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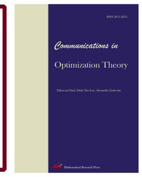

RECONSTRUCTION OF AN UNKNOWN DISTURBANCE IN A SYSTEM OF DIFFERENTIAL EQUATIONS BASED ON MEASUREMENTS OF PHASE STATES

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Dedicated to the memory of Professor Jack Warga on the occasion of his 100th birthday

Abstract. A system, which is nonlinear with respect to phase coordinates and linear with respect to disturbance, is considered. The problem of the dynamical reconstruction of unknown disturbances acting on the system is investigated. It is assumed that inaccurate measurements of the phase states are available at discrete times. An algorithm that is stable under informational noises and computational errors is designed. This algorithm is based on constructions of feedback control theory. The general construction is illustrated by two examples.

Keywords. Dynamical reconstruction; Differential equations; Disturbance; Feedback control theory; Measurements of phase states.

1. Introduction and Problem Statement

Let a dynamical system described by the nonlinear system of differential equations

$$\dot{x}(t) = f(t, y(t), x(t), u(t)) \tag{1.1}$$

with the initial condition

$$x(0) = x_0$$

operate on a time interval $T = [0, \vartheta]$, $0 < \vartheta < +\infty$, where, $y \in \mathbb{R}^n$, $x \in \mathbb{R}^N$, $u \in \mathbb{R}^r$, $f(t,y,x,u) = f_1(t,y,x) + Bu$, f_1 is a given function satisfying the Lipschitz condition with constant L, u is a disturbance, B is a constant matrix of corresponding dimension, and $y(\cdot)$ is a parametric function. The problem under consideration consists in the following. Some unknown disturbance $u(\cdot) \in L_2(T;\mathbb{R}^r)$ acts on system (1.1). At discrete, frequent enough, times

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^{m} \quad (\tau_0 = 0, \tau_m = \vartheta,\ \tau_{i+1} = \tau_i + \delta)$$

the phase states $x(\tau_i) = x(\tau_i; x_0, y(\cdot), u(\cdot)), i \in [0 : m-1]$ of system (1.1) are measured with an error. The results of these measurements, vectors $\xi_i^h \in \mathbb{R}^N$, satisfy the inequalities

$$|x(\tau_i) - \xi_i^h|_N \leq h. \tag{1.2}$$

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It is assumed that the parametric function is known not exactly. Namely, instead of the function $y(\cdot)$, we know a function $\phi^\nu(\cdot) \in L_\infty(T;\mathbb{R}^n)$ with the property

$$|y(t) - \phi^\nu(t)|_n \le \nu \quad \text{for a.a.} \quad t \in T. \tag{1.3}$$

Here, $h \in (0,1)$ and $\nu = \nu(h)$ are the levels of the measurement accuracy, the symbol $|\cdot|_n$ stands for the Euclidean norm in the space $\mathbb{R}^n$. It is necessary to design an algorithm for the approximate reconstruction of the unknown disturbance on the basis of inaccurate measurements of $x(\tau_i)$. In other words, the task is as follows: given the current measurements of $x(\tau_i)$, to design a feedback algorithm that generates in real time a function $u^h = u^h(\cdot)$ that approximates the unknown disturbance (in the $L_2(T;\mathbb{R}^r)$-metric) generating the solution $x(\cdot)$ of system (1.1).

The problem described above belongs to the class of dynamical inverse problems. There are a lot of monographs and papers devoted to reconstruction problems, including problems for dynamical systems (see, for example [1, 4, 16]). One of the approaches to solving dynamical reconstruction problems was developed in [2, 5, 8, 9, 10, 11, 12, 13, 14]. This approach is based on a combination of the methods of feedback control theory [3] (for example, the method of extremal shift) and the methods of ill-posed problems [15]. In the case when the disturbance $u(\cdot)$ is subject to a priori constraints, the problem in question can be solved by means of constructions from [5, 12]. In the present paper, we consider the case when instantaneous constraints on the disturbance are absent. Accordingly, $u(\cdot)$ is assumed to be a square integrable function. Other dynamical reconstruction problems with solution algorithms based on suitable modifications of the extremal shift method were discussed, for example, in [2, 5, 8, 9, 10, 11, 12, 13, 14]. More specifically, systems of ordinary differential equations were considered in [2, 5, 8, 10]; systems with memory in [11]; stochastic differential equations in [14], and systems with distributed parameters in [9, 13].

2. Method for Solving the Problem

Let us proceed to describe the method for solving the problem under consideration. As mentioned above, it is based on the constructions of feedback control theory. Namely, a dynamical reconstruction problem is replaced by a problem of feedback control for an appropriate dynamical system.

First, we consider the case of discrete measurements of the states. For any $h \in (0,1)$, let us fix a family Δ_h of partitions of the interval T by times $\tau_{h,i}$:

$$\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}, \quad \tau_{h,0} = 0, \quad \tau_{h,m_h} = \vartheta, \quad \tau_{h,i+1} = \tau_{h,i} + \delta(h), \quad \delta(h) \in (0,1). \tag{2.1}$$

Along with system (1.1), we consider the auxiliary system

$$\dot{w}^h(t) = f_1(\tau_i, \phi^\nu(t), \xi_i^h) + Bu_i^h \quad \text{for a.a.} \quad t \in [\tau_i, \tau_{i+1}) \quad (\tau_i = \tau_{h,i}, \quad i \in [0 : m_h - 1]) \tag{2.2}$$

with the initial state $w^h(0) = \xi_0^h$. The law $U(\cdot,\cdot,\cdot) : T \times \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^r$ of forming the control u_i^h is constructed in such a way that, for concordant parameters h and δ, the piece-wise functions

$$u^h(t) = u_i^h = U(\tau_i, \xi_i^h, w^h(\tau_i)) \quad \text{for a.a.} \quad \mathrm{t} \in [\tau_i, \tau_{i+1}) \quad (i \in [0 : m_h - 1]) \tag{2.3}$$

approximate the unknown disturbance generating $x(\cdot)$.

