psi}_1(t) = -\frac{\partial \mathscr{H}}{\partial x_1} = 6\lambda_0 [x_1(t) - s(t)]; \\ \dot{\psi}_2(t) = -\frac{\partial \mathscr{H}}{\partial x_2} = -\psi_1(t); \\ \psi_1(1) = \lambda_0 \nabla \varphi_0(x,u,t) = \lambda_0 \frac{1}{2}[x_1(1) - 3];\ \ \psi_2(1) = 0. \end{aligned}\right\} \tag{5.15}$$

It is worth noting that the term $\lambda_0 [x_2(t) - s'(t)]^2$ is absent in (5.14)–(5.15), because the constraint $J_1(u(\cdot)) \leq 0$ is not active at $u_0(\cdot)$, and therefore $\lambda_1 = 0$. Let us set now $\lambda_0 := 1$, which provides

$$\dot{\psi}_{01}(t) = 6[x_{01}(t) - s(t)] = 6t^2,\ \ \psi_{01}(1) = \frac{1}{2}[3 - 3] = 0;$$

$$\psi_{01} = 6 \int_0^t \tau^2 d\tau + c = 2t^3 + c = 2t^3 - 2 = 2(t^3 - 1),$$

$$\dot{\psi}_{02}(t) = 2(1 - t^3),\ \ \psi_{02}(1) = 0,$$

$$\psi_{02} = 2 \int_0^t [1 - \tau^3] d\tau + c = 2t - \frac{1}{2}t^4 - \frac{3}{2}.$$

It can be readily seen that $\psi_{02}(t) < 0 \ \forall t \in [0,1[$, and, hence, the control $u_0(t) \equiv 4$ satisfies the next minimum condition (see (5.14)):

$$\mathscr{H}(x_0(t), u_0(t), \psi_0(t), t) = \min_v \{\mathscr{H}(x_0(t), v, \psi_0(t), t) \mid v \in [-4,4]\} \ \ \forall t \in T; \tag{5.16}$$

which obviously amounts to the relation

$$\psi_{02}(t) u_0(t) = \min_v \{\psi_{02}(t) v \mid v \in [-4,4]\} \ \ \forall t \in T. \tag{5.16$'$}$$

Hence, $u_0(\cdot)$ satisfies the PMP.

(B) Nevertheless, let us now show that $u_0(\cdot)$ is not a globally optimal control for the problem (5.11)–(5.13). To this end, let us first compute the value of the cost functional (see (5.11)) at the control $u_0(\cdot)$

$$J_0(u_0) \triangleq \frac{1}{4}[x_{01}(1) - 3]^2 - 3\int_0^1 [x_{01}(t) - s(t)]^2 dt = 0 - 3\int_0^1 t^4 dt = -\frac{3}{5} =: \zeta_0. \tag{5.17}$$

Furthermore, in order to apply Theorems 4.1, let us represent the data of the OC problem (5.11)–(5.13) in the relevant form as follows

$$g_{10}(x) = \frac{1}{4}[x_1 - 3]^2, \ \ h_{10} \equiv 0, \ \ g_0(x,t) \equiv 0, \ \ h_0(x,t) = 3[x_1 - s(t)]^2,$$
$$g_{11}(x) \equiv 0, \ \ h_{11}(x) = \frac{3}{4}x_2^2, \ \ g_1(x,t) = [x_2 - s'(t)]^2, \ \ h_1(x,t) \equiv 0.$$

Therefore, thanks to (3.1)–(3.1$'$), we have

$$\left.\begin{aligned} G_0(x(\cdot),u(\cdot)) &= \frac{1}{4}[x_1(t_1) - 3]^2, \ \ F_0(x(\cdot)) = 3\int_0^1 [x_1(t) - s(t)]^2 dt; \\ G_1(x(\cdot),u(\cdot)) &= \int_0^1 [x_2(t) - s'(t)]^2 dt, \ \ F_1(x(\cdot)) = \frac{3}{4}[x_2(t_1)]^2. \end{aligned}\right\} \tag{5.18}$$

