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INFINITE-DIMENSIONAL MULTIOBJECTIVE OPTIMAL CONTROL IN CONTINUOUS TIME

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Dedicated to the memory of Professor Jack Warga on the occasion of his 100th birthday

Abstract. This paper studies multiobjective optimal control problems in the continuous-time framework when the space of states and the space of controls are infinite-dimensional and with lighter smoothness assumptions than the usual ones. The paper generalizes to the multiobjective case existing results for single-objective optimal control problems in that framework. The dynamics are governed by differential equations and a finite number of terminal equality and inequality constraints are present. Necessary conditions of Pareto optimality are provided namely Pontryagin maximum principles in the strong form. Sufficient conditions are also provided.

Keywords. Multiobjective optimization; Pontryagin maximum principle; Pareto optimality; Piecewise continuous functions.

1. Introduction

In this paper, we study multiobjective optimal control problems, with open loop information structure, in the continuous-time framework, when the space of states and the space of controls are infinite-dimensional. We derive necessary conditions and sufficient conditions of Pareto optimality. We rely on lighter smoothness assumptions than the usual ones. The paper extends to the multiobjective case, results obtained for single-objective optimal control problems in infinite dimension.

In the continuous-time framework, some results of multiobjective optimal control problems can be found in Bellaassali and Jourani [3], Zhu [22], Bonnel and Kaya [6], Gramatovici [10], de Oliveira and Nunes Silva [20] and the references therein. Differential games were widely used in economic theory; see, e.g., [7, 8, 15, 18, 21] and Pareto optimality plays a central role in analyzing these problems. In the discrete-time framework, results on infinite-horizon multiobjective optimal control problems can be found in Hayek [11, 12, 13], Ngo-Hayek [17].

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Bachir and Blot [1, 2] extended infinite-horizon single-objective optimal control problems in the discrete-time framework, to the case of infinite-dimensional spaces of states and controls and Hayek [14] extended these results to multiobjective optimal control problems.

In this paper we rely on the results of Blot and Yilmaz [4] and [5] to study multiobjective optimal control problems in an infinite-dimensional setting and in continuous time. We obtain necessary conditions of Pareto optimality under the form of Pontryagin Principles and we provide sufficient conditions of Pareto optimality.

We start by providing necessary conditions of optimality for Mayer multiobjective optimal control problems, and we deduce necessary conditions for Bolza problems with lighter smoothness assumptions. The Hadamard differential of a mapping between Banach spaces, which is stronger than the Gâteaux differential but weaker than the Fréchet differential, has been applied many times in the literature. In finite dimension, the Hadamard differential coincides with the Fréchet differential , but for infinite-dimensional spaces the Fréchet differential is much stronger, even for Lipschitz functions.

We provide different results relying on different constraint qualifications namely to obtain non trivial multipliers associated to the objective functions. For the sufficient conditions we follow Mangasarian [16] and Seierstadt-Sydsaeter [19] and we rely on weaker assumptions than the usual ones namely the concavity at a point and the quasi-concavity at a point.

The plan of this paper is as follows. Section 2 is devoted to definitions and assumptions. In section 3, the problems are presented: multiobjective optimal control problems governed by a differential equation when the space of states and the space of controls are infinite-dimensional, in the continuous-time framework. The notions of Pareto optimality and weak Pareto optimality are defined. In section 4, necessary conditions of Pareto optimality are provided namely Pontryagin maximum principles in the strong form for a Bolza problem. Sufficient conditions are given in section 5.

2. Definitions and assumptions

$\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+$ the set of non-negative real numbers.

When X and Y are Hausdorff spaces, $C^0(X,Y)$ denotes the space of continuous mappings from X into Y.

Let Y be a Hausdorff space and $T \in \mathbb{R}_+^* =]0,+\infty[$. As in [4], a function $u : [0,T] \to Y$ is called piecewise continuous when there exists a subdivision $0 = \tau_0 < \tau_1 < \ldots < \tau_k < \tau_{k+1} = T$ such that

- for all $i \in \{0,\ldots,k\}$, u is continuous on $]\tau_i, \tau_{i+1}[$,
- for all $i \in \{0,\ldots,k\}$, the right-hand limit $u(\tau_i+)$ exists in Y,
- for all $i \in \{1,\ldots,k+1\}$, the left-hand limit $u(\tau_i-)$ exists in Y.

The space of piecewise continuous mappings from $[0,T]$ to Y is denoted by $PC^0([0,T],Y)$.

A function $u \in PC^0([0,T],Y)$ is called a normalized piecewise continuous function when moreover u is right continuous on $[0,T[$ and when $u(T-) = u(T)$ cf. [4].

We denote by $NPC^0([0,T],Y)$ the space of such functions.

As in [4], when Y is a real Banach space, a function $x : [0,T] \to Y$ is called piecewise continuously differentiable when $x \in C^0([0,T],Y)$ and there exists a subdivision $(\tau_i)_{0 \leq i \leq k+1}$ of $[0,T]$ such that the following conditions are fulfilled.

- For all $i \in \{0,...,k\}$, x is continuously differentiable on $]\tau_i, \tau_{i+1}[$.
- For all $i \in \{0,...,k\}$, $x'(\tau_i+)$ exists in Y.
- For all $i \in \{1,...,k+1\}$, $x'(\tau_i-)$ exists in Y.

The $(\tau_i)_{1 \leq i \leq k+1}$ are the corners of the function x.
We denote by $PC^1([0,T],Y)$ the space of such functions.
When G is an open subset of Y, $PC^1([0,T],G)$ is the set of functions $x \in PC^1([0,T],Y)$ such that $x([0,T]) \subset G$.
When $x \in PC^1([0,T],Y)$ and $(\tau_i)_{0 \leq i \leq k+1}$ are the corners of the function x, we define the function $\underline{d}x : [0,T] \to Y$, called the extended derivative of x, by setting

$$\underline{d}x(t) := \begin{cases} x'(t) & \text{if} \quad t \in [0,T] \setminus \{\tau_i : i \in \{0,...,k+1\}\}, \\ x'(\tau_i+) & \text{if} \quad t = \tau_i, i \in \{0,...,k\}, \\ x'(T-) & \text{if} \quad t = T. \end{cases} \tag{2.1}$$

Notice that, contrary to the usual derivative of x, the extended derivative of x is defined on $[0,T]$ all over. Note that $\underline{d}x \in NPC^0([0,T],Y)$ and we have the following relation between x, $\underline{d}x$ and the Riemann integral:

$$\text{for all } a \leq t \text{ in } [0,T], x(t) - x(a) = \int_a^t \underline{d}x(s)ds.$$

Besides, $\underline{d}$ is a bounded linear operator from $PC^1([0,T],Y)$ into $NPC^0([0,T],Y)$.
All these properties motivated the authors of [4] to introduce the notion of extended derivative for piecewise continuously differentiable functions.
When X and Y are real normed vector spaces, $\mathscr{L}(X,Y)$ denotes the space of the bounded linear mappings from X into Y and X^* denotes the topological dual of X.
We denote by $\| \cdot \|_{\mathscr{L}}$ the usual norm of $\mathscr{L}(X,Y)$.
Let G be a non-empty open subset of X, let $\mathfrak{f} : G \to Y$ be a mapping and let $x \in G$.
The mapping $\mathfrak{f}$ is called Gâteaux differentiable at x when there exists $D_G\mathfrak{f}(x) \in \mathscr{L}(X,Y)$ such that for all $h \in X$, $\lim_{t \to 0+} \frac{\mathfrak{f}(x+th)-\mathfrak{f}(x)}{t} = D_G\mathfrak{f}(x) \cdot h$.
Moreover, $D_G\mathfrak{f}(x)$ is called the Gâteaux differential of $\mathfrak{f}$ at x.
We say that $\mathfrak{f}$ is Hadamard differentiable at x when there exists $D_H\mathfrak{f}(x) \in \mathscr{L}(X,Y)$ such that for each K compact in X, $\lim_{t \to 0+} \sup_{h \in K} \| \frac{\mathfrak{f}(x+th)-\mathfrak{f}(x)}{t} - D_H\mathfrak{f}(x) \cdot h \| = 0$.
Moreover, $D_H\mathfrak{f}(x)$ is called the Hadamard differential of $\mathfrak{f}$ at x.
When $\mathfrak{f}$ is Hadamard differentiable at x, $\mathfrak{f}$ is also Gâteaux differentiable at x and $D_H\mathfrak{f}(x) = D_G\mathfrak{f}(x)$. But the converse is false in general when the dimension of X is greater than 2.
Notice that Hadamard differentiability and Gâteaux differentiability always coincide for locally Lipschitz functions in any normed vector space.
When it exists, $D_F\mathfrak{f}(x)$ denotes the Fréchet differential of $\mathfrak{f}$ at x. When $\mathfrak{f}$ is Fréchet differentiable at x, $\mathfrak{f}$ is Hadamard differentiable at x and $D_F\mathfrak{f}(x) = D_H\mathfrak{f}(x)$. But the converse is false in general when the dimension of X is infinite.
When X is a finite product of n real normed spaces, $X = \prod_{1 \leq i \leq n} X_i$, if $k \in \{1,...,n\}$, $D_{F,k}\mathfrak{f}(x)$ (respectively $D_{H,k}\mathfrak{f}(x)$, respectively $D_{G,k}\mathfrak{f}(x)$) denotes the partial Fréchet (respectively Hadamard, respectively Gâteaux) differential of $\mathfrak{f}$ at x with respect to the k-th vector variable.

More information on these notions of differentials can be found in [9].
Next, we introduce definitions of notions of concavity at a point in infinite dimension cf. Mangasarian [16] for the finite dimension. This concepts will be used for sufficient conditions.
Let $\mathfrak{g} : G \to \mathbb{R}$ be a mapping. The mapping $\mathfrak{g}$ is said to be concave at x when for all $y \in G$, for all $t \in [0,1]$ s.t. $(1-t)x+ty \in G$, $\mathfrak{g}((1-t)x+ty) \geq (1-t)\mathfrak{g}(x)+t\mathfrak{g}(y)$.
When $\mathfrak{g}$ is Gâteaux differentiable at x, the function $\mathfrak{g}$ is said to be pseudo-concave at x when for all $y \in G$, $[D_G\mathfrak{g}(x)\cdot(y-x) \leq 0 \Rightarrow \mathfrak{g}(y) \leq \mathfrak{g}(x)]$.
The mapping $\mathfrak{g}$ is said to be quasi-concave at x when for all $y \in G$, for all $t \in [0,1]$ s.t. $(1-t)x+ty \in G$, $[\mathfrak{g}(x) \leq \mathfrak{g}(y) \Rightarrow \mathfrak{g}(x) \leq \mathfrak{g}((1-t)x+ty)]$.
When $\mathfrak{g}$ is Gâteaux differentiable at x and $\mathfrak{g}$ is quasi-concave at x, we have, for all $y \in G$, $[\mathfrak{g}(y) \geq \mathfrak{g}(x) \Rightarrow D_G\mathfrak{g}(x)\cdot(y-x) \geq 0]$.