In the case of continuous measurements of the states, the auxiliary system has the form

$$\dot{w}_1^h(t) = f_1(t, \phi^\nu(t), \xi^h(t)) + Bu^h(t), \quad t \in T, \quad w_1^h(0) = \xi^h(0), \tag{2.4}$$

and the control $u^h(\cdot)$ is defined by the rule

$$u^h(t) = U(\xi^h(t), w_1^h(t)), \quad t \in T. \tag{2.5}$$

It should be noted that the same solution of system (1.1) can be produced by multiple disturbances. Let $\mathbb{U}(x(\cdot), y(\cdot))$ be the set of all functions from $L_2(T;\mathbb{R}^r)$ generating the solution $x(\cdot)$ of system (1.1), i.e.

$$\mathbb{U}(x(\cdot), y(\cdot)) = \{\tilde{u}(\cdot) \in L_2(T;\mathbb{R}^r) : \dot{x}(t) - f_1(t, y(t), x(t)) = B\tilde{u}(t) \quad \text{for a.a.} \quad t \in T\}.$$

Let $u_*(\cdot)$ be an element of the set $\mathbb{U}(x(\cdot), y(\cdot))$ of minimal $L_2(T;\mathbb{R}^r)$-norm, i.e.

$$u_*(\cdot) = \arg \min_{u(\cdot) \in \mathbb{U}(x(\cdot), y(\cdot))} |u(\cdot)|_{L_2(T;\mathbb{R}^r)}.$$

Note that the set $\mathbb{U}(x(\cdot), y(\cdot))$ is convex and closed in $L_2(T;\mathbb{R}^r)$. Therefore, the element $u_*(\cdot)$ is unique. According to the approach conventional in the theory of ill-posed problems which is we reconstruct $u_*(\cdot)$.

In what follows, the constants c_j, C_j, k_j and $k^{(j)}$ used in the paper depend on the structure of system (1.1) and do not depend on the parameters h, α, δ and ν.

3. Solving Algorithm in the Case of Continuous Measurements of the States

Consider the case of continuous measurements of $x(\cdot)$. We fix some function $\alpha = \alpha(h)$: $(0,1) \to (0,1)$.

Before starting the work of the algorithm, we fix values $h \in (0,1)$ and $\alpha = \alpha(h)$. Define the control $u^h(\cdot)$ in system (2.4) according to rule (2.5), in which we set

$$U(\xi^h(t), w_1^h(t)) = -\alpha^{-1} B'(w_1^h(t) - \xi^h(t)). \tag{3.1}$$

Here, the prime denotes transposition. As the input of system (2.4), we take the control $u^h(\cdot)$ of form (2.5), (3.1) for all $t \in T$.

Let

$$\varepsilon(t) = 0,5|w_1^h(t) - y(t)|_N^2.$$

Lemma 3.1. *Let $\alpha(h) \to 0$, $h\alpha^{-2}(h) \to 0$ and $\nu(h)\alpha^{-1}(h) \to 0$ as $h \to 0$. Then there exists a number $h_0 \in (0,1)$ such that the inequalities*

$$\varepsilon(t) \le c^{(0)}(\alpha(h) + \nu(h) + h\alpha^{-1}(h)), \tag{3.2}$$

$$\int_0^{\vartheta} |u^h(s)|_r^2 ds \le (1_h \alpha^{-1}(h))(1 + c^{(1)} h\alpha^{-1}(h)) \int_0^{\vartheta} |u_*(s)|_r^2 ds + c^{(2)}(h\alpha^{-2}(h) + \nu(h)\alpha^{-1}(h)) \tag{3.3}$$

hold for all $h \in (0, h_0), t \in T$.

Proof. Note that (see (2.5) and (3.1))

$$u^h(t) = \arg\min\{2(w_1^h(t) - \xi^h(t), Bu) + \alpha|u|_r^2 : u \in \mathbb{R}^r\}.$$

Differentiating the function $\varepsilon(t)$, we get

$$2\dot{\varepsilon}(t) + \alpha\{|u^h(t)|_r^2 - |u_*(t)|_r^2\} = \tag{3.4}$$

$$2(w_1^h(t) - x(t), f_1(t, \phi^\nu(t), \xi^h(t)) - f_1(t, y(t), x(t)) + 2B(u^h(t) - u_*(t)) + \alpha\{|u^h(t)|_r^2 - |u_*(t)|_r^3\}.$$

In turn, in virtue of (4.25), (2.5) and (3.1), we obtain the inequality

$$|u^h(t)|_r \le b_* h\alpha^{-1} + b_*\alpha^{-1}(2\varepsilon(t))^{1/2}. \tag{3.5}$$

Here, $\alpha = \alpha(h)$ and b_* is the Euclidean norm of the matrix B'. Let

$$\gamma(t) = 2\varepsilon(t) + \alpha\int_0^t |u^h(s)|_r^2 ds.$$

From (4.25) and (1.3), we derive

$$2(w_1^h(t)-x(t), f_1(t,\phi^\nu(t),\xi^h(t)) - f_1(t,y(t),x(t))) \le 2L\{|\phi^\nu(t)-y(t)|_n + \tag{3.6}$$
$$|x(t)-\xi^h(t)|_N\} \le 2L\{h+\nu\}|w_1^h(t)-x(t)|_N = 2L(h+\nu)(2\varepsilon(t))^{1/2}.$$

Then, using (4.25) and (3.5), we have

$$2(w_1^h(t)-x(t), B(u^h(t)-u_*(t))) + \alpha\{|u^h(t)|_r^2 - |u_*(t)|_r^2\} \le \tag{3.7}$$
$$2(w_1^h(t)-\xi^h(t), B(u^h(t)-u_*(t))) + \alpha\{|u^h(t)|_r^2 - |u_*(t)|_r\} + 2hb_*\{|u^h(t)|_r + |u_*(t)|_r\} \le$$
$$2hb_*\{|u^h(t)|_r + |u_*(t)|_r\} \le 2hb_*|u_*(t)|_r + 2b_*^2h^2\alpha^{-1} + 2hb_*^2\alpha^{-1}(2\varepsilon(t))^{1/2}.$$