Finally, all above yields that the cost functional $J_\sigma(x(\cdot),u(\cdot)) = G_\sigma(x(\cdot),u(\cdot)) - F_\sigma(x(\cdot))$ of the penalized problem $(\mathscr{P}_\sigma)$–(2.9) is defined by

$$\left.\begin{aligned} G_\sigma(x(\cdot),u(\cdot)) &= \frac{1}{4}[x_1(t_1) - 3]^2 + \sigma \max\{\frac{3}{4}[x_2(t_1)]^2; \int_0^1 [x_2(t) - s'(t)]^2 dt\}, \\ F_\sigma(x(\cdot)) &= 3\int_0^1 [x_1(t) - s(t)]^2 dt + \frac{3}{4}[x_2(t_1)]^2. \end{aligned}\right\} \tag{5.19}$$

It is easy to see that $G_\sigma(\cdot)$, $F_\sigma(\cdot)$ are state-convex.

In order to verify the GOCs (4.6)–(4.7), according to Theorems 4.1, we have to find a pair $(y(\cdot),\beta), \ y(\cdot) \in PC(T), \ \beta \in I\!R$, satisfying the equation (4.5): $F_\sigma(y(\cdot)) = \beta - \zeta_0$, and some control $u(\cdot) \in \mathscr{U}$ with which we intend to break down (4.7).

To this end, let us consider the control $u_*(t) \equiv -4$, which provides for the state $x_*(\cdot) = (x_1^*(\cdot), x_2^*(\cdot)) \in AC_2(T):$

$$x_2^*(t) = -4t, \ \ x_2^*(1) = -4, \ \ x_1^*(t) = 1 - 2t^2, \ \ x_1^*(1) = -1. \tag{5.20}$$

Further, one can compute the value $J_1(u_*)$:

$$\left.\begin{array}{c} J_1(u_*) := \int\limits_0^1 [x_2^*(t) - s'(t)]^2 dt - \frac{3}{4}[x_2^*(1)]^2 = \\ \int\limits_0^1 [-4t - 2t]^2 dt - \frac{3}{4}[-4]^2 = \int\limits_0^1 36t^2 dt - 12 = 0, \end{array}\right\}$$

so that the process $(x_*(\cdot), u_*(\cdot))$ is feasible in Problem (5.11)–(5.13).

Now let us set $\sigma := 1$ and compute $G_\sigma(x_*(\cdot), u_*(\cdot))$.

$$\left.\begin{array}{c} G_\sigma(x_*(\cdot), u_*(\cdot)) = \frac{1}{4}[-1-3]^2 + \max\{\frac{3}{4}[-4]^2; \int\limits_0^1 [-4t - 2t]^2 dt\} = \\ = 4 + \max\{12; \int\limits_0^1 36t^2 dt\} = 4 + 12 = 16. \end{array}\right\} \tag{5.21}$$

Furthermore, we have to produce $y(t) = (y_1(\cdot), y_2(\cdot)) \in PC(T)$ (or $AC_2(T)$) and $\beta \in I\!R$ for computing the term $\beta + \langle\langle \nabla F_\sigma(y(\cdot)), x(\cdot, u) - y(\cdot)\rangle\rangle$ in (4.7), where

$$\langle\langle \nabla F_\sigma(y(\cdot)), x(\cdot)\rangle\rangle = \frac{3}{2}\langle y_2(t_1), x_2(t_1)\rangle + 3\int\limits_0^1 2\langle y_1(t) - s(t), x_1(t)\rangle dt. \tag{5.22}$$

Let us use further the function $y(\cdot) = (y_1(t), y_2(t)) \in C(T)$ given by the formulae

$$y_1(t) = 1 - \frac{3}{2}t^2, \;\; y_2(t) = -4t, \;\; y_1(1) = -\frac{1}{2}, \;\; y_2(1) = -4.$$