3. The multiobjective optimal control problems

Let $T \in]0,+\infty[$, let E be a real Banach space, Ω a non-empty subset of E, U a Hausdorff topological space and $\xi_0 \in \Omega$. We consider the functions $f : [0,T]\times\Omega\times U \to E$, $f_i^0 : [0,T]\times\Omega\times U \to \mathbb{R}$ when $i \in \{1,...,l\}$, $g_i^0 : \Omega \to \mathbb{R}$ when $i \in \{1,...,l\}$, $g^\alpha : \Omega \to \mathbb{R}$ when $\alpha \in \{1,...,m\}$ and $h^\beta : \Omega \to \mathbb{R}$ when $\beta \in \{1,...,q\}$, when $(l,m,q) \in (\mathbb{N}^*)^3$. For all $i \in \{1,...,l\}$ we consider also the function $J_i : PC^1([0,T],\Omega)\times NPC^0([0,T],U) \to \mathbb{R}$ defined by, for all $(x,u) \in PC^1([0,T],\Omega)\times NPC^0([0,T],U)$, $J_i(x,u) := g_i^0(x(T)) + \int_0^T f_i^0(t,x(t),u(t))dt$.
With these elements, we can build the following multiobjective Bolza problem

$$(\mathscr{B})\begin{cases} \text{Maximize} & (J_1(x,u),...,J_l(x,u)) \\ \text{subject to} & x \in PC^1([0,T],\Omega), u \in NPC^0([0,T],U) \\ & \forall t \in [0,T],\ \underline{d}x(t) = f(t,x(t),u(t)),\ x(0)=\xi_0 \\ & \forall \alpha \in \{1,...,m\},\ \ g^\alpha(x(T)) \geq 0 \\ & \forall \beta \in \{1,...,q\},\ \ h^\beta(x(T)) = 0. \end{cases}$$

Our problem is a reformulation of the multiobjective classical Bolza problem where the controlled dynamical system is formulated as follows: $x'(t) = f(t,x(t),u(t))$ when $x'(t)$ exists, and the control function $u \in PC^0([0,T],U)$. In [4], the authors explain that the formulation with the extended derivative is equivalent to the classical one, for the single-objective Bolza problem. By using the same reasoning, we remark that this formulation is also equivalent for the multiobjective Bolza problem.
When for all $i \in \{1,...,l\}$, $f_i^0 = 0$, $(\mathscr{B})$ is called a multiobjective Mayer problem and it is denoted by $(\mathscr{M})$.
We denote by $Adm(\mathscr{B})$ (respectively $Adm(\mathscr{M})$) the set of the admissible processes of $(\mathscr{B})$ (respectively $(\mathscr{M})$).
It is clear that $Adm(\mathscr{B}) = Adm(\mathscr{M})$. When (x,u) is an admissible process for $(\mathscr{B})$, we consider the following constraint qualifications, when the functions defining the terminal constraints and the terminal parts of the criterion are Hadamard differentiable at $x(T)$.

$$(QC_0)\begin{cases} \text{If } (b_i)_{1\leq i\leq l} \in \mathbb{R}_+^l, (c_\alpha)_{1\leq\alpha\leq m} \in \mathbb{R}_+^m, (d_\beta)_{1\leq\beta\leq q} \in \mathbb{R}^q \ \text{ satisfy} \\ (\forall \alpha \in \{1,...,m\},\ c_\alpha g^\alpha(x(T)) = 0), \text{and} \\ \sum_{i=1}^l b_i D_H g_i^0(x(T)) + \sum_{\alpha=1}^m c_\alpha D_H g^\alpha(x(T)) + \sum_{\beta=1}^q d_\beta D_H h^\beta(x(T)) = 0, \\ \text{then } (\forall i \in \{1,...,l\},\ b_i = 0),\ (\forall \alpha \in \{1,...,m\},\ c_\alpha = 0) \text{ and} \\ (\forall \beta \in \{1,...,q\},\ d_\beta = 0). \end{cases}$$

and

$$(QC_1)\begin{cases} \text{If } (c_\alpha)_{1\leq\alpha\leq m}\in\mathbb{R}^m_+, (d_\beta)_{1\leq\beta\leq q}\in\mathbb{R}^q \text{ satisfy} \\ (\forall\alpha\in\{1,...,m\},\ c_\alpha g^\alpha(x(T))=0), \text{and} \\ \sum_{\alpha=1}^m c_\alpha D_H g^\alpha(x(T))+\sum_{\beta=1}^q d_\beta D_H h^\beta(x(T))=0, \text{then} \\ (\forall\alpha\in\{1,...,m\},\ c_\alpha=0) \text{ and } (\forall\beta\in\{1,...,q\},\ d_\beta=0). \end{cases}$$

Definition 3.1. An admissible process $(\overline{x},\overline{u})$ for $(\mathscr{B})$ is a Pareto optimal solution for $(\mathscr{B})$ when there does not exist an admissible process (x,u) for $(\mathscr{B})$ such that for all $i\in\{1,...,l\}$, $J_i(x,u)\geq J_i(\overline{x},\overline{u})$ and for some $i_0\in\{1,...,l\}$, $J_{i_0}(x,u)>J_{i_0}(\overline{x},\overline{u})$.

Definition 3.2. An admissible process $(\overline{x},\overline{u})$ for $(\mathscr{B})$ is a weak Pareto optimal solution for $(\mathscr{B})$ when there does not exist an admissible process (x,u) for $(\mathscr{B})$ such that for all $i\in\{1,...,l\}$, $J_i(x,u)>J_i(\overline{x},\overline{u})$.

Now, we formulate a list of conditions which will become the assumptions of our theorems. Let (x_0,u_0) be an admissible process for $(\mathscr{B})$.

Conditions on the vector field.

- **(Av1)** $f\in C^0([0,T]\times\Omega\times U,E)$, for all $(t,\xi,\zeta)\in[0,T]\times\Omega\times U$, $D_{G,2}f(t,\xi,\zeta)$ exists, for all $(t,\zeta)\in[0,T]\times U$, $D_{F,2}f(t,x_0(t),\zeta)$ exists and $[(t,\zeta)\mapsto D_{F,2}f(t,x_0(t),\zeta)]\in C^0([0,T]\times U,\mathscr{L}(E,E))$.
- **(Av2)** For all non-empty compact $K\subset\Omega$, for all non-empty compact $M\subset U$, $\sup_{(t,\xi,\zeta)\in[0,T]\times K\times M}\|D_{G,2}f(t,\xi,\zeta)\|_{\mathscr{L}}<+\infty$.

Conditions on the integrands of the criterion.

- **(Ai1)** For all $i\in\{1,...,l\}$, $f_i^0\in C^0([0,T]\times\Omega\times U,\mathbb{R})$, for all $(t,\xi,\zeta)\in[0,T]\times\Omega\times U$, $D_{G,2}f_i^0(t,\xi,\zeta)$ exists, for all $(t,\zeta)\in[0,T]\times U$, $D_{F,2}f_i^0(t,x_0(t),\zeta)$ exists and $[(t,\zeta)\mapsto D_{F,2}f_i^0(t,x_0(t),\zeta)]\in C^0([0,T]\times U,E^*)$.
- **(Ai2)** For all $i\in\{1,...,l\}$, for all non-empty compact $K\subset\Omega$, for all non-empty compact $M\subset U$, $\sup_{(t,\xi,\zeta)\in[0,T]\times K\times M}\|D_{G,2}f_i^0(t,\xi,\zeta)\|_{\mathscr{L}}<+\infty$.

Conditions on the functions defining the terminal constraints and terminal parts of the criterion.

- **(At1)** For all $i\in\{1,...,l\}$, g_i^0 is Hadamard differentiable at $x_0(T)$.
- **(At2)** For all $\alpha\in\{1,...,m\}$, g^α is Hadamard differentiable at $x_0(T)$.
- **(At3)** For all $\beta\in\{1,...,q\}$, h^β is continuous on a neighborhood of $x_0(T)$ and Hadamard differentiable at $x_0(T)$.

4. Necessary conditions of Pareto optimality

Definition 4.1. The Hamiltonian of $(\mathscr{B})$ is the function $\mathscr{H}_B:[0,T]\times\Omega\times U\times E^*\times\mathbb{R}^l\to\mathbb{R}$ defined by $\mathscr{H}_B(t,x,u,p,\theta):=\sum_{i=1}^l\theta_i f_i^0(t,x,u)+p\cdot f(t,x,u)$.

Definition 4.2. The Hamiltonian of $(\mathscr{M})$ is the function $\mathscr{H}_M:[0,T]\times\Omega\times U\times E^*\to\mathbb{R}$ defined by $\mathscr{H}_M(t,x,u,p):=p\cdot f(t,x,u)$.

Notice that the Hamiltonian of the multiobjective Mayer problem is the same as the Hamiltonian of the single-objective Mayer problem.

Theorem 4.3. *(Pontryagin Principle for the Bolza problem)*
When (x_0,u_0) is a Pareto optimal solution of $(\mathscr{B})$, under (AI1), (AI2), (A∨1), (A∨2), (AT1), (AT2) and (AT3), there exists $(\theta_i)_{1\leq i\leq l}\in\mathbb{R}^l$, $(\lambda_\alpha)_{1\leq\alpha\leq m}\in\mathbb{R}^m$, $(\mu_\beta)_{1\leq\beta\leq q}\in\mathbb{R}^q$ and an adjoint function $p\in PC^1([0,T],E^*)$ which satisfy the following conditions.

