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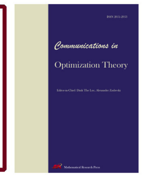

TO THE DYNAMIC RECONSTRUCTION PROBLEM WITH NON-CONVEX GEOMETRICAL RESTRICTIONS ON THE CONTROLS

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Dedicated to the memory of Professor Rafail Fedorovich Gabasov

Abstract. In the paper, the problem of dynamic reconstruction of controls and trajectories for deterministic control-affine systems is considered. The reconstruction is performed in real time using known discrete inaccurate measurements of the observed trajectory of the system. This trajectory is generated by an unknown measurable control that satisfies known geometrical restrictions. In this paper, the case of non-convex restrictions is considered. In the previous works, the authors of this paper considered only convex restrictions. To describe the motion of the system with non-convex control restrictions, generalized dynamics and controls are introduced. A well-posed statement of the dynamic reconstruction problem is given. An algorithm for construction of approximations of the solution is proposed and justified. It is based on the variational approach developed by the authors. This approach uses auxiliary variational problems with regularized integral residual functionals. The key feature of the approach is that the integrants of the functionals in the auxiliary variational problems are d.c. functions. The suggested algorithm reduces the dynamic reconstruction problem to integration of Hamiltonian systems of ordinary linear differential equations. The results of numerical simulation are demonstrated.

Keywords. Dynamic reconstruction; Calculus of variations; D.c. functions; Non-convex restrictions; Hamiltonian systems.

1. INTRODUCTION

Forward and inverse problems of the mathematical control theory for dynamic systems have many applications in a wide spectrum of modern areas of science and engineering. The algorithms for construction of optimal feedback and programmed controls were used in robotics, navigation, cosmonautics, economics, medicine, and many other applied areas. There exist numerous fundamental and empirical approaches to solving optimal control problems.

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Tn this paper, we consider inverse problems for dynamic controlled systems. Namely, the dynamic reconstruction problem is considered. That is the problem of finding unknown controls and trajectory of a dynamic controlled system by inaccurate data on the observed motion. The reconstruction is done in real time with the arrival of new data. Control-affine deterministic dynamical systems are under consideration. The admissible controls are measurable functions with values in a given compact set. Discrete inaccurate measurements of the observed trajectory of the system, that arrive in real time, are known.

There exists a variety of mathematical approaches to solving inverse problems. For example, an approach, based on the classical least squares method, is suggested in [14]. In [21], inverse problems are solved empirically by analyzing the response of the dynamical system to special controls. Another method, developed in [9], relies on an optimization approach based on gradient-type methods. In this paper, inverse problems are formulated as operator equations and are reduced to the minimization of the corresponding discrepancy functionals. The adjoint variables are used to calculate the gradients. An inverse parameter identification problem for data, oscillating in time, is considered in [3]. In this paper, the parameter identification problem is reduced to Fourier-regularized parameter optimization problem constrained by ODEs that describe the dynamics.

There exists a series of methods that use variants of Tikhonov regularization [26]. For example, the work [18] suggests an optimization method based on discrete linear operator equations that reduces inverse problems to minimization of Tikhonov-regularized functionals in which an additional penalty term leads to smoothness of the minimizer. The ideas of Tikhonov regularization are also developed in [16] to solve inverse problems of identification.

A method for solving the dynamic reconstruction problem was proposed in [11]. This approach consists of minimizing a Tikhonov-regularized residual between the dynamics and measurements of the observed states of the system. It uses the extreme aiming procedure, which has roots in the works of the school of N. N. Krasovskii on the theory of positional optimal control [12].

More detailed reviews of modern methods for solving inverse problems can be found in [19, 17].

The authors of this paper have suggested an original variational approach [22, 23] to solving inverse dynamic reconstruction problems. It provides a method for construction of approximations of the desired controls with the use of constructions from auxiliary variational problems with integral regularized residual functionals. The key feature of the method is the use of functionals which's integrants are d.c. functions [20].

Let us briefly summarise the previous results, obtained by the authors of this paper. In the works [22, 23], there was described and justified an algorithm for construction of approximations of the solution of the dynamic reconstruction problem. These approximating controls generate the trajectories of the dynamics that converge uniformly to the observed trajectory. The drawback of this method is that the constructed control approximations are bounded, but not necessary satisfy the given geometrical control restrictions. Thus, they are not necessary admissible controls. In [24], a development of this algorithm was suggested. It allows to construct admissible piecewise-constant controls satisfying given convex geometrical restrictions.

In this paper, we consider the case of non-convex geometrical restrictions on the controls. In such case, so-called sliding modes often appear [27]. To describe the dynamics of the system with sliding modes, generalised controls and dynamics are introduced. A well-posed dynamic reconstruction problem is stated for generalised dynamics. An algorithm for construction of admissible (that is, satisfying non-convex geometrical restrictions) approximations of the problems's solution is suggested and justified. The results of numerical simulation are shown on the example of a mechanical model.

2. Materials and Methods

2.1. **Dynamics.** Dynamic deterministic controlled systems of the following form are considered:

$$\begin{gathered}\frac{dx(t)}{dt} = G(t,x(t))u(t) + f(t,x(t)),\\ x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad G(\cdot):[0,T]\times\mathbb{R}^n \to \mathbb{R}^{n\times m}, \quad f(\cdot):[0,T]\times\mathbb{R}^n \to \mathbb{R}^n,\\ m \geq n, \quad t\in[0,T], \quad T<\infty,\end{gathered} \tag{2.1}$$

where $x(\cdot)$ is the state variables vector and $u(\cdot)$ is the vector of the controls.

The admissible controls are measurable functions satisfying the restrictions

$$u(t) \in \mathbf{U} \subset \mathbb{R}^m \quad \text{a. e. on } [0,T], \tag{2.2}$$

where the control restriction set $\mathbf{U}$ is compact.

Remark 2.1. Note that the set $\mathbf{U}$ is not necessarily convex. Previously [22, 23], only convex control restriction sets were considered.

2.2. **Generalised and averaged controls.** In some cases, non-convex control restriction (2.2) can lead to appearance of sliding modes (for references on sliding modes see, for example, [27], and [7, Ch. 2.2], and [28, Ch. III.2]). To describe the behavior of dynamics (2.1) with sliding modes, generalised controls are introduced, following R. V. Gamkrelidze [7, Ch. 2.1] and J. Warga [28, Ch. III.2].

Definition 2.2. A *generalised control* is a measurable function $\mu(\cdot|du):[0,T]\to \mathrm{rpm}(\mathbf{U})$ with values in the set of regular probability Borel measures on $\mathbf{U}$ with the topology, induced by the weak* topology of the space $\mathbb{C}^*(\mathbf{U})$ [28, Ch. III.2].

Generalised controls play the role of a mathematical model of sliding modes in real observed system. Instead of dynamics (2.1), we consider the generalised dynamics

$$\begin{gathered}\frac{dx(t)}{dt} = G(t,x(t))\int\limits_{\mathbf{U}} u\mu(t|du) + f(t,x(t)),\\ \mu(t|du):[0,T]\to \mathrm{rpm}(\mathbf{U}).\end{gathered} \tag{2.3}$$

Definition 2.3. We will refer to the result of integration of a generalised control as an *averaged control* $u(\cdot):[0,T]\to \mathrm{Conv}(\mathbf{U})$:

$$t \to u(t) = \int\limits_{\mathbf{U}} u\mu(t|du), \tag{2.4}$$

where the notation $\mathrm{Conv}(\cdot)$ means the convex hull of a set.

All values of any averaged control belong to the convex hull of $\mathbf{U}$ [7, Ch. 2.1]. Any averaged control $u(\cdot)$ is a measurable function [7, Ch. 2.1].

Introduction of generalised controls is equivalent to convexification of the geometrical restrictions on the controls $\mathbf{U}$ (2.2). Indeed, instead of dynamics (2.3) with generalized controls, we can consider dynamics (2.1) with measurable controls satisfying the convex geometrical restrictions

$$u(t) \in \mathrm{Conv}(\mathbf{U}) \quad \text{a. e. on } [0,T]. \tag{2.5}$$

The following propositions explain the relation between generalized (2.3) and convexified (2.1), (2.5) dynamics.

Proposition 2.4. *A trajectory of (2.3), generated by a generalised control, coincides with the trajectory of (2.1), (2.5) that is generated by the corresponding averaged control* [7, Ch. 2.1].

Proposition 2.5. *Only one averaged control (2.4) corresponds to a generalised control, while an averaged control can correspond to more than one generalised controls* [7, Ch. 2.1].

Definition 2.6. All generalised controls that correspond to the same averaged control are *equivalent*.

The notations, introduced in this section, are needed for regularization of the problem of reconstruction of an unknown control that generates the observed trajectory (section 2.5).

2.3. **Input data.** In this section, the input data for the dynamic reconstruction problem is described.

It is assumed that some trajectory of system (2.3), generated by an unknown generalized control, is being observed. It is called the basic trajectory $x^*(\cdot) : [0,T] \to \mathbb{R}^n$.

Discrete inaccurate measurements of the basic trajectory arrive in real time. The measurements have error $\delta > 0$ and arrive with regular time step $h^\delta > 0$:

$$\{y_k^\delta : \ \|y_k^\delta - x^*(t_k)\| \le \delta, \quad t_k = kh^\delta, \quad k = 0,\dots,N, \quad N = \lceil T/h^\delta \rceil\}. \tag{2.6}$$

The notation $\|\cdot\|$ means the Euclidean norm.

2.4. **Assumptions.** We assume that the input data (2.1)–(2.6) satisfy the following assumptions.

