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ASYMPTOTIC ANALYSIS AND OPEN-LOOP SOLUTIONS OF ONE CLASS OF PARTIAL CHEAP CONTROL ZERO-SUM DIFFERENTIAL GAMES WITH STATE AND CONTROL DELAYS

VALERY Y. GLIZER

The Galilee Research Center for Applied Mathematics, ORT Braude College of Engineering, Karmiel, Israel
Independent Center for Studies in Control Theory and Applications, Haifa, Israel

Dedicated to the memory of Professor Jack Warga on the occasion of his 100th birthday

Abstract. A finite-horizon zero-sum linear-quadratic differential game with delays in the state variable and the players' control variables is considered. The feature of the game is that the cost of some (but, in general, not all) control coordinates of the minimizing player (the minimizer) in the cost functional is much smaller than the cost of the other control coordinates of this player, the cost of the control of the maximizing player (the maximizer) and the state cost. This smallness is expressed by a positive small multiplier (a small parameter) for the square of the weighted L^2-norm of the corresponding block of the minimizer's vector-valued control in the cost functional. By proper transformations, the originally formulated game is converted to an equivalent zero-sum differential game which does not contain delays any more. The parameter-free open-loop solvability condition of the new (undelayed) game is derived. An asymptotic analysis of the open-loop saddle point solution to this game (as the small parameter tends to zero) is carried out. This analysis yields the boundedness of the game's open-loop saddle point and the parameter-free open-loop quasi saddle point of this game. Along with the parameter-dependent undelayed game, another finite-horizon zero-sum linear-quadratic differential game (the degenerate game), obtained from the original one by replacing the small minimizer's control cost with zero, is considered. Its open-loop saddle point and value are derived. Relation between solutions of both games is established. Illustrative example is presented.

Keywords. Degenerate differential game; Finite-horizon zero-sum linear-quadratic differential game; Partial cheap control game; Open-loop saddle point solution.

1. INTRODUCTION

A cheap control problem is an extremal control problem in which a control cost of at least one decision maker is much smaller than a state cost in at least one cost functional of the problem. If the cost of only part of control coordinates is small, the extremal control problem is called a partial cheap control problem. Complete/partial cheap control problems appear in many topics

E-mail address: valgl120@gmail.com.
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of optimal control, H_∞ control and differential games theories. For example, such problems appear in the following topics: (1) solution of singular optimal control problems by regularization (see, e.g., [5, 9, 10, 11, 12, 13]); (2) solution of singular H_∞ control problems by regularization (see, e.g., [14, 15, 16]); (3) solution of singular differential games by regularization (see, e.g., [17, 18, 19, 20, 21, 37]); (4) limitation analysis for optimal regulators and filters (see, e.g., [7, 13, 22, 32, 36, 38]); (5) extremal control problems with high gain control in dynamics (see, e.g., [30, 44]); (6) inverse optimal control problems (see, e.g., [34]); (7) robust optimal control of systems with uncertainties/disturbances (see, e.g., [39, 40]); (8) guidance/interception problems (see, e.g., [38, 41, 42]).

The Hamilton boundary-value problem and the Hamilton–Jacobi–Bellman–Isaacs equation, associated with a complet/partial cheap control problem by solvability conditions, are singularly perturbed because of the smallness of the control cost.

In the present paper, we considered one class of partial cheap control finite-horizon zero-sum linear-quadratic differential games with delays in the state variable and the players' control variables.

Complete cheap control games with undelayed dynamics have been extensively investigated in the literature (see, e.g., [17, 18, 23, 24, 35, 37, 38, 39, 40]). Complete cheap control games with delayed dynamics have been investigated much less (see [19, 21]). In these two works, the case where the integrand in the cost functional contains a quadratic state cost was considered. The latter feature allows (subject to some additional condition on the state cost) to apply the boundary function method [43] to asymptotic analysis and solution of the corresponding singularly perturbed problem, appearing in the solvability conditions of the game. The optimal control of the player with a small (cheap) control cost in the cost functional has an impulse-like behaviour, meaning its unboundedness as the control cost tends to zero.

In the present paper, we considered another (than in [19, 21]) class of zero-sum linear-quadratic differential games. Namely, we consider the game with linear dynamics, having multiple point-wise and distributed delays in the state variable and controls of both players. The feature of the game is that the cost of some (but, in general, not all) control coordinates of the minimizing player (the minimizer) in the quadratic cost functional is much smaller than the cost of the other control coordinates of this player, the cost of the control of the maximizing player (the maximizer) and the state cost. Thus, the considered game is a partial cheap control game. Moreover, the integral part of the cost functional does not contain a quadratic state cost. This yields inapplicability of the boundary function method [43] and its generalization [19, 21] to asymptotic analysis and solution of the corresponding singularly perturbed problem, appearing in the solvability conditions of the game. In the present paper, another approach to the construction of asymptotic expansions for the open-loop saddle point and the value of the considered partial cheap control game is proposed. Based on the asymptotic expansion for the open-loop saddle point, its boundedness with respect to the small control cost is established.

Along with the partial cheap control game, in the present paper, we consider one more zero-sum linear-quadratic differential game. This game is obtained from the partial cheap control game by replacing the small cost of the corresponding minimizer's control coordinates with zero. This new game is called a degenerate game and is similar to the continuous/discrete time system obtained from a singularly perturbed system by replacing a small parameter of singular perturbation with zero. The open-loop saddle point and value of the degenerate game are derived. The relations between the open-loop saddle points and the values of the partial cheap control game and the degenerate game are established.

This paper is organised as follows. In Section 2, the problems of the paper (the partial cheap control differential game and the degenerate differential game with state and controls delays in dynamics) are rigorously formulated, main definitions are presented and the objectives of the paper are stated. In Section 3, a proper transformation of these games is carried out. Due to this transformation, the initially formulated games are converted to equivalent games (partial cheap control and degenerate ones) with undelayed dynamics. Section 4 presents the open-loop solvability condition of the partial cheap control game with undelayed dynamics. Asymptotic analysis of this game is done in Section 5. In Section 6, the open-loop saddle point solution and the value of the degenerate game with undelayed dynamics are derived. Illustrative example is solved in Section 7. Conclusions are presented in Section 8.

The following main notations and notions are used in the paper:

(1) For an $n\times m$-matrix A, $(n\geq 1,\ m\geq 1)$, its norm is defined as: $\|A\| \stackrel{\triangle}{=} \sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|$, where a_{ij}, $(i=1,...n;\ j=1,...,m)$ are the entries of A.
(2) The upper index $''T''$ denotes the transposition either of a vector x (x^T) or of a matrix A (A^T).
(3) I_n denotes the identity matrix of dimension n.
(4) $L^2[t_1,t_2;\mathbb{R}^n]$ denotes the linear space of all functions $x(\cdot):[t_1,t_2]\to\mathbb{R}^n$ square integrable in the interval $[t_1,t_2]$.
(5) $\mathrm{col}(x,y)$, where $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, denotes the column block-vector of the dimension $n+m$ with the upper block x and the lower block y, i.e., $\mathrm{col}(x,y)=(x^T,y^T)^T$.
(6) $\mathrm{diag}(A,B)$, where A and B are matrices of the dimensions $n\times n$ and $m\times m$, is a block-diagonal matrix with the upper left-hand block A and the lower right-hand block B.
(7) $O_{n_1\times n_2}$ is used for the zero matrix of the dimension $n_1\times n_2$, excepting the cases where the dimension of the zero matrix is obvious. In such cases, the notation 0 is used for the zero matrix.

2. Problem statement and main definitions

2.1. **Initial game formulation.** The following differential equation describes the dynamics of the game:

$$\begin{aligned}\frac{dz(t)}{dt}&=\sum_{i=0}^{N_z}\mathscr{A}_i(t)z(t-h_{z,i})+\int_{-h_z}^{0}\mathscr{G}(t,\tau)z(t+\tau)d\tau\\&+\sum_{j=0}^{N_u}\mathscr{B}_j(t)u(t-h_{u,j})+\int_{-h_u}^{0}\mathscr{P}(t,\eta)u(t+\eta)d\eta\\&+\sum_{k=0}^{N_v}\mathscr{C}_k(t)v(t-h_{v,k})+\int_{-h_v}^{0}\mathscr{Q}(t,\zeta)v(t+\zeta)d\zeta+f(t),\quad t\in[0,t_f],\end{aligned}\tag{2.1}$$

where $z(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^r$, $(r\leq n)$, $v(t)\in\mathbb{R}^s$, (u and v are players' controls); $0=h_{z,0}<h_{z,1}<...<h_{z,N_z}=h_z$ and $0=h_{u,0}<h_{u,1}<...<h_{u,N_u}=h_u$, $0=h_{v,0}<h_{v,1}<...<h_{v,N_u}=h_v$ are given constant time delays in the state and the players' controls, respectively; $t_f>0$ is a given time-instant; $\mathscr{A}_i(t)$, $(i = 0,1,...,N_z)$, $\mathscr{G}(t,\tau)$, $\mathscr{B}_j(t)$, $(j = 0,1,...,N_u)$, $\mathscr{P}(t,\eta)$, $\mathscr{C}_k(t)$, $(k = 0,1,...,N_v)$, and $\mathscr{Q}(t,\zeta)$ are given matrices of corresponding dimensions; $f(t) \in \mathbb{R}^n$ is a given vector.

The initial conditions for the equation (2.1) are

$$z(\tau) = \varphi_z(\tau), \quad \tau \in [-h_z,0); \quad z(0) = \varphi_{0,z}, \tag{2.2}$$

$$u(\eta) = \varphi_u(\eta), \quad \eta \in [-h_u,0), \tag{2.3}$$

$$v(\zeta) = \varphi_v(\zeta), \quad \zeta \in [-h_v,0), \tag{2.4}$$

where $\varphi_z(\tau) \in L^2[-h_z,0;\mathbb{R}^n]$, $\varphi_u(\eta) \in L^2[-h_u,0;\mathbb{R}^r]$, $\varphi_v(\zeta) \in L^2[-h_v,0;\mathbb{R}^s]$ and $\varphi_{0,z} \in \mathbb{R}^n$ are given functions and vector.

The cost functional, to be minimized by $u(t)$ (the minimizer) and maximized by $v(t)$ (the maximizer), is

$$\mathscr{J}_\varepsilon\big(u(t),v(t)\big) = \frac{1}{2}z^T(t_f)\mathscr{F}z(t_f) + \frac{1}{2}\int_0^{t_f}\big[u^T(t)R_u(t,\varepsilon)u(t) - v^T(t)R_v(t)v(t)\big]dt, \tag{2.5}$$

where the constant matrix $\mathscr{F}$ is symmetric and positive semi-definite; the time-dependent matrix $R_v(t)$ is symmetric and positive definite for all $t \in [0,t_f]$; $\varepsilon > 0$ is a small parameter. The matrix $R_u(t,\varepsilon)$ has the form

$$R_u(t,\varepsilon) = \mathrm{diag}\big(R_{u,1}(t)\ ,\ \varepsilon R_{u,2}(t)\big), \quad t \in [0,t_f], \tag{2.6}$$

the matrices $R_{u,1}(t)$ and $R_{u,2}(t)$ are of the dimensions $q \times q$, $(0 \le q < r)$, and $(r-q)\times(r-q)$, respectively. Moreover, for all $t \in [0,t_f]$, these matrices are symmetric and positive definite.

Remark 2.1. The form (2.6) of the matrix $R_u(t,\varepsilon)$ and the smallness of the parameter $\varepsilon > 0$ mean that the game (2.1)-(2.5) is a zero-sum differential game with a partial cheap control of a minimizer (or, briefly, a partial cheap control game).