(NN) $((\theta_i)_{1\leq i\leq l},(\lambda_\alpha)_{1\leq\alpha\leq m},(\mu_\beta)_{1\leq\beta\leq q})\neq 0$
(Si) *For all $i\in\{1,...,l\}$, $\theta_i\geq 0$ and for all $\alpha\in\{1,...,m\}$, $\lambda_\alpha\geq 0$.*
(Sℓ) *For all $\alpha\in\{1,...,m\}$, $\lambda_\alpha g^\alpha(x_0(T))=0$.*
(TC) $\sum_{i=1}^l\theta_i D_H g_i^0(x_0(T))+\sum_{\alpha=1}^m\lambda_\alpha D_H g^\alpha(x_0(T))+\sum_{\beta=1}^q\mu_\beta D_H h^\beta(x_0(T))=p(T)$.
(AE) $\underline{d}p(t)=-D_{F,2}\mathscr{H}_B(t,x_0(t),u_0(t),p(t),(\theta_i)_{1\leq i\leq l})$ *for all $t\in[0,T]$.*
(MP) *For all $t\in[0,T]$, for all $\zeta\in U$,*
$\mathscr{H}_B(t,x_0(t),u_0(t),p(t),(\theta_i)_{1\leq i\leq l})\geq\mathscr{H}_B(t,x_0(t),\zeta,p(t),(\theta_i)_{1\leq i\leq l})$.
(CH) $\bar{\mathscr{H}}_B:=[t\mapsto\mathscr{H}_B(t,x_0(t),u_0(t),p(t),(\theta_i)_{1\leq i\leq l})]\in C^0([0,T],\mathbb{R})$.

(NN) is a condition of non nullity, (Si) is a sign condition, (Sℓ) is a slackness condition, (TC) is the transversality condition, (AE) is the adjoint equation, (MP) is the maximum principle and (CH) is a condition of continuity on the Hamiltonian.

Proof. The proof will be done through the following steps. First, we transform the multiobjective Bolza problem into a multiobjective Mayer problem. Then, we reduce this last one to a single-objective Mayer problem. Finally, we use a new Pontryagin Maximum Principle for the single-objective Mayer problem established in [5], and so we deduce our Theorem 4.3. That is why, we introduce the following elements, for all $t\in[0,T]$, for all $X=(\sigma_1,...,\sigma_l,x)\in\mathbb{R}^l\times\Omega$, for all $u\in U$, $F(t,X,u):=(f_1^0(t,x,u),...,f_l^0(t,x,u),f(t,x,u))$, $G_i^0(X):=\sigma_i+g_i^0(x)$ for all $i\in\{1,...,l\}$, $G^\alpha(X):=g^\alpha(x)$ for all $\alpha\in\{1,...,m\}$, $H^\beta(X):=h^\beta(x)$ for all $\beta\in\{1,...,q\}$.
Then, we can introduce the following multiobjective Mayer problem

$$(\mathscr{M}\mathscr{B})\begin{cases}\text{Maximize} & (G_1^0(X(T)),...,G_l^0(X(T)))\\ \text{subject to} & X\in PC^1([0,T],\mathbb{R}^l\times\Omega),u\in NPC^0([0,T],U)\\ & \forall t\in[0,T],\ \underline{d}X(t)=F(t,X(t),u(t)),\ X(0)=(0,\xi_0)\\ & \forall\alpha\in\{1,...,m\},\ \ G^\alpha(X(T))\geq 0\\ & \forall\beta\in\{1,...,q\},\ \ H^\beta(X(T))=0.\end{cases}$$

Lemma 4.4. *For each $(x,u)\in Adm(\mathscr{B})$, by setting for all $t\in[0,T]$, for all $i\in\{1,...,l\}$, $\sigma_i(t):=\int_0^t f_i^0(s,x(s),u(s))ds$, we have $((\sigma_1,...,\sigma_l,x),u)\in Adm(\mathscr{M}\mathscr{B})$.*

Proof. Let $(x,u)\in Adm(\mathscr{B})$. Since $u\in NPC^0([0,T],U)$ and $x\in PC^1([0,T],\Omega)$, by using (AI1), we have, for each $i\in\{1,...,l\}$, $[t\mapsto f_i^0(t,x(t),u(t))]\in NPC^0([0,T],\mathbb{R})$.
Consequently, for each $i\in\{1,...,l\}$, $\sigma_i\in PC^1([0,T],\mathbb{R})$ and for all $t\in[0,T]$,
$\underline{d}\sigma_i(t)=f_i^0(t,x(t),u(t))$.
Hence, $(\sigma_1,...,\sigma_l,x)\in PC^1([0,T],\mathbb{R}^l\times\Omega)$ and for all $t\in[0,T]$,

$$\begin{aligned}\underline{d}(\sigma_1,...,\sigma_l,x)(t)&=(\underline{d}\sigma_1(t),...,\underline{d}\sigma_l(t),\underline{d}x(t))\\&=(f_1^0(t,x(t),u(t)),...,f_l^0(t,x(t),u(t)),f(t,x(t),u(t)))\\&=F(t,(\sigma_1,...,\sigma_l,x)(t),u(t))\end{aligned}$$

Moreover, we have, for all $\alpha\in\{1,...,m\}$, $G^\alpha((\sigma_1,...,\sigma_l,x)(T))=g^\alpha(x(T))\geq 0$ and for all $\beta\in\{1,...,q\}$, $H^\beta((\sigma_1,...,\sigma_l,x)(T))=h^\beta(x(T))=0$.

Therefore, since $(\sigma_1,...,\sigma_l,x)(0) = (\sigma_1(0),...,\sigma_l(0),x(0)) = (0,\xi_0)$, we have $((\sigma_1,...,\sigma_l,x),u) \in Adm(\mathscr{MB})$. □

Hence, by setting for all $i \in \{1,...,l\}$, for all $t \in [0,T]$, $\sigma_i^0(t) := \int_0^t f_i^0(s,x_0(s),u_0(s))ds$, by using the Lemma 4.4, we have $(X_0,u_0) := ((\sigma_1^0,...,\sigma_l^0,x_0),u_0) \in Adm(\mathscr{MB})$.

Lemma 4.5. (X_0,u_0) *is a Pareto optimal solution of the multiobjective problem* $(\mathscr{MB})$.

Proof. We proceed by contradiction, we assume that (X_0,u_0) is not a Pareto optimal solution for $(\mathscr{MB})$ i.e. there exists $(X,u) = ((\sigma_1,...,\sigma_l,x),u) \in PC^1([0,T],\mathbb{R}^l \times \Omega) \times NPC^0([0,T],U)$ admissible process for $(\mathscr{MB})$ s.t. for all $i \in \{1,...,l\}$,
$G_i^0(X(T)) \geq G_i^0(X_0(T))$ and for some $i_0 \in \{1,...,l\}$, $G_{i_0}^0(X(T)) > G_{i_0}^0(X_0(T))$.
Since $X \in PC^1([0,T],\mathbb{R}^l \times \Omega)$ and $\forall t \in [0,T]$, $\underline{d}X(t) := F(t,X(t),u(t))$, we have $x \in PC^1([0,T],\Omega)$ and for all $i \in \{1,...,l\}$, $\sigma_i \in PC^1([0,T],\mathbb{R})$ s.t.

$$\forall t \in [0,T],\ \underline{d}x(t) = f(t,x(t),u(t)) \text{ and } \underline{d}\sigma_i(t) = f_i^0(t,x(t),u(t)).$$

Moreover, we have also for all $\alpha \in \{1,...,m\}$, $g^\alpha(x(T)) \geq 0$ and for all $\beta \in \{1,...,q\}$, $h^\beta(x(T)) = 0$. Consequently, we have $(x,u) \in Adm(\mathscr{B})$.
Moreover, for all $t \in [0,T]$, we have $\sigma_i(t) = \int_0^t f_i^0(s,x(s),u(s))ds$. Then, for all $i \in \{1,...,l\}$,

$$\begin{aligned} \int_0^T f_i^0(s,x(s),u(s))ds + g_i^0(x(T)) &= G_i^0(X(T)) \\ &\geq G_i^0(X_0(T)) \\ &= \int_0^T f_i^0(s,x_0(s),u_0(s))ds + g_i^0(x_0(T)) \end{aligned}$$

and for $i_0 \in \{1,...,l\}$,

$$\begin{aligned} \int_0^T f_{i_0}^0(s,x(s),u(s))ds + g_{i_0}^0(x(T)) &= G_{i_0}^0(X(T)) \\ &> G_{i_0}^0(X_0(T)) \\ &= \int_0^T f_{i_0}^0(s,x_0(s),u_0(s))ds + g_{i_0}^0(x_0(T)). \end{aligned}$$

But this contradicts the Pareto optimality of (x_0,u_0) for the multiobjective problem $(\mathscr{B})$. □

Lemma 4.6. *For all* $i \in \{1,...,l\}$, (X_0,u_0) *is a solution of the following single-objective Mayer problem*

$$(\mathscr{MB}_i)\left\{\begin{array}{ll} \text{Maximize} & G_i^0(X(T)) \\ \text{subject to} & X \in PC^1([0,T],\mathbb{R}^l \times \Omega), u \in NPC^0([0,T],U) \\ & \forall t \in [0,T],\ \underline{d}X(t) = F(t,X(t),u(t)),\ X(0) = (0,\xi_0) \\ & \forall k \in \{1,...,l\}, k \neq i,\ \ G_k^0(X(T)) \geq G_k^0(X_0(T)) \\ & \forall \alpha \in \{1,...,m\},\ \ G^\alpha(X(T)) \geq 0 \\ & \forall \beta \in \{1,...,q\},\ \ H^\beta(X(T)) = 0. \end{array}\right.$$

Proof. Let $i \in \{1,...,l\}$. We proceed by contradiction, we assume that (X_0,u_0) is not a solution of $(\mathscr{MB}_i)$ i.e. there exists (X,u) an admissible process of $(\mathscr{MB}_i)$ s.t. $G_i^0(X(T)) > G_i^0(X_0(T))$. This can be rewritten $(X,u) \in Adm(\mathscr{MB})$ satisfies $G_i^0(X(T)) > G_i^0(X_0(T))$ and for all $k \in \{1,...,l\}$, $k \neq i$, $G_k^0(X(T)) \geq G_k^0(X_0(T))$.
Therefore, (X_0,u_0) is not a Pareto optimal solution of the multiobjective problem $(\mathscr{MB})$. This is a contradiction. □