A.1 There exist constants $d_0 > 0$, $\delta_0 > 0$, $h_0 > 0$ and a compact $\Psi \subset \mathbb{R}^n$ such that for any accuracy $\delta \in (0, \delta_0]$ and any measurement step $h^\delta \in (0, h_0]$

$$\bigcup_{k=0,\dots,N} B_{d_0}[y_k^\delta] \subset \Psi,$$

where $B_r[x]$ is the closed ball of the radius r with the center in x.

A.2 The matrix function $G(\cdot)$ and the vector function $f(\cdot)$ from dynamics (2.1) are Lipschitz continuous on $D_0 \triangleq [0,T] \times \Psi$ with the Lipschitz constant $L_{D_0} > 0$:

$$\forall (t_1,x_1),(t_2,x_2) \in D_0 \quad \|G(t_2,x_2) - G(t_1,x_1)\|_2 \le L_{D_0}\|(t_2,x_2) - (t_1,x_1)\|,$$
$$\|f(t_2,x_2) - f(t_1,x_1)\| \le L_{D_0}\|(t_2,x_2) - (t_1,x_1)\|.$$

The notation $\|\cdot\|_2$ means the spectral Matrix norm induced by the Euclidean norm

$$\|A\|_2 \triangleq \max_{x \ne 0} \frac{\|Ax\|}{\|x\|}.$$

A.3 Rang of $G(t,x)$ equals n for $(t,x) \in D_0$.

2.5. **Regularisation of the control reconstruction problem.** Since sliding modes are allowed, we assume that the basic trajectory is the observed trajectory of (2.3), generated by an unknown generalized control. It follows from definitions 2.3–2.6 that all equivalent generalised controls will generate the same trajectory (2.3). Therefore, the problem of reconstruction of a unknown generalised control is ill-posed. Then, instead, we will find the unique (see Proposition 2.5) averaged control that corresponds to all such generalised controls.

It follows from Proposition 2.4 that the desired averaged control generates the trajectory $x^*(\cdot)$ of (2.1) and satisfy (2.5). So, we can consider the "convex" dynamic reconstruction problem for dynamics (2.1) with restrictions (2.5). Convex dynamic reconstruction problems have been studied in [22, 23, 24].

It was shown in [23] that convex problem is still ill-posed since more than one averaged control may generate $x^*(\cdot)$ (and all such averaged controls correspond to different sets of generalised controls). In turn, to regularize the convex problem, a notation of the normal control is introduced.

Definition 2.7. *The normal control* $u^*(\cdot) : [0,T] \to \mathbb{R}^m$ is the measurable control, generating the basic trajectory $x^*(\cdot)$ of (2.1), that has the minimal norm in $L^2([0,T],\mathbb{R}^m)$ space.

It was proved [23] that for a basic trajectory $x^*(\cdot)$, satisfying assumptions **A.1–A.3**, there exists the unique normal control.

An additional assumption is introduced:

A.4 The normal control $u^*(\cdot)$ satisfies restrictions (2.5).

Remark 2.8. In particular, assumption **A.4** is always true in the case of coinciding control and state variables dimensions $m = n$ and in the case of ball geometrical restrictions on the controls

$$\mathbf{U} = B_R[\bar{0}], \quad R > 0.$$

2.6. **The dynamic control reconstruction problem.** The following *dynamic reconstruction problem (the DRP)* is considered in this paper:

For any $\delta \in (0,\delta_0]$, $h^\delta \in (0,h_0]$ and the set of measurements $\{y_k^\delta,\ k = 1,\ldots,i \le N\}$ (2.6), obtained up to the moment t_i, to construct a piece-wise constant control $u^\delta(\cdot) : [0,t_i] \to \mathbb{R}^m$ such that at the terminal instant $t_N = T$ the following conditions are fulfilled:

B.1 The functions $u^\delta(\cdot) : [0,T] \to \mathbb{R}^m$ are admissible controls (that is, they satisfy the restrictions (2.2)).

B.2 Each control $u^\delta(\cdot)$ generates a trajectory $x^\delta(\cdot) : [0,T] \to \mathbb{R}^n$ of system (2.1) with the boundary conditions $x^\delta(0) = y_0^\delta$ such that these trajectories converge uniformly to the basic trajectory:

$$\lim_{\delta \to 0} \|x^\delta(\cdot) - x^*(\cdot)\|_{\mathbb{C}([0,T],\mathbb{R}^n)} = 0.$$

The notation $\|f(\cdot)\|_{\mathbb{C}([0,T],\mathbb{R}^n)} = \max\limits_{t\in[0,T]} \|f(t)\|$ means the norm in the space of continuous functions $\mathbb{C}([0,T],\mathbb{R}^n)$.

B.3 The functions $u^\delta(\cdot)$ converge to the normal control in the sense

$$\int\limits_0^T \langle g(\tau), u^\delta(\tau) - u^*(\tau)\rangle d\tau \overset{\delta\to 0}{\longrightarrow} 0 \quad \forall g(\cdot) \in \mathbb{C}([0,T],\mathbb{R}^m). \tag{2.7}$$

The notation $\langle \cdot, \cdot \rangle$ means the scalar product.

2.7. **Previous results.** In the works [22, 23, 24] an algorithm for construction of approximations of the normal control was described and justified. The constructed approximations are piecewise-constant functions satisfying convex restrictions (2.5). The description and justification of the new algorithm for solving the DRP, suggested in this paper, rely on the material of [22, 23, 24]. This section offers a brief review of these results.

First, let us describe the previously suggested algorithm [22, 23, 24]. It is stepwise, and each step of the algorithm is performed for the interval $[t_{k-1}, t_k] = [(k-1)h^\delta, kh^\delta]$, $k = 1, \ldots, N$ when the measurement y_k^δ arrives. On each step the following procedures are performed:

(1) **Interpolation of the measurements.** A third-order spline interpolation $y^\delta(\cdot) : [0, t_k] \to \mathbb{R}^n$ of the discrete measurements (2.6) is constructed. It is continued on each step on the interval $[t_{k-1}, t_k]$ and is continuously differentiable on $[0, t_k]$.

(2) **Solving auxiliary variational problem.** The constructed function $y^\delta(\cdot)$ is used to state an auxiliary variational problem. It consists of finding a pair of functions $x_k(\cdot) : [t_{k-1}, t_k] \to \mathbb{R}^n$, $u_k(\cdot) : [t_{k-1}, t_k] \to \mathbb{R}^m$ such that:

   (a) They are continuously differentiable functions that satisfy the dynamics' equation (2.1) and $u_k(\cdot)$ has such structure that there exists a function $s_k(\cdot) \in \mathbb{C}^1([t_{k-1}, t_k], \mathbb{R}^n)$ such that

$$u_k(t) = -\frac{1}{\alpha^2} G^\top(t_{k-1}, y_{k-1}^\delta) s_k(t), \quad t \in [t_{k-1}, t_k]. \tag{2.8}$$

   (b) They satisfy the boundary conditions

$$\begin{aligned} k = 1: &\quad x_1(0) = y_0^\delta, \quad s_1(0) = 0, \\ k = 2, \ldots, N: &\quad x_k(t_{k-1}) = y_{k-1}^\delta, \quad s_k(t_{k-1}) = s_{k-1}(t_{k-1}). \end{aligned}$$

   (c) They provide the unique stationary point of the functional

$$I(x(\cdot), u(\cdot)) = \int_{t_{k-1}}^{t_k} \frac{\|x(t) - y_k^\delta(t)\|^2}{2} - \frac{\alpha^2 \|u(t)\|^2}{2} dt,$$

   where $\alpha > 0$ is a small regularizing parameter.

Remark 2.9. In the suggested method just stationary point of the functional is used. So, there is no need to find extremum of the functional. It is an original feature of the method.

The conditions for the stationary point can be written in the form of a Hamiltonian system of non-linear ODEs [8]. This system is linearized and after linearization has the form

$$\begin{gathered} \frac{dx_k(t)}{dt} = -\frac{1}{\alpha^2} Q_k s_k(t) + f_k, \\ \frac{ds_k(t)}{dt} = x_k(t) - y^\delta(t), \\ u_k(t) = -\frac{1}{\alpha^2} G_k^\top s_k(t), \\ t \in [t_{k-1}, t_k], \\ k = 1: \quad x_1(0) = y_0^\delta, \quad s(0) = 0, \\ k = 2, \dots, N: \quad x_k(t_{k-1}) = y_{k-1}^\delta, \quad s_k(t_{k-1}) = s_{k-1}(t_{k-1}), \end{gathered} \tag{2.9}$$

where

$$Q_k \triangleq G_k G_k^\top, \quad G_k \triangleq G(t_{k-1}, y_{k-1}^\delta), \quad f_k \triangleq f(t_{k-1}, y_{k-1}^\delta). \tag{2.10}$$

The function $s_k(\cdot)$ is the vector of the adjoint variables, which plays the role of the function $s_k(\cdot)$ from condition (2.8).

(3) **Construction of an auxiliary approximation of the normal control.** The solution $s_k(\cdot)$ of system (2.9) is used to construct the auxiliary piecewise-continuous control

$$\tilde{u}^\delta(t) \triangleq \{\tilde{u}_k^\delta(t),\ t \in [t_{k-1}, t_k]\}, \quad \tilde{u}_k^\delta(t) = -\frac{1}{\alpha^2} G_k s_k(t). \tag{2.11}$$

The following theorem about properties of the auxiliary controls (2.11) was proved in [22, Theorem 1], [23, Theorem 1].