In what follows, we assume:

A1. The matrix-valued functions $\mathscr{A}_i(t)$, $(i = 0,1,...,N_z)$, $\mathscr{B}_j(t)$, $(j = 0,1,...,N_u)$, $\mathscr{C}_k(t)$, $(k = 0,1,...,N_v)$, $R_{u,1}(t)$, $R_{u,2}(t)$, $R_v(t)$, and the vector-valued function $f(t)$ are continuous in the interval $[0,t_f]$.

A2. For any $t \in [0,t_f]$, the matrix-valued function $\mathscr{G}(t,\tau)$ is piece-wise continuous with respect to $\tau \in [-h_z,0]$, and $\mathscr{G}(t,\tau)$ is continuous with respect to $t \in [0,t_f]$ uniformly in $\tau \in [-h_z,0]$.

A3. For any $t \in [0,t_f]$, the matrix-valued function $\mathscr{P}(t,\eta)$ is piece-wise continuous with respect to $\eta \in [-h_u,0]$, and $\mathscr{P}(t,\eta)$ is continuous with respect to $t \in [0,t_f]$ uniformly in $\eta \in [-h_u,0]$.

A4. For any $t \in [0,t_f]$, the matrix-valued function $\mathscr{Q}(t,\zeta)$ is piece-wise continuous with respect to $\zeta \in [-h_v,0]$, and $\mathscr{Q}(t,\zeta)$ is continuous with respect to $t \in [0,t_f]$ uniformly in $\zeta \in [-h_v,0]$.

Remark 2.2. By virtue of the results of the work [8], for any given pair of players' controls $\big(u(t)\ ,\ v(t)\big)$, $\big(u(t) \in L^2[0,t_f;\mathbb{R}^r]\ ,\ v(t) \in L^2[0,t_f;\mathbb{R}^s]\big)$, the initial-value problem (2.1)-(2.4) has the unique absolutely continuous solution $z = z\big(t;u(t),v(t)\big)$, $t \in [0,t_f]$.

Based on the work [2], we present the following definition of the open-loop saddle point solution to the game (2.1)-(2.5).

Definition 2.3. For a given $\varepsilon > 0$, the pair $\big(u^*(t,\varepsilon)\ ,\ v^*(t,\varepsilon)\big)$, $\big(u^*(t,\varepsilon) \in L^2[0,t_f;\mathbb{R}^r]\ ,\ v^*(t,\varepsilon) \in L^2[0,t_f;\mathbb{R}^s]\big)$ is called an open-loop saddle point solution (or, briefly, an open-loop saddle point) of the game (2.1)-(2.5) if this pair satisfies the inequality

$$\begin{aligned}\mathscr{J}_\varepsilon\big(u^*(t,\varepsilon),v(t)\big) \le \mathscr{J}_\varepsilon\big(u^*(t,\varepsilon),v^*(t,\varepsilon)\big) \le \mathscr{J}_\varepsilon\big(u(t),v^*(t,\varepsilon)\big)\\ \forall\, u(t) \in L^2[0,t_f;\mathbb{R}^r],\ v(t) \in L^2[0,t_f;\mathbb{R}^s].\end{aligned} \tag{2.7}$$

If the open-loop saddle point $\big(u^*(t,\varepsilon)\ ,\ v^*(t,\varepsilon)\big)$ exists for the game (2.1)-(2.5), this game is called open-loop solvable. The value $\mathscr{J}_\varepsilon^* \stackrel{\triangle}{=} \mathscr{J}_\varepsilon\big(u^*(t,\varepsilon),v^*(t,\varepsilon)\big)$ is called a value of the game.

2.2. **Degenerate game.** Along with the partial cheap control game (2.1)-(2.5), we consider the game obtained from (2.1)-(2.5) by setting formally $\varepsilon = 0$ in its cost functional. Thus, the new game consists of the equation of dynamics (2.1), the initial conditions (2.2)-(2.4) and the cost functional

$$\mathscr{J}_0\big(u(t),v(t)\big) = \frac{1}{2}z^T(t_f)\mathscr{F}z(t_f) + \frac{1}{2}\int_0^{t_f}\big[u^T(t)R_u(t,0)u(t) - v^T(t)R_v(t)v(t)\big]dt, \tag{2.8}$$

where, due to (2.6),

$$R_u(t,0) = \mathrm{diag}\big(R_{u,1}(t)\ ,\ O_{(r-q)\times(r-q)}\big), \quad t \in [0,t_f]. \tag{2.9}$$

This cost functional is minimized by the control $u(t)$ and maximized by the control $v(t)$.

In what follows, we call the game (2.1)-(2.4), and (2.8) a degenerate game. In this game the weight matrix $R_u(t,0)$ for the minimizer's control cost is singular. Therefore, the game itself is singular, i.e., it can be solved neither using the Issacs MinMax principal, nor using the Bellman-Isaacs equation [2, 29].

Consider the pair of functions $\big(u_0^*(t)\ ,\ v_0^*(t)\big)$, where $u_0^*(t) \in L^2[0,t_f;\mathbb{R}^r]$, $v_0^*(t) \in L^2[0,t_f;\mathbb{R}^s]$.

Definition 2.4. The pair $\big(u_0^*(t)\ ,\ v_0^*(t)\big)$ is called an open-loop saddle point solution (or, briefly, an open-loop saddle point) of the game (2.1)-(2.4), and (2.8) if this pair satisfies the inequality

$$\begin{aligned}\mathscr{J}_0\big(u_0^*(t),v(t)\big) \le \mathscr{J}_0\big(u_0^*(t),v_0^*(t)\big) \le \mathscr{J}_0\big(u(t),v_0^*(t)\big),\\ \forall\, u(t) \in L^2[0,t_f;\mathbb{R}^r],\ v(t) \in L^2[0,t_f;\mathbb{R}^s].\end{aligned} \tag{2.10}$$

If the open-loop saddle point $\big(u_0^*(t)\ ,\ v_0^*(t)\big)$ exists for the game (2.1)-(2.4), and (2.8), this game is called open-loop solvable. The value $\mathscr{J}_0^* \stackrel{\triangle}{=} \mathscr{J}_0\big(u_0^*(t),v_0^*(t)\big)$ is called a value of the game.

2.3. **Objectives of the paper.** The objectives of the paper are the following:
(i) to construct and justify an asymptotic expansion with respect to $\varepsilon > 0$ of the saddle point solution to the partial cheap control game (2.1)-(2.5);
(ii) to construct and justify an asymptotic expansion with respect to $\varepsilon > 0$ of the game value for the game (2.1)-(2.5);
(iii) to establish the existence of the open-loop saddle point solution of the game (2.1)-(2.4), and (2.8), and to obtain this solution;
(iv) to establish the relation between the solution of the game (2.1)-(2.5) and the solution of the game (2.1)-(2.4), and (2.8) as $\varepsilon \to +0$.

3. Transformation of the games (2.1)-(2.5) and (2.1)-(2.4), (2.8)

By $\Psi(t)$ we denote the $n\times n$-matrix-valued solution to the problem

$$\frac{d\Psi(t)}{dt} = -\sum_{i=0}^{N_z}\Psi(t+h_{z,i})\mathscr{A}_i(t+h_{z,i})$$
$$-\int_{-h_z}^{0}\Psi(t-\tau)\mathscr{G}(t-\tau,\tau)d\tau,\quad t\in[0,t_f),$$
$$\Psi(t_f)=I_n,\qquad \Psi(t)=0,\quad t>t_f. \tag{3.1}$$

Also, let us denote

$$B_j(t+h_{u,j}) \triangleq \begin{cases}\Psi(t+h_{u,j})\mathscr{B}_j(t+h_{u,j}), & 0\le t\le t_f-h_{u,j},\\ 0, & t>t_f-h_{u,j},\end{cases}\quad j=0,1,\dots,N_u, \tag{3.2}$$

$$P(t-\eta,\eta) \triangleq \begin{cases}\Psi(t-\eta)\mathscr{P}(t-\eta,\eta), & -h_u\le\eta\le 0,\ 0\le t-\eta\le t_f,\\ 0, & -h_u\le\eta\le 0,\ t-\eta>t_f,\end{cases} \tag{3.3}$$

$$C_k(t+h_{v,k}) \triangleq \begin{cases}\Psi(t+h_{v,k})\mathscr{C}_k(t+h_{v,k}), & 0\le t\le t_f-h_{v,k},\\ 0, & t>t_f-h_{v,k},\end{cases}\quad k=0,1,\dots,N_v, \tag{3.4}$$

$$Q(t-\zeta,\zeta) \triangleq \begin{cases}\Psi(t-\zeta)\mathscr{Q}(t-\zeta,\zeta), & -h_v\le\zeta\le 0,\ 0\le t-\zeta\le t_f,\\ 0, & -h_v\le\zeta\le 0,\ t-\zeta>t_f,\end{cases} \tag{3.5}$$

$$\mathscr{K}_u(t) \triangleq \sum_{j=0}^{N_u}B_j(t+h_{u,j})+\int_{-h_u}^{0}P(t-\eta,\eta)d\eta,\qquad t\in[0,t_f], \tag{3.6}$$

$$\mathscr{K}_v(t) \triangleq \sum_{k=0}^{N_v}C_k(t+h_{v,k})+\int_{-h_v}^{0}Q(t-\zeta,\zeta)d\zeta,\qquad t\in[0,t_f]. \tag{3.7}$$

Based on the Halanay transformation for a linear system with state delays (see [28]), the transformation for a linear nonhomogeneous system (see [25, 26]) and the Kwon-Pearson-Artstein transformation for a linear system with control delays (see [1, 33]), let us make the following change of the state variable in the games (2.1)-(2.5) and (2.1)-(2.4), (2.8):

$$x(t)=w(t)+\sum_{j=1}^{N_u}\int_{t-h_{u,j}}^{t}B_j(\sigma+h_{u,j})u(\sigma)d\sigma+\int_{-h_u}^{0}d\eta\left(\int_{t+\eta}^{t}P(\sigma-\eta,\eta)u(\sigma)d\sigma\right)$$
$$+\sum_{k=1}^{N_v}\int_{t-h_{v,k}}^{t}C_k(\kappa+h_{v,k})u(\kappa)d\kappa+\int_{-h_v}^{0}d\zeta\left(\int_{t+\zeta}^{t}Q(\kappa-\zeta,\zeta)u(\kappa)d\kappa\right), \tag{3.8}$$

where

$$w(t)=\Psi(t)z(t)+\sum_{i=1}^{N_z}\int_{t}^{t+h_{z,i}}\Psi(s)\mathscr{A}_i(s)z(s-h_{z,i})ds$$
$$+\int_{t}^{t+h_z}\left(\int_{t}^{s}\Psi(\sigma)\mathscr{G}\Big(\sigma,s-\sigma-h_z\Big)d\sigma\right)z(s-h)ds+\int_{t}^{t_f}\Psi(\xi)f(\xi)d\xi,\quad t\in[0,t_f], \tag{3.9}$$

and $x(t)$ is a new state variable.

Using the results of [1, 25, 28] and the equations (3.2)-(3.7), we obtain the following assertion as a direct extension of the results of [22] to the case of a system controlled by two decision makers.