For each $X \in \mathbb{R}^l \times \Omega$, for each $i \in \{2,...,l\}$, we set $\mathfrak{G}_i(X) = G_i^0(X) - G_i^0(X_0(T))$. The problem $(\mathscr{M}\mathscr{B}_1)$ can be rewritten as follows.

$$(\mathscr{M}\mathscr{B}_1)\begin{cases} \text{Maximize} & G_1^0(X(T)) \\ \text{subject to} & X \in PC^1([0,T],\mathbb{R}^l \times \Omega), u \in NPC^0([0,T],U) \\ & \forall t \in [0,T],\, \underline{d}X(t) = F(t,X(t),u(t)),\ X(0) = (0,\xi_0) \\ & \forall i \in \{2,...,l\},\ \mathfrak{G}_i(X(T)) \geq 0 \\ & \forall \alpha \in \{1,...,m\},\ G^\alpha(X(T)) \geq 0 \\ & \forall \beta \in \{1,...,q\},\ H^\beta(X(T)) = 0. \end{cases}$$

Lemma 4.7. *The assumptions of Theorem 2.4 in* [5] *for the single-objective Mayer problem $(\mathscr{M}\mathscr{B}_1)$ with the solution (X_0,u_0) are verified.*

Proof. We consider the linear functions $i \in \{1,...,l\}$, $w_i^1 : \mathbb{R}^l \times E \to \mathbb{R}$ defined by, $w_i^1(\sigma_1,...,\sigma_l,\xi) = \sigma_i$ and $w^2 : \mathbb{R}^l \times E \to E$, defined by, $w^2(\sigma_1,...,\sigma_l,\xi) = \xi$.
Since $G_1^0 = w^1_{1|\mathbb{R}^l\times\Omega} + g_1^0 \circ w^2_{|\mathbb{R}^l\times\Omega}$, by using the property of the chain rule of Hadamard differentiable functions, see [9] p.267, and (AT1), we have G_1^0 is Hadamard differentiable at $X_0(T)$, and we have

$$D_H G_1^0((\sigma_1^0,...,\sigma_l^0,x_0)(T)) = w_1^1 + D_H g_1^0(x_0(T)) \circ w^2. \tag{4.1}$$

Besides, by using the same reasoning, for all $i \in \{2,...,l\}$, G_i^0 and $\mathfrak{G}_i$ are Hadamard differentiable at $X_0(T)$, and we have

$$D_H G_i^0((\sigma_1^0,...,\sigma_l^0,x_0)(T)) = D_H\mathfrak{G}_i(X_0(T)) = w_i^1 + D_H g_i^0(x_0(T)) \circ w^2. \tag{4.2}$$

Next, for all $\alpha \in \{1,...,m\}$, since $G^\alpha = g^\alpha \circ w^2_{|\mathbb{R}^l\times\Omega}$, by using the property of the chain rule of Hadamard differentiable functions, see [9] p.267, and (AT2), we have G^α is Hadamard differentiable at $X_0(T)$, and we have

$$D_H G^\alpha((\sigma_1^0,...,\sigma_l^0,x_0)(T)) = D_H g^\alpha(x_0(T)) \circ w^2. \tag{4.3}$$

Moreover, for all $\beta \in \{1,...,q\}$, since $H^\beta = h^\beta \circ w^2_{|\mathbb{R}^l\times\Omega}$, by using the property of chain rule of Hadamard differentiable functions, see [9] p.267, and (AT3), we have

$$D_H H^\beta((\sigma_1^0,...,\sigma_l^0,x_0)(T)) = D_H h^\beta(x_0(T)) \circ w^2. \tag{4.4}$$

Since h^β is continuous on a neighborhood V_0^β of $x_0(T)$ in Ω and $w^2_{|\mathbb{R}^l\times\Omega} \in C^0(\mathbb{R}^l \times \Omega, \Omega)$, there exists W_0^β of $X_0(T)$ in $\mathbb{R}^l \times \Omega$ s.t. $w_{2|W_0^\beta} \in C^0(W_0^\beta, V_0^\beta)$. Hence, we have $H^\beta_{|W_0^\beta} \in C^0(W_0^\beta, \mathbb{R})$.
Consequently, for $(\mathscr{M}\mathscr{B}_1)$ with the solution (X_0,u_0), the assumptions on the functions defining the terminal constraints and the terminal parts of the criterion of Theorem 2.4 in [5] are verified.
We consider the continuous function $\chi : [0,T] \times \mathbb{R}^l \times \Omega \times U \to [0,T] \times \mathbb{R}^l \times \Omega$ defined by $\chi(t,\sigma,\xi,\zeta) = (t,\xi,\zeta)$.
We remark that $F := (f_1^0 \circ \chi,...,f_l^0 \circ \chi, f \circ \chi)$.
By using (AI1) and (AV1), we have, for all $i \in \{1,...,l\}$, $f_i^0 \circ \chi \in C^0([0,T] \times \mathbb{R}^l \times \Omega \times U, \mathbb{R})$ and $f \circ \chi \in C^0([0,T] \times \mathbb{R}^l \times \Omega \times U, E)$.
Consequently, we have $F \in C^0([0,T] \times \mathbb{R}^l \times \Omega \times U, \mathbb{R}^l \times E)$.

By using (AI1) and (AV1), we have, for all $(t,\sigma,\xi,\zeta)\in[0,T]\times\mathbb{R}^l\times\Omega\times U$, $D_{G,2}F(t,(\sigma,\xi),\zeta)$ exists and

$$\left.\begin{array}{l} D_{G,2}F(t,(\sigma,\xi),\zeta) \\ =(D_{G,2}f_1^0(t,\xi,\zeta)\circ w^2,...,D_{G,2}f_l^0(t,\xi,\zeta)\circ w^2,D_{G,2}f(t,\xi,\zeta)\circ w^2). \end{array}\right\} \tag{4.5}$$

For all $t\in[0,T]$ and $\zeta\in U$, since
$F(t,\cdot,\zeta):=(f_1^0(t,\cdot,\zeta)\circ w^2_{|\mathbb{R}^l\times\Omega},...,f_l^0(t,\cdot,\zeta)\circ w^2_{|\mathbb{R}^l\times\Omega},f(t,\cdot,\zeta)\circ w^2_{|\mathbb{R}^l\times\Omega})$, by using (AI1) and (AV1), we have $D_{F,2}F(t,X_0(t),\zeta)$ exists and

$$\left.\begin{array}{l} D_{F,2}F(t,X_0(t),\zeta) \\ =(D_{F,2}f_1^0(t,x_0(t),\zeta)\circ w^2,...,D_{F,2}f_l^0(t,x_0(t),\zeta)\circ w^2,D_{F,2}f(t,x_0(t),\zeta)\circ w^2). \end{array}\right\}$$

Consequently, by using (AI1) and (AV1), we have

$$[(t,\zeta)\mapsto D_{F,2}F(t,X_0(t),\zeta)]\in C^0([0,T]\ \times U,\mathscr{L}(\mathbb{R}^l\times E,\mathbb{R}^l\times E)).$$

Let K be a non-empty compact set s.t. $K\subset\mathbb{R}^l\times\Omega$ and M be a non-empty compact set s.t. $M\subset U$.
We consider the linear continuous function $\varpi:\mathbb{R}^l\times\Omega\to\Omega$, defined by, for all $(\sigma,\xi)\in\mathbb{R}^l\times\Omega$, $\varpi(\sigma,\xi):=\xi$.
Since K is a non-empty compact set, $\tilde{K}=\varpi(K)$ is a non empty compact set s.t. $\tilde{K}\subset\Omega$.
Consequently, by using (AI2) and (AV2), we have

$$\text{for all } i\in\{1,...,l\}\ \sup_{(t,\xi,\zeta)\in[0,T]\times\tilde{K}\times M}\|D_{G,2}f_i^0(t,\xi,\zeta)\|_{\mathscr{L}}<+\infty,$$

and

$$\sup_{(t,\xi,\zeta)\in[0,T]\times\tilde{K}\times M}\|D_{G,2}f(t,\xi,\zeta)\|_{\mathscr{L}}<+\infty.$$

Therefore, by using (4.5), we have

$$\begin{array}{l} \sup_{(t,(\sigma,\xi),\zeta)\in[0,T]\times K\times U}\|D_{G,2}F(t,(\sigma,\xi),\zeta)\|_{\mathscr{L}} \\ \leq\sum_{i=1}^l\sup_{(t,\xi,\zeta)\in[0,T]\times\tilde{K}\times M}\|D_{G,2}f_i^0(t,\xi,\zeta)\|_{\mathscr{L}}+\sup_{(t,\xi,\zeta)\in[0,T]\times\tilde{K}\times M}\|D_{G,2}f(t,\xi,\zeta)\|_{\mathscr{L}} \\ <+\infty. \end{array}$$

Hence, for $(\mathscr{MB}_1)$ with the solution (X_0,u_0), the assumptions on the vector field of Theorem 2.4 in [5] are verified. Therefore, we have proven the lemma. □

By using the Lemma 4.7, and by applying Theorem 2.4 in [5], we obtain that, there exists $(\theta_i)_{1\leq i\leq l}\in\mathbb{R}^l$, $(\lambda_\alpha)_{1\leq\alpha\leq m}\in\mathbb{R}^m$, $(\mu_\beta)_{1\leq\beta\leq q}\in\mathbb{R}^q$ and an adjoint function $P\in PC^1([0,T],(\mathbb{R}^l\times E)^*)$ which satisfy the following conditions.

(i) $((\theta_i)_{1\ \leq i\leq l},(\lambda_\alpha)_{1\leq\alpha\leq m},\ (\mu_\beta)_{1\leq\beta\leq q})\neq 0$
(ii) For all $i\in\{1,...,l\}$, $\theta_i\geq 0$ and for all $\alpha\in\{1,...,m\}$, $\lambda_\alpha\geq 0$.
(iii) For all $\alpha\in\{1,...,m\}$, $\lambda_\alpha G^\alpha(X_0(T))=0$.
(iv) $\sum_{i=1}^l\theta_i D_H G_i^0(X_0(T))+\sum_{\alpha=1}^m\lambda_\alpha D_H G^\alpha(X_0(T))+\sum_{\beta=1}^q\mu_\beta D_H H^\beta(X_0(T))=P(T)$.
(v) $\underline{d}P(t)=-D_{F,2}\mathscr{H}_{MB_1}(t,X_0(t),u_0(t),P(t))$ for all $t\in[0,T]$.
(vi) For all $t\in[0,T]$, for all $\zeta\in U$,
$\mathscr{H}_{MB_1}(t,X_0(t),u_0(t),P(t))\geq\mathscr{H}_{MB_1}(t,X_0(t),\zeta,P(t))$.
(vii) $\tilde{\mathscr{H}}_{MB_1}:=[t\mapsto\mathscr{H}_{MB_1}(t,X_0(t),u_0(t),P(t))]\in C^0([0,T],\mathbb{R})$.