Then we get the next value of $F_\sigma(y(\cdot))$ (see (5.19))

$$\frac{1}{3}F_\sigma(y(\cdot)) = \int\limits_0^1 [1 - \frac{3}{2}t^2 - t^2 - 1]^2 dt + \frac{1}{4}[-4]^2 = 4 + \frac{25}{4}\int\limits_0^1 t^4 dt = 4 + \frac{5}{4} = 5\frac{1}{4}, \;\; F_\sigma(y(\cdot)) = 15\frac{3}{4};$$

which provides the value of β (see (4.6))

$$\beta = F_\sigma(y(\cdot)) + \zeta_0 = 15\frac{3}{4} - \frac{3}{5} = 15\frac{3}{20}. \tag{5.23}$$

Besides, thanks to (5.22), it allows us to compute the corresponding term in (4.7)

$$\frac{1}{3}\langle\langle \nabla F_\sigma(y(\cdot)), x_*(\cdot) - y(\cdot)\rangle\rangle = \frac{1}{2}\langle y_2(t_1), x_2^*(t_1) - y_2(t_1)\rangle + 2\int\limits_0^1 \langle y_1(t) - t^2 - 1, x_1^*(t) - y_1(t)\rangle dt =$$

$$= \frac{1}{2}\langle -4, -4+4\rangle + 2\int\limits_0^1 \left(-\frac{5}{2}t^2\right)\left(-\frac{1}{2}t^2\right) dt = 2 \cdot \frac{5}{4} \cdot \int\limits_0^1 t^4 dt = \frac{1}{2},$$

so that $\langle\langle \nabla F_\sigma(y(\cdot)), x_*(\cdot) - y(\cdot)\rangle\rangle = \frac{3}{2}$. Then from the latter chain, (5.21), and (5.23), it follows that

$$G_\sigma(x_*(\cdot), u_*(\cdot)) = 16 < 16\frac{13}{20} = 15\frac{3}{20} + \frac{3}{2} = \beta + \langle\langle \nabla F_\sigma(y(\cdot)), x_*(\cdot) - y(\cdot)\rangle\rangle,$$

so that the inequality (4.7) turns out to be broken down. Hence, in virtue of Theorems 4.1, it means that the control $u_0(t) \equiv 4$ is not a global solution to the OC problem (5.11)–(5.13).

Moreover, it is confirmed by the next chain

$$J_0(u_*) \stackrel{\triangle}{=} \frac{1}{4}[x_1^*(1)-3]^2 - 3\int_0^1 [x_1^*(t)-s(t)]^2 dt = \frac{1}{4}[-1-3]^2 - 3\int_0^1 [1-2t^2-t^2-1]^2 dt =$$

$$= 4 - 3\int_0^1 [-3t^2]^2 dt = 4 - \frac{27}{5} = -\frac{7}{5} < -\frac{3}{5} = \zeta_0 \stackrel{\triangle}{=} J_0(u_0),$$

which shows us that the control $u_*(t) \equiv -4$ turned out to be better for the OC problem (5.11)–(5.13) than the control $u_0(\cdot)$ fulfilling the PMP (5.14)–(5.16). This example obviously demonstrated the evident advantages of the GOCs of Theorems 4.1 and 5.1 with respect to the PMP which is a particular case of the GOCs (4.5)–(4.6) (see Remark 4.5 and Example 4.4). #

6. Sufficiency of the GOCs

In this section we look for a possibility to transform the necessary GOCs (4.5)–(4.6) into some sufficient optimality conditions for the original Problem $(\mathcal{P})$–(2.7).

Theorem 6.1. *Let for a feasible in Problem $(\mathcal{P})$ process $(z(\cdot), w(\cdot))$, $z(t) = x(t, w)$, $t \in T$, $w(\cdot) \in \mathcal{U}$, $\pi(z(\cdot), w(\cdot)) = 0$, the assumption $(\mathcal{A}_0)$–(5.2) takes place*

$$(\mathcal{A}_0): \qquad J_0(x(\cdot, v), v(\cdot)) > \zeta - \varepsilon, \;\; \varepsilon > 0, \tag{5.2$''$}$$

with $\zeta = J_0(z(\cdot), w(\cdot))$ and a control $v(\cdot) \in \mathcal{U}$.