Proposition 3.1. *Let the assumptions A1-A4 be satisfied. Let for given $u(t) \in L^2[0,t_f;\mathbb{R}^r]$ and $v(t) \in L^2[0,t_f;\mathbb{R}^s]$, the absolutely continuous function $z(t)$, $t \in [0,t_f]$ be the solution of the initial-value problem (2.1)-(2.4). Then, the function $x(t)$, given by (3.8)-(3.9), is the unique absolutely continuous solution of the initial-value problem*

$$\frac{dx(t)}{dt} = \mathscr{K}_u(t)u(t) + \mathscr{K}_v(t)v(t), \quad t \in [0,t_f], \tag{3.10}$$

$$x(0) = x_0, \tag{3.11}$$

where

$$x_0 = w_0 + \sum_{j=1}^{N_u} \int_{-h_{u,j}}^{0} B_j(\eta + h_{u,j})\varphi_u(\eta)d\eta + \int_{-h_u}^{0} d\eta \left(\int_{\eta}^{0} P(\sigma - \eta, \eta)\varphi_u(\sigma)d\sigma \right)$$
$$+ \sum_{k=1}^{N_v} \int_{-h_{v,k}}^{0} C_k(\kappa + h_{v,k})\varphi_v(\kappa)d\kappa + \int_{-h_v}^{0} d\zeta \left(\int_{\zeta}^{0} Q(\kappa - \zeta, \zeta)\varphi_v(\kappa)d\kappa \right), \tag{3.12}$$

and

$$w_0 \stackrel{\triangle}{=} \Psi(0)\varphi_{0,z} + \sum_{i=1}^{N_z} \int_{-h_{z,i}}^{0} \Psi(\tau + h_{z,i})\mathscr{A}_i(\tau + h_{z,i})\varphi_z(\tau)d\tau$$
$$+ \int_{-h_z}^{0} \left(\int_{0}^{\tau + h_z} \Psi(\sigma)\mathscr{G}\Big(\sigma, \tau - \sigma\Big)d\sigma \right) \varphi_z(\tau)d\tau + \int_{0}^{t_f} \Psi(\xi)f(\xi)d\xi. \tag{3.13}$$

Moreover,

$$x(t_f) = z(t_f). \tag{3.14}$$

Using (3.14), the cost functionals (2.5) and (2.8) become, respectively,

$$J_\varepsilon\big(u(t),v(t)\big) = \frac{1}{2}x^T(t_f)\mathscr{F}x(t_f) + \frac{1}{2}\int_0^{t_f} \big[u^T(t)R_u(t,\varepsilon)u(t) - v^T(t)R_v(t)v(t)\big]dt \tag{3.15}$$

and

$$J_0\big(u(t),v(t)\big) = \frac{1}{2}x^T(t_f)\mathscr{F}x(t_f) + \frac{1}{2}\int_0^{t_f} \big[u^T(t)R_u(t,0)u(t) - v^T(t)R_v(t)v(t)\big]dt. \tag{3.16}$$

Thus the state transformation (3.8)-(3.9) in the games (2.1)-(2.5) and (2.1)-(2.4), (2.8) yields two new zero-sum differential games which do not contain delays in their states and controls. The first new undelayed game consists of the dynamics equation (3.10) with the state initial condition (3.11), and the cost functional (3.15). The second new undelayed game consists of the same dynamics equation and the same state initial condition, while it has another cost functional. Namely, the cost functional of the second undelayed game is (3.16). In both games, $u(t)$ is a minimizing control, while $v(t)$ is a maximizing control. Like the game (2.1)-(2.4), (2.8), the second new undelayed game (3.10)-(3.11), (3.16) is singular.

Remark 3.2. Open-loop saddle point solutions (or briefly, open-loop saddle-points) and game values of the games (3.10)-(3.11), (3.15) and (3.10)-(3.11), (3.16) are defined quite similarly to Definitions 2.3 and 2.4, respectively.

Definition 3.3. We say that two zero-sum differential games, considered in the same time-interval, are equivalent to each other, if they have the same open-loop saddle point solutions (open-loop saddle points) and the same game values.

Lemma 3.4. *Let the assumptions A1-A4 be satisfied. Let, for a given* $\varepsilon > 0$*, one of the games (2.1)-(2.5) or (3.10)-(3.11), (3.15) be open-loop solvable. Then, the other game also is open-loop solvable, and these games are equivalent to each other.*

Proof. Let the functions $u(t) \in L^2[0,t_f;\mathbb{R}^r]$ and $v(t) \in L^2[0,t_f;\mathbb{R}^s]$ be any prechosen. Then, due to Proposition 3.1, the outcomes of the games (2.1)-(2.5) and (3.10)-(3.11), (3.15), generated by the pair of the players' controls $\big(u(t),v(t)\big)$, are equal to each other, i.e., $\mathscr{J}_\varepsilon\big(u(t),v(t)\big) = J_\varepsilon\big(u(t),v(t)\big)$. This equality directly yields the statements of the lemma. □

Lemma 3.5. *Let the assumptions A1-A4 be satisfied. Let one of the degenerate games (2.1)-(2.4), (2.8) or (3.10)-(3.11), (3.16) be open-loop solvable. Then, the other game also is open-loop solvable, and these games are equivalent to each other.*

Proof. The lemma is proven similarly to Lemma 3.4. □

Remark 3.6. Due to Lemma 3.4, the initially formulated game (2.1)-(2.5) and the game (3.10)-(3.11), (3.15) are equivalent to each other. In what follows of the paper, we deal with the game (3.10)-(3.11), (3.15) and we call this game the Original Partial Cheap Control Game (OPCCG). Similarly, due to Lemma 3.5, the games (2.1)-(2.4), (2.8) and (3.10)-(3.11), (3.16) are equivalent to each other. Therefore, in what follows of the paper, we deal with the game (3.10)-(3.11), (3.16) and we call it the Original Degenerate Game (ODG).

4. Solvability condition of the original partial cheap control game

In what follows, we assume:
A5. There exists an $n \times n$-matrix F, such that

$$F^T F = \mathscr{F}, \tag{4.1}$$

and the matrix $W_v(0)$ is positive definite, where

$$W_v(t) \stackrel{\triangle}{=} I_n + F \int_{t_f}^{t} S_v(\sigma) d\sigma F^T, \qquad S_v(t) \stackrel{\triangle}{=} \mathscr{K}_v(t) R_v^{-1}(t) \mathscr{K}_v^T(t), \quad t \in [0,t_f]. \tag{4.2}$$

Consider the following terminal-value problem for the Riccati differential equation with respect to the matrix-valued function $N(t)$:

$$\frac{dN(t)}{dt} = -N(t)S_v(t)\big)N(t), \quad t \in [0,t_f], \quad N(t_f) = \mathscr{F}, \tag{4.3}$$

Also, for a given $\varepsilon > 0$, consider the following terminal-value problem for the Riccati differential equation with respect to the matrix-valued function $M(t,\varepsilon)$:

$$\frac{dM(t,\varepsilon)}{dt} = M(t,\varepsilon)\big(S_u(t,\varepsilon) - S_v(t)\big)M(t,\varepsilon), \quad t \in [0,t_f], \quad M(t_f,\varepsilon) = \mathscr{F}, \tag{4.4}$$

where

$$S_u(t,\varepsilon) \stackrel{\triangle}{=} \mathscr{K}_u(t)R_u^{-1}(t,\varepsilon)\mathscr{K}_u^T(t), \quad t \in [0,t_f]. \tag{4.5}$$

Lemma 4.1. *Let the assumptions A1-A5 be satisfied. Then, the problems (4.3) and (4.4) have the unique solutions $N(t)$ and $M(t,\varepsilon)$ in the entire interval $[0,t_f]$.*

Proof. For any given $t \in (0,t_f]$, the matrix $W_v(t)$ can be represented as:

$$W_v(t) = W_v(0) + F\int_0^t S_v(\sigma)d\sigma F^T. \tag{4.6}$$

Since the matrix $R_v(t)$ is positive definite for all $t \in [0,t_f]$, then the matrix $S_v(t)$ is at least positive semi-definite for all $t \in [0,t_f]$. Therefore, the matrix

$$F\int_0^t S_v(\sigma)d\sigma F^T$$

is positive semi-definite for all $t \in [0,t_f]$. The latter, along with the equation (4.6) and the assumption A5, implies the positive definiteness of the matrix $W_v(t)$ for all $t \in [0,t_f]$.

For a given $\varepsilon > 0$, consider the matrix

$$W_u(t,\varepsilon) \stackrel{\triangle}{=} W_v(t) - F\int_{t_f}^t S_u(\sigma,\varepsilon)d\sigma F^T, \quad t \in [0,t_f]. \tag{4.7}$$

Since the matrix $R_u(t,\varepsilon)$ is positive definite for all $t \in [0,t_f]$ and $\varepsilon > 0$, then the matrix $S_u(t,\varepsilon)$ is at least positive semi-definite for all $t \in [0,t_f]$ and $\varepsilon > 0$. Therefore, the matrix

$$F\int_t^{t_f} S_u(\sigma,\varepsilon)d\sigma F^T$$

is positive semi-definite for all $t \in [0,t_f]$ and $\varepsilon > 0$. Hence, due to the equation (4.7) and the positive definiteness of the matrix $W_v(t)$ for all $t \in [0,t_f]$, the matrix $W_u(t,\varepsilon)$ is positive definite for all $t \in [0,t_f]$ and $\varepsilon > 0$.

Using the positive definiteness (and, therefore, invertibility) of the matrices $W_v(t)$, $t \in [0,t_f]$ and $W_u(t,\varepsilon)$, $t \in [0,t_f]$, $\varepsilon > 0$, let us construct the following matrix-valued functions:

$$N(t) \stackrel{\triangle}{=} F^T W_v^{-1}(t)F, \quad t \in [0,t_f], \tag{4.8}$$

$$M(t,\varepsilon) \stackrel{\triangle}{=} F^T W_u^{-1}(t,\varepsilon)F, \quad t \in [0,t_f], \quad \varepsilon > 0. \tag{4.9}$$

Now, it is verified by direct substitution of (4.8) and (4.9) into (4.3) and (4.4), respectively, that $N(t)$ is the solution of the terminal-value problem (4.3) and $M(t,\varepsilon)$ is the solution of the terminal-value problem (4.4) in the entire interval $[0,t_f]$. The uniqueness of these solutions follows immediately from the quadratic dependence of the right-hand sides of the differential equations in (4.3) and (4.4) on $N(t)$ and $M(t)$, respectively. This completes the proof of the lemma. □

Based on Lemma 4.1, as well as on the results of the works [2] (Section 6.5), [6] (Section 9.4) and [3], we immediately have the following assertion.

Lemma 4.2. *Let the assumptions A1-A5 be satisfied. Then, for any given* $\varepsilon > 0$*, the OPCCG is open-loop solvable. The components of its open-loop saddle point solution* $\big(u^*(t,\varepsilon)\ ,\ v^*(t,\varepsilon)\big)$ *have the form*

$$\begin{aligned} u^*(t,\varepsilon) &= -R_u^{-1}(t,\varepsilon)\mathscr{K}_u^T(t)M(0,\varepsilon)x_0, \quad t\in[0,t_f],\\ v^*(t,\varepsilon) &= R_v^{-1}(t)\mathscr{K}_v^T(t)M(0,\varepsilon)x_0, \quad t\in[0,t_f]. \end{aligned} \tag{4.10}$$

The OPCCG value J_ε^* *has the form*

$$J_\varepsilon^* = \frac{1}{2}x_0^T M(0,\varepsilon)x_0. \tag{4.11}$$

5. Asymptotic analysis of the original partial cheap control game

5.1. **Asymptotic expansion of the open-loop saddle point solution to the original partial cheap control game.** We start the construction of the asymptotic expansion of the open-loop saddle point solution to the OPCCG with the asymptotic analysis of the matrix $M(0,\varepsilon)$ for all sufficiently small $\varepsilon > 0$.

Let us partition the matrix $\mathscr{K}_u(\sigma)$, $\sigma\in[0,t_f]$ into blocks as:

$$\mathscr{K}_u(\sigma) = \Big(\mathscr{K}_{u,1}(\sigma)\ ,\ \mathscr{K}_{u,2}(\sigma)\Big), \quad \sigma\in[0,t_f], \tag{5.1}$$

where the matrices $\mathscr{K}_{u,1}(\sigma)$ and $\mathscr{K}_{u,2}(\sigma)$ have the dimensions $n\times q$ and $n\times(r-q)$, respectively.