In this case, from (3.6) and (3.7), we obtain the estimate

$$\dot\gamma(t) - \alpha|u_*(t)|_r^2 \le 2h(2\varepsilon(t))^{1/2} + 2b_*h|u_*(t)|_r + 2b_*^2h^2\alpha^{-1} + \tag{3.8}$$
$$2b_*^2h\alpha^{-1}(2\varepsilon(t))^{1/2} + 2L(h+\nu)|u_*(t)|_r(2\varepsilon(t))^{1/2}.$$

Using the inequality $ab \le 0,5(a^2+b^2)$, we derive the inequalities

$$2(h+\nu)L(2\varepsilon(t))^{1/2} \le 2(h+\nu)L^2 + (h+\nu)\varepsilon(t), \tag{3.9}$$
$$2b_*^2h\alpha^{-1}(2\varepsilon(t))^{1/2} \le 2b_*^4h\alpha^{-1} + h\alpha^{-1}\varepsilon(t).$$

Inequalities (3.8) and (3.9) imply

$$\dot\gamma(t) \le \alpha|u_*(t)|_r^2 + 2b_*h|u_*(t)|_r + 2b_*^2h^2\alpha^{-1} + (h+\nu+h\alpha^{-1})\varepsilon(t) + 2h(L^2 + b_*^4\alpha^{-1}) + 2\nu L^2. \tag{3.10}$$

Using the inclusion $\alpha(h) \in (0,1)$, and the inequality $2\varepsilon(t) \le \gamma(t)$, we derive from (3.10) the inequality

$$\dot\gamma(t) \le \alpha|u_*(t)|_r^2 + 2b_*h|u_*(t)|_r + c_1(\nu + h\alpha^{-1}) + c_2(\nu + h\alpha^{-1})\gamma(t) + 2\nu L^2. \tag{3.11}$$

Applying the Gronwall lemma and inequality (3.11), we obtain

$$\gamma(t) \le \{c_1\vartheta(\nu + h\alpha^{-1}) + \alpha\int_0^\vartheta |u_*(s)|_r^2 ds + 2\nu\vartheta L^2 + \tag{3.12}$$
$$2b_*h\int_0^\vartheta |u_*(s)|_r ds\}\exp\{c_2t(\nu + h\alpha^{-1})\} \le c_3(\alpha + \nu + h\alpha^{-1})\exp\{c_2t(\nu + h\alpha^{-1})\}.$$

Inequality (3.2) follows from inequality (3.12). Let us prove inequality (3.3). In virtue of the convergence $h\alpha^{-2}(h) \to 0$ as $h \to 0$, there exists a number $h_0 \in (0,1)$ such that the inequalities

$$\exp\{c_2h\vartheta\alpha^{-1}(h)\} \le 1 + c_4h\alpha^{-1}(h), \quad h\alpha^{-1}(h) \le \alpha(h) \le 1 \tag{3.13}$$

hold for all $h \in (0,h_0)$. Note that

$$\int_0^{\vartheta} |u_*(s)|_r ds \le c_5. \tag{3.14}$$

Hence, taking into account (3.13), (3.14), and (3.12), we have

$$\alpha \int_0^t |u^h(s)|_r^2 ds \le \{c_6(h+\nu)\alpha^{-1} + (\alpha+h)\int_0^t |u_*(s)|_r^2 ds\}(1+c_4 h\alpha^{-1}) \le \tag{3.15}$$

$$(h+\alpha(h))(1+c_4 h\alpha^{-1}(h))\int_0^t |u_*(s)|_r^2 ds + c_7(h\alpha^{-1}(h)+\nu).$$

Inequality (3.3) follows from inequality (3.15). The lemma is proved. □

From Lemma 3.1 and Theorem 1.2.1. [5, p. 23], we obtain

Theorem 3.2. *Let the conditions of Lemma 3.1 be fulfilled. Then the convergence $u^h(\cdot) \to u_*(\cdot)$ in $L_2(T;\mathbb{R}^r)$ as $h \to 0$ takes place.*

4. Solving Algorithm in the Case of Discrete Measurements of the States

Here we describe an algorithm for solving the problem in the case of discrete measurements of $x(\cdot)$. Consider family Δ_h (2.1) and the function $\alpha(h) : (0,1) \to (0,1)$.

Before starting the process, we fix a value $h \in (0,1)$, numbers $\nu = \nu(h)$ and $\alpha = \alpha(h)$ and partition $\Delta_h = \{\tau_{h,i}\}_{i=0}^{m_h}$ (2.1). The work of the algorithm is decomposed into $m-1$ $(m = m_h)$ identical steps. At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{h,i}$, the following actions are fulfilled. First, at the time τ_i, the vector u_i^h is calculated by formula (2.3), in which

$$U(\tau_i, \xi_i^h, w^h(\tau_i)) = -\alpha^{-1}B'(w^h(\tau_i) - \xi_i^h). \tag{4.1}$$

Then, for all $t \in \delta_i$, the control $u^h(t)$ of form (2.3), (4.1) is taken as the input of system (2.2). As a result, under the action of such control, system (2.2) passes from the state $w^h(\tau_i)$ to the state $w^h(\tau_{i+1})$. The work of the algorithm stops at time ϑ.

It turns out that, under appropriate relations between h, $\delta(h)$, $\nu(h)$, $\alpha(h)$, the function $u^h(\cdot)$ approximates $u_*(\cdot)$. Before proceeding to the proof of this fact, we present two auxiliary lemmas used below.

Remark 4.1. In the case when the right-hand part of system (1.1) does not depend on the parametric function $y(\cdot)$, i.e. $f = f_1(t,x) + Bu$, an algorithm for solving of the problem under discussion is given in [10]. In this paper, we consider the auxiliary system

$$\dot{w}^h(t) = f_1(\tau_i, \xi_i^h) + Bu_i^h + v_i^h, \quad t \in \delta_i.$$

The controls u_i^h and v_i^h are calculated by the formulas

$$u_i^h = \alpha(h)^{-1}B'(\xi_i^h - w^h(\tau_i)),$$

$$v_i^h = c\delta(h)\alpha^{-2}(h)(\xi_i^h - w^h(\tau_i)),$$

where c is a positive constant. It was proved in [10] that, under appropriate relations between parameters $h, \alpha(h)$ and $\delta(h)$, the convergence of $u^h(\cdot)$ to $u_*(\cdot)$ in $L_2(T;\mathbb{R}^r)$ takes place. In the proof of this convergence, the function $v^h(\cdot)$ was important. In this paper, in contradiction with [10], first, we consider a system with a right-hand part f depending on some parametric function. Second, we prove that it is possible to set $v^h(t) = v_i^h = 0$ for a.a. $t \in [\tau_i, \tau_{i+1})$, $i \in [0 : m_h - 1]$ in the auxiliary system.