Where the function $\mathscr{H}_{MB_1} : [0,T] \times (\mathbb{R}^l \times \Omega) \times U \times (\mathbb{R}^l \times E)^* \to \mathbb{R}$ is the Hamiltonian of the problem $(\mathscr{M}\mathscr{B}_1)$, defined by

$$\mathscr{H}_{MB_1}(t,(\sigma_1,...,\sigma_l,x),u,P) = P \cdot F(t,(\sigma_1,...,\sigma_l,x),u).$$

We consider the linear continuous function $\psi : (\mathbb{R}^l \times E)^* \to E^*$ defined by, for all $\mathfrak{l} \in (\mathbb{R}^l \times E)^*$, for all $\xi \in E$, $\psi(\mathfrak{l}) \cdot \xi = \mathfrak{l} \cdot (0,\xi)$.
We set $p = \psi \circ P$. Since $\psi \in \mathscr{L}((\mathbb{R}^l \times E)^*, E^*)$, we have $p \in PC^1([0,T],E^*)$ and for all $t \in [0,T]$, $\underline{d}p(t) = \psi \cdot \underline{d}P(t)$.
By using (iv), (4.1), (4.2), (4.3) and (4.4), we have, for each $i \in \{1,...,l\}$,
$P(T) \cdot (e_i,0) = \theta_i$ where $(e_i)_{1 \le i \le l}$ is the canonical basis of $\mathbb{R}^l$ and $\forall \xi \in E$,

$$\begin{aligned}
&p(T) \cdot \xi = P(T) \cdot (0,\xi) \\
&= (\textstyle\sum_{i=1}^{l} \theta_i D_H G_i^0(X_0(T)) + \sum_{\alpha=1}^{m} \lambda_\alpha D_H G^\alpha(X_0(T)) \\
&+ \textstyle\sum_{\beta=1}^{q} \mu_\beta D_H H^\beta(X_0(T))) \cdot (0,\xi) \\
&= (\textstyle\sum_{i=1}^{l} \theta_i D_H g_i^0(x_0(T)) + \sum_{\alpha=1}^{m} \lambda_\alpha D_H g^\alpha(x_0(T)) + \sum_{\beta=1}^{q} \mu_\beta D_H h^\beta(x_0(T))) \cdot \xi.
\end{aligned}$$

Hence (TC) is verified.
For all $i \in \{1,...,l\}$, we consider the linear continuous function $\varphi_i : (\mathbb{R}^l \times E)^* \to \mathbb{R}$ defined by, $\forall \mathfrak{l} \in (\mathbb{R}^l \times E)^*$, $\varphi_i(\mathfrak{l}) = \mathfrak{l} \cdot (e_i,0)$. We set $p_0^i = \varphi_i \circ P$.
Since $\varphi_i \in \mathscr{L}((\mathbb{R}^l \times E)^*, \mathbb{R})$ we have $p_0^i \in PC^1([0,T],\mathbb{R})$ and

$$\underline{d}p_0^i(t) = \varphi_i \cdot \underline{d}P(t) = \underline{d}P(t) \cdot (e_i,0) = 0.$$

Moreover, since $p_0^i(T) = \theta_i$, we have $\forall t \in [0,T]$, $p_0^i(t) = \theta_i$.
Besides, $\forall \xi \in E$, $\forall t \in [0,T]$,

$$\begin{aligned}
\underline{d}p(t) \cdot \xi &= \underline{d}P(t) \cdot (0,\xi) \\
&= -P(t) \cdot D_{F,2}F(t,X_0(t),u_0(t)) \cdot (0,\xi) \\
&= -\textstyle\sum_{i=1}^{l} p_0^i(t) D_{F,2} f_i^0(t,x_0(t),u_0(t)) \cdot \xi - p(t) \cdot D_{F,2} f(t,x_0(t),u_0(t)) \cdot \xi \\
&= -\textstyle\sum_{i=1}^{l} \theta_i D_{F,2} f_i^0(t,x_0(t),u_0(t)) \cdot \xi - p(t) \cdot D_{F,2} f(t,x_0(t),u_0(t)) \cdot \xi \\
&= -D_{F,2}\mathscr{H}_B(t,x_0(t),u_0(t),p(t),(\theta_i)_{1 \le i \le l}) \cdot \xi
\end{aligned}$$

Consequently (AE) is verified.
Furthermore, we have, $\forall (t,\zeta) \in [0,T] \times U$,

$$\mathscr{H}_{MB_1}(t,X_0(t),\zeta,P(t)) = \mathscr{H}_B(t,x_0(t),\zeta,p(t),(\theta_i)_{1 \le i \le l}).$$

Consequently, by using (vi) and (vii), we have proved (MP) and (CH).
By using (i), (ii) and (iii), we have respectively (NN), (Si) and (Sℓ).
Hence the proof of Theorem 4.3 is complete. □

Corollary 4.8. *In this setting and under the assumptions of Theorem 4.3, if moreover we assume that (QC$_1$) is fulfilled for $(x,u) = (x_0,u_0)$, then, for all $t \in [0,T]$, $((\theta_i)_{1 \le i \le l}, p(t))$ is never equal to zero.*

Proof. We proceed by contradiction by assuming that there exists $t_1 \in [0,T]$ such that $((\theta_i)_{1 \le i \le l}, p(t_1)) = (0,0)$. Since (AE) becomes a homogeneous linear equation, and by using the uniqueness of the Cauchy problem ((AE), $p(t_1) = 0$), we obtain that p is equal to zero on $[0,T]$, in particular we have $p(T) = 0$.
Hence, by using (TC), (Si), (Sℓ), (QC$_1$), we obtain that $(\forall \alpha \in \{1,...,m\}, \lambda_\alpha = 0)$ and $(\forall \beta \in$

$\{1,...,q\}$, $\mu_\beta = 0$).
Therefore, since $(\theta_i)_{1\leq i\leq l} = 0$, we have $((\theta_i)_{1\ \leq i\leq l}, (\lambda_\alpha)_{1\leq\alpha\leq m}, (\mu_\beta)_{1\leq\beta\leq q}) = 0$ which contradicts (NN). $\square$

Corollary 4.9. *In this setting and under the assumptions of Theorem 4.3, when for all $i \in \{1,...,l\}$, $f_i^0 = 0$, if moreover we assume that (QC_0) is fulfilled for $(x,u) = (x_0,u_0)$, then, for all $t \in [0,T]$, $p(t)$ is never equal to zero.*

Proof. We proceed by contradiction by assuming that there exists $t_1 \in [0,T]$ such $p(t_1) = 0$.
Since (AE) is an homogeneous linear equation, and by using the uniqueness of the Cauchy problem ((AE), $p(t_1) = 0$), we obtain that p is equal to zero on $[0,T]$, in particular we have $p(T) = 0$.
Consequently, by using (TC), (Si), (Sℓ), (QC$_0$), we obtain that
$((\theta_i)_{1\ \leq i\leq l}, (\lambda_\alpha)_{1\leq\alpha\leq m}, (\mu_\beta)_{1\leq\beta\leq q}) = 0$ which is a contradiction with (NN). $\square$

As in [5], we introduce another condition

(A$\vee$3) U is a subset of a real normed vector space Y, there exists $\hat{t} \in [0,T]$ s.t. U is a neighborhood of $u_0(\hat{t})$ in Y, $D_{G,3}f(\hat{t},x_0(\hat{t}),u_0(\hat{t}))$ exists and it is surjective.

We introduce a new condition of linear independence.

(ALIB) U is a subset of a real normed vector space Y s.t. U is a neighborhood of $u_0(T)$ in Y, $D_{G,3}f(T,x_0(T),u_0(T))$ exists and
$((D_H g^\alpha(x_0(T)) \circ D_{G,3}f(T,x_0(T),u_0(T)))_{1\leq\alpha\leq m}$,
$(D_H h^\beta(x_0(T)) \circ D_{G,3}f(T,x_0(T),u_0(T)))_{1\leq\beta\leq q})$ are linearly independent.

Corollary 4.10. *In this setting and under the assumptions of Theorem 4.3, if moreover we assume that (QC_1) is fulfilled for $(x,u) = (x_0,u_0)$ and (A$\vee$3), then $(\theta_i)_{1\leq i\leq l} \neq 0$.*

Proof. We proceed by contradiction, we assume that $(\theta_i)_{1\leq i\leq l} = 0$.
Since $D_{G,3}f(\hat{t},x_0(\hat{t}),u_0(\hat{t}))$ exists, $D_{G,3}\mathscr{H}_B(\hat{t},x_0(\hat{t}),u_0(\hat{t}),p(\hat{t}),0)$ exists and

$$D_{G,3}\mathscr{H}_B(\hat{t},x_0(\hat{t}),u_0(\hat{t}),p(\hat{t}),0) = p(\hat{t}) \circ D_{G,3}f(\hat{t},x_0(\hat{t}),u_0(\hat{t})).$$

Therefore, by using (MP), we have $p(\hat{t}) \circ D_{G,3}f(\hat{t},x_0(\hat{t}),u_0(\hat{t})) = 0$.
Since $D_{G,3}f(\hat{t},x_0(\hat{t}),u_0(\hat{t}))$ is surjective, we have $p(\hat{t}) = 0$.
Therefore, we have $((\theta_i)_{1\leq i\leq l}, p(\hat{t})) = 0$.
This is a contradiction with the Corollary 4.8, therefore $(\theta_i)_{1\leq i\leq l} \neq 0$. $\square$

Corollary 4.11. *In the setting and under the assumptions of Theorem 4.3, if moreover we assume (ALIB) is fulfilled, then $(\theta_i)_{1\leq i\leq l} \neq 0$.*

Proof. We proceed by contradiction, we assume that $(\theta_i)_{1\leq i\leq l} = 0$.
Since $D_{G,3}f(T,x_0(T),u_0(T))$ exists, $D_{G,3}\mathscr{H}_B(T,x_0(T),u_0(T),p(T),0)$ exists and

$$D_{G,3}\mathscr{H}_B(T,x_0(T),u_0(T),p(T),0) = p(T) \circ D_{G,3}f(T,x_0(T),u_0(T)).$$