Theorem 2.10. *Let assumptions* **A.1**–**A.4** *hold for the input data (2.1)–(2.6). Let the approximation parameters $\delta \le \delta_0$, $h^\delta \le h_0$, α tend to zero in the following agreement:*

$$\begin{gathered} h^\delta = h^\delta(\delta), \quad \alpha = \alpha(\delta), \\ \lim_{\delta \to 0} h^\delta = 0, \quad \lim_{\delta \to 0} \alpha = 0, \quad \lim_{\delta \to 0} \frac{\delta}{h^\delta} = 0, \quad \frac{\alpha}{(h^\delta)^2} = K_0 > 0. \end{gathered} \tag{2.12}$$

Then, the constructed functions $\tilde{u}^\delta(\cdot)$ (2.11) satisfy the following conditions:

$\mathscr{B}$**.1** *They are bounded essentially and uniformly with respect to the parameters δ, h^δ, α.*

$\mathscr{B}$**.2** *The trajectories of system (2.1), generated by these controls, converge uniformly to the basic trajectory.*

$\mathscr{B}$**.3** *These controls converge to the normal control in the sense (2.7).*

Remark 2.11. The results $\mathscr{B}$**.2** and $\mathscr{B}$**.3** of Theorem 2.10 coincide with the DRP conditions **B.2** and **B.3**. Yet, the result $\mathscr{B}$**.1** is weaker than condition **B.1**. Indeed, it says than the approximations are bounded, but does not guarantee that they satisfy the restrictions (2.2), and thus, they may not be admissible controls. This fact was the motivation to introduce the next two procedures in the algorithm [24].

(4) **Averaging the auxiliary approximation.** The auxiliary function $\tilde{u}_k^\delta(\cdot)$ is averaged be the formula

$$\bar{u}_k^\delta \triangleq \frac{1}{h^\delta} \int_{t_{k-1}}^{t_k} \tilde{u}_k^\delta(t)\,dt. \tag{2.13}$$

(5) **"Cut-off" of the averaged control.** The averaged control $\bar{u}_k^\delta$ is "cut-off":

$$\hat{u}_k^\delta \triangleq \begin{cases} \bar{u}_k^\delta & , \quad \bar{u}_k^\delta \in \mathrm{Conv}(\mathbf{U}); \\ \hat{u} \in \mathrm{Conv}(\mathbf{U}) : \ \|\hat{u} - \bar{u}_k^\delta\| = \min\limits_{u \in \mathrm{Conv}(\mathbf{U})} \|u - \bar{u}_k^{\alpha,\delta}\| & , \quad \bar{u}_k^\delta \notin \mathrm{Conv}(\mathbf{U}). \end{cases}, \qquad (2.14)$$
$$t \in [t_{k-1}, t_k], \quad k = 1, \ldots, N.$$

The following theorem about properties of the "cut-off" controls (2.14) was proved in [24, Theorem 2].

Theorem 2.12. *Let assumptions* **A.1**–**A.4** *hold for the input data (2.1)–(2.6). Let the approximation parameters* $\delta \le \delta_0$, $h^\delta \le h_0$, α *tend to zero in the agreement (2.12).*

Then, the constructed functions $\hat{u}^\delta(\cdot)$ *(2.14) satisfy the following conditions:*

$\mathfrak{B}$**.1** *They satisfy the convex restrictions on the controls (2.5).*

$\mathfrak{B}$**.2** *The trajectories of system (2.1), generated by these controls, converge uniformly to the basic trajectory.*

$\mathfrak{B}$**.3** *These controls converge almost everywhere on* $[0,T]$ *to the normal control.*

Remark 2.13. The result $\mathfrak{B}$**.2** coincides with the DRP condition **B.2**. Moreover, it will be proved further in this paper (3.11) that from the result $\mathfrak{B}$**.3** follows the DRP condition **B.3**. Yet, the result $\mathfrak{B}$**.1** is still weaker than condition **B.1**. Indeed, it says than the approximations $\hat{u}^\delta(\cdot)$ satisfy only convexified control restrictions (2.5), but does not guarantee that they satisfy the restrictions (2.2), and thus, they still may not be admissible controls. This fact was the motivation to introduce another additional procedure into the algorithm, which will be explained and justified further in this paper.

The algorithm 1–5 is described in details in [23] (parts 1–3) and in [24] (parts 4,5).

2.8. **New algorithm for solving the DRP.** In this paper, a special case of non-convex geometrical restrictions on the controls is considered. Namely, a set of 2^m discrete points, which are the vertices of an m-dimensional orthotope:

$$\mathbf{U} = \{\underline{u}_1, \overline{u}_1\} \times \ldots \times \{\underline{u}_m, \overline{u}_m\}, \quad \underline{u}_j < \overline{u}_j, \quad \|\underline{u}_j\|, \|\overline{u}_j\| \le U > 0, \qquad (2.15)$$

where $A \times B$ is the Cartesian product of sets A and B. Consider the DRP **B.1–B.3** for the control restriction set (2.15).

The previously developed algorithm, described in Section 2.7, is modified. An additional procedure is done on each step. The procedures 1–5 remain the same. After obtaining the value $\hat{u}_k^\delta$ on the k-th step (procedure 5), a piecewise-constant function $v_k^\delta(\cdot) : [t_{k-1}, t_k] \to \mathbf{U}$ is constructed:

$$v_{k,j}^\delta = \begin{cases} \overline{u}_j, & t \in [t_{k-1}, \tau_{k,j}]; \\ \underline{u}_j, & t \in (\tau_{k,j}, t_k]. \end{cases},$$
$$\tau_{k,j} = t_k - \frac{h^\delta(\overline{u}_j - \hat{u}_{k,j}^\delta)}{\overline{u}_j - \underline{u}_j}, \qquad (2.16)$$
$$j = 1, \ldots, m, \quad t \in [t_{k-1}, t_k], \quad k = 1, \ldots, N.$$

The procedure is graphically explained on Fig. 1. At the end of the reconstruction process (when

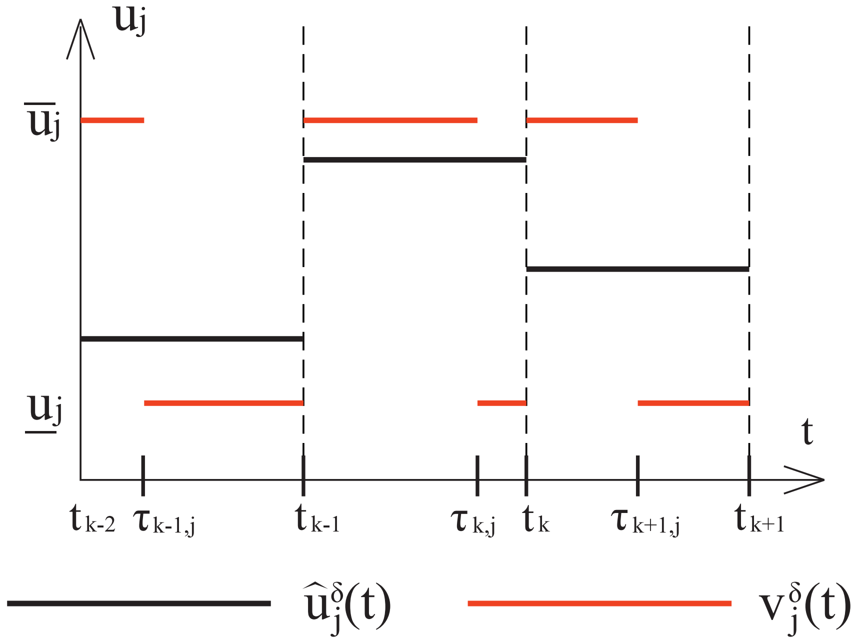


FIGURE 1. Construction of admissible approximating control $v_j^\delta(\cdot)$.

all functions $v_k^\delta(\cdot),\ k=1,\ldots,N$ (2.16) have been constructed) the function $v^\delta(\cdot):[0,T]\to\mathbf{U}$ can be defined piecewise:

$$v^\delta(t)=\{v_k^\delta(t),\quad t\in[t_{k-1},t_k],\quad k=1,\ldots,N\}. \tag{2.17}$$

The functions $v^\delta(\cdot)$ (2.17) are proposed as admissible approximations of the averaged normal control.

3. Results and Discussion

3.1. **The main result.** The main result of this work is the following theorem.

Theorem 3.1. *Let assumptions* **A.1**–**A.4** *hold for the input data (2.1)–(2.6). Let the approximation parameters* $\delta\le\delta_0,\ h^\delta\le h_0,\ \alpha$ *tend to zero in the agreement (2.12).*

Then, the constructed functions $v^\delta(\cdot)$ *(2.16), (2.17) satisfy conditions* **B.1**–**B.3**.

To prove this theorem, we first formulate an auxiliary proposition and prove an auxiliary lemma.

Proposition 3.2. *Let assumptions* **A.1**–**A.4** *hold for the input data (2.1)–(2.6). Let the approximation parameters* $\delta\le\delta_0,\ h^\delta\le h_0,\ \alpha$ *tend to zero in the agreement (2.12).*

Then, the values $\bar{u}_k^\delta$ *(2.11) satisfy the condition*

$$\|\bar{u}_k^\delta-\bar{u}_k^*\|\le r_{\hat{u}}(\delta,h^\delta,\alpha),\quad k=1,\ldots,N, \tag{3.1}$$

where

$$\bar{u}_k^*\triangleq\frac{1}{h^\delta}\int\limits_{t_{k-1}}^{t_k}u^*(t)dt.$$

The function $r_{\hat{u}}(\delta, h^\delta, \alpha)$ *is*

$$r_{\hat{u}}(\delta,h^\delta,\alpha) \triangleq \Big(\frac{L_{D_0}K}{\lambda_*^2}\Big(2TK_0\sqrt{\lambda^*}n+2K+3\lambda_*\Big)+24\frac{1}{\lambda_*^{1.5}}K_0\Big)(\delta+(K+1)h^\delta)$$
$$+TK_0^2Kn\sqrt{\lambda^*}\frac{1}{\lambda_*^2}\left(96\delta\alpha+48Kh^\delta\alpha+\left(\frac{48}{\sqrt{\lambda_*}}+\frac{96}{\lambda_*}\right)(K_0\delta h^\delta\alpha+K\alpha^2)\right)+2K\frac{1}{\lambda_*}\frac{\delta}{h^\delta},$$

where

$$K \triangleq \max_{(t,x)\in D_0,\ u\in \mathrm{Conv}(\mathbf{U})} \|G(t,x)u+f(t,x)\|. \tag{3.2}$$

The parameters λ_* *and* λ^* *are respectively the minimum and the maximum eigenvalues of the matrix function* $Q(t,x) = G(t,x)G^\top(t,x),\ (t,x)\in D_0$.