Suppose, in addition, that some penalty parameter $\sigma > 0$ is given.

Besides, assume that for every pair $(y(\cdot), \beta)$, $\beta \in I\!R$, $y(\cdot) \in AC_n(T)$, satisfying the equation

$$F_\sigma(y(\cdot)) = \beta - \zeta + \varepsilon, \tag{5.3}$$

the following (principal) inequality holds true

$$G_\sigma(x(\cdot, u), u) \geq \beta + \langle\langle \nabla F_\sigma(y(\cdot)), x(\cdot, u) - y(\cdot) \rangle\rangle \quad \forall u(\cdot) \in \mathcal{U}. \tag{6.1}$$

Then, the control $w(\cdot) \in \mathcal{U}$ turns out to be an ε- globally optimal to Problem $(\mathcal{P}_\sigma)$–(2.9),(2.10) as well as to the original Problem $(\mathcal{P})$–(2.7).

Proof. (A) Supposing that despite of the assertion of Theorem 6.1, we found a control $\bar{u}(\cdot) \in \mathcal{U}$, such that $(\bar{x}(\cdot), \bar{u}(\cdot)) \in \mathcal{F}(\mathcal{P}))$, $\bar{x}(t) = x(t, \bar{u})$, $t \in T$, $\pi(\bar{x}(\cdot), \bar{u}(\cdot)) = 0$ and

$$J_\sigma(\bar{u}) + \varepsilon = J_0(\bar{u}) + \varepsilon < J_0(w) = J_0(w) + \sigma\pi(z(\cdot), w(\cdot)) = J_\sigma(w),$$

meanwhile the conditions (5.3) and (6.1) take place.

Further on the proof completely coincides with the proof of Theorem 5.1 till the part (C).

(C) Because the pair $(y(\cdot), \beta) \in AC_n(T) \times I\!R$ (constructed by the same manner as in parts (A) and (B) of the proof of Theorem 5.1) satisfies the equation (5.3) and $(\bar{x}(\cdot), \bar{u}(\cdot)) \in \mathcal{F} = \mathcal{F}(\mathcal{P}))$, $\bar{x}(t) = x(t, \bar{u})$, $t \in T$ the inequality (6.1) must hold with $\bar{u}(\cdot)$, i.e.

$$0 \geq \beta - G_\sigma(\bar{x}(\cdot), \bar{u}(\cdot)) + \langle\langle \nabla F_\sigma(y(\cdot)), \bar{x}(\cdot) - y(\cdot) \rangle\rangle.$$

Furthermore, employing the same transformations as in the proof of Theorem 5.1, the latter inequality with the help of presentations (5.8′) and the assumption ($\mathscr{A}_0$)–(5.2″) leads us to the following absurdity

$$0 \geq \frac{1-\alpha}{\alpha}\,[J_\sigma(x(\cdot,v),v(\cdot)) - \zeta + \varepsilon] > 0.$$

Therefore, the conjecture made in the beginning of the part (A) is incorrect. Hence, we obtain

$$J_0(w) = J_\sigma(w) \leq J_\sigma(u) + \varepsilon = J_0(u) + \varepsilon \quad \forall u(\cdot) \in \mathscr{U}: \ \ (x(\cdot,u),u(\cdot)) \in \mathscr{F}(P)),$$

which provides that $w(\cdot)$ is ε-globally optimal control to the original Problem ($\mathscr{P}$)–(2.7). #

Remark 6.2. It is not difficult to point out that in Theorem 6.1 the value of penalty parameter $\sigma > 0$ is not precise, but fixed. However, according to the assumptions of Theorem 6.1, the process $(z(\cdot),w(\cdot))$ is feasible: $J_i(w) \leq 0,\ i \in I,\ \pi(z(\cdot),w(\cdot)) = 0$, which implies that the value $\sigma > 0$ must be sufficiently large to ensure the feasibility of $z(\cdot),w(\cdot))$. The condition can be guaranteed, say, during an implementation of a special local search method for Problem ($\mathscr{P}_\sigma$) (see [31]) with the help of corresponding stopping criteria [28, 29, 30, 31].