Using the equations (2.6) and (5.1), as well as the invertibility of the matrices $R_{u,1}(t)$ and $R_{u,2}(t)$, we obtain for all $\sigma\in[0,t_f]$ and $\varepsilon > 0$

$$\begin{aligned} F\mathscr{K}_u(\sigma)R_u^{-1}(\sigma,\varepsilon)\mathscr{K}_u^T(\sigma)F^T &= F\mathscr{K}_{u,1}(\sigma)R_{u,1}^{-1}(\sigma)\mathscr{K}_{u,1}^T(\sigma)F^T\\ &+\frac{1}{\varepsilon}F\mathscr{K}_{u,2}(\sigma)R_{u,2}^{-1}(\sigma)\mathscr{K}_{u,2}^T(\sigma)F^T. \end{aligned} \tag{5.2}$$

Consider the matrix

$$K_{u,2} \stackrel{\triangle}{=} F\int_0^{t_f}\mathscr{K}_{u,2}(\sigma)R_{u,2}^{-1}(\sigma)\mathscr{K}_{u,2}^T(\sigma)d\sigma F^T. \tag{5.3}$$

Since the matrix $R_{u,2}(\sigma)$ is symmetric and positive definite for all $\sigma\in[0,t_f]$, then the matrix $K_{u,2}$ is symmetric and positive semi-definite.

In what follows, we assume

A6. The matrix $K_{u,2}$ has zero eigenvalue of the algebraic multiplicity k, $(n-r+q\le k<n)$.

Hence, by virtue of [4], there exists an orthogonal $n\times n$-matrix L, ($L^T = L^{-1}$), such that the following equality is valid:

$$D_{u,2} \stackrel{\triangle}{=} LK_{u,2}L^T = \begin{pmatrix} 0 & 0\\ 0 & \Theta_{u,2}\end{pmatrix}, \tag{5.4}$$

where the block $\Theta_{u,2}$ is of dimension $(n-k)\times(n-k)$, and it is a nonsingular (positive definite) matrix. In particular, for a properly chosen orthogonal matrix L, this block has the form

$$\Theta_{u,2} = \mathrm{diag}(\mu_1,\mu_2,\ldots,\mu_{n-k}), \tag{5.5}$$

μ_1, μ_2,..., μ_{n-k} are all non-zero (and, therefore, positive) eigenvalues (not obligatory different) of the matrix $K_{u,2}$.

Since the integrand in (5.3) is at least positive semi-definite and the matrix $R_{u,2}(t)$ is symmetric and positive definite for all $t \in [0,t_f]$, the equations (5.3) and (5.4) yield

$$LF\mathscr{K}_{u,2}(t) = \begin{pmatrix} 0 \\ \mathscr{E}_{u,2}(t) \end{pmatrix}, \quad t \in [0,t_f], \tag{5.6}$$

where the block $\mathscr{E}_{u,2}(t)$ is of dimension $(n-k)\times(r-q)$.

Note that, due to (5.3)-(5.4) and (5.6),

$$\int_0^{t_f} \mathscr{E}_{u,2}(\sigma)R_{u,2}^{-1}(\sigma)\mathscr{E}_{u,2}^T(\sigma)d\sigma = \Theta_{u,2}. \tag{5.7}$$

Taking into account that the matrix L is orthogonal and using the equations (4.2), (4.5), (4.7), (4.9), and (5.1)-(5.5), we can represent the matrix $M(0,\varepsilon)$ as:

$$M(0,\varepsilon) = (LF)^T\Bigg(I_n + \frac{1}{\varepsilon}D_{u,2} + LF\int_0^{t_f}\big(\mathscr{K}_{u,1}(\sigma)R_{u,1}^{-1}(\sigma)\mathscr{K}_{u,1}^T(\sigma) - S_v(\sigma)\big)d\sigma(LF)^T\Bigg)^{-1}LF. \tag{5.8}$$

Let us partition the matrices $LF\int_0^{t_f}\mathscr{K}_{u,1}(\sigma)R_{u,1}^{-1}(\sigma)\mathscr{K}_{u,1}^T(\sigma)d\sigma(LF)^T$ and $LF\int_0^{t_f}S_v(\sigma)d\sigma(LF)^T$ into blocks as:

$$LF\int_0^{t_f}\mathscr{K}_{u,1}(\sigma)R_{u,1}^{-1}(\sigma)\mathscr{K}_{u,1}^T(\sigma)d\sigma(LF)^T = \begin{pmatrix} \Omega_{u,11} & \Omega_{u,12} \\ \Omega_{u,12}^T & \Omega_{u,13} \end{pmatrix},$$
$$LF\int_0^{t_f}S_v(\sigma)d\sigma(LF)^T = \begin{pmatrix} \Lambda_{v,1} & \Lambda_{v,2} \\ \Lambda_{v,2}^T & \Lambda_{v,3} \end{pmatrix}, \tag{5.9}$$

where the blocks $\Omega_{u,11}$ and $\Lambda_{v,1}$ are of the dimension $k\times k$; the blocks $\Omega_{u,12}$ and $\Lambda_{v,2}$ are of the dimension $k\times(n-k)$; the blocks $\Omega_{u,13}$ and $\Lambda_{v,3}$ are of the dimension $(n-k)\times(n-k)$; the matrices $\Omega_{u,11}$, $\Omega_{u,13}$, $\Lambda_{v,1}$ and $\Lambda_{v,3}$ are symmetric.

Using the equations (5.4) and (5.9), we can represent the matrix $M(0,\varepsilon)$, given by (5.8), in the form

$$M(0,\varepsilon) = (LF)^T\Gamma^{-1}(\varepsilon)LF, \tag{5.10}$$

where

$$\Gamma(\varepsilon) = \begin{pmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2^T & (1/\varepsilon)\Gamma_3(\varepsilon) \end{pmatrix},$$
$$\Gamma_1 = I_k + \Omega_{u,11} - \Lambda_{v,1}, \quad \Gamma_2 = \Omega_{u,12} - \Lambda_{v,2},$$
$$\Gamma_3(\varepsilon) = \Theta_{u,2} + \varepsilon\big(I_{n-k} + \Omega_{u,13} - \Lambda_{v,3}\big). \tag{5.11}$$

Since the matrix $\Theta_{u,2}$ is positive definite, it is invertible.

Let ε_1 be a given number satisfying the inequality

$$0 < \varepsilon_1 < \min\left\{1, \frac{1}{\big\|\big(I_{n-k} + \Omega_{u,13} - \Lambda_{v,3}\big)\Theta_{u,2}^{-1}\big\|}\right\}. \tag{5.12}$$

Based on the results of [31] (Chapter IV), we obtain that, for any $\varepsilon \in [0, \varepsilon_1]$, the matrix $\Gamma_3(\varepsilon)$ is invertible. Moreover,

$$\Gamma_3^{-1}(\varepsilon) = \Theta_{u,2}^{-1} \sum_{l=0}^{+\infty} (-1)^l \varepsilon^l \left[\left(I_{n-k} + \Omega_{u,13} - \Lambda_{v,3} \right) \Theta_{u,2}^{-1} \right]^l, \quad \varepsilon \in [0, \varepsilon_1]. \tag{5.13}$$

Due to this expansion, we directly have that the matrix $\Gamma_3^{-1}(\varepsilon)$ is bounded uniformly in $\varepsilon \in [0, \varepsilon_1]$ and the following inequality is valid:

$$\left\| \Gamma_3^{-1}(\varepsilon) - \Theta_{u,2}^{-1} \right\| \le a_1 \varepsilon, \quad \varepsilon \in [0, \varepsilon_1], \tag{5.14}$$

where

$$a_1 = \frac{\|\Theta_{u,2}^{-1}\| \left\| \left(I_{n-k} + \Omega_{u,13} - \Lambda_{v,3} \right) \Theta_{u,2}^{-1} \right\|}{1 - \varepsilon_1 \left\| \left(I_{n-k} + \Omega_{u,13} - \Lambda_{v,3} \right) \Theta_{u,2}^{-1} \right\|}. \tag{5.15}$$

In what follows, we assume

A7. The matrix Γ_1, given in the equation (5.11), is invertible, i.e., $\det\left(\Gamma_1\right) \neq 0$.

Now, applying the Frobenius formula (see e.g. [27]) to the calculation of $\Gamma^{-1}(\varepsilon)$, we obtain the existence of a positive number $\varepsilon_2 \le \varepsilon_1$ (below, the number ε_2 is introduced more precisely), such that for all $\varepsilon \in (0, \varepsilon_2]$

$$\Gamma^{-1}(\varepsilon) = \Phi(\varepsilon) = \begin{pmatrix} \Phi_1(\varepsilon) & \Phi_2(\varepsilon) \\ \Phi_2^T(\varepsilon) & \Phi_3(\varepsilon) \end{pmatrix}, \tag{5.16}$$

where

$$\begin{aligned} \Phi_1(\varepsilon) &= \left(\Gamma_1 - \varepsilon \Gamma_2 \Gamma_3^{-1}(\varepsilon) \Gamma_2^T \right)^{-1}, \\ \Phi_2(\varepsilon) &= -\varepsilon \left(\Gamma_1 - \varepsilon \Gamma_2 \Gamma_3^{-1}(\varepsilon) \Gamma_2^T \right)^{-1} \Gamma_2 \Gamma_3^{-1}(\varepsilon), \\ \Phi_3(\varepsilon) &= \varepsilon \Gamma_3^{-1}(\varepsilon) + \varepsilon^2 \Gamma_3^{-1}(\varepsilon) \Gamma_2^T \left(\Gamma_1 - \varepsilon \Gamma_2 \Gamma_3^{-1}(\varepsilon) \Gamma_2^T \right)^{-1} \Gamma_2 \Gamma_3^{-1}(\varepsilon). \end{aligned} \tag{5.17}$$

Let us estimate the matrices $\Phi_1(\varepsilon)$, $\Phi_2(\varepsilon)$ and $\Phi_3(\varepsilon)$ for all sufficiently small $\varepsilon > 0$. We start with $\Phi_1(\varepsilon)$. First of all, let us estimate the product of the matrices $\Gamma_2 \Gamma_3^{-1}(\varepsilon) \Gamma_2^T \Gamma_1^{-1}$. Using the inequality (5.14), we obtain

$$\left\| \Gamma_3^{-1}(\varepsilon) \right\| \le \left\| \Theta_{u,2}^{-1} \right\| + a_1 \varepsilon, \quad \varepsilon \in [0, \varepsilon_1]. \tag{5.18}$$

Due to this inequality, we have

$$\left\| \Gamma_2 \Gamma_3^{-1}(\varepsilon) \Gamma_2^T \Gamma_1^{-1} \right\| \le \left\| \Gamma_2 \right\| \left(\left\| \Theta_{u,2}^{-1} \right\| + a_1 \varepsilon \right) \left\| \Gamma_2^T \Gamma_1^{-1} \right\|, \quad \varepsilon \in [0, \varepsilon_1]. \tag{5.19}$$

Let ε_2 be a given number satisfying the inequality

$$0 < \varepsilon_2 < \min \left\{ \varepsilon_1, \frac{1}{\left\| \Gamma_2 \right\| \left(\left\| \Theta_{u,2}^{-1} \right\| + a_1 \varepsilon_1 \right) \left\| \Gamma_2^T \Gamma_1^{-1} \right\|} \right\}. \tag{5.20}$$