Moreover,

$$\lim_{\delta,h^\delta,\alpha\to 0}\|r_{\hat{u}}(\delta,h^\delta,\alpha)\| = 0.$$

Note that the matrix function $Q(\cdot)$ is symmetric for each $t\in[0,T]$ [13, ch. 1, p. 6] and continuous on D_0 (Assumption **A.2**). Therefore, its eigenvalues $\{\lambda_1(\cdot),\dots,\lambda_n(\cdot)\}$ are also continuous on D_0 [13, ch. 8, p. 8]. Therefore, λ_* and λ^* exist.

The proof of the proposition is a part of the proof of Theorem 2 from [24, formulae (9)–(21)].

Lemma 3.3. *Let assumptions* **A.1**–**A.4** *hold for the input data (2.1)–(2.6). Then, the following relation is true for the values* $\hat{u}_k^\delta$ *(2.14) and* $\bar{u}_k^\delta$ *(2.13):*

$$\|\hat{u}_k^\delta - u\| \le \|\bar{u}_k^\delta - u\|, \quad k\in\{1,\dots,N\}, \quad u\in\mathrm{Conv}(\mathbf{U}). \tag{3.3}$$

Proof. If $\hat{u}_k^\delta = u$, then the lemma is true. If $\bar{u}_k^\delta \in \mathrm{Conv}(\mathbf{U})$, then $\bar{u}_k^\delta = \hat{u}_k^\delta$ by the construction (2.14) and lemma is true. So, consider the case $\hat{u}_k^\delta \neq u$, $\bar{u}_k^\delta \notin \mathrm{Conv}(\mathbf{U})$.

Assume in contradiction to (3.3) that

$$\|\hat{u}_k^\delta - u\| > \|\bar{u}_k^\delta - u\|. \tag{3.4}$$

Then, let ω_k be a point on the line $\overleftrightarrow{\hat{u}_k^\delta, u}$ such that it is closest to the point $\bar{u}_k^\delta$. In other words, $\overline{\bar{u}_k^\delta, \omega_k} \perp \overline{\hat{u}_k^\delta, u}$. In other words,

$$\exists l_k\in\mathbb{R}:\quad \omega_k = u + l_k(\hat{u}_k^\delta - u),$$
$$\langle(\bar{u}_k^\delta - \omega_k),(\hat{u}_k^\delta - u)\rangle = 0$$
$$\Rightarrow$$
$$\langle(\bar{u}_k^\delta - u - l_k(\hat{u}_k^\delta - u)),(\hat{u}_k^\delta - u)\rangle = 0$$
$$\Rightarrow$$
$$l_k = \frac{\langle(\bar{u}_k^\delta - u),(\hat{u}_k^\delta - u)\rangle}{\|\hat{u}_k^\delta - u\|^2} = \frac{\cos\big(\widehat{(\bar{u}_k^\delta - u),(\hat{u}_k^\delta - u)}\big)\|\bar{u}_k^\delta - u\|}{\|\hat{u}_k^\delta - u\|} < 1,$$

since we assumed (3.4).

But $l_k < 1$ means that ω_k belongs to the line segment $\overline{\hat{u}_k^\delta, u}$. But by the construction (2.14), $\hat{u}_k^\delta \in \mathrm{Conv}(\mathbf{U})$ and $u\in\mathrm{Conv}(\mathbf{U})$. Therefore, $\omega_k\in\mathrm{Conv}(\mathbf{U})$. But ω_k is the point on $\overline{\hat{u}_k^\delta, u}$, closest to the point $\bar{u}_k^\delta$ by the definition of ω_k. Therefore, ω_k is closer to $\bar{u}_k^\delta$ than to $\hat{u}_k^\delta$. But by the construction (2.14), $\hat{u}_k^\delta$ is the point of Conv($\mathbf{U}$), closest to $\bar{u}_k^\delta$. So, we got a contradiction and (3.4) is false. □

Now, we can prove theorem 3.1.

Proof. Condition **B.1** is fulfilled by the construction (2.16), (2.17).

First, let us prove the fulfilment of condition **B.3**. Fix an arbitrary $\varphi(\cdot) \in \mathbb{C}([0,T],\mathbb{R}^m)$, following the definition of convergence (2.7):

$$\int_0^T \langle \varphi(t), v^\delta(t) - u^*(t)\rangle dt \stackrel{\pm \hat{u}^\delta(t)}{=} \sum_{k=1,\dots,N} \left[\int_{t_{k-1}}^{t_k} \langle \varphi(t), v_k^\delta(t) - \hat{u}_k^\delta \rangle dt \right] + \int_0^T \langle \varphi(t), \hat{u}^\delta(t) - u^*(t)\rangle dt, \tag{3.5}$$

where $\hat{u}_k^\delta$ is the "cut-off" auxiliary control (2.14), constructed in the procedure 5 (2.14) on the k-th step of the algorithm. In expression (3.5),

$$\int_{t_{k-1}}^{t_k} \varphi_j(t)(v_{k,j}^\delta(t) - \hat{u}_{k,j}^\delta) dt \stackrel{\pm \varphi_j(t_{k-1})}{=} \int_{t_{k-1}}^{t_k} \varphi_j(t_{k-1})(v_{k,j}^\delta(t) - \hat{u}_{k,j}^\delta)dt + \int_{t_{k-1}}^{t_k} (\varphi_j(t) - \varphi_j(t_{k-1}))(v_{k,j}^\delta(t) - \hat{u}_{k,j}^\delta)dt, \quad j = 1,\dots,m. \tag{3.6}$$

Evaluate the first term in (3.6). By the definition (2.16),

$$\begin{aligned} &\int_{t_{k-1}}^{t_k} \varphi_j(t_{k-1})(v_{k,j}^\delta(t) - \hat{u}_{k,j}^\delta)dt \\ &= \int_{t_{k-1}}^{\tau_{k,j}} \varphi_j(t_{k-1})(\overline{u}_j - \hat{u}_{k,j}^\delta)dt + \int_{\tau_{k,j}}^{t_k} \varphi_j(t_{k-1})(\underline{u}_j - \hat{u}_{k,j}^\delta)dt \\ &= \varphi_j(t_{k-1})\Big((\tau_{k,j} - t_{k-1})(\overline{u}_j - \hat{u}_{k,j}^\delta) + (t_k - \tau_{k,j})(\underline{u}_j - \hat{u}_{k,j}^\delta)\Big) \\ &= \varphi_j(t_{k-1}) \left(\frac{h^\delta(\hat{u}_{k,j}^\delta - \underline{u}_j)(\overline{u}_j - \hat{u}_{k,j}^\delta)}{\overline{u}_j - \underline{u}_j} + \frac{h^\delta(\overline{u}_j - \hat{u}_{k,j}^\delta)(\underline{u}_j - \hat{u}_{k,j}^\delta)}{\overline{u}_j - \underline{u}_j} \right) = 0 \end{aligned} \tag{3.7}$$

Thus, the first term in (3.6) equals zero. In the second term of (3.6),

$$|\varphi_j(t) - \varphi_j(t_{k-1})| \le \omega_\varphi(|t - t_{k-1}|) \le \omega_\varphi(h^\delta),$$

where $\omega_\varphi(\cdot)$ is the modulus of continuity of $\varphi(\cdot)$. So, by the construction of $v^\delta_{k,j}(t)$ (2.16) and $\hat{u}^\delta_{k,j}$ (2.14), we have in (3.6) that

$$\left\| \int\limits_{t_{k-1}}^{t_k} (\varphi_j(t)-\varphi_j(t_{k-1}))(v^\delta_{k,j}(t)-\hat{u}^\delta_{k,j})dt \right\| \le h^\delta \omega_\varphi(h^\delta)2U, \tag{3.8}$$

where the constant U is from (2.15).

Thus, substituting (3.8) and (3.7) into (3.6), we get

$$\left\| \int\limits_{t_{k-1}}^{t_k} \varphi_j(t)(v^\delta_{k,j}(t)-\hat{u}^\delta_{k,j})dt \right\| \le h^\delta \omega_\varphi(h^\delta)2U. \tag{3.9}$$

Apply estimate (3.9) to the first term (the sum) of (3.5):

$$\begin{aligned}\left\| \sum_{k=1,\dots,N} \left[\int\limits_{t_{k-1}}^{t_k} \langle \varphi(t), v^\delta_k(t)-\hat{u}^\delta_k\rangle dt \right] \right\| &\le \frac{T}{h^\delta}\sqrt{m}\, h^\delta \omega_\varphi(h^\delta)2U \\ &= T\sqrt{m}\,\omega_\varphi(h^\delta)2U \xrightarrow{h^\delta\to 0} 0.\end{aligned} \tag{3.10}$$

Now, we consider the second term of (3.5):

$$\int\limits_0^T \langle \varphi(t), \hat{u}^\delta - u^*(t)\rangle dt.$$

We will prove that it converges to zero under agreement (2.12). Indeed, consider the functions

$$\psi^\delta(t) \triangleq \langle \varphi(t), \hat{u}^\delta(t)-u^*(t)\rangle, \quad \delta\in(0,\delta_0].$$

By Theorem 2.12,

$$\hat{u}^\delta(t) \xrightarrow[{[0,T]}]{\text{a.e.}} u^*(t) \quad \Rightarrow \quad \hat{u}^\delta(t)-u^*(t) \xrightarrow[{[0,T]}]{\text{a.e.}} \bar{0} \quad \Rightarrow \quad \|\hat{u}^\delta(t)-u^*(t)\| \xrightarrow[{[0,T]}]{\text{a.e.}} 0$$
$$\Rightarrow \quad |\langle \varphi(t), \hat{u}^\delta(t)-u^*(t)\rangle| \le (\max_{t\in[0,T]}\|\varphi(t)\|)\|\hat{u}^\delta(t)-u^*(t)\| \xrightarrow[{[0,T]}]{\text{a.e.}} 0$$
$$\Rightarrow \quad \psi^\delta(t) = \langle \varphi(t), \hat{u}^\delta(t)-u^*(t)\rangle \xrightarrow[{[0,T]}]{\text{a.e.}} 0,$$

where the notation $\xrightarrow[{[0,T]}]{\text{a.e.}}$ means "converges almost everywhere on $[0,T]$ under agreement (2.12)".