Lemma 4.2. [5, p. 47] *Let* $x_1(\cdot) \in L_\infty(T_*;\mathbb{R}^n)$ *and* $y_1(\cdot) \in W(T_*;\mathbb{R}^n)$*,* $T_* = [a,b]$*,* $-\infty < a < b < +\infty$*, be such that*

$$\Big|\int_a^t x_1(\tau)\,d\tau\Big|_n \le \varepsilon, \quad |y_1(t)|_n \le K \quad \forall\, t \in T_*.$$

Then, for all $t \in T_*$*, the inequality*

$$\Big|\int_a^t (x_1(\tau), y_1(\tau))\,d\tau\Big| \le \varepsilon(K + \mathrm{var}(T_*; y_1(\cdot)))$$

is valid.

Lemma 4.3. [6] *Let a nonnegative function* $\phi(t)$*,* $t \in T$*, satisfy the inequalities*

$$\phi(\tau_{i+1}) \le \phi(\tau_i)(1 + p\delta) + \int_{\tau_i}^{\tau_{i+1}} |G(\tau)|\,d\tau$$

for all $i \in [0 : m-1]$*, where* $\tau_i \in \Delta$*,* $\delta = \tau_{i+1} - \tau_i$*,* $p = \mathrm{const} > 0$*,* $G(\cdot) \in L_\infty(T;\mathbb{R})$*. Then, the inequalities*

$$\phi(\tau_i) \le \Big(\phi(0) + \int_0^{\tau_i} |G(\tau)|\,d\tau\Big)\exp(p\tau_i), \quad i \in [0 : m],$$

take place.

Here, the symbol $\mathrm{var}(T_*; y_1(\cdot))$ means the variation of the function $y_1(\cdot)$ over the interval T_*, the symbol $(\cdot,\cdot)$ means the scalar product in the corresponding finite-dimensional Euclidean space,the symbol $|\cdot|$ means the modulo of a number, and the symbol $W(T_*;\mathbb{R}^n)$ means the set of functions $z(\cdot) : T_* \to \mathbb{R}^n$ of bounded variation.

Lemma 4.4. *Let* $\alpha(h) \to 0$ *and* $\delta(h)\alpha^{-2}(h) \to 0$ *as* $h \to 0$*. Then there exists a number* $h_1 \in (0,1)$ *such that the inequalities*

$$\varepsilon_*(t) \le C_1\rho_1(\alpha,\delta,h,\nu), \tag{4.2}$$

$$\int_0^{\vartheta} |u^h(\tau)|_r^2\,d\tau \le (1 + C_2\alpha\delta^{-2})\int_0^{\vartheta} |u(\tau)|_r^2\,d\tau + C_3\rho(\alpha,\delta,h,\nu)\alpha^{-1} \tag{4.3}$$

hold for all $h \in (0,h_1), t \in T$*. Here,* $\alpha = \alpha(h), \delta = \delta(h), \nu = \nu(h)$*,*

$$\varepsilon_*(t) = 0,5|w^h(t) - x(t)|_N^2, \quad \rho_1(\alpha,\delta,h,\nu) = \rho(\alpha,\delta,h,\nu) + \alpha + \delta + \delta^2\nu^2,$$

$$\rho(\alpha,\delta,h,\nu) = \alpha^2\delta + h^2\delta^{-1} + h + \alpha^2\nu^2\delta^{-1}.$$

Proof. Estimate the change of the function $\varepsilon_*(t)$, $t \in T$. It is easily seen that the equality

$$\varepsilon_*(t) = 0,5|w^h(\tau_i) - x(\tau_i) + \int_{\tau_i}^{t} \{f^i(\tau) + B^i(\tau)\}\, d\tau|_N^2$$

is true for $t \in \delta_i = [\tau_i, \tau_{i+1}), i \in [0 : m-1]$. Here, $m = m_h, \tau_i = \tau_{h,i}$,

$$f^i(t) = f_1(\tau_i, \phi^\nu(t), \xi_i^h) - f_1(t, y(t), x(t)), \quad B^i(t) = B(u_i^h - u(t)) \quad \text{for a.a.} \quad \mathrm{t} \in \delta_i.$$

In this case, the equality

$$\varepsilon_*(t) = \varepsilon_*(\tau_i) + \sum_{j=1}^{5} \nu_i^{(j)}(t)$$

takes place for all $t \in \delta_i$. Here,

$$\nu_i^{(1)}(t) = (w^h(\tau_i) - x(\tau_i), \int_{\tau_i}^{t} f^i(\tau)\, d\tau), \;\; \nu_i^{(2)}(t) = 0,5\Big|\int_{\tau_i}^{t} f^i(\tau)\, d\tau\Big|_N^2,$$

$$\nu_i^{(3)}(t) = (w^h(\tau_i) - x(\tau_i), \int_{\tau_i}^{t} B^i(\tau)\, d\tau),$$

$$\nu_i^{(4)}(t) = \Big(\int_{\tau_i}^{t} f^i(\tau)\, d\tau, \int_{\tau_i}^{t} B^i(\tau)\, d\tau\Big), \;\; \nu_i^{(5)}(t) = 0,5\Big|\int_{\tau_i}^{t} B^i(\tau)\, d\tau\Big|_N^2, \quad t \in \delta_i.$$