Then, using the inequalities (5.19) and (5.20), we obtain (similarly to (5.14)-(5.15)) the following inequality:

$$\left\| \Phi_1(\varepsilon) - \Gamma_1^{-1} \right\| \le a_2 \varepsilon, \quad \varepsilon \in [0, \varepsilon_2], \tag{5.21}$$

where

$$a_2 = \frac{\|\Gamma_1^{-1}\| \|\Gamma_2\| \left(\|\Theta_{u,2}^{-1}\| + a_1\varepsilon_1\right) \|\Gamma_2^T\Gamma_1^{-1}\|}{1 - \varepsilon_2\|\Gamma_2\| \left(\|\Theta_{u,2}^{-1}\| + a_1\varepsilon_1\right) \|\Gamma_2^T\Gamma_1^{-1}\|}. \tag{5.22}$$

Using the expressions for $\Phi_1(\varepsilon)$, $\Phi_2(\varepsilon)$, $\Phi_3(\varepsilon)$ (see the equation (5.17)) and the inequalities (5.14), (5.18), and (5.21), we obtain by a routine algebra

$$\begin{aligned} \|\varepsilon^{-1}\Phi_2(\varepsilon) + \Gamma_1^{-1}\Gamma_2\Theta_{u,2}^{-1}\| &\le \|\Phi_1(\varepsilon) - \Gamma_1^{-1}\| \|\Gamma_2\| \|\Gamma_3^{-1}(\varepsilon) - \Theta_{u,2}^{-1}\| \\ + \|\Phi_1(\varepsilon) - \Gamma_1^{-1}\| \|\Gamma_2\Theta_{u,2}^{-1}\| + \|\Gamma_1^{-1}\Gamma_2\| \|\Gamma_3^{-1}(\varepsilon) - \Theta_{u,2}^{-1}\| &\le a_3\varepsilon + a_4\varepsilon^2, \quad \varepsilon \in (0, \varepsilon_2], \\ \|\varepsilon^{-1}\Phi_3(\varepsilon) - \Theta_{u,2}^{-1}\| &\le \|\Gamma_3^{-1}(\varepsilon) - \Theta_{u,2}^{-1}\| + \varepsilon\|\Gamma_3^{-1}(\varepsilon)\| \|\Gamma_2\| \|\varepsilon^{-1}\Phi_2(\varepsilon)\| \\ &\le a_5\varepsilon + a_6\varepsilon^2 + a_7\varepsilon^3 + a_8\varepsilon^4, \quad \varepsilon \in (0, \varepsilon_2], \end{aligned} \tag{5.23}$$

$$\begin{aligned} \|\Phi_2(\varepsilon)\| &\le \|\Gamma_1^{-1}\Gamma_2\Theta_{u,2}^{-1}\|\varepsilon + a_3\varepsilon^2 + a_4\varepsilon^3, \quad \varepsilon \in (0, \varepsilon_2], \\ \|\Phi_3(\varepsilon)\| &\le \|\Theta_{u,2}^{-1}\|\varepsilon + a_5\varepsilon^2 + a_6\varepsilon^3 + a_7\varepsilon^4 + a_8\varepsilon^5, \quad \varepsilon \in (0, \varepsilon_2], \end{aligned} \tag{5.24}$$

where

$$\begin{aligned} a_3 = a_1\|\Gamma_1^{-1}\Gamma_2\| + a_2\|\Gamma_2\Theta_{u,2}^{-1}\|, \quad a_4 = a_1a_2\|\Gamma_2\|, \\ a_5 = a_1 + \|\Gamma_1^{-1}\Gamma_2\Theta_{u,2}^{-1}\| \|\Theta_{u,2}^{-1}\| \|\Gamma_2\|, \quad a_6 = \left(a_3\|\Theta_{u,2}^{-1}\| + a_1\|\Gamma_1^{-1}\Gamma_2\Theta_{u,2}^{-1}\|\right)\|\Gamma_2\|, \\ a_7 = \left(a_1a_3 + a_4\|\Theta_{u,2}^{-1}\|\right)\|\Gamma_2\|, \quad a_8 = a_1a_4\|\Gamma_2\|. \end{aligned} \tag{5.25}$$

Using the estimates (5.21), (5.23), and (5.24), we are going to construct and justify asymptotic expansions of the components $u^*(t,\varepsilon)$ and $v^*(t,\varepsilon)$ of the open-loop saddle point solution to the OPCCG. Let us start with the $u^*(t,\varepsilon)$. Substitution of (2.6), (5.1) and (5.10) into the expression for $u^*(t,\varepsilon)$ (see the equation (4.10)) yields

$$\begin{aligned} u^*(t,\varepsilon) &= -\begin{pmatrix} R_{u,1}^{-1}(t) & 0 \\ 0 & \varepsilon^{-1}R_{u,2}^{-1}(t) \end{pmatrix} \begin{pmatrix} \mathscr{K}_{u,1}^T(t) \\ \mathscr{K}_{u,2}^T(t) \end{pmatrix} (LF)^T\Gamma^{-1}(\varepsilon)LFx_0 \\ &= \begin{pmatrix} u_1^*(t,\varepsilon) \\ u_2^*(t,\varepsilon) \end{pmatrix}, \quad t \in [0, t_f], \end{aligned} \tag{5.26}$$

where

$$\begin{aligned} u_1^*(t,\varepsilon) &= -R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)(LF)^T\Gamma^{-1}(\varepsilon)LFx_0, \quad t \in [0, t_f], \\ u_2^*(t,\varepsilon) &= -\varepsilon^{-1}R_{u,2}^{-1}(t)\mathscr{K}_{u,2}^T(t)(LF)^T\Gamma^{-1}(\varepsilon)LFx_0, \quad t \in [0, t_f]. \end{aligned} \tag{5.27}$$

Thus, we have represented the component $u^*(t,\varepsilon)$ of the open-loop saddle point solution to the OPCCG in the form of the block-vector with the upper block $u_1^*(t,\varepsilon)$ and the lower block $u_2^*(t,\varepsilon)$. We start the asymptotic analysis of $u^*(t,\varepsilon)$ with its upper block.

Consider the following block-matrix of the dimension $n\times n$:

$$\Phi_0 \overset{\triangle}{=} \begin{pmatrix} \Gamma_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.28}$$

Using this matrix, let us construct the vector-valued function

$$u_{10}^*(t) \overset{\triangle}{=} -R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)(LF)^T\Phi_0 LFx_0, \quad t\in[0,t_f]. \tag{5.29}$$

Now, using the equations (5.16), (5.17), (5.27), (5.28), and the inequalities (5.21) and (5.24), we have the inequality

$$\left\|u_1^*(t,\varepsilon)-u_{10}^*(t)\right\| \le \sum_{l=1}^{5} c_{1l}\varepsilon^l \quad \forall\, t\in[0,t_f],\ \varepsilon\in(0,\varepsilon_2], \tag{5.30}$$

where

$$\begin{aligned}
c_{11} &= \left(\max_{t\in[0,t_f]}\left\|R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)(LF)^T\right\|\right)\left(a_2+2\left\|\Gamma_1^{-1}\Gamma_2\Theta_{u,2}^{-1}\right\|+\left\|\Theta_{u,2}^{-1}\right\|\right)\left\|LFx_0\right\|,\\
c_{12} &= \left(\max_{t\in[0,t_f]}\left\|R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)(LF)^T\right\|\right)\left(2a_3+a_5\right)\left\|LFx_0\right\|,\\
c_{13} &= \left(\max_{t\in[0,t_f]}\left\|R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)(LF)^T\right\|\right)\left(2a_4+a_6\right)\left\|LFx_0\right\|,\\
c_{14} &= \left(\max_{t\in[0,t_f]}\left\|R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)(LF)^T\right\|\right)\left\|LFx_0\right\|a_7,\\
c_{15} &= \left(\max_{t\in[0,t_f]}\left\|R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)(LF)^T\right\|\right)\left\|LFx_0\right\|a_8.
\end{aligned} \tag{5.31}$$

The inequality (5.30) means that $u_{10}^*(t)$ is the zero-order asymptotic expansion with respect to $\varepsilon>0$ of $u_1^*(t,\varepsilon)$, and this expansion is uniform in $t\in[0,t_f]$.

Proceed to the asymptotic analysis of $u_2^*(t,\varepsilon)$. From (5.6), we directly have

$$\mathscr{K}_{u,2}^T(t)(LF)^T = \left(0\ ,\ \mathscr{E}_{u,2}^T(t)\right), \quad t\in[0,t_f]. \tag{5.32}$$

Substitution of (5.16) and (5.32) into the expression for $u_2^*(t,\varepsilon)$ (see the equation (5.27)) yields after a routine matrix algebra

$$u_2^*(t,\varepsilon) = -\left(\varepsilon^{-1}R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^T(t)\Phi_2^T(\varepsilon)\ ,\ \varepsilon^{-1}R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^T(t)\Phi_3(\varepsilon)\right)LFx_0, \quad t\in[0,t_f]. \tag{5.33}$$

Consider the vector-valued function

$$u_{20}^*(t) \overset{\triangle}{=} \left(R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^T(t)\Theta_{u,2}^{-1}\Gamma_2^T\Gamma_1^{-1}\ ,\ -R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^T(t)\Theta_{u,2}^{-1}\right)LFx_0, \quad t\in[0,t_f]. \tag{5.34}$$

Now, using the inequalities in (5.23), we have the inequality

$$\left\|u_2^*(t,\varepsilon)-u_{20}^*(t)\right\| \le \sum_{l=1}^{4} c_{2l}\varepsilon^l \quad \forall\, t\in[0,t_f],\ \varepsilon\in(0,\varepsilon_2], \tag{5.35}$$

where

$$
\begin{aligned}
c_{21} &= \left(\max_{t\in[0,t_f]} \left\|R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^{T}(t)\right\|\right)\left(a_3+a_5\right)\left\|LFx_0\right\|,\\
c_{22} &= \left(\max_{t\in[0,t_f]} \left\|R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^{T}(t)\right\|\right)\left(a_4+a_6\right)\left\|LFx_0\right\|,\\
c_{23} &= \left(\max_{t\in[0,t_f]} \left\|R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^{T}(t)\right\|\right)\left\|LFx_0\right\|a_7,\\
c_{24} &= \left(\max_{t\in[0,t_f]} \left\|R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^{T}(t)\right\|\right)\left\|LFx_0\right\|a_8.
\end{aligned}
\tag{5.36}
$$

The inequality (5.35) means that $u_{20}^{*}(t)$ is the zero-order asymptotic expansion with respect to $\varepsilon>0$ of $u_2^{*}(t,\varepsilon)$, and this expansion is uniform in $t\in[0,t_f]$.

Consider the vector-valued function

$$
u_0^{*}(t) \overset{\triangle}{=} \mathrm{col}\left(u_{10}^{*}(t)\ ,\ u_{20}^{*}(t)\right), \quad t\in[0,t_f]. \tag{5.37}
$$

Based on the equation (5.26) and the inequalities (5.30) and (5.35), we directly have the following inequality:

$$
\left\|u^{*}(t,\varepsilon)-u_0^{*}(t)\right\| \le \sum_{l=1}^{5} c_{u,l}\varepsilon^{l} \quad \forall\, t\in[0,t_f],\ \varepsilon\in(0,\varepsilon_2], \tag{5.38}
$$

where

$$
c_{u,1}=c_{11}+c_{21},\quad c_{u,2}=c_{12}+c_{22},\quad c_{u,3}=c_{13}+c_{23},\quad c_{u,4}=c_{14}+c_{24},\quad c_{u,5}=c_{15}. \tag{5.39}
$$

The inequality (5.38) means that $u_0^{*}(t)$ is the zero-order asymptotic expansion with respect to $\varepsilon>0$ of $u^{*}(t,\varepsilon)$, and this expansion is uniform in $t\in[0,t_f]$.