So, the functions $\psi^\delta(\cdot)$ converge almost everywhere to identical zero. Moreover,

$$|\psi^\delta(t)| \le K \triangleq \max_{t\in[0,T]}\|\varphi(t)\|2\sqrt{m}U, \quad \delta\in(0,\delta_0].$$

Then, by the Lebesgue theorem on passage to the limit under the integral sign [10, Ch. V, §5.5],

$$\int\limits_0^T \psi^\delta(t)dt = \int\limits_0^T \langle \varphi(t), \hat{u}^\delta_k(t)-u^*(t)\rangle dt \to 0 \tag{3.11}$$

under agreement (2.12). Which means by the definition (2.7) that $\hat{u}^\delta_k(\cdot)$ converge to $u^*(\cdot)$ in the sense (2.7) under agreement (2.12).

Return to expression (3.5). Both terms of it converge to zero (3.10), (3.11). Therefore,

$$\int_0^T \langle \varphi(t), v^\delta(t) - u^*(t) \rangle dt \xrightarrow{\delta \to 0} 0$$

under agreement (2.12). In other words, $v^\delta(t)$ converges in the sense (2.7) to $u^*(\cdot)$ and condition **B.3** is fulfilled.

Let us now prove the fulfillment of condition **B.2**. To do it we will estimate the residual

$$\|\check{x}^\delta(t) - x^*(t)\|.$$

It will be done in a series of successive nested estimates. To structurise the proof, the estimated expressions will be denoted as $A_1,\ A_2,\ \ldots,\ B_1,\ B_2,\ldots,\ C_1,\ C_2,\ldots$.

Since $\check{x}^\delta(\cdot)$ and $x^*(\cdot)$ are the trajectories of (2.1), generated by the controls $v^\delta(\cdot)$ and $u^*(\cdot)$,

$$\begin{aligned} &\|\check{x}^\delta(t) - x^*(t)\| \\ &= \left\| y_0 + \int_0^t \left[G(t,\check{x}^\delta(t))v^\delta(t) + f(t,\check{x}^\delta(t))\right]dt - x^*(0) - \int_0^t \left[G(t,x^*(t))u^*(t) + f(t,x^*(t))\right]dt \right\| \\ &\overset{\pm\int_0^t G(t,x^*(t))v^\delta(t)dt}{\leq} \underbrace{\left\| \int_0^t \left[\Big(G(t,\check{x}^\delta(t)) - G(t,x^*(t))\Big) v^\delta(t) + f(t,\check{x}^\delta(t)) - f(t,x^*(t)) \right] dt \right\|}_{A_1(t)} \\ &\quad + \underbrace{\left\| \int_0^t \left[G(t,x^*(t))(v^\delta(t) - u^*(t)) \right] dt \right\|}_{A_2(t)} + \underbrace{\|y_0 - x^*(0)\|}_{A_3}. \end{aligned} \tag{3.12}$$

We will successively estimate the terms $A_1(t),\ A_2(t),\ A_3$ of (3.12).

Estimation of $A_1(t)$ from (3.12). The matrix function $G(\cdot)$ and the vector function $f(\cdot)$ are Lipschitz-continuous (2.4) by Assumption **A.2**. Moreover, the functions $v^\delta(\cdot)$ and $\hat{u}^\delta(\cdot)$ are bounded by construction (2.14), (2.16):

$$\|v^\delta(t)\| \leq \sqrt{m}U, \quad \|\hat{u}^\delta(t)\| \leq \sqrt{m}U, \quad t \in [0,T]. \tag{3.13}$$

Therefore,

$$\begin{aligned} A_1(t) &= \left\| \int_0^t \left[\Big(G(t,\check{x}^\delta(t)) - G(t,x^*(t))\Big) v^\delta(t) + f(t,\check{x}^\delta(t)) - f(t,x^*(t)) \right] dt \right\| \\ &\leq \int_0^t L_{D_0} \left\| \check{x}^\delta(t) - x^*(t) \right\| (\sqrt{m}U + 1)dt. \end{aligned} \tag{3.14}$$

Estimation of $A_2(t)$ from (3.12). Break the integral into the sum:

$$A_2(t) = \left\| \int_0^t \Big[G(t,x^*(t))(v^\delta(t) - u^*(t)) \Big] dt \right\|$$

$$\leq \underbrace{\sum_{k=1,\dots,i-1} \left[\left\| \int_{t_{k-1}}^{t_k} \Big[G(t,x^*(t))(v_k^\delta - u^*(t)) \Big] dt \right\| \right]}_{B_1(t)} + \underbrace{\left\| \int_{t_{i-1}}^{t} \Big[G(t,x^*(t))(v_i^\delta - u^*(t)) \Big] dt \right\|}_{B_2(t)}, \quad (3.15)$$

$$i = \lfloor \frac{t}{h^\delta} \rfloor,$$

where $\lfloor \cdot \rfloor$ means the floor function.

Estimation of $B_1(t)$ from (3.15). Let us add $\pm G_k$ (2.10) in each term of the sum:

$$B_1(t) = \sum_{k=1,\dots,i-1} \left[\left\| \int_{t_{k-1}}^{t_k} \Big[G(t,x^*(t))(v_k^\delta - u^*(t)) \Big] dt \right\| \right]$$

$$\stackrel{\pm G_k}{\leq} \underbrace{\sum_{k=1,\dots,i-1} \left[\int_{t_{k-1}}^{t_k} \|G(t,x^*(t)) - G_k\| \, \|v_k^\delta - u^*(t)\| dt \right]}_{C_1(t)} + \underbrace{\sum_{k=1,\dots,i-1} \left[\|G_k\| \left\| \int_{t_{k-1}}^{t_k} v_k^\delta - u^*(t) dt \right\| \right]}_{C_2(t)} \quad (3.16)$$

Estimation of $C_1(t)$ from (3.16). Since the definition (2.10),

$$\|G(t,x^*(t)) - G_k\| = \left\| G(t,x^*(t)) - G(t,y_{k-1}^\delta) \right\|$$

$$\stackrel{\pm G(t,y^\delta(t))}{=} \left\| G(t,x^*(t)) - G(t,y^\delta(t)) \right\| + \left\| G(t,y^\delta(t)) - G(t,y_{k-1}^\delta) \right\|. \quad (3.17)$$

It was shown in [23, formulae (3.1), (3.5), (3.6)] that

$$\|x^*(t) - y^\delta(t)\| \leq 14(\delta + h^\delta K), \qquad \left\| \frac{dy^\delta(t)}{dt} \right\| \leq Y_1^\delta \triangleq 27K + 26\frac{\delta}{h^\delta}, \quad t \in [0,T],$$

where K is defined in (3.2).

The matrix function $G(\cdot)$ is Lipschitz continuous (2.4). Therefore, we can estimate (3.17):

$$\|G(t,x^*(t)) - G_k\| \leq L_{D_0}(14(\delta + h^\delta K) + h^\delta Y_1^\delta), \quad t \in [t_{k-1}, t_k], \quad k = 1, \dots, N.$$

Thus, using the controls boundness (3.13),

$$C_1(t) = \sum_{k=1,\dots,i-1} \left[\int_{t_{k-1}}^{t_k} \|G(t,x^*(t)) - G_k\| \, \|v_k^\delta - u^*(t)\| dt \right]$$
$$\leq \frac{T}{h^\delta} h^\delta L_{D_0}(14(\delta + h^\delta K) + h^\delta Y_1^\delta) 2\sqrt{m} U. \quad (3.18)$$

Estimation of $C_2(t)$ from (3.16). First, note that

$$\left\| \int\limits_{t_{k-1}}^{t_k} v_k^\delta - u^*(t)dt \right\| \overset{\pm\hat{u}_k^\delta}{\leq} \underbrace{\left\| \int\limits_{t_{k-1}}^{t_k} v_k^\delta - \hat{u}_k^\delta dt \right\|}_{D_1} + \underbrace{\left\| \int\limits_{t_{k-1}}^{t_k} \hat{u}_k^\delta - u^*(t)dt \right\|}_{D_2}. \tag{3.19}$$

Estimation of D_1 from (3.19). One can prove in the same way it was done in (3.7) that, by the definition (2.16),

$$\int\limits_{t_{k-1}}^{t_k} v_{k,j}^\delta - \hat{u}_{k,j}^\delta dt = 0.$$

And therefore,

$$D_1 = 0. \tag{3.20}$$

Estimation of D_2 from (3.19). It follows from Proposition 3.1 Lemma 3.3 that

$$\|\hat{u}_k^\delta - \bar{u}_k^*\| \leq r_{\hat{u}}(\delta, h^\delta, \alpha), \quad k = 1, \dots, N.$$

But then,

$$\left\| \int\limits_{t_{k-1}}^{t_k} \hat{u}_k^\delta - u^*(t)dt \right\| = \left\| h^\delta(\hat{u}_k^\delta - \bar{u}_k^*) \right\| \leq h^\delta r_{\hat{u}}(\delta, h^\delta, \alpha), \quad k = 1, \dots, N. \tag{3.21}$$

Substituting estimates for D_1 (3.20) and D_2 (3.21) into expression (3.19),

$$\begin{aligned} C_2(t) &= \sum_{k=1,\dots,i-1} \left[\|G_k\| \left\| \int\limits_{t_{k-1}}^{t_k} v_k^\delta - u^*(t)dt \right\| \right] \\ &\leq \frac{T}{h^\delta} K h^\delta r_{\hat{u}}(\delta, h^\delta, \alpha), \end{aligned} \tag{3.22}$$

where the constant K is defined in (3.2).