Consequently, by using (MP), we have $p(T) \circ D_{G,3}f(T,x_0(T),u_0(T)) = 0$.
That is why, thanks to (TC) and $(\theta_i)_{1\leq i\leq l} = 0$, we obtain that

$$\left.\begin{array}{l} \sum_{\alpha=1}^m \lambda_\alpha D_H g^\alpha(x_0(T)) \circ D_{G,3}f(T,x_0(T),u_0(T)) \\ +\sum_{\beta=1}^q \mu_\beta D_H h^\beta(x_0(T)) \circ D_{G,3}f(T,x_0(T),u_0(T)) = 0. \end{array}\right\}$$

Hence, thanks to (ALIB), we have $((\lambda_\alpha)_{1\le\alpha\le m},(\mu_\beta)_{1\le\beta\le q})=0$.
Consequently, since $(\theta_i)_{1\le i\le l}=0$, we have $((\theta_i)_{1\le i\le l},(\lambda_\alpha)_{1\le\alpha\le m},(\mu_\beta)_{1\le\beta\le q})=0$ which contradicts (NN). ☐

For each $j\in\{1,...,l\}$, we consider the following condition:

(AF)$_j^0$ U is a subset of a real normed vector space Y s.t. U is a neighborhood of $u_0(T)$ in Y, $D_{G,3}f(T,x_0(T),u_0(T))$ exists, $\forall i\in\{1,...,l\}$, $i\neq j$ $D_{G,3}f_i^0(T,x_0(T),u_0(T))$ exists and $((D_Hg_i^0(x_0(T))\circ D_{G,3}f(T,x_0(T),u_0(T))+D_{G,3}f_i^0(T,x_0(T),u_0(T)))_{i\neq j}$, $(D_Hg^\alpha(x_0(T))\circ D_{G,3}f(T,x_0(T),u_0(T)))_{1\le\alpha\le m}$, $(D_Hh^\beta(x_0(T))\circ D_{G,3}f(T,x_0(T),u_0(T)))_{1\le\beta\le q})$ are linearly independent.

Corollary 4.12. *In this setting and under the assumptions of Theorem 4.3, let $j\in\{1,...,l\}$, if (AF)$_j^0$ is satisfied, then $\theta_j\neq 0$ i.e. there exists $((\theta'_i)_{1\le i\le l},(\lambda'_\alpha)_{1\le\alpha\le m},(\mu'_\beta)_{1\le\beta\le q},p')\in\mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^q\times PC^1([0,T],E^*)$ with $\theta'_j=1$, that verify the conclusions of Theorem 4.3.*
Moreover, if $D_{G,3}f_j^0(T,x_0(T),u_0(T))$ exists and (AF)$_j^0$ is satisfied, then we have: $((\theta_i)_{1\le i\le l},(\lambda_\alpha)_{1\le\alpha\le m},(\mu_\beta)_{1\le\beta\le q},p)\in\mathbb{R}^l\times\mathbb{R}^m\times\mathbb{R}^q\times PC^1([0,T],E^*)$ with $\theta_j=1$, that verify the conclusions of Theorem 4.3 are unique.

Proof. We assume that (AF)$_j^0$ is satisfied.
We proceed by contradiction, we assume that $\theta_j=0$.
Since $D_{G,3}f(T,x_0(T),u_0(T))$ exists and for all $i\neq j$, $D_{G,3}f_i^0(T,x_0(T),u_0(T))$ exists, $D_{G,3}\mathscr{H}_B(T,x_0(T),u_0(T),p(T),(\theta_i)_{1\le i\le l})$ exists and

$$\left.\begin{array}{l} D_{G,3}\mathscr{H}_B(T,x_0(T),u_0(T),p(T),(\theta_i)_{1\le i\le l}) \\ =p(T)\circ D_{G,3}f(T,x_0(T),u_0(T))+\sum_{i\neq j}\theta_i D_{G,3}f_i^0(T,x_0(T),u_0(T)). \end{array}\right\}$$

Consequently, by using (MP), we have

$$p(T)\circ D_{G,3}f(T,x_0(T),u_0(T))+\sum_{i\neq j}\theta_i D_{G,3}f_i^0(T,x_0(T),u_0(T))=0.$$

That is why, thanks to (TC) and $\theta_j=0$, we obtain that

$$\left.\begin{array}{l} \sum_{i\neq j}\theta_i(D_Hg_i^0(x_0(T))\circ D_{G,3}f(T,x_0(T),u_0(T))+D_{G,3}f_i^0(T,x_0(T),u_0(T))) \\ +\sum_{\alpha=1}^m\lambda_\alpha D_Hg^\alpha(x_0(T))\circ D_{G,3}f(T,x_0(T),u_0(T)) \\ +\sum_{\beta=1}^q\mu_\beta D_Hh^\beta(x_0(T))\circ D_{G,3}f(T,x_0(T),u_0(T))=0. \end{array}\right\}$$

Hence, thanks to (AF)$_j^0$, we have $((\theta_i)_{i\neq j},(\lambda_\alpha)_{1\le\alpha\le m},(\mu_\beta)_{1\le\beta\le q})=0$.
Consequently, since $\theta_j=0$, we have $((\theta_i)_{1\le i\le l},(\lambda_\alpha)_{1\le\alpha\le m},(\mu_\beta)_{1\le\beta\le q})=0$ which contradicts (NN). Thus $\theta_j\neq 0$.
We set $\forall i\in\{1,...,l\}$, $\theta'_i=\frac{\theta_i}{\theta_j}$, $\forall\alpha\in\{1,...,m\}$, $\lambda'_\alpha:=\frac{\lambda_\alpha}{\theta_j}$, $\forall\beta\in\{1,...,q\}$, $\mu'_\beta:=\frac{\mu_\beta}{\theta_j}$ and $p':=\frac{1}{\theta_j}p$.
Since the set of $((\overline{\theta}_i)_{1\le i\le l},(\overline{\lambda}_\alpha)_{1\le\alpha\le m},(\overline{\mu}_\beta)_{1\le\beta\le q},\overline{p})\in\mathbb{R}^{l+m+q}\times PC^1([0,T],E^*)$ verifying the conclusions of Theorem 4.3 is a cone, we have that $((\theta'_i)_{1\le i\le l},(\lambda'_\alpha)_{1\le\alpha\le m},(\mu'_\beta)_{1\le\beta\le q},p')$ verifies the conclusions of Theorem 4.3 with $\theta'_j=1$.
Now, we assume that $D_{G,3}f_j^0(T,x_0(T),u_0(T))$ exists and (AF)$_j^0$ is satisfied.
Let $((\theta_i^1)_{1\le i\le l},(\lambda_\alpha^1)_{1\le\alpha\le m},(\mu_\beta^1)_{1\le\beta\le q},p^1)\in\mathbb{R}^{l+m+q}\times PC^1([0,T],E^*)$ and

$((\theta_i^2)_{1\leq i\leq l},(\lambda_\alpha^2)_{1\leq\alpha\leq m},(\mu_\beta^2)_{1\leq\beta\leq q},p^2)\in\mathbb{R}^{l+m+q}\times PC^1([0,T],E^*)$ s.t. the conclusions of the Theorem 4.3 are verified with $\theta_j^1=\theta_j^2=1$.
Then, we have, for all $\ell\in\{1,2\}$,

$$\left.\begin{array}{l} p^\ell(T)\circ D_{G,3}f(T,x_0(T),u_0(T))+D_{G,3}f_j^0(T,x_0(T),u_0(T))+\\ \sum_{i\neq j}\theta_i^\ell D_{G,3}f_i^0(T,x_0(T),u_0(T))=0.\end{array}\right\}$$

Therefore, we have

$$(p^1(T)-p^2(T))\circ D_{G,3}f(T,x_0(T),u_0(T))+\sum_{i\neq j}(\theta_i^1-\theta_i^2)D_{G,3}f_i^0(T,x_0(T),u_0(T))=0.$$

By using (TC), we have

$$\left.\begin{array}{l} \sum_{i\neq j}(\theta_i^1-\theta_i^2)(D_Hg_i^0(x_0(T))\circ D_{G,3}f(T,x_0(T),u_0(T))+\\ D_{G,3}f_i^0(T,x_0(T),u_0(T)))\\ +\sum_{\alpha=1}^m(\lambda_\alpha^1-\lambda_\alpha^2)D_Hg^\alpha(x_0(T))\circ D_{G,3}f(T,x_0(T),u_0(T))\\ +\sum_{\beta=1}^q(\mu_\beta^1-\mu_\beta^2)D_Hh^\beta(x_0(T))\circ D_{G,3}f(T,x_0(T),u_0(T))=0.\end{array}\right\}$$

Hence, by using $(\text{AF})_j^0$, $\forall(i,\alpha,\beta)\in\{1,...,l\}\times\{1,...,m\}\times\{1,...,q\}$, $\theta_i^1=\theta_i^2$, $\lambda_\alpha^1=\lambda_\alpha^2$ and $\mu_\beta^1=\mu_\beta^2$.
Therefore, $p^1(T)=p^2(T)$; that is why, by using (AE), we have : $p^1=p^2$. □

5. Sufficient conditions of Pareto optimality

Let $(\overline{x},\overline{u})\in PC^1([0,T],\Omega)\times NPC^0([0,T],U)$, we consider the following conditions.

- **(St1)** For all $i\in\{1,...,l\}$ g_i^0 is concave at $\overline{x}(T)$ and Hadamard differentiable at $\overline{x}(T)$.
- **(St1-bis)** For all $i\in\{1,...,l\}$ g_i^0 is pseudo-concave at $\overline{x}(T)$ and Hadamard differentiable at $\overline{x}(T)$.
- **(St2)** For all $\alpha\in\{1,...,m\}$, g^α is quasi-concave at $\overline{x}(T)$ and Hadamard differentiable at $\overline{x}(T)$.
- **(St3)** For all $\beta\in\{1,...,q\}$, h^β and $-h^\beta$ are quasi-concave at $\overline{x}(T)$ and Hadamard differentiable at $\overline{x}(T)$.
- **(Si1)** For all $i\in\{1,...,l\}$, $f_i^0\in C^0([0,T]\times\Omega\times U,\mathbb{R})$.
- **(Si2)** For all $t\in[0,T]$, for all $i\in\{1,...,l\}$, $D_{F,2}f_i^0(t,\overline{x}(t),\overline{u}(t))$ exists and $[t\mapsto D_{F,2}f_i^0(t,\overline{x}(t),\overline{u}(t))]\in NPC^0([0,T],E^*)$.
- **(Sv1)** $f\in C^0([0,T]\times\Omega\times U,E)$.
- **(Sv2)** For all $t\in[0,T]$ $D_{F,2}f(t,\overline{x}(t),\overline{u}(t))$ exists and $[t\mapsto D_{F,2}f(t,\overline{x}(t),\overline{u}(t))]\in NPC^0([0,T],\mathscr{L}(E,E))$.