To complete this subsection, we should construct and justify the asymptotic expansion for the $v^{*}(t,\varepsilon)$ component of the open-loop saddle point solution to the OPCCG.

Consider the vector-valued function

$$
v_0^{*}(t) \overset{\triangle}{=} R_v^{-1}(t)\mathscr{K}_v^{T}(t)(LF)^{T}\Phi_0 LFx_0, \quad t\in[0,t_f]. \tag{5.40}
$$

Using the expression for $v^{*}(t,\varepsilon)$ (see the equations (4.10)), as well as the equations (5.10), (5.16), (5.28), and the inequalities (5.21) and (5.24), we have the inequality

$$
\left\|v^{*}(t,\varepsilon)-v_0^{*}(t)\right\| \le \sum_{l=1}^{5} c_{v,l}\varepsilon^{l} \quad \forall\, t\in[0,t_f],\ \varepsilon\in(0,\varepsilon_2], \tag{5.41}
$$

where

$$
\begin{aligned}
c_{v,1} = \left(\max_{t\in[0,t_f]} \left\|R_v^{-1}(t)\mathscr{K}_v^T(t)(LF)^T\right\|\right)\left(a_2 + 2\left\|\Gamma_1^{-1}\Gamma_2\Theta_{u,2}^{-1}\right\| + \left\|\Theta_{u,2}^{-1}\right\|\right)\left\|LFx_0\right\|,\\
c_{v,2} = \left(\max_{t\in[0,t_f]} \left\|R_v^{-1}(t)\mathscr{K}_v^T(t)(LF)^T\right\|\right)\left(2a_3 + a_5\right)\left\|LFx_0\right\|,\\
c_{v,3} = \left(\max_{t\in[0,t_f]} \left\|R_v^{-1}(t)\mathscr{K}_v^T(t)(LF)^T\right\|\right)\left(2a_4 + a_6\right)\left\|LFx_0\right\|,\\
c_{v,4} = \left(\max_{t\in[0,t_f]} \left\|R_v^{-1}(t)\mathscr{K}_v^T(t)(LF)^T\right\|\right)\left\|LFx_0\right\|a_7,\\
c_{v,5} = \left(\max_{t\in[0,t_f]} \left\|R_v^{-1}(t)\mathscr{K}_v^T(t)(LF)^T\right\|\right)\left\|LFx_0\right\|a_8.
\end{aligned}
\tag{5.42}
$$

The inequality (5.41) means that $v_0^*(t)$ is the zero-order asymptotic expansion with respect to $\varepsilon > 0$ of $v^*(t,\varepsilon)$, and this expansion is uniform in $t \in [0,t_f]$.

5.2. **Asymptotic expansion of the value for the original partial cheap control game.** Consider the following value:

$$
J_0^* \stackrel{\triangle}{=} \frac{1}{2}x_0^T(LF)^T\Phi_0 LFx_0. \tag{5.43}
$$

Using this value, as well as the equations (4.11), (5.10), (5.16), (5.28) and the inequalities (5.21) and (5.24), we obtain the inequality

$$
\left|J_\varepsilon^* - J_0^*\right| \le \sum_{l=1}^{5}\alpha_l\varepsilon^l \quad \forall\, \varepsilon \in (0,\varepsilon_2], \tag{5.44}
$$

where

$$
\begin{aligned}
\alpha_1 = \left(a_2 + \left\|\Gamma_1^{-1}\Gamma_2\Theta_{u,2}^{-1}\right\| + \left\|\Theta_{u,2}^{-1}\right\|\right)\left\|LFx_0\right\|^2, \quad \alpha_2 = \left(a_3 + a_5\right)\left\|LFx_0\right\|^2,\\
\alpha_3 = \left(a_4 + a_6\right)\left\|LFx_0\right\|^2, \quad \alpha_4 = a_7\left\|LFx_0\right\|^2, \quad \alpha_5 = a_8\left\|LFx_0\right\|^2.
\end{aligned}
\tag{5.45}
$$

The inequality (5.44) means that J_0^* is the zero-order asymptotic expansion with respect to $\varepsilon > 0$ of the OPCCG value J_ε^*.

5.3. **Open-loop quasi saddle point of the original partial cheap control game.** For the OPCCG, let us consider the following pair of controls: $\left(u_0^*(t)\ ,\ v_0^*(t)\right)$. Let $\bar{x}^*(t)$, $t \in [0,t_f]$ be the solution of the initial-value problem (3.10)-(3.11) generated by this pair of controls. Thus,

$$
\bar{x}^*(t_f) = x_0 + \int_0^{t_f}\left[\mathscr{K}_u(t)u_0^*(t) + \mathscr{K}_v(t)v_0^*(t)\right]dt. \tag{5.46}
$$

By $x^*(t,\varepsilon)$, $t\in[0,t_f]$, we denote the solution of the initial-value problem (3.10)-(3.11) generated by the open-loop saddle point solution $\big(u^*(t,\varepsilon)\ ,\ v^*(t,\varepsilon)\big)$ to the OPCCG. Thus,

$$x^*(t_f,\varepsilon)=x_0+\int_0^{t_f}\big[\mathscr{K}_u(t)u^*(t,\varepsilon)+\mathscr{K}_v(t)v^*(t,\varepsilon)\big]dt. \tag{5.47}$$

The equations (5.46) and (5.47), along with the inequalities (5.38) and (5.41), yield

$$\big\|x^*(t_f,\varepsilon)-\bar{x}^*(t_f)\big\|\le\sum_{l=1}^{5}c_{x,l}\varepsilon^l\quad\forall\varepsilon\in(0,\varepsilon_2], \tag{5.48}$$

where

$$c_{x,l}=c_{u,l}\int_0^{t_f}\big\|\mathscr{K}_u(t)\big\|dt+c_{v,l}\int_0^{t_f}\big\|\mathscr{K}_v(t)\big\|dt,\quad l=1,\dots,5. \tag{5.49}$$

Now, using the equations (2.6), (3.15), (5.37) and the inequalities (5.38), (5.41), and (5.48), we obtain after a routine algebra the following inequality in the OPCCG:

$$\big|J_\varepsilon\big(u^*(t,\varepsilon),v^*(t,\varepsilon)\big)-J_\varepsilon\big(u_0^*(t),v_0^*(t)\big)\big|\le g_x(\varepsilon)+g_u(\varepsilon)+g_v(\varepsilon)\quad\forall\,\varepsilon\in(0,\varepsilon_2], \tag{5.50}$$

where

$$\begin{aligned}
g_x(\varepsilon)&=\big\|\mathscr{F}\bar{x}^*(t_f)\big\|\left(\sum_{l=1}^{5}c_{x,l}\varepsilon^l\right)+\frac{1}{2}\|\mathscr{F}\|\left(\sum_{l=1}^{5}c_{x,l}\varepsilon^l\right)^2,\\
g_u(\varepsilon)&=\left(\int_0^{t_f}\big\|R_{u,1}(t)u_{10}^*(t)\big\|dt+\varepsilon\int_0^{t_f}\big\|R_{u,2}(t)u_{20}^*(t)\big\|dt\right)\|\left(\sum_{l=1}^{5}c_{u,l}\varepsilon^l\right)\\
&\quad+\frac{1}{2}\left(\int_0^{t_f}\big\|R_{u,1}(t)\big\|dt+\varepsilon\int_0^{t_f}\big\|R_{u,2}(t)\big\|dt\right)\|\left(\sum_{l=1}^{5}c_{u,l}\varepsilon^l\right)^2,\\
g_v(\varepsilon)&=\left(\int_0^{t_f}\big\|R_v(t)v_0^*(t)\big\|dt\right)\left(\sum_{l=1}^{5}c_{v,l}\varepsilon^l\right)+\frac{1}{2}\left(\int_0^{t_f}\big\|R_v(t)\big\|dt\right)\left(\sum_{l=1}^{5}c_{v,l}\varepsilon^l\right)^2,
\end{aligned} \tag{5.51}$$

and

$$\lim_{\varepsilon\to+0}\frac{g_x(\varepsilon)+g_u(\varepsilon)+g_v(\varepsilon)}{\varepsilon}=c_{x,1}\big\|\mathscr{F}\bar{x}^*(t_f)\big\|+c_{u,1}\int_0^{t_f}\big\|R_{u,1}(t)u_{10}^*(t)\big\|dt+c_{v,1}\int_0^{t_f}\big\|R_v(t)v_0^*(t)\big\|dt. \tag{5.52}$$

Remark 5.1. Definition 2.3 and Remark 3.2, along with the inequality (5.50) and the equations (5.51)-(5.52), mean that the ε-independent pair of the controls $\big(u_0^*(t)\ ,\ v_0^*(t)\big)$, $t\in[0,t_f]$ is an open-loop quasi saddle point of the OPCCG for all sufficiently small $\varepsilon>0$.

Since the OPCCG value J_ε^* equals to $J_\varepsilon\big(u^*(t,\varepsilon),v^*(t,\varepsilon)\big)$, then the inequalities (5.44) and (5.50) yield the inequality

$$\big|J_\varepsilon\big(u_0^*(t),v_0^*(t)\big)-J_0^*\big|\le\sum_{l=1}^{5}\alpha_l\varepsilon^l+g_x(\varepsilon)+g_u(\varepsilon)+g_v(\varepsilon)\quad\forall\,\varepsilon\in(0,\varepsilon_2]. \tag{5.53}$$

6. OPEN-LOOP SADDLE POINT SOLUTION OF THE ORIGINAL DEGENERATE GAME

We start the derivation of the open-loop saddle point solution to the ODG with computing the value $J_0\big(u_0^*(t),v_0^*(t)\big)$ in this game.

From the equations (2.6), (3.15), and (3.16), we directly have the following inequality for all $\varepsilon > 0$:

$$\Big|J_\varepsilon\big(u_0^*(t),v_0^*(t)\big) - J_0\big(u_0^*(t),v_0^*(t)\big)\Big| \le \frac{\varepsilon}{2}\int_0^{t_f} \Big|\big(u_{20}^*(t)\big)^T R_{u,2}(t)u_{2,0}^*(t)\Big| dt. \tag{6.1}$$

Since the values $J_0\big(u_0^*(t),v_0^*(t)\big)$ and J_0^* are independent of $\varepsilon > 0$, then the inequalities (5.53) and (6.1) yield immediately the following equality

$$J_0\big(u_0^*(t),v_0^*(t)\big) = J_0^*. \tag{6.2}$$

Remember that J_0^* is given by the equation (5.43).

Let us substitute $u(t) = u_0^*(t)$ into the initial-value problem (3.10)-(3.11) and the functional (3.16). This substitution yields the optimal control problem consisting of the equation of dynamics

$$\frac{dx(t)}{dt} = \mathscr{K}_u(t)u_0^*(t) + \mathscr{K}_v(t)v(t), \quad t \in [0,t_f], \quad x(0) = x_0, \tag{6.3}$$

and the performance index

$$\widetilde{J}\big(v(t)\big) \overset{\triangle}{=} J_0\big(u_0^*(t),v(t)\big)$$
$$= \frac{1}{2}x^T(t_f)\mathscr{F}x(t_f) + \frac{1}{2}\int_0^{t_f}\Big[\big(u_0^*(t)\big)^T R_u(t,0)u_0^*(t) - v^T(t)R_v(t)v(t)\Big]dt \to \sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]}. \tag{6.4}$$

Lemma 6.1. *Let the assumptions A1-A7 be satisfied. Then, the optimal value $\widetilde{J}^*$ of the cost functional $\widetilde{J}\big(v(t)\big)$ in the optimal control problem (6.3)-(6.4) is*

$$\widetilde{J}^* = J_0^*. \tag{6.5}$$

Proof. For any given $v(t) \in L^2[0,t_f;\mathbb{R}^s]$ and any $\varepsilon > 0$, we have the following inequality along trajectories of the initial-value problem (3.10)-(3.11):

$$J_0\big(u^*(t,\varepsilon),v(t)\big) \le J_\varepsilon\big(u^*(t,\varepsilon),v(t)\big). \tag{6.6}$$

Remember that the functional $J_\varepsilon\big(u(t),v(t)\big)$ is given by (3.15).