Now, we can apply the estimates for $C_1(t)$ (3.18) and $C_2(t)$ (3.22) to $B_1(t)$ (3.16):

$$B_1(t) \leq T L_{D_0}(14(\delta + h^\delta K) + h^\delta Y_1^\delta) 2\sqrt{m} U + T K r_{\hat{u}}(\delta, h^\delta, \alpha). \tag{3.23}$$

Estimation of $B_2(t)$ from (3.15). We estimate the norm of the integrant:

$$B_2(t) = \left\| \int\limits_{t_{k-1}}^{t} \left[G(t, x^*(t))(v_k^\delta - u^*(t)) \right] dt \right\| \leq h^\delta K 2\sqrt{m} U. \tag{3.24}$$

Apply the estimates for $B_1(t)$ (3.23) and $B_2(t)$ (3.24) to $A_2(t)$ (3.15):

$$A_2(t) \leq T L_{D_0}(14(\delta + h^\delta K) + h^\delta Y_1^\delta) 2\sqrt{m} U + T K r_{\hat{u}}(\delta, h^\delta, \alpha) + h^\delta K 2\sqrt{m} U. \tag{3.25}$$

Estimation of A_3 from (3.12). All measurements have the error δ (2.6). Therefore,

$$A_3 = \|y_0 - x^*(0)\| \leq \delta. \tag{3.26}$$

Finally, apply the estimates for $A_1(t)$ (3.14), $A_2(t)$ (3.25) and A_3 (3.26) to the residual (3.12):

$$\|\check{x}^\delta(t) - x^*(t)\| \leq \int_0^t \left\|\check{x}^\delta(t) - x^*(t)\right\| L_{D_0}(\sqrt{m}U + 1)dt + r_{\check{x}}(\delta, h^\delta, \alpha), \tag{3.27}$$

$$r_{\check{x}}(\delta, h^\delta, \alpha) \triangleq TL_{D_0}(14(\delta + h^\delta K) + h^\delta Y_1^\delta)2\sqrt{m}U + TKr_{\hat{u}}(\delta, h^\delta, \alpha) + h^\delta K2\sqrt{m}U + \delta.$$

The inequality (3.27) allows to apply the Grönwall lemma [2], from which it follows that

$$\|\check{x}^\delta(t) - x^*(t)\| \leq r_{\check{x}}(\delta, h^\delta, \alpha)\exp(TL_{D_0}(\sqrt{m}U + 1)), \quad t \in [0, T].$$

It follows from Proposition 3.1 that

$$\lim_{\delta, h^\delta, \alpha \to 0} \|r_{\check{x}}(\delta, h^\delta, \alpha)\| = 0$$

when the parameters δ, h^δ, α tend to zero in the agreement (2.12). Therefore,

$$\lim_{\delta, h^\delta, \alpha \to 0} \|\check{x}^\delta(t) - x^*(t)\| = 0, \quad t \in [0, T].$$

Thus, condition **B.2** is fulfilled and the theorem is proved. □

3.2. **Example.** A numerical simulation of solving the DRP by the suggested algorithm was done. The so-called car quarter model from the paper [6] was considered. This model represents the dynamics of a wheel damper with two springs (see Fig. 2). The dynamics are

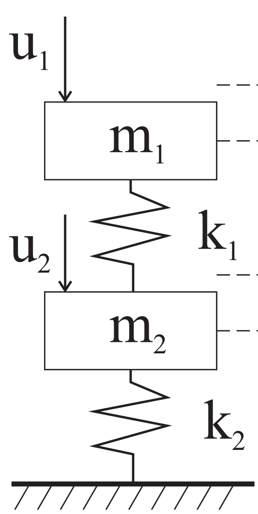


FIGURE 2. The car quarter model.

$$\begin{pmatrix} \dfrac{dx_1(t)}{dt} \\ \dfrac{dx_2(t)}{dt} \end{pmatrix} = \begin{pmatrix} x_3(t) \\ x_4(t) \end{pmatrix},$$

$$\begin{pmatrix} \dfrac{dx_3(t)}{dt} \\ \dfrac{dx_4(t)}{dt} \\ \dfrac{dx_5(t)}{dt} \end{pmatrix} = \begin{pmatrix} -\dfrac{x_1(t)}{m_1} + \dfrac{x_2(t)}{m_1} & 0 & \dfrac{1}{m_1} & 0 \\ \dfrac{x_1(t)}{m_2} - \dfrac{x_2(t)}{m_2} & -\dfrac{x_2(t)}{m_1} & 0 & \dfrac{1}{m_2} \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} k_1(t) \\ k_2(t) \\ u_1(t) \\ u_2(t) \end{pmatrix}, \tag{3.28}$$

$$t \in [0,1], \quad k_1(t) \in \{0, 0.5\}, \quad k_2(t) \in \{0, 0.8\}, \quad u_1 \in \{0,1\}, \quad u_2 \in \{0,1\}.$$

In this model, the state variables $x_1(t)$ and $x_2(t)$ are the values of the springs' deflections, $x_3(t)$ and $x_4(t)$ are the vertical speeds of the two bodies, attached to the springs' ends, and $x_5(t)$ is the force impulse. The controls are the springs' stiffness coefficients $k_1(t)$ and $k_2(t)$ and the external forces $u_1(t)$ and $u_2(t)$, applied to the bodies (see Fig. 2).

To simulate the input data of the problem (the inaccurate measurements (2.6)), the basic trajectory $(x_1^*(\cdot), x_2^*(\cdot), x_3^*(\cdot), x_4^*(\cdot), x_5^*(\cdot))$ was constructed numerically for the controls

$$k_1(t) = 0.5, \quad k_2(t) = \begin{cases} 0.8, & t \in [0, 0.5T]; \\ 0, & t \in (0.5T, T]. \end{cases},$$

$$u_1(t) = \begin{cases} 0.1, & t \in [0, 0.3T]; \\ 0.5, & t \in (0.3T, 0.6T]; \\ 0.9, & t \in (0.6T, T]. \end{cases}, \quad u_2(t) = 0.5 + 0.5\cos(4t). \tag{3.29}$$

The function $k_2(\cdot)$ simulates the break of the spring.

Note that the controls (3.29) are averaged controls with values from the convex hull of $\mathbf{U} = \{0, 0.5\} \times \{0, 0.8\} \times \{0,1\} \times \{0,1\}$. It is assumed that each of the controls corresponds to a set of generalized controls, which represents sliding modes in dynamics (3.28).

The calculated basic trajectory was perturbed several times with different maximal error δ in discrete points with different steps h^δ to simulate inaccurate measurements (2.6) for various approximation parameters.

Remark 3.4. Note that, generally speaking, the dynamics (3.28) don't satisfy Assumption **A.3** since the first two rows of the matrix $G(\cdot)$ are zero rows. Yet, the first two equations of (3.28) can be omitted in the application of the algorithm, because it won't affect the constructions. The measurements of the trajectories $x_1^*(\cdot)$ and $x_2^*(\cdot)$ are still needed. This regularization is explained in details in [25].

The result of numerical simulations for the parameters

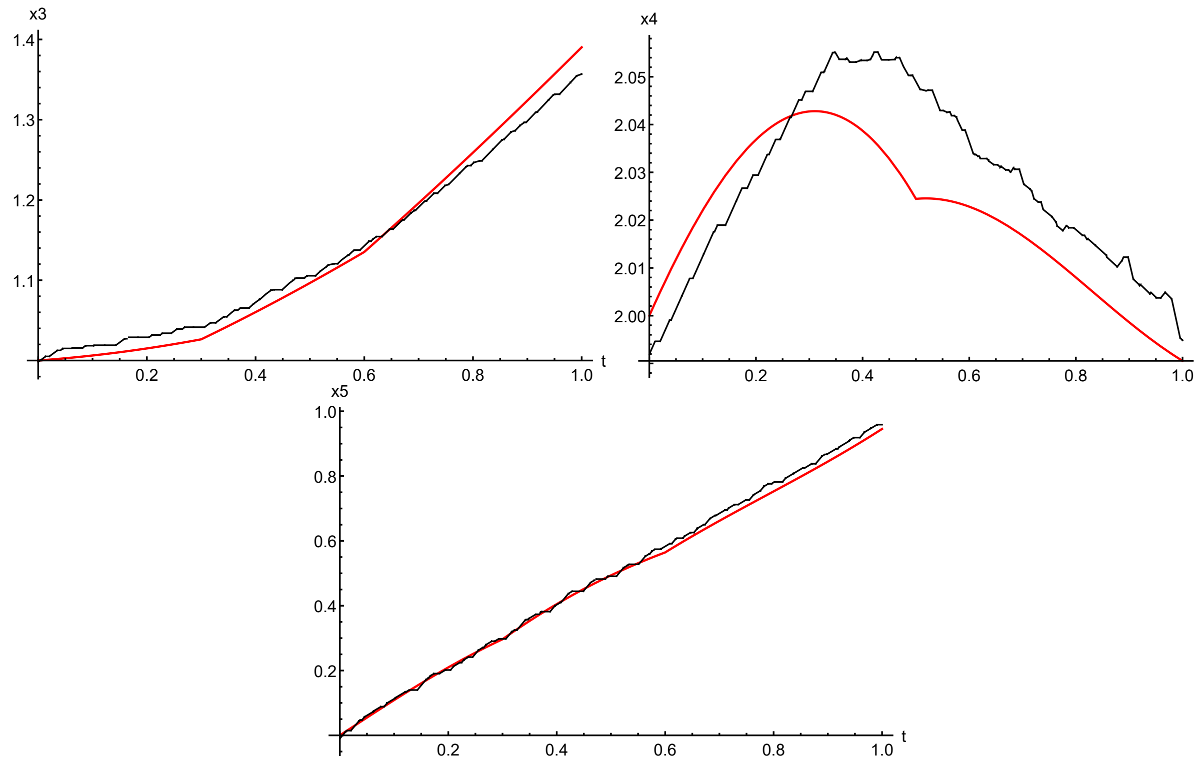
$$\delta = 0.01, \ h^\delta = 0.02, \ (N = 50), \ \alpha = 0.01$$

are shown on Fig. 3 (the trajectories $\check{x}^\delta(t)$) and Fig. 4 (the controls $v^\delta(t) = (\check{k}_1^\delta(t), \check{k}_2^\delta(t), \check{u}_1^\delta(t), \check{u}_2^\delta(t))$.