Throughout the proof of this lemma, $\alpha = \alpha(h)$, $\nu = \nu(h)$, $\delta = \delta(h)$. Note that, for all i (see (4.1)), the inequalities

$$|u_i^h|_r = \Big|\frac{B'(\xi_i^h - w^h(\tau_i))}{\alpha}\Big|_r \le \frac{b_*}{\alpha} Q_i \tag{4.4}$$

are true, where $Q_i = h + (2\varepsilon_*(\tau_i))^{1/2}$, b_* is the Euclidean norm of the matrix B^i. For all $t \in [\tau_i, \tau_{i+1}]$, we derive the estimate

$$|f^i(t)|_N \le L\{\delta + h + \nu + |y(t) - y(\tau_i)|_N\} \le L Q_t^{(i)}, \tag{4.5}$$

$$Q_t^{(i)} = \delta + h + \nu + \int_{\tau_i}^{t} |\dot{y}(\tau)|_N\, d\tau.$$

In turn, in virtue of (4.5), we obtain the inequality

$$\nu_i^{(1)}(t) \le L(2\varepsilon_*(\tau_i))^{1/2} \delta Q_t^{(i)} \le \frac{\delta^2}{4\alpha^2}\varepsilon_*(\tau_i) + k_1 \alpha^2 (Q_t^{(i)})^2, \;\; t \in \delta_i. \tag{4.6}$$

Denote $U_{ip}(t) = \int_{\tau_i}^{t} |u(\tau)|_r^p\, d\tau, \quad p = 1, 2$. We have

$$\nu_i^{(2)}(t) \le k_2 \delta^2 (Q_t^{(i)})^2, \quad t \in \delta_i. \tag{4.7}$$

In addition, one can show that the inequality (see (4.4))

$$|\int_{\tau_i}^{t} B^i(\tau)\, d\tau|_N \le b_*\{U_{i1}(t) + \delta |u_i^h|_N\} \le b_*\{b_* \delta \alpha^{-1} Q_i + U_{i1}(t)\}, \quad \mathrm{t} \in \delta_i \tag{4.8}$$

holds. In virtue of (4.4) and (4.8), we deduce that

$$\nu_i^{(3)}(t) \leq \int_{\tau_i}^{t} (w^h(\tau_i) - \xi_i^h, B^i(\tau))\,d\tau + hb_*\{b_*\delta\alpha^{-1}(h + (2\varepsilon_*(\tau_i))^{1/2}) + U_{i1}(t)\} \leq \quad (4.9)$$

$$\int_{\tau_i}^{t} (w^h(\tau_i) - \xi_i^h, B^i(\tau))\,d\tau + k_3\{(1 + \frac{\delta}{\alpha})h^2 + hU_{i1}(t)\} + \frac{\delta^2}{4\alpha^2}\varepsilon_*(\tau_i).$$

It is easily seen that the inequalities

$$\nu_i^{(4)}(t) \leq k_4\{\frac{\delta^2}{\alpha}Q_t^{(i)}(h + \varepsilon_*^{1/2}(\tau_i)) + \delta Q_t^{(i)}U_{i1}(t)\} \leq \quad (4.10)$$

$$\frac{\delta^2}{4\alpha^2}\varepsilon_*(\tau_i) + k_5\{\delta^2(Q_t^{(i)})^2 + \frac{\delta^2}{\alpha^2}h^2 + \delta U_{i2}(t)\},$$

$$\nu_i^{(5)}(t) \leq 0,5b_*^2\left(U_{i1}(t) + \delta b_*\frac{Q_i}{\alpha}\right)^2 \leq 4b_*^4\frac{\delta^2}{\alpha^2}\varepsilon_*(\tau_i) + k_6\{h^2\frac{\delta^2}{\alpha^2} + \delta U_{i2}(t)\} \quad (4.11)$$

hold for $t \in [\tau_i, \tau_{i+1}]$. Note that $\delta(h)\alpha^{-2}(h) \to 0$ as $h \to 0$. Therefore, by using relations (4.6), (4.7), (4.9)–(4.11), the inequality

$$\varepsilon_*(t) \leq \varepsilon_*(\tau_i) + (1 + 4b_*^4\frac{\delta^2}{\alpha^2})\varepsilon_*(\tau_i) + k_7\{\delta U_{i2}(t) + (\alpha^2 + \delta^2)(Q_t^{(i)})^2 + h^2 + hU_{i1}(t)\} \quad (4.12)$$

takes place for $t \in [\tau_i, \tau_{i+1})$. Then, we have

$$(\alpha^2 + \delta^2)\sum_{i=0}^{m-1}(Q_{\tau_{i+1}}^{(i)})^2 \leq k_8\alpha^2\sum_{i=0}^{m-1}\{h^2 + \delta^2 + \nu^2 + \delta Q_{i,\tau_{i+1}}\} \leq \quad (4.13)$$

$$k_9\{\alpha^2\delta + \alpha^2(h^2 + \nu^2)\delta^{-1}\},$$

where

$$Q_{i,t} = \int_{\tau_i}^{t} |\dot{x}(\tau)|_N^2\,d\tau, \quad t \in [\tau_i, \tau_{i+1}].$$

Let

$$\mu(t) = 2\varepsilon_*(t) + \alpha\int_0^t \{|u^h(\tau)|_r^2 - |u(\tau)|_r^2\}\,d\tau.$$

Due to (4.12), the rule for finding the control $u^h(\cdot)$ implies the inequality

$$\mu(t) \leq \mu(\tau_i) + 1(1 + 4b_*^4\frac{\delta^2}{\alpha^2})\varepsilon_*(\tau_i) + k_7\{\delta U_{i2}(t) + (\alpha^2 + \delta^2)(Q_t^{(i)})^2 + h^2 + hU_{i1}(t)\}. \quad (4.14)$$

Introduce

$$\gamma_*(t) = 2\varepsilon_*(t) + \alpha\int_0^t |u^h(\tau)|_r^2\,d\tau.$$

Using (4.14) and the inequality $(1 + \delta\alpha^{-1})\delta\alpha^{-1} \leq$ const for $t \in [\tau_i, \tau_{i+1}]$, we derive the estimate

$$\gamma_*(t) \leq \{1 + 4b_*^4\frac{\delta^2}{\alpha^2}\}\gamma_*(\tau_i) + (\alpha + k_{10}\delta)U_{i2}(t) + k_{11}\{(\alpha^2 + \delta^2)(Q_t^{(i)})^2 + h^2 + hU_{i1}(t)\}. \quad (4.15)$$