Using the inequality (6.6), as well as the fact that the pair $\big(u^*(t,\varepsilon)\ ,\ v^*(t,\varepsilon)\big)$ is the open-loop saddle point solution to the OPCCG, we obtain the following chain of the inequalities and the equalities along trajectories of the initial-value problem (3.10)-(3.11):

$$J_0\big(u^*(t,\varepsilon),v_0^*(t)\big) \le \sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]} J_0\big(u^*(t,\varepsilon),v(t)\big) \le \sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]} J_\varepsilon\big(u^*(t,\varepsilon),v(t)\big)$$
$$= J_\varepsilon\big(u^*(t,\varepsilon),v^*(t,\varepsilon)\big) = J_\varepsilon^*. \tag{6.7}$$

Let $\widetilde{x}^*(t,\varepsilon)$, $t\in[0,t_f]$ be the solution of the initial-value problem (3.10)-(3.11) generated by the controls $u(t)=u^*(t,\varepsilon)$ and $v(t)=v_0^*(t)$, $t\in[0,t_f]$. Thus,

$$\widetilde{x}^*(t_f,\varepsilon)=x_0+\int_0^{t_f}\big[\mathscr{K}_u(t)u^*(t,\varepsilon)+\mathscr{K}_v(t)v_0^*(t)\big]dt. \tag{6.8}$$

Using this equation and the equation (5.47), we have similarly to the inequality (5.48)

$$\big\|x^*(t_f,\varepsilon)-\widetilde{x}^*(t_f,\varepsilon)\big\|\le\sum_{l=1}^{5}\widetilde{c}_{x,l}\varepsilon^l\quad\forall\varepsilon\in(0,\varepsilon_2], \tag{6.9}$$

where

$$\widetilde{c}_{x,l}=c_{v,l}\int_0^{t_f}\big\|\mathscr{K}_v(t)\big\|dt,\quad l=1,...,5, \tag{6.10}$$

and $c_{v,l}$, $(l=1,...,5)$ are given by (5.42).

The inequalities (5.48) and (6.9) yield

$$\big\|\widetilde{x}^*(t_f,\varepsilon)\big\|\le\big\|\bar{x}^*(t_f)\big\|+\sum_{l=1}^{5}\big(c_{x,l}+\widetilde{c}_{x,l}\big)\varepsilon^l\quad\forall\varepsilon\in(0,\varepsilon_2]. \tag{6.11}$$

Further, using the equations (2.6), (3.15), (3.16), (5.26), and the inequalities (5.35), (5.41), (6.9), and (6.11), we obtain after a routine algebra the following inequality

$$\Big|J_\varepsilon\big(u^*(t,\varepsilon),v^*(t,\varepsilon)\big)-J_0\big(u^*(t,\varepsilon),v_0^*(t)\big)\Big|\le\gamma_1\varepsilon\quad\forall\,\varepsilon\in(0,\varepsilon_2], \tag{6.12}$$

where $\gamma_1>0$ is some constant independent of ε.

From the inequalities (5.44) and (6.12), and the chain of the inequalities and the equalities (6.7), we directly have the following inequality along trajectories of the initial-value problem (3.10)-(3.11):

$$\left|\sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]}J_0\big(u^*(t,\varepsilon),v(t)\big)-J_0^*\right|\le\gamma_2\varepsilon\quad\forall\,\varepsilon\in(0,\varepsilon_2], \tag{6.13}$$

where $\gamma_2>0$ is some constant independent of ε.

Now, we are going to establish a relation between the value $\sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]}J_0\big(u^*(t,\varepsilon),v(t)\big)$, appearing in the inequality (6.13), and the value $\sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]}J_0\big(u_0^*(t),v(t)\big)$, which is the optimal value of the cost functional in the optimal control problem (6.3)-(6.4). Based on the results of [6] (Section 9.4), we obtain that the value $\sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]}J_0\big(u^*(t,\varepsilon),v(t)\big)$ is attained for

$$v(t)=v_1(t,\varepsilon)\overset{\triangle}{=}R_v^{-1}(t)\mathscr{K}_v^T(t)N(t)x_1(t,\varepsilon)-R_v^{-1}(t)\mathscr{K}_v^T(t)y_1(t,\varepsilon),\quad t\in[0,t_f], \tag{6.14}$$

where the matrix-valued function $N(t)$ is given by (4.8); for a given $\varepsilon\in(0,\varepsilon_2]$, the vector-valued function $y_1(t,\varepsilon)$ is the unique solution of the terminal-valued problem

$$\frac{dy_1(t)}{dt}=-N(t)S_v(t)y_1(t)+N(t)\mathscr{K}_u(t)u^*(t,\varepsilon),\quad t\in[0,t_f],\ \ y_1(t_f)=0; \tag{6.15}$$

for a given $\varepsilon\in(0,\varepsilon_2]$, the vector-valued function $x_1(t,\varepsilon)$ is the unique solution of the initial-value problem

$$\frac{dx_1(t)}{dt}=\mathscr{K}_u(t)u^*(t,\varepsilon)+S_v(t)N(t)x_1(t)-S_v(t)y_1(t,\varepsilon),\quad t\in[0,t_f],\ \ x_1(0)=x_0; \tag{6.16}$$

the vector x_0 is the initial value of the unknown vector-valued function in the problem (3.10)-(3.11); the matrix-valued function $S_v(t)$ is given in (4.2). Thus,

$$\sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]} J_0\big(u^*(t,\varepsilon),v(t)\big) = J_0\big(u^*(t,\varepsilon),v_1(t,\varepsilon)\big), \quad \varepsilon\in(0,\varepsilon_2]. \tag{6.17}$$

Similarly, we obtain that the value $\sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]} J_0\big(u_0^*(t),v(t)\big)$ is attained for

$$v(t) = v_2(t) \overset{\triangle}{=} R_v^{-1}(t)\mathscr{K}_v^T(t)N(t)x_2(t) - R_v^{-1}(t)\mathscr{K}_v^T(t)y_2(t), \quad t\in[0,t_f], \tag{6.18}$$

where the vector-valued function $y_2(t)$ is the unique solution of the terminal-valued problem

$$\frac{dy_2(t)}{dt} = -N(t)S_v(t)y_2(t) + N(t)\mathscr{K}_u(t)u_0^*(t), \quad t\in[0,t_f], \;\; y_1(t_f)=0; \tag{6.19}$$

the vector-valued function $x_2(t)$ is the unique solution of the initial-value problem

$$\frac{dx_2(t)}{dt} = \mathscr{K}_u(t)u_0^*(t) + S_v(t)N(t)x_2(t) - S_v(t)y_2(t), \quad t\in[0,t_f], \;\; x_2(0)=x_0. \tag{6.20}$$

Thus,

$$\sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]} J_0\big(u_0^*(t),v(t)\big) = J_0\big(u_0^*(t),v_2(t)\big). \tag{6.21}$$

Using the inequality (5.38), as well as that $y_1(t,\varepsilon)$ and $y_2(t)$ are the solutions of the terminal-value problems (6.15) and (6.19), respectively, we directly have

$$\big\|y_1(t,\varepsilon)-y_2(t)\big\| \le \gamma_3\varepsilon \quad \forall\, t\in[0,t_f],\; \varepsilon\in(0,\varepsilon_2], \tag{6.22}$$

where $\gamma_3>0$ is some constant independent of ε.

Furthermore, using the inequalities (5.38) and (6.22), as well as that $x_1(t,\varepsilon)$ and $x_2(t)$ are the solutions of the initial-value problems (6.16) and (6.20), respectively, we have immediately

$$\big\|x_1(t,\varepsilon)-x_2(t)\big\| \le \gamma_4\varepsilon \quad \forall\, t\in[0,t_f],\; \varepsilon\in(0,\varepsilon_2], \tag{6.23}$$

where $\gamma_4>0$ is some constant independent of ε.

The equations (6.14) and (6.18), and the inequalities (6.22) and (6.23) yield the inequality

$$\big\|v_1(t,\varepsilon)-v_2(t)\big\| \le \gamma_5\varepsilon \quad \forall\, t\in[0,t_f],\; \varepsilon\in(0,\varepsilon_2], \tag{6.24}$$

where $\gamma_5>0$ is some constant independent of ε.

Now, using the equations (3.16), (6.17), (6.21), and the inequalities (5.38), (6.23), and (6.24), we obtain

$$\left|\sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]} J_0\big(u^*(t,\varepsilon),v(t)\big) - \sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]} J_0\big(u_0^*(t),v(t)\big)\right| \le \gamma_6\varepsilon \quad \forall\,\varepsilon\in(0,\varepsilon_2], \tag{6.25}$$

where $\gamma_6>0$ is some constant independent of ε.

Finally, the inequalities (6.13) and (6.25) imply the equality

$$J_0^* = \sup_{v(t)\in L^2[0,t_f;\mathbb{R}^s]} J_0\big(u_0^*(t),v(t)\big), \tag{6.26}$$

which means the fulfilment of the equality (6.5). Thus, the lemma is proven. □

Let us substitute $v(t) = v_0^*(t)$ into the initial-value problem (3.10)-(3.11) and the functional (3.16). This substitution yields the optimal control problem consisting of the equation of dynamics

$$\frac{dx(t)}{dt} = \mathscr{K}_u(t)u(t) + \mathscr{K}_v(t)v_0^*(t), \quad t \in [0, t_f], \quad x(0) = x_0, \tag{6.27}$$

and the performance index

$$\widehat{J}\big(u(t)\big) \stackrel{\triangle}{=} J_0\big(u(t), v_0^*(t)\big)$$
$$= \frac{1}{2}x^T(t_f)\mathscr{F}x(t_f) + \frac{1}{2}\int_0^{t_f}\left[u(t)^T R_u(t,0)u(t) - \big(v_0^*(t)\big)^T R_v(t)v_0^*(t)\right]dt \to \inf_{u(t)\in L^2[0,t_f;\mathbb{R}^r]}. \tag{6.28}$$

Lemma 6.2. *Let the assumptions A1-A7 be satisfied. Then, the optimal value $\widehat{J}^*$ of the cost functional $\widehat{J}\big(u(t)\big)$ in the optimal control problem (6.27)-(6.28) is*

$$\widehat{J}^* = J_0^*. \tag{6.29}$$

Proof. Since $\det R_u(t,0) = 0$, $t \in [0, t_f]$, the optimal control problem (6.27)-(6.28) is singular (see e.g. [10]), and the optimal value $\widehat{J}^*$ of its cost functional can be derived similarly to this work. To obtain $\widehat{J}^*$, first, we replace approximately the singular optimal control problem (6.27)-(6.28) with the regular one consisting of the system (6.27) and the new (regular) cost functional

$$\widehat{J}_\varepsilon\big(u(t)\big) \stackrel{\triangle}{=} J_\varepsilon\big(u(t), v_0^*(t)\big)$$
$$= \frac{1}{2}x^T(t_f)\mathscr{F}x(t_f) + \frac{1}{2}\int_0^{t_f}\left[u(t)^T R_u(t,\varepsilon)u(t) - \big(v_0^*(t)\big)^T R_v(t)v_0^*(t)\right]dt \tag{6.30}$$

to be minimized by $u(t) \in L^2[0, t_f; \mathbb{R}^r]$ along trajectories of the system (6.27). In (6.30), $\varepsilon > 0$ is a small parameter of the regularization.