The result of numerical simulations for the parameters

$$\delta = 0.001, \ h^\delta = 0.01, \ (N = 100), \ \alpha = 0.001$$

are shown on Fig. 5 (the trajectories $\check{x}^\delta(t)$) and Fig. 6 (the controls $v^\delta(t) = (\check{k}_1^\delta(t), \check{k}_2^\delta(t), \check{u}_1^\delta(t), \check{u}_2^\delta(t))$.



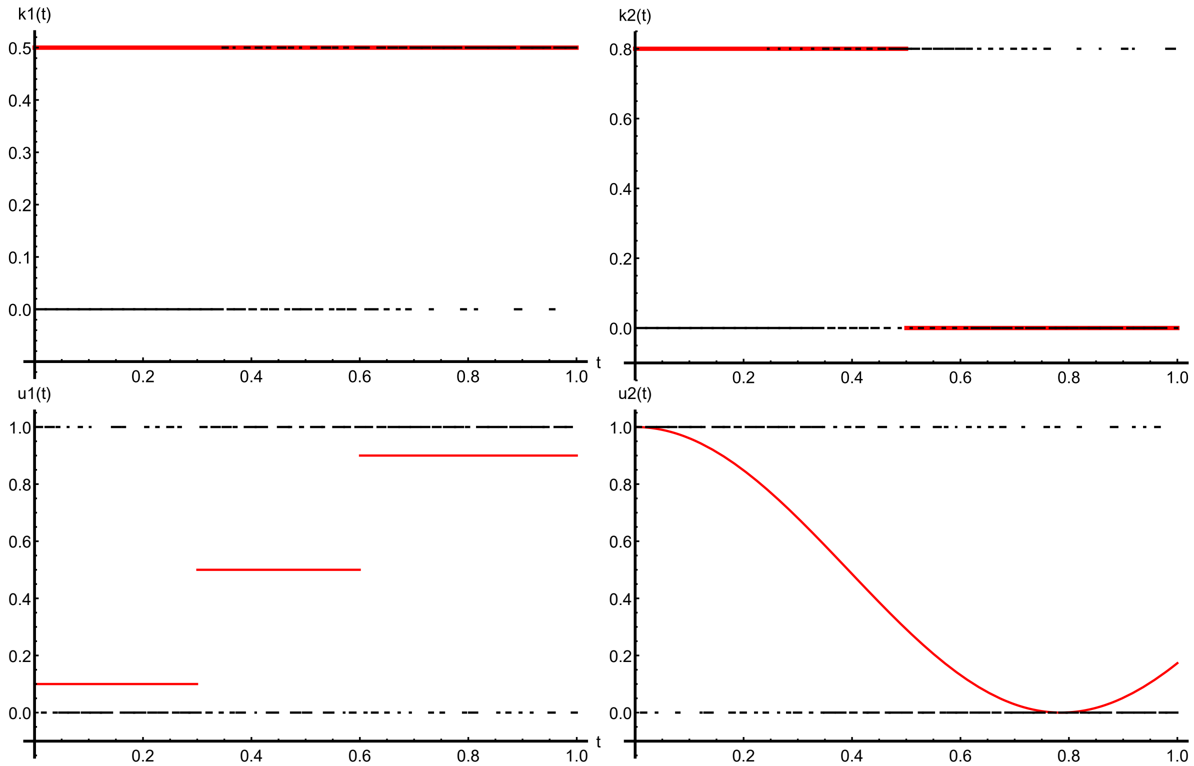
The legend:
——- the basic trajectory $x^*(t)$ ——- the reconstructed trajectory $\check{x}^\delta(t)$

FIGURE 3. Reconstructed trajectories for the parameters $\delta = 0.01$, $h^\delta = 0.02$, $(N = 50)$, $\alpha = 0.01$.

Remark 3.5. It can be seen, in particular, on the graphs for $\check{k}_1^\delta(t)$ in Fig. 4 and Fig. 6 that the reconstructed controls are not everywhere close to the controls (3.29) that generated the basic trajectory. It is explained by the fact that the normal control does not necessarily coincide with the controls (3.29). In the example, the norm of the approximation $\|\check{k}_1^\delta(\cdot)\|_{\mathbb{L}^2} \approx 0.331$, while the norm of the control (3.29) $\|k_1(\cdot)\|_{\mathbb{L}^2} = 0.5$.

3.3. **Discussion.** The method, suggested in this paper, allows to construct approximations of the normal control in the dynamic reconstruction problem in the form of piecewise-constant functions that converge to the normal control.

Previously [22, 23], another algorithm was developed, which allows to construct piecewise-continuous approximations of the normal control that converge in the sense (2.7). The drawback of this approach is that it can't guarantee that the approximations satisfy the given geometrical restrictions on the controls. This problem motivated another development of the method that is described and justified in [24]. It allows to construct piecewise-constant approximations of the normal control that satisfy the given geometrical restrictions, but only in the case of convex restrictions. These approximations converge almost everywhere to the normal control and the trajectories, generated by these approximating controls, converge uniformly to the basic trajectory. Yet, this approach is unsuitable for the case of non-convex geometrical restrictions



The legend:
——- controls (3.29) that generated the basic trajectory
——- the reconstructed admissible controls $v^{\delta}(t) = (\check{k}_1^{\delta}(t), \check{k}_2^{\delta}(t), \check{u}_1^{\delta}(t), \check{u}_2^{\delta}(t))$ (2.17)

FIGURE 4. Reconstructed admissible controls for the parameters $\delta = 0.01,\ h^{\delta} = 0.02,\ (N = 50),\ \alpha = 0.01.$

on the controls. Note that the convergence in the sense (2.7) is equivalent to weak* convergence in the space $\mathbb{C}^*([0,T]\times\mathbb{R}^m,\mathbb{R})$.

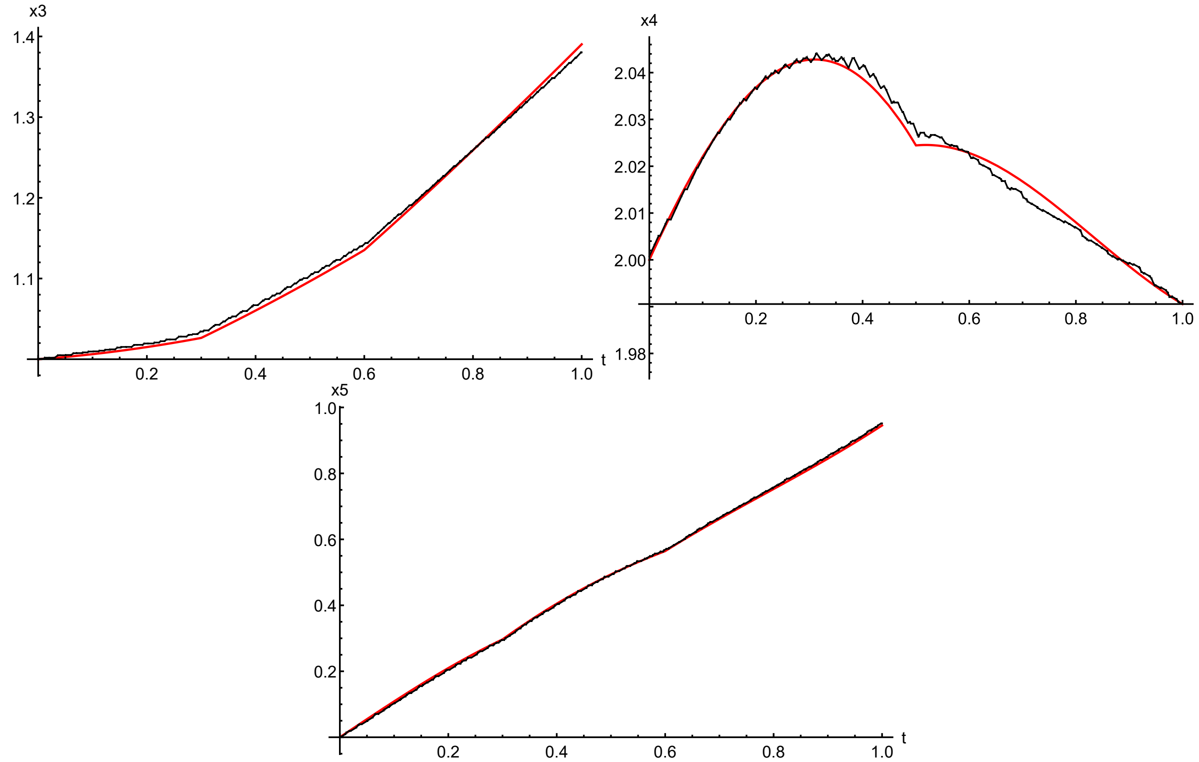
In comparison with the results of works [22, 23, 24], the approximations, suggested in this paper, satisfy the given geometrical restrictions even in the case of non-convex restrictions (2.15).