Hence, taking into account (4.13)), (4.15), and Lemma 4.3, we get

$$\gamma_*(\tau_{i+1}) \le [\gamma_*(0) + (\alpha + k_{10}\delta)U^{(i+1)} + k_{12}\rho]\exp\{4b_*^4\frac{\delta}{\alpha^2}\tau_{i+1}\}, \quad i \in [0:m-1]. \tag{4.16}$$

Here,

$$\rho = \rho(\alpha,\delta,h,\nu), \quad U^{(i+1)} = \int_0^{\tau_{i+1}} |u(\tau)|_r^2\,d\tau.$$

In virtue of the inequality $\gamma_*(0) \le h^2$ (see (1.2) as $i = 0$), we derive

$$\gamma_*(\tau_i) \le [k_{13}\rho + (\alpha + k_{10}\delta U^{(i)}]\exp\{\frac{4b_*^4\delta}{\alpha^2}\tau_i\}. \tag{4.17}$$

Note that $\delta(h)\alpha^{-2}(h) \to 0$ as $h \to 0$. Then there exists a number $h^* \in (0,1)$ such that the inequality

$$\exp\{4b_*^4\vartheta\delta\alpha^{-2}\} \le 1 + k_{14}\delta\alpha^{-2} \tag{4.18}$$

takes place for $h \in (0,h^*)$. In this case, from (4.18) and (4.17), we obtain the estimate

$$2\varepsilon_*(\tau_i) \le \gamma_*(\tau_i) \le k_{15}\rho + (\alpha + k_{10}\delta)(1 + k_{14}\delta\alpha^{-2})U^{(i)}. \tag{4.19}$$

This estimate is valid for all $h \in (0,h^*), i \in [0:m]$. In turn, using (4.19), we deduce that

$$\int_0^{\tau_i} |u^h(\tau)|_r^2\,d\tau \le (1 + k_{10}\delta\alpha^{-1})(1 + k_{14}\delta\alpha^{-2})U^{(i)} + k_{15}\rho\alpha^{-1} \le \tag{4.20}$$

$$(1 + k_{16}\frac{\delta}{\alpha^2})U^{(i)} + k_{15}\rho\alpha^{-1}, \quad \mathrm{i} \in [0:m], \quad h \in (0,h^*).$$

Inequality (4.3) follows from (4.20) for $i = m$. Let us prove inequality (4.2). Note that for all $t \in [\tau_i, \tau_{i+1}]$ the estimate

$$(2\varepsilon_*(t))^{1/2} \le (2\varepsilon_*(\tau_i))^{1/2} + I_{t,i} + |\int_{\tau_i}^{t} B\{u_i^h - u(\tau)\}\,d\tau|_N \tag{4.21}$$

is valid. Here (see (4.5)),

$$I_{t,i} = \int_{\tau_i}^{t} |f_1(\tau, y(\tau), x(\tau)) - f_1(\tau_i, \xi_i^h, \phi^\nu(\tau))|_n\,d\tau \le \delta L\{h + \delta + \nu + \int_{\tau_i}^{t} |\dot{x}(\tau)|_N\,d\tau\}. \tag{4.22}$$

Therefore, from (4.21), (4.22), and the inequality

$$\max_{i\in[0:m-1]} \int_{\tau_i}^{\tau_{i+1}} |Bu(t)|_N\,dt \le k_{17}\delta^{1/2},$$

we derive

$$(2\varepsilon_*(t))^{1/2} \le (2\varepsilon_*(\tau_i))^{1/2} + k_{18}\{\delta\nu + \delta|u_i^h|_r + \delta^{1/2}\}.$$

Consequently,

$$2\varepsilon_*(t) \le k_{19}\{\varepsilon_*(\tau_i) + \delta + \delta^2|u_i^h|_r^2 + \delta^2\nu^2\}, \quad t \in [\tau_i, \tau_{i+1}]. \tag{4.23}$$

Next, using (4.4), we get

$$\delta^2|u_i^h|_r^2 \le 2b_*^2\delta^2\alpha^{-2}(h^2+2\varepsilon_*(\tau_i)) \le k_{20}(h^2+\varepsilon_*(\tau_i)). \tag{4.24}$$

Then, in virtue of (4.23) and (4.24), we obtain

$$2\varepsilon_*(t) \le k_{21}\{\varepsilon_*(\tau_i)+\delta+\delta^2\nu^2\} \quad t\in[\tau_i,\tau_{i+1}]. \tag{4.25}$$

Hence, taking into account (4.25) and (4.19), we get

$$2\varepsilon_*(t) \le k_{22}[\rho+\delta+\delta^2\nu^2+(\alpha+k_{10}\delta)(1+k_{14}\delta\alpha^{-2})\int_0^t |u(\tau)|_r^2\,d\tau] \le k_{23}\rho_1(\alpha,\delta,h,\nu), \quad t\in T.$$

Inequality (4.2) follows from the latter inequality. The lemma is proved. □

From Lemma 4.4 and Theorem 1.2.1 [5, p. 23], we obtain

Theorem 4.5. *Let the conditions of Lemma 4.2 be fulfilled. Let also $\nu(h)\to 0$,*

$$\rho_1(\alpha(h),\delta(h),h,\nu(h))\to 0, \quad \rho(\alpha(h),\delta(h),h,\nu(h))\alpha^{-1}(h)\to 0 \quad \text{as} \quad h\to 0. \tag{4.26}$$

Then the convergence $u^h(\cdot)\to u_*(\cdot)$ in $L_2(T;\mathbb{R}^r)$ as $h\to 0$ takes place.

Remark 4.6. Conditions (4.26) take place if

$$\delta(h)\alpha^{-2}(h)\to 0,\ h^2(\delta(h)\alpha(h))^{-1}\to 0,\ h\alpha^{-1}(h)\to 0,\ \alpha(h)\nu^2(h)\delta^{-1}(h)\to 0 \text{ as } h\to 0.$$

5. The Convergence Rate of the Algorithm

Under some additional conditions, one can obtain the convergence rate of the algorithm (see Lemma 4.3).