7. Conclusion

We addressed a nonconvex OC problem where the objective and inequality-constraints were given by Bolza functionals, which have been represented by the difference of two state-convex functions. Obviously such problem are nonconvex, i.e. may possess a large number of stationary (say, in the sense of the PMP) and local pitfalls. In order to avoid some singularities of the problem, the original OC problem was reduced to an auxiliary penalized problem with the help of Exact Penalization Theory.

Moreover this reduction enables us to represent the objective of the penalized problem as the difference of two state-convex Bolza functionals. Using this property, we successfully developed the new Global Optimality Conditions (the GOC's) having the co-called constructive (algorithmic) property allowing to escape any local pitfall with improving of the cost functional: once the GOC's are violated, one has a feasible control which is better than this one in question. It worth noting that first one got the necessary GOC's (Theorem 4.1), and after that the sufficient ones (Theorem 6.1).

But the principal value of the GOC's consists in the crucial impact on the construction of new local and global search methods and algorithms, based on the constructive property of the GOC's. A number of such methods and algorithms were developed not only for finite-dimensional nonconvex problems [27, 29, 30, 31], where they demonstrated its effectiveness, but also for OC problems of state-high-dimension cases (e.g. till 20×20: state$\times$controls) [28].

This paper opened the way to attack the similar OC problems with inequality and equality constraints [3, 4, 5, 9, 17, 20, 26, 38, 39], under condition to develop and substantiate corresponding local and global search algorithms.

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REFERENCES

[1] J. Burke, An exact penalization viewpoint of constrained optimization, SIAM J. Control Optim. 29 (1991), 968-998.
[2] R. Byrd, G. Lopez-Calva, J. Nocedal, A line search exact penalty method using steering rules, Math. Program. Ser. A 133 (2012), 39-73.
[3] F.L. Chernousko, I.M. Ananievski, S.A. Reshmin, Control of Nonlinear Dynamical Systems: Methods and Applications, Springer, Berlin, 2008.
[4] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley-Interscience, New York (1983).
[5] V.F. Demyanov, Extremum's Conditions and Variational Calculus, High School Edition, Moscow, 2005. [In Russian].
[6] G. Di Pillo, S. Lucidi, F. Rinaldi, An approach to constrained global optimization based on exact penalty functions, J. Global Optim. 54 (2012), 251-260.
[7] I. Eremin, The penalty method in convex programming, Soviet Math. Dokl. 8 (1966), 459-462.
[8] C.A. Floudas, P.M. Pardalos (eds.) Frontiers in global optimization, Kluwer Academic Publishers, Dordrecht, 2004.
[9] R. Gabasov, F.M. Kirillova, Optimization of Linear Systems, Plenum, New York, 1979.
[10] J.-B. Hiriart-Urruty, Generalized differentiability, duality and optimization for problems dealing with difference of convex functions, In: J. Ponstein, (ed.) Convexity and Duality in Optimization, Lect. Notes Econ. Math. Syst. 256 (1985), 37-69.
[11] J.-B. Hiriart-Urruty, Optimisation et Analyse Convex, Presses Universitaires de France, Paris, 1998.
[12] J.-B. Hiriart-Urruty, C. Lemaréchal, Convex analysis and minimization algorithms, Springer-Verlag, Berlin, 1993.
[13] R. Horst, H. Tuy, Global Optimization. Deterministic approaches, Springer-Verlag, Berlin, 1993.
[14] A. Kruger, Error bounds and metric subregularity, Optimization 64 (2015), 49-79.
[15] J. Nocedal, S.J. Wright, Numerical optimization, Springer, New York, 2006.
[16] J.S. Pang, Three modelling paradigms in mathematical programming, Math. Program. Ser. B 125 (2010), 297–323.
[17] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, New York, 1976.
[18] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[19] R.T. Rockafellar, R.J.-.B. Wets, Variational Analysis, Springer, New York, 1998.
[20] V.A. Srochko, Iterative Solution of Optimal Control Problems. Fizmatlit, Moscow, 2000. (in Russian)
[21] A.S. Strekalovsky, Global optimality conditions in nonconvex optimization, J. Optim. Theory Appl. 173 (2017), 770-792.
[22] A.S. Strekalovsky, Global optimality conditions and exact penalization, Optim. Lett. 13 (2019), 597-615.
[23] A.S. Strekalovsky, On global optimality conditions for d.c. minimization problems with d.c. constraints. J. Appl. Numer. Optim. 3 (2021), 175-196.
[24] A.S. Strekalovsky, Optimal control problems with terminal functionals represented as a difference of two convex functions, Comput. Math. Math. Phys. 47 (2007), 1788–1801.
[25] A.S. Strekalovsky, Maximizing a state convex Lagrange functional in optimal control, Autom. Remote Control 73 (2012), 949–961.
[26] A.S. Strekalovsky, Global optimality conditions for optimal control problems with functions of A.D. Alexandrov, J. Optim. Theory Appl. 159 (2013), 297–321,
[27] A.S. Strekalovsky, M.V. Yanulevich, Global search in a nonconvex optimal control problem, J. Comput. Syst. Sci. Int. 52 (2013), 30–45.
[28] A.S. Strekalovsky, M.V. Yanulevich, On global search in nonconvex optimal control problems, J Glob. Optim. 65 (2016), 119–135.
[29] A.S. Strekalovsky, On solving optimization problems with hidden nonconvex structures. In: Themistocles,M., Floudas, Ch. A., Butenko, S. (eds.) Optimization in Science and Engineering, pp. 465–502, Springer, New York, 2014.