4. CONCLUSION

This paper offers a development of an approach to solving dynamic reconstruction problems, previously published in [22, 23, 24]. Namely, a modification of previously described [22, 23, 24] algorithm is suggested and justified. This modification allows to solve the dynamic reconstruction problem for one case on non-convex geometrical restrictions on the controls. Previously, only convex geometrical restrictions on the controls were considered. The results of numerical simulation of solving a dynamic reconstruction problem by the suggested algorithm are shown.

Dedication

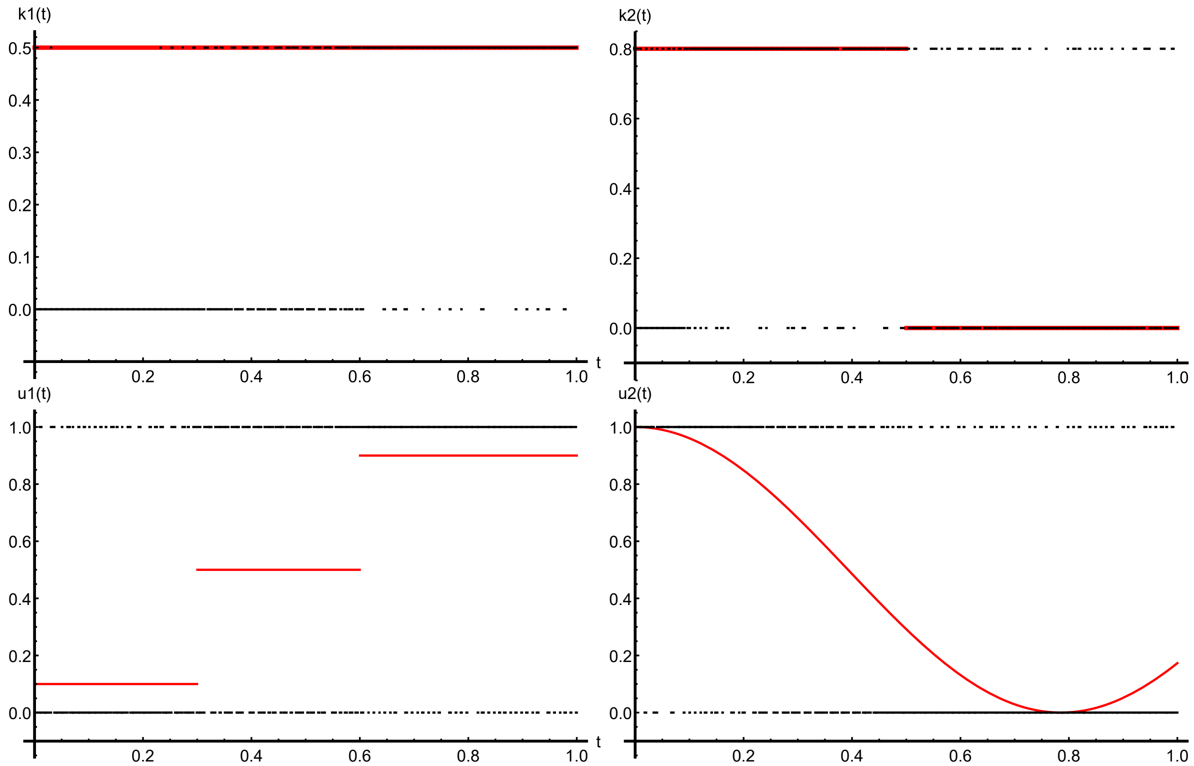
The paper is dedicated to the memory of R. F. Gabasov. He and his students made a great contribution to the theory of optimal control. In particular, we would like to mention a great contribution to the development of applications of the Pontryagin's maximum principle [15], done



The legend:
——- the basic trajectory $x^*(t)$ ——- the reconstructed trajectory $\check{x}^\delta(t)$

FIGURE 5. Reconstructed trajectories for the parameters $\delta = 0.001$, $h^\delta = 0.01$, $(N = 100)$, $\alpha = 0.001$.

by R. F. Gabasov, F. M. Kirillova, and their students [4]. In the early 1970s, R. F. Gabasov and F. M. Kirillova suggested and justified a new approach to solving linear programming problems. Their works were a stepping stone for such direction as constructive optimization methods [5]. The ideas, proposed in their works, are still relevant and are used to develop modern applied methods. For the list of corresponding references, we suggest the bibliography work [1].



The legend:

—— controls (3.29) that generated the basic trajectory

—— the reconstructed admissible controls $v^{\delta}(t) = (\check{k}_1^{\delta}(t), \check{k}_2^{\delta}(t), \check{u}_1^{\delta}(t), \check{u}_2^{\delta}(t))$ (2.17)

FIGURE 6. Reconstructed admissible controls for the parameters $\delta = 0.001$, $h^{\delta} = 0.01$, $(N = 100)$, $\alpha = 0.001$.

REFERENCES

[1] V.V. Alsevitch, N.M. Dmitruk, A.I. Kalinin, Rafail Gabasov. 80 anniversary: bibliography, BGU, Minsk, 2015. [In Russian]

[2] E.A. Barbashin, Introduction to the Theory of Stability, Wolters-Noordhoff Publishing, Groningen, 1970.

[3] M.C. D'Autilia, I. Sgura, B. Bozzini, Parameter identification in ODE models with oscillatory dynamics: a Fourier regularization approach, Inverse Probl. 33 (2017) 124009.

[4] R.F. Gabasov, F.M. Kirillova, The qualitative theory of optimal processes, M. Dekker, New York, 1976.

[5] R.F. Gabasov, F.M. Kirillova, Constructive optimization methods. In 5 parts. P.1: Linear programming, BGU, Minsk, 1983. [In Russian]

[6] R.F. Gabasov, F.M. Kirillova, Vo Thi Thanh Ha, Optimal real-time control of multidimensional dynamic plant, Autom Remote Control 76 (2015) 98–110.

[7] R.V. Gamkrelidze, Principles of Optimal Control Theory, Springer, New York, 1978.

[8] A.D. Ioffe, V.M. Tikhomirov, Theory of Extremal Problems, North-Holland Publishing Company, New York, 1979.

[9] S.I. Kabanikhin, O.I. Krivorotko, Identification of biological models described by systems of nonlinear differential equations, J. Inverse Ill-posed Probl. 23 5 (2015) 519-527.

[10] A.N. Kolmogorov, S.V. Fomin, Elements of the Theory of Functions and Functional Analysis. Vol 2, Graylock Press, New York , 1957.

[11] A.V. Kryazhimskij, Yu.S. Osipov, Modelling of a control in a dynamic system, Eng. cybernetics 21 (1983) 38-47.

[12] N.N. Krasovskii, A.I. Subbotin, Game-theoretical Control Problems, Springer, New York, 1987.

[13] H. Neudecker, J.R. Magnus, Matrix Differential Calculus with Applications in Statistics and Econometrics, John Wiley & Sons Ltd, New Jersey, 2019.

[14] D.V. Ngoc, H. Marcelo, Ang Jr., Dynamic model identification for industrial robots, Acta Polytechnica Hungarica 6 (2009) 51-68.

[15] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, The Mathematical Theory of Optimal Processes. VIII, John Wiley & Sons, New York/London, 1962.

[16] C. Ren, N. Wang, Q. Liu, Ch. Liu, Dynamic force identification problem based on a novel improved Tikhonov regularization, Math. Probl. Eng. (2019) 6095184.

[17] P.C. Sabatier, Past and future of inverse problems, J. Math. Phys. 41 (2000) 4082-4124.

[18] U. Schmitt, A. K. Louis, C. Wolters, M. Vauhkonen, Efficient algorithms for the regularization of dynamic inverse problems. Inverse Theory, Inverse Probl. 18 (2002) 659-676.

[19] N. Schuster, M. Burger, B. Hahn, Dynamic inverse problems: modelling-regularization-numerics, Inverse Probl. 34 (2018) 040301.

[20] A.S. Strekalovsky, Elements of Nonconvex Optimization, Nauka, Novosibirsk, 2003. [In Russian].

[21] Y.R. Sturz, L.M. Affolter, R.S. Smith, Parameter identification of the KUKA LBR iiwa robot including constraints on physical feasibility, IFAC-PapersOnLine 50 (2017) 6863-6868.

[22] N.N. Subbotina, E.A. Krupennikov, Weak* approximations for the solution of a dynamic reconstruction problem, Trudy Inst. Mat. i Mekh. UrO RAN, 27 (2021) 208–220. [In Russian]

[23] N.N. Subbotina, E.A. Krupennikov, Weak* Solution to a Dynamic Reconstruction Problem, Proc. Steklov Inst. Math. 315 (2021) 233-246.

[24] N.N. Subbotina, E.A. Krupennikov, Variational approach to construction of piecewise-constant control approximations in the problem of dynamic reconstruction, Proceedings in Mathematics & Statistics, Springer, 2022. [In print].

[25] N.N. Subbotina, E.A. Krupennikov, On Inverse Problems for Mechanical Systems, Materials of 20th International Conference "Aviation and Cosmonautics", Springer Nature, 2022. [In print].

[26] A.N. Tikhonov, Solution of incorrectly formulated problems and the regularization method, Soviet Math. 4 (1963) 1035-1038.

[27] V.I. Utkin, Sliding regimes in problems of optimisation and control, Nauka, Moscow, 1981. [In Russian].

[28] J. Warga, Optimal Control of Differential and Functional Equations, Elsevier, Amsterdam, 1972.