Lemma 5.1. *Let $u(\cdot)$ be a function of bounded variation. Let also $N\ge r$, $\operatorname{rank} B = r$, and conditions of Lemma 4.2 hold. Then it is possible to find a positive constant C_4 such that the inequality*

$$\int_0^{\vartheta} |u^h(\tau)-u(\tau)|_r^2\,d\tau \le C_4\rho_0(\alpha,\delta,h,\nu)+C_3\rho(\alpha,\delta,h,\nu)\alpha^{-1}$$

takes place for all $h\in(0,h_1)$.

Here, the constant C_3 is from (4.3),

$$\rho_0(\alpha,\delta,h,\nu)=\alpha^{1/2}+\delta^{1/2}+h\delta^{-1/2}+\alpha\nu\delta^{-1/4}+\nu\delta.$$

Proof. Using the Lipschitz property of the function f_1, we conclude that the inequality

$$\Big|\int_{t_1}^{t_2} B\{u^h(t)-u(t)\}\,dt\Big|_N = \Big|\int_{t_1}^{t_2}[\dot w^h(\tau)-\dot x(\tau)-f_1(\tau,\xi^h(\tau),\phi^\nu(\tau))+f_1(\tau,y(\tau),x(\tau))]\,d\tau\Big|_N \le$$

$$|\mu_h(t_2)-\mu_h(t_1)|_N+k^{(1)}\{\int_{t_1}^{t_2}\{|\xi^h(\tau)-x(\tau)|_N+|\phi^\nu(\tau)-y(\tau)|_n\}\,d\tau+\delta\}\le$$

$$|\mu_h(t_2)-\mu_h(t_1)|_N+k^{(2)}\int_{t_1}^{t_2}\{|\mu_h(\tau)|_N+h+\delta+\nu\}\,d\tau$$

is fulfilled for any $t_1, t_2 \in T$, $t_1 < t_2$. Here,

$$\mu_h(t) = w^h(t) - y(t), \quad \xi^h(\tau) = \xi_i^h \quad \text{for a.a.} \quad \tau \in [\tau_i, \tau_{i+1}).$$

In addition, in virtue of Lemma 4.4 (see (4.2)), we have

$$|\mu_h(t)|_N = (2\varepsilon_*(t))^{1/2} \le C_1^{1/2} \rho_1^{1/2}(\alpha, \delta, h, \nu).$$

This inequality implies

$$\Big| \int_{t_1}^{t_2} \{u^h(t) - u(t)\}\, dt|_r \le k^{(3)} \Big| \int_{t_1}^{t_2} B\{u^h(t) - u(t)\}\, dt\Big|_N \le \tag{5.1}$$

$$k^{(4)}\{\rho_1^{1/2}(\alpha, \delta, h, \nu) + h + \delta + \nu\} \le k^{(5)} \rho_0(\alpha, \delta, h, \nu).$$

From Lemma 4.4 (see (4.3)), we derive

$$\int_0^\vartheta |u^h(\tau) - u(\tau)|_r^2 d\tau = \int_0^\vartheta |u^h(\tau)|_r^2 d\tau - 2\int_0^\vartheta (u^h(\tau), u(\tau)) d\tau + \int_0^\vartheta |u(\tau)|_r^2 d\tau \le$$

$$+ (2 + C_2 \alpha \delta^{-2}) \int_0^\vartheta |u(\tau)|_r^2 d\tau - 2\int_0^\vartheta (u^h(\tau), u(\tau)) d\tau + C_3 \rho(\alpha, \delta, h, \nu)\alpha^{-1} =$$

$$2\int_0^\vartheta (u(\tau) - u^h(\tau), u(\tau))\, d\tau + C_2 \alpha \delta^{-2} \int_0^\vartheta |u(\tau)|_r^2\, d\tau + C_3 \rho(\alpha, \delta, h, \nu)\alpha^{-1}, \quad t \in T.$$

In virtue of Lemma 4.2 and (5.1), we obtain

$$\sup_{t \in T} \Big| \int_0^t (u(\tau) - u^h(\tau), u(\tau))\, d\tau \Big| \le k^{(6)} \rho_0(\alpha, \delta, h, \nu).$$

Consequently, the inequality

$$\int_0^\vartheta |u^h(\tau) - u(\tau)|_r^2 d\tau \le 2k^{(6)} \rho_0(\alpha, \delta, h, \nu) + C_3 \rho(\alpha, \delta, h, \nu)\alpha^{-1} \tag{5.2}$$

takes place for all $h \in (0,1)$, $t \in T$. The statement of the lemma follows from this inequality. The lemma is proved. □

It is easily seen that the following lemma takes place.

Lemma 5.2. *Let the conditions of Lemma 4.4 be fulfilled. Let also $\chi \in (0, 1/2)$, $\delta(h) = h$, $\alpha(h) = h^{1/2-\chi}(h)$, $\nu(h) = h^{1/2}$. Then there exists a number $h_2 \in (0, h_1)$ such that the inequalities*

$$\rho(\alpha(h), \delta(h), h, \nu(h))\alpha^{-1}(h) \le C_5 h^{1/2-\chi},$$

$$\rho_1(\alpha(h), \delta(h), h, \nu(h)) \le C_6 h^{1/2-\chi}, \quad \rho_0(\alpha(h), \delta(h), h, \nu(h)) \le C_7 h^{1/4-\chi/2}$$

hold for all $h \in (0, h_2)$.

Lemmas 5.1 and 5.2 imply

Corollary 5.3. *Let the conditions of Lemma 5.2 be fulfilled. Then the inequality*

$$\int_0^{\vartheta} |u^h(t) - u(t)|_r^2 dt \le C_8 h^{1/4-\chi/2}$$

takes place.

6. Examples

Example 6.1. Let $y(t) = \dot{z}(t)$ in system (1.1), where a function $z(\cdot)$ is unknown. But some properties of this function are known: $\dot{z}(0) = 0$, $\ddot{z}(\cdot) \in L_2(T;\mathbb{R}^n)$. At times τ_i, the values of this function are measured with an error h, i.e. vectors $\psi_i^h \in \mathbb{R}^n$,

$$|\psi_i^h - z(\tau_i)|_n \le h$$

are calculated. The problem consists in reconstructing a disturbance, which generates the solution $x(\cdot)$ of system (1.1), on the base of measurements of $z(\tau_i)$ and $x(\tau_i)$.