[30] A.S. Strekalovsky, I.M. Minarchenko, A local search method for optimization problem with d.c. inequality constraints, Appl. Math. Model. 58 (2018) 229–244.

[31] A.S. Strekalovsky, Local Search for Nonsmooth DC Optimization with DC Equality and Inequality, In: Bagirov, A.M., Gaudioso, M., Karmitsa, N., Mäkelä, M.M., Taheri, S. (Eds.) Constraints Numerical Nonsmooth Optimization: State of the Art Algorithms, pp. 229-261, Springer International Publishing, 2020.

[32] H. Tuy, Concave programming under linear constraints, Dokl. Akad. Nauk SSSR 159 (1964) 32-35.

[33] H. Tuy, D.C. optimization: theory, methods and algorithms, In: Horst, R., Pardalos, P.M. (eds.) Handbook of Global optimization, 149-216, Kluwer Academic Publisher, Dordrecht, 1995.

[34] H. Tuy, Convex Analysis and Global Optimization, Kluwer Academic Publishers, Dordrecht, 1998.

[35] H. Tuy, On a decomposition method for nonconvex global optimization, Optim. Lett. 1 (2007) 245–258.

[36] H. Tuy, On duality bound methods for nonconvex global optimization, J. Global Optim. 37 (2007) 321–323.

[37] H. Tuy, On solving nonconvex optimization problems by reducing the duality gap, J. Global Optim. 32 (2005) 349–365.

[38] F.P. Vasil'ev, Optimization Methods. Factorial Press, Moscow, 2002. (in Russian)

[39] O.V. Vasiliev, Optimization Methods. Word Federation Publishing Company, Atlanta, 1996.

[40] W. Zangwill, Non-linear programming via penalty functions, Manage. Sci. 13 (1967) 344-358.

[41] W. Zangwill, Nonlinear Programming: A Unified Approach, Prentice-Hall, 1969.

[42] A.J. Zaslavski, Optimization on Metric and Normed Spaces, Springer Optimization and Its Applications, Springer, New York, 2010. doi: 10.1007/978-0-387-88621-3.

[43] A.J. Zaslavski, Exact penalty property in optimization with mixed constraints via variational analysis, SIAM J. Optim. 23 (2013), 170-187.