Theorem 5.1. *When $(\overline{x},\overline{u})\in Adm(\mathscr{B})$, under (St1), (St2), (St3), (Si1), (Si2) (Sv1), (Sv2) if there exists $((\theta_i)_{1\leq i\leq l},(\lambda_\alpha)_{1\leq\alpha\leq m},(\mu_\beta)_{1\leq\beta\leq q},p)$ belongs to $\mathbb{R}^{l+m+q}\times PC^1([0,T],E^*)$ verifying the conclusions (NN), (Si), (Sℓ) and (TC) of Theorem 4.3 with $(x_0,u_0)=(\overline{x},\overline{u})$ and if the following condition is satisfied*

- **(Shb1)** *for each $(x,u)\in Adm(\mathscr{B})$, for all $t\in[0,T]$ almost everywhere for the canonical measure of Borel on $[0,T]$,*

$$\mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),(\theta_i)_{1\leq i\leq l})-\mathscr{H}_B(t,x(t),u(t),p(t),(\theta_i)_{1\leq i\leq l})\geq \underline{d}p(t)\cdot(x(t)-\overline{x}(t)),$$

then we have:
if $(\theta_i)_{1\leq i\leq l}\neq 0$, then $(\overline{x},\overline{u})$ is a weak Pareto optimal solution of $(\mathscr{B})$,
if for all $i\in\{1,...,l\}$, $\theta_i\neq 0$, then $(\overline{x},\overline{u})$ is a Pareto optimal solution of $(\mathscr{B})$.

Proof. Let $(x,u) \in Adm(\mathscr{B})$. By using (ST1), we have

$$\begin{array}{l}\sum_{i=1}^{l} \theta_i J_i(x,u) = \sum_{i=1}^{l} \theta_i g_i^0(x(T)) + \int_0^T \sum_{i=1}^{l} \theta_i f_i^0(t,x(t),u(t))dt \\ \leq \sum_{i=1}^{l} \theta_i g_i^0(\overline{x}(T)) + \sum_{i=1}^{l} \theta_i D_H g_i(\overline{x}(T)) \cdot (x(T) - \overline{x}(T)) + \\ \int_0^T \sum_{i=1}^{l} \theta_i f_i^0(t,x(t),u(t))dt.\end{array}$$

By using (TC), we have

$$\left.\begin{array}{l}\sum_{i=1}^{l} \theta_i D_H g_i^0(\overline{x}(T)) \cdot (x(T) - \overline{x}(T)) \\ = p(T) \cdot (x(T) - \overline{x}(T)) - \sum_{\alpha=1}^{m} \lambda_\alpha D_H g^\alpha(\overline{x}(T)) \cdot (x(T) - \overline{x}(T)) \\ - \sum_{\beta=1}^{q} \mu_\beta D_H h^\beta(\overline{x}(T)) \cdot (x(T) - \overline{x}(T)).\end{array}\right\} \tag{5.1}$$

Furthermore, by using (Si) and (Sℓ), we have for each $\alpha \in \{1,...,m\}$, $\lambda_\alpha g^\alpha(\overline{x}(T)) \leq \lambda_\alpha g^\alpha(x(T))$. Consequently, by using (ST2), we have for all $\alpha \in \{1,...,m\}$, $\lambda_\alpha D_H g^\alpha(\overline{x}(T)) \cdot (x(T) - \overline{x}(T)) \geq 0$.
Besides, thanks to (ST3), we have for all $\beta \in \{1,...,q\}$, $\mu_\beta D_H h^\beta(\overline{x}(T)) \cdot (x(T) - \overline{x}(T)) = 0$.
Hence, by using (5.1) and (SHB1), we have

$$\begin{array}{l}\sum_{i=1}^{l} \theta_i D_H g_i^0(\overline{x}(T)) \cdot (x(T) - \overline{x}(T)) \\ \leq p(T) \cdot (x(T) - \overline{x}(T)) \\ = \int_0^T \underline{d}(p(t) \cdot (x(t) - \overline{x}(t)))dt \\ = \int_0^T \underline{d}p(t) \cdot (x(t) - \overline{x}(t))dt + \int_0^T p(t) \cdot \underline{d}(x(t) - \overline{x}(t))dt \\ \leq \int_0^T (\mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),(\theta_i)_{1\leq i\leq l}) - \mathscr{H}_B(t,x(t),u(t),p(t),(\theta_i)_{1\leq i\leq l}))dt + \\ \int_0^T (p(t) \cdot f(t,x(t),u(t)) - p(t) \cdot f(t,\overline{x}(t),\overline{u}(t)))dt \\ = \int_0^T \sum_{i=1}^{l} \theta_i f_i^0(t,\overline{x}(t),\overline{u}(t))dt - \int_0^T \sum_{i=1}^{l} \theta_i f_i^0(t,x(t),u(t))dt.\end{array}$$

Therefore, we have

$$\begin{array}{l}\sum_{i=1}^{l} \theta_i J_i(x,u) \leq \sum_{i=1}^{l} \theta_i g_i^0(\overline{x}(T)) + \int_0^T \sum_{i=1}^{l} \theta_i f_i^0(t,\overline{x}(t),\overline{u}(t))dt \\ - \int_0^T \sum_{i=1}^{l} \theta_i f_i^0(t,x(t),u(t))dt + \int_0^T \sum_{i=1}^{l} \theta_i f_i^0(t,x(t),u(t))dt \\ = \sum_{i=1}^{l} \theta_i J_i(\overline{x},\overline{u}).\end{array}.$$

Consequently, $(\overline{x},\overline{u})$ is a solution of the following single optimization problem :

$$(\mathscr{P}_\theta)\left\{\begin{array}{ll}\text{Maximize} & \sum_{i=1}^{l} \theta_i J_i(x,u) \\ \text{subject to} & (x,u) \in Adm(\mathscr{B}).\end{array}\right.$$

Now, we assume that $(\theta_i)_{1\leq i\leq l} \neq 0$.
We want to prove that $(\overline{x},\overline{u})$ is a weak Pareto optimal solution. We proceed by contradiction, we assume that $(\overline{x},\overline{u})$ is not a weak Pareto optimal solution i.e. there exists $(x,u) \in Adm(\mathscr{B})$ such that for all $i \in \{1,...,l\}$, $J_i(x,u) > J_i(\overline{x},\overline{u})$.
Consequently, we have $\sum_{i=1}^{l} \theta_i J_i(x,u) > \sum_{i=1}^{l} \theta_i J_i(\overline{x},\overline{u})$. This is a contradiction with $(\overline{x},\overline{u})$ is a solution of $(\mathscr{P}_\theta)$.
Next, we assume that for all $i \in \{1,...,l\}$, $\theta_i \neq 0$.
We want to prove that $(\overline{x},\overline{u})$ is a Pareto optimal solution. We proceed by contradiction, we assume that $(\overline{x},\overline{u})$ is not a Pareto optimal solution i.e. there exists $(x,u) \in Adm(\mathscr{B})$ such that for all $i \in \{1,...,l\}$, $J_i(x,u) \geq J_i(\overline{x},\overline{u})$ and for some $i_0 \in \{1,...,l\}$, $J_{i_0}(x,u) > J_{i_0}(\overline{x},\overline{u})$. Hence, we obtain that $\sum_{i=1}^{l} \theta_i J_i(x,u) > \sum_{i=1}^{l} \theta_i J_i(\overline{x},\overline{u})$. This is a contradiction with $(\overline{x},\overline{u})$ is a solution of $(\mathscr{P}_\theta)$. □

Theorem 5.2. *When $(\overline{x},\overline{u}) \in Adm(\mathscr{B})$, under (ST1), (ST2), (ST3), (SI1), (SI2), (SV1), (SV2) if there exists $((\theta_i)_{1\leq i\leq l},(\lambda_\alpha)_{1\leq\alpha\leq m},(\mu_\beta)_{1\leq\beta\leq q},p)$ belongs to $\mathbb{R}^{l+m+q}\times PC^1([0,T],E^*)$ verifying all the conclusions of Theorem 4.3 with $(x_0,u_0) = (\overline{x},\overline{u})$ and if the following condition is satisfied*

(SHB2) *for all $(t,\xi)\in[0,T]\times\Omega$,*
$\hat{\mathscr{H}}_B(t,\xi,p(t),(\theta_i)_{1\leq i\leq l}) = \max_{\zeta\in U}\mathscr{H}_B(t,\xi,\zeta,p(t),(\theta_i)_{1\leq i\leq l})$ *exists, and for all $t\in[0,T]$,* $[\xi\mapsto\hat{\mathscr{H}}_B(t,\xi,p(t),(\theta_i)_{1\leq i\leq l})]$ *is concave at $\overline{x}(t)$ and Gâteaux differentiable at $\overline{x}(t)$,*

then we have:
if $(\theta_i)_{1\leq i\leq l}\neq 0$, then $(\overline{x},\overline{u})$ is a weak Pareto optimal solution of $(\mathscr{B})$,
if for all $i\in\{1,...,l\}$, $\theta_i\neq 0$, then $(\overline{x},\overline{u})$ is a Pareto optimal solution of $(\mathscr{B})$.