To solve this problem, it is possible to apply the algorithm described in Section 5. In this case, we reconstruct a function $u_*(\cdot)$, an element of the set

$$\mathbb{U}(x(\cdot),z(\cdot)) = \{u(\cdot) \in L_2(T;\mathbb{R}^r) : \dot{x}(t) = f_1(t,\dot{z}(t),x(t)) + Bu(t) \quad \text{for a.a.} \quad t \in T\}.$$

with minimal $L_2(T;\mathbb{R}^r)$-norm.

We fix two functions $\alpha = \alpha(h) : (0,1) \to (0,1)$ and $\alpha_1 = \alpha_1(h) : (0,1) \to (0,1)$. Together with model (2.2), we introduce an additional auxiliary system of the form

$$\dot{w}_1^h(t) = u_1^h(t), \quad t \in T.$$

The initial state of this system is $w_1^h(0) = 0$. The control $u_1^h(\cdot)$ is calculated by the rule

$$u_1^h(t) = u_i^h = -\frac{w_1^h(\tau_i) - \psi_i^h}{\alpha_1(h)}, \quad t \in [\tau_i, \tau_{i+1}).$$

From [7] (see Theorem 5) we derive the inequality

$$\sup_{t\in T} |u_1^h(t) - \dot{z}(t)|_n \le \nu(h), \tag{6.1}$$

where

$$\nu(h) = C_9\{\alpha_1(h) + (h + \delta(h))\alpha_1^{-1}(h)\}.$$

In this case, it is possible to assume

$$\phi^\nu(t) = u_1^h(t), \quad t \in T.$$

Let

$$\alpha(h) \to 0, \quad \alpha_1(h) \to 0, \quad h^2(\delta(h)\alpha(h))^{-1} \to 0, \quad \alpha(h)\alpha_1^2(h)\delta^{-1}(h) \to 0, \tag{6.2}$$
$$\delta(h)\alpha^{-2}(h) \to 0, \quad h\alpha^{-1}(h) \to 0, \quad \alpha(h)(h^2 + \delta^2(h))(\alpha_1(h)\delta(h))^{-1} \to 0 \quad \text{as} \quad h \to 0.$$

Then, in virtue of Theorem 4.5, the convergence $u^h(\cdot) \to u_*(\cdot)$ in $L_2(T;\mathbb{R}^r)$ as $h \to 0$ takes place. Relations (6.2) are fulfilled if, for example,

$$\delta(h) = C_{10}h, \quad \alpha(h) = C_{11}h^\mu, \quad (\mu = \text{const} \in (1/2,1)) \quad \alpha_1(h) = C_{12}h^{1/4}.$$

In this case,

$$\nu^2(h) \le C_{13}h^{1/2}, \quad \alpha(h)\nu^2(h)\delta^{-1}(h) \le C_{14}h^{\mu-1/2}.$$

Note that the function $z(\cdot)$ is measured at discrete times. But, the role of function $\phi^\nu(\cdot)$ is played by the function $u_1^h(\cdot)$ defined for all $t \in T$ by means of the vectors ψ_i^h.

Example 6.2. Consider the system with time delay

$$\dot{x}(t) = f_1(t, x(t-\tau), x(t)) + Bu(t), \quad t \in T = [0, \vartheta] \tag{6.3}$$

with the initial state

$$x(s) = x_0(s) \quad s \in [-\tau, 0].$$

The state of the system depends on a function $u(\cdot) \in L_2(T; \mathbb{R}^r)$. Here, $x \in \mathbb{R}^n$; $\vartheta \in (0, +\infty)$; $\tau = \text{const} \in (0, +\infty)$ is a time delay; B is constant $n \times r$-matrix; f_1 is a given $n \times n$-matrix satisfying the Lipschitz condition; $x_0(\cdot)$ is a given continuous function.

Let $u = u(\cdot)$ be a disturbance generating the solution of system (6.3) denoted by $x(\cdot)$. The problem is to identify in "real time" a priory unknown $u(\cdot)$ through results of inaccurate measurements of $x(\cdot)$ at discrete times $\tau_i \in \Delta = \{\tau_j\}_{j=0}^m$, $\tau_{j+1} = \tau_j + \delta$. These results are vectors $\xi_i^h \in \mathbb{R}^n$ such that

$$|x(\tau_i) - \xi_i^h|_n \le h.$$

To solve this problem, it is also possible to apply the algorithm described in Section 5. Indeed, let

$$y(t) = x(t - \tau).$$

It is easily seen that

$$|x(t) - \xi_i^h|_n \le C_{15}(h + \delta^{1/2}(h))$$

for $t \in [\tau_i, \tau_{i+1})$, $i \in [0 : m-1]$, $m = m_h$, $\tau_i = \tau_{h,i}$. We assume that the function $x_0(\cdot)$ is differentiable and $\dot{x}_0(\cdot) \in L_2([-\tau, 0]; \mathbb{R}^n)$. In additional, for simplicity, we assume that numbers $k_h = \tau/m_h$ are natural. Then, for all $s \in [-i\delta(h), -(i-1)\delta(h)), i \in [1 : k_h]$, the inequalities

$$|x_0(s) - x_0(-i\delta)|_n \le C_{16}\delta^{1/2}(h)$$

hold. If $\nu(h) = \max\{C_{15}, C_{16}\}(h + \delta^{1/2}(h))$ and

$$\phi^\nu(t) = x_0((i - k_h)\delta(h), \quad \text{if} \quad t \in [i\delta(h), (i+1)\delta(h)), i \in [0 : k_h - 1]$$

$$\xi_{i-k_h}^h, \quad \text{if} \quad t \in [i\delta(h), (i+1)\delta(h), i \in [k_h, m_h - 1],$$

then inequalities (1.3) are fulfilled, the auxiliary system has form (2.2), and the control $u^h(\cdot)$ is defined by rule (2.3), (4.1).

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