Proof. Notice that (SHB2) implies (SHB1).
Indeed, let $(x,u)\in Adm(\mathscr{B})$.
We set $\theta = (\theta_i)_{1\leq i\leq l}$. For all $t\in[0,T]$, for all $\varepsilon>0$ small enough, we have $\overline{x}(t)+\varepsilon(\overline{x}(t)-x(t))\in\Omega$, therefore by using (MP)

$$\begin{aligned}&\tfrac{1}{\varepsilon}(\hat{\mathscr{H}}_B(t,\overline{x}(t)+\varepsilon(\overline{x}(t)-x(t)),p(t),\theta)-\hat{\mathscr{H}}_B(t,\overline{x}(t),p(t),\theta))\\ &\geq\tfrac{1}{\varepsilon}(\mathscr{H}_B(t,\overline{x}(t)+\varepsilon(\overline{x}(t)-x(t)),\overline{u}(t),p(t),\theta)-\mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),\theta)).\end{aligned}$$

Hence, since (SHB2), (SI2) and (SV2), when $\varepsilon\to 0$ we have $D_{G,2}\hat{\mathscr{H}}_B(t,\overline{x}(t),p(t),\theta)\cdot(\overline{x}(t)-x(t))\geq D_{G,2}\mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),\theta)\cdot(\overline{x}(t)-x(t))$. Hence, by using (AE), we have

$$-D_{G,2}\hat{\mathscr{H}}_B(t,\overline{x}(t),p(t),\theta)\cdot(x(t)-\overline{x}(t))\geq \underline{d}p(t)\cdot(x(t)-\overline{x}(t)). \tag{5.2}$$

Besides, for all $\varepsilon>0$ small enough, we have $\overline{x}(t)+\varepsilon(x(t)-\overline{x}(t))\in\Omega$, hence by using (MP) and (SHB2), we have

$$\begin{aligned}&\tfrac{1}{\varepsilon}(\hat{\mathscr{H}}_B(t,\overline{x}(t)+\varepsilon(x(t)-\overline{x}(t)),p(t),\theta)-\hat{\mathscr{H}}_B(t,\overline{x}(t),p(t),\theta))\\ &\geq\hat{\mathscr{H}}_B(t,x(t),p(t),\theta)-\hat{\mathscr{H}}_B(t,\overline{x}(t),p(t),\theta)\\ &\geq\mathscr{H}_B(t,x(t),u(t),p(t),\theta)-\mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),\theta).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),\theta)-\mathscr{H}_B(t,x(t),u(t),p(t),\theta)\\ &\geq\tfrac{1}{\varepsilon}(\hat{\mathscr{H}}_B(t,\overline{x}(t),p(t),\theta)-\hat{\mathscr{H}}_B(t,\overline{x}(t)+\varepsilon(x(t)-\overline{x}(t)),p(t),\theta)).\end{aligned}$$

Consequently, when $\varepsilon\to 0$, from (5.2), we have
$\mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),\theta)-\mathscr{H}_B(t,x(t),u(t),p(t),\theta)\geq\underline{d}p(t)\cdot(x(t)-\overline{x}(t))$.
Hence, the assumptions of the Theorem 5.1 are verified and the conclusions follow. □

Theorem 5.3. *When $(\overline{x},\overline{u}) \in Adm(\mathscr{B})$, under (ST1), (ST2), (ST3), (SI1), (SI2), (SV1), (SV2) if there exists $((\theta_i)_{1\leq i\leq l},(\lambda_\alpha)_{1\leq\alpha\leq m},(\mu_\beta)_{1\leq\beta\leq q},p)$ belongs to $\mathbb{R}^{l+m+q}\times PC^1([0,T],E^*)$ verifying all the conclusions of Theorem 4.3 with $(x_0,u_0) = (\overline{x},\overline{u})$ and if the following condition is satisfied*

(SHB3) *U is a subset of a real normed vector space Y s.t. for all $t\in[0,T]$, U is a neighborhood of $\overline{u}(t)$, and for all $t\in[0,T]$,*
$[(\xi,\zeta)\mapsto\mathscr{H}_B(t,\xi,\zeta,p(t),(\theta_i)_{1\leq i\leq l})$ *is Gâteaux differentiable at $(\overline{x}(t),\overline{u}(t))$ and concave at $(\overline{x}(t),\overline{u}(t))$,*

then we have:
if $(\theta_i)_{1\le i\le l} \neq 0$, *then* $(\overline{x},\overline{u})$ *is a weak Pareto optimal solution of* $(\mathscr{B})$,
if for all $i \in \{1,...,l\}$, $\theta_i \neq 0$, *then* $(\overline{x},\overline{u})$ *is a Pareto optimal solution of* $(\mathscr{B})$.

Proof. Notice that (SHB3) implies (SHB1).
Indeed, let $(x,u) \in Adm(\mathscr{B})$, let $t \in [0,T]$, since $[(\xi,\zeta) \mapsto \mathscr{H}_B(t,\xi,\zeta,p(t),(\theta_i)_{1\le i\le l})]$ is Gâteaux differentiable and concave at $(\overline{x}(t),\overline{u}(t))$, we have
$\mathscr{H}_B(t,x(t),u(t),p(t),(\theta_i)_{1\le i\le l}) - \mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),(\theta_i)_{1\le i\le l})$
$\le D_{G,(2,3)}\mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),(\theta_i)_{1\le i\le l}) \cdot (x(t)-\overline{x}(t),u(t)-\overline{u}(t))$. Therefore, by using (AE) and (MP), we have
$D_{G,(2,3)}\mathscr{H}_B(t,\overline{x}(t),\overline{u}(t),p(t),(\theta_i)_{1\le i\le l}) \cdot (x(t)-\overline{x}(t),u(t)-\overline{u}(t)) = -\underline{d}p(t)\cdot(x(t)-\overline{x}(t))$. Hence, (SHB1) is verified. Therefore,the assumptions of the Theorem 5.1 are verified and the conclusions follow. □

For a Mayer multiobjective problem, we can replace condition (ST1) by (ST1-bis) which is a weaker assumption.

Theorem 5.4. *When* $(\overline{x},\overline{u}) \in Adm(\mathscr{M})$, *under (ST1-bis), (ST2), (ST3), (SV1) if there exists* $((\theta_i)_{1\le i\le l},(\lambda_\alpha)_{1\le\alpha\le m},(\mu_\beta)_{1\le\beta\le q},p) \in \mathbb{R}^{l+m+q} \times PC^1([0,T],E^*)$ *verifying the conclusions (NN), (Si), (Sℓ) and (TC) of Theorem 4.3 with* $(x_0,u_0) = (\overline{x},\overline{u})$ *and if the following condition is satisfied*

(SHM1) *for each* $(x,u) \in Adm(\mathscr{M})$, *for all* $t \in [0,T]$ *almost everywhere for the canonical measure of Borel on* $[0,T]$,

$$\mathscr{H}_M(t,\overline{x}(t),\overline{u}(t),p(t)) - \mathscr{H}_M(t,x(t),u(t),p(t)) \ge \underline{d}p(t)\cdot(x(t)-\overline{x}(t)),$$

then we have:
if $(\theta_i)_{1\le i\le l} \neq 0$, *then* $(\overline{x},\overline{u})$ *is a weak Pareto optimal solution of* $(\mathscr{M})$,
if for all $i \in \{1,...,l\}$, $\theta_i \neq 0$, *then* $(\overline{x},\overline{u})$ *is a Pareto optimal solution of* $(\mathscr{M})$.

Proof. Let $(x,u) \in Adm(\mathscr{M})$. By using (TC), we have

$$\left.\begin{array}{l} \sum_{i=1}^{l}\theta_i D_H g_i^0(\overline{x}(T))\cdot(x(T)-\overline{x}(T)) \\ = p(T)\cdot(x(T)-\overline{x}(T)) - \sum_{\alpha=1}^{m}\lambda_\alpha D_H g^\alpha(\overline{x}(T))\cdot(x(T)-\overline{x}(T)) \\ -\sum_{\beta=1}^{q}\mu_\beta D_H h^\beta(\overline{x}(T))\cdot(x(T)-\overline{x}(T)). \end{array}\right\} \tag{5.3}$$

Moreover, by using (Si) and (Sℓ), we have for each $\alpha \in \{1,...,m\}$, $\lambda_\alpha g^\alpha(\overline{x}(T)) \le \lambda_\alpha g^\alpha(x(T))$. Consequently, by using (ST2), we have for all $\alpha \in \{1,...,m\}$, $\lambda_\alpha D_H g^\alpha(\overline{x}(T))\cdot(x(T)-\overline{x}(T)) \ge 0$.
Moreover, thanks to (ST3), we have for all $\beta \in \{1,...,q\}$, $\mu_\beta D_H h^\beta(\overline{x}(T))\cdot(x(T)-\overline{x}(T)) = 0$.
Hence

$$\begin{array}{l} \sum_{i=1}^{l}\theta_i D_H g_i^0(\overline{x}(T))\cdot(x(T)-\overline{x}(T)) \\ \le p(T)\cdot(x(T)-\overline{x}(T)) \\ = p(0)\cdot(x(0)-\overline{x}(0)) + \int_0^T \underline{d}(p(t)\cdot(x(t)-\overline{x}(t)))dt \\ = \int_0^T \underline{d}p(t)\cdot(x(t)-\overline{x}(t))dt + \int_0^T p(t)\cdot\underline{d}(x(t)-\overline{x}(t))dt \\ \le \int_0^T (\mathscr{H}_M(t,\overline{x}(t),\overline{u}(t),p(t)) - \mathscr{H}_M(t,x(t),u(t),p(t)))dt + \\ \int_0^T (\mathscr{H}_M(t,x(t),u(t),p(t)) - \mathscr{H}_M(t,\overline{x}(t),\overline{u}(t),p(t)))dt \\ = 0 \end{array}$$

where we have used (SHM1).

Therefore, thanks to (ST1-bis), we have $\sum_{i=1}^{l} \theta_i g_i^0(x(T)) \leq \sum_{i=1}^{l} \theta_i g_i^0(\overline{x}(T))$.
Hence, $(\overline{x}, \overline{u})$ is a solution of the following single-objective optimization problem :

$$(\mathscr{P}_\theta) \begin{cases} \text{Maximize} & \sum_{i=1}^{l} \theta_i J_i(x,u) \\ \text{subject to} & (x,u) \in Adm(\mathscr{M}). \end{cases}$$

Now, by doing the same reasoning as at the end of the proof in Theorem 5.1, we obtain the conclusion of the theorem. □

Remark 5.5. When the Bolza problem $(\mathscr{B})$ is a Mayer problem $(\mathscr{M})$, we can replace the assumption (ST1) by (ST1-bis) for Theorems 5.2 and 5.3. By using Theorem 5.4 instead of Theorem 5.1, the proofs of these results are analogous to the proofs of Theorems 5.2 and 5.3.

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