Based on the results of [6] (Section 9.4), we obtain that the solution (the optimal control) of the problem (6.27), (6.30) has the form

$$u(t) = \hat{u}(t,\varepsilon) \stackrel{\triangle}{=} -R_u^{-1}(t,\varepsilon)\mathscr{K}_u^T(t)\Big(\widehat{N}(t,\varepsilon)\hat{x}(t,\varepsilon) + \hat{y}(t,\varepsilon)\Big), \quad t \in [0, t_f], \tag{6.31}$$

where, for a given $\varepsilon \in (0, \varepsilon_2]$, the matrix-valued function $\widehat{N}(t,\varepsilon)$ is the unique solution of the terminal-value problem

$$\frac{d\widehat{N}(t)}{dt} = \widehat{N}(t)S_u(t,\varepsilon)\widehat{N}(t), \quad t \in [0, t_f], \quad \widehat{N}(t_f) = \mathscr{F}; \tag{6.32}$$

for a given $\varepsilon \in (0, \varepsilon_2]$, the vector-valued function $\hat{y}(t,\varepsilon)$ is the unique solution of the terminal-valued problem

$$\frac{d\hat{y}(t)}{dt} = \widehat{N}(t,\varepsilon)S_u(t,\varepsilon)\hat{y}(t) - \widehat{N}(t,\varepsilon)\mathscr{K}_v(t)v_0^*(t), \quad t \in [0, t_f], \ \ \hat{y}(t_f) = 0; \tag{6.33}$$

for a given $\varepsilon \in (0, \varepsilon_2]$, the vector-valued function $\hat{x}(t,\varepsilon)$ is the unique solution of the initial-value problem

$$\frac{d\hat{x}(t)}{dt} = -S_u(t,\varepsilon)\Big(\widehat{N}(t,\varepsilon)\hat{x}(t) + \hat{y}(t,\varepsilon)\Big) + \mathscr{K}_v(t)v_0^*(t), \quad t \in [0, t_f], \ \ \hat{x}(0) = x_0; \tag{6.34}$$

the vector x_0 is the initial value of the unknown vector-valued function in the problem (3.10)-(3.11); the matrix-valued function $S_u(t,\varepsilon)$ is given by (4.5). Thus, for a given $\varepsilon \in (0,\varepsilon_2]$,

$$\widehat{J}_\varepsilon^* \stackrel{\triangle}{=} \min_{u(t)\in L^2[0,t_f;\mathbb{R}^r]} \widehat{J}_\varepsilon\big(u(t)\big) = J_\varepsilon\big(\hat{u}(t,\varepsilon), v_0^*(t)\big)$$
$$= \frac{1}{2}\hat{x}^T(t_f,\varepsilon)\mathscr{F}\hat{x}(t_f,\varepsilon) + \frac{1}{2}\int_0^{t_f}\Big[\hat{u}^T(t,\varepsilon)R_u(t,\varepsilon)\hat{u}(t,\varepsilon) - \big(v_0^*(t)\big)^T R_v(t)v_0^*(t)\Big]dt. \tag{6.35}$$

For a given $\varepsilon \in (0,\varepsilon_2]$, consider the matrix-valued function

$$\widehat{W}_u(t,\varepsilon) \stackrel{\triangle}{=} I_n - F\int_{t_f}^t S_u(\sigma,\varepsilon)d\sigma F^T, \quad t \in [0,t_f], \tag{6.36}$$

where the matrix F is defined in the assumption A5 (see the equation (4.1)).

Since the matrix $S_u(t,\varepsilon)$ is at least positive semi-definite for all $t \in [0,t_f]$ and $\varepsilon \in (0,\varepsilon_2]$, the matrix $\widehat{W}_u(t,\varepsilon)$ is positive definite (and, therefore, invertible) for all $t \in [0,t_f]$ and $\varepsilon \in (0,\varepsilon_2]$. Hence, for a given $\varepsilon \in (0,\varepsilon_2]$, the unique solution of the terminal-value problem (6.32) has the form

$$\widehat{N}(t,\varepsilon) = F^T\widehat{W}_u^{-1}(t)F, \quad t \in [0,t_f]. \tag{6.37}$$

Let us consider the expression $\Big(\widehat{N}(t,\varepsilon)\hat{x}(t,\varepsilon) + \hat{y}(t,\varepsilon)\Big)$. Differentiating this expression and using the equations (6.32)-(6.34), we obtain for a given $\varepsilon \in (0,\varepsilon_2]$

$$\frac{d}{dt}\Big(\widehat{N}(t,\varepsilon)\hat{x}(t,\varepsilon) + \hat{y}(t,\varepsilon)\Big) = \frac{d\widehat{N}(t,\varepsilon)}{dt}\hat{x}(t,\varepsilon) + \widehat{N}(t,\varepsilon)\frac{d\hat{x}(t,\varepsilon)}{dt} + \frac{d\hat{y}(t,\varepsilon)}{dt}$$
$$= \widehat{N}(t,\varepsilon)S_u(t,\varepsilon)\widehat{N}(t,\varepsilon)\hat{x}(t,\varepsilon) - \widehat{N}(t,\varepsilon)S_u(t,\varepsilon)\widehat{N}(t,\varepsilon)\hat{x}(t,\varepsilon) - \widehat{N}(t,\varepsilon)S_u(t,\varepsilon)\hat{y}(t,\varepsilon)$$
$$+\widehat{N}(t,\varepsilon)\mathscr{K}_v(t)v_0^*(t) + \widehat{N}(t,\varepsilon)S_u(t,\varepsilon)\hat{y}(t,\varepsilon) - \widehat{N}(t,\varepsilon)\mathscr{K}_v(t)v_0^*(t) = 0, \quad t \in [0,t_f].$$

Thus,

$$\Big(\widehat{N}(t,\varepsilon)\hat{x}(t,\varepsilon) + \hat{y}(t,\varepsilon)\Big) = c = const, \quad t \in [0,t_f].$$

Substitution of $t = t_f$ into the left-hand side of this equation and use of the terminal conditions for $\widehat{N}(t,\varepsilon)$ and $\hat{y}(t,\varepsilon)$ (see the equations (6.32)-(6.33)) yield that $c = \mathscr{F}\hat{x}(t_f,\varepsilon)$, implying

$$\Big(\widehat{N}(t,\varepsilon)\hat{x}(t,\varepsilon) + \hat{y}(t,\varepsilon)\Big) = \mathscr{F}\hat{x}(t_f,\varepsilon), \quad t \in [0,t_f]. \tag{6.38}$$

The latter, along with the equation (6.31), yields

$$\hat{u}(t,\varepsilon) = -R_u^{-1}(t,\varepsilon)\mathscr{K}_u^T(t)\mathscr{F}\hat{x}(t_f,\varepsilon), \quad t \in [0,t_f]. \tag{6.39}$$

Substituting (6.38) into (6.34), we obtain the following initial-value problem for the functional-differential equation with respect to $\hat{x}(t,\varepsilon)$:

$$\frac{d\hat{x}(t,\varepsilon)}{dt} = -S_u(t,\varepsilon)\mathscr{F}\hat{x}(t_f,\varepsilon) + \mathscr{K}_v(t)v_0^*(t), \quad t \in [0,t_f], \quad \hat{x}(0,\varepsilon) = x_0. \tag{6.40}$$

Integrating the functional-differential equation in (6.40) from $t = 0$ to $t = t_f$ and using the corresponding initial condition yield

$$\hat{x}(t_f,\varepsilon) = x_0 - \int_0^{t_f} S_u(t,\varepsilon)dt\mathscr{F}\hat{x}(t_f,\varepsilon) + \int_0^{t_f}\mathscr{K}_v(t)v_0^*(t)dt. \tag{6.41}$$

Now our aim is, using the equation (4.1), to find the expression $F\hat{x}(t_f,\varepsilon)$, because, due to (6.39), just $F\hat{x}(t_f,\varepsilon)$ (but not the pure $\hat{x}(t_f,\varepsilon)$) appears in the equation (6.35). Thus, multiplying the equation (6.41) from the left by the matrix F and using (6.36), we obtain after a routine algebra the following linear equation with respect to $F\hat{x}(t_f,\varepsilon)$:

$$\widehat{W}_u(0,\varepsilon)\Big(F\hat{x}(t_f,\varepsilon)\Big) = F\left(x_0 + \int_0^{t_f}\mathscr{K}_v(t)v_0^*(t)dt\right).$$

This equation has the unique solution

$$F\hat{x}(t_f,\varepsilon) = \widehat{W}_u^{-1}(0,\varepsilon)F\left(x_0 + \int_0^{t_f}\mathscr{K}_v(t)v_0^*(t)dt\right). \tag{6.42}$$

Let us analyze the asymptotic behaviour of the matrix $\widehat{W}_u^{-1}(0,\varepsilon)$ for all sufficiently small $\varepsilon > 0$. Using the equation (6.36), we obtain (quite similarly to the equation (5.10)) the following representation for the matrix $\widehat{W}_u^{-1}(0,\varepsilon)$:

$$\widehat{W}_u^{-1}(0,\varepsilon) = L^T\widehat{\Gamma}^{-1}(\varepsilon)L, \tag{6.43}$$

where the matrix L is the same as in the equation (5.4); the matrix $\widehat{\Gamma}(\varepsilon)$ has the block form

$$\begin{gathered}\widehat{\Gamma}(\varepsilon) = \begin{pmatrix}\widehat{\Gamma}_1 & \widehat{\Gamma}_2\\ \widehat{\Gamma}_2^T & (1/\varepsilon)\widehat{\Gamma}_3(\varepsilon)\end{pmatrix},\\ \widehat{\Gamma}_1 = I_k + \Omega_{u,11},\quad \widehat{\Gamma}_2 = \Omega_{u,12},\\ \widehat{\Gamma}_3(\varepsilon) = \Theta_{u,2} + \varepsilon\big(I_{n-k} + \Omega_{u,13}\big);\end{gathered} \tag{6.44}$$

the matrices $\Omega_{u,11}$, $\Omega_{u,12}$, $\Omega_{u,13}$ are defined by the equations (5.1) and (5.9); the matrix $\Theta_{u,2}$ is defined by the equations (5.1), (5.3), and (5.4).

Similarly to the inequality (5.14), we obtain the existence of a positive number $\varepsilon_3 \le \varepsilon_2$ such that, for all $\varepsilon \in (0,\varepsilon_3]$, the following inequality is valid:

$$\big\|\widehat{\Gamma}_3^{-1}(\varepsilon) - \Theta_{u,2}^{-1}\big\| \le \delta_1\varepsilon, \tag{6.45}$$

where $\delta_1 > 0$ is some constant independent of ε.

Based on the inequality (6.45), we have (quite similarly to the equations (5.16)-(5.17)) the following block representation of the matrix $\widehat{\Gamma}^{-1}(\varepsilon)$ for all $\varepsilon \in (0,\varepsilon_4]$, where $0 < \varepsilon_4 \le \varepsilon_3$ (below, the number ε_4 is introduced more precisely):

$$\widehat{\Gamma}^{-1}(\varepsilon) = \widehat{\Phi}(\varepsilon) = \begin{pmatrix}\widehat{\Phi}_1(\varepsilon) & \widehat{\Phi}_2(\varepsilon)\\ \widehat{\Phi}_2^T(\varepsilon) & \widehat{\Phi}_3(\varepsilon)\end{pmatrix}, \tag{6.46}$$

where

$$\begin{aligned}\widehat{\Phi}_1(\varepsilon) &= \big(\widehat{\Gamma}_1 - \varepsilon\widehat{\Gamma}_2\widehat{\Gamma}_3^{-1}(\varepsilon)\widehat{\Gamma}_2^T\big)^{-1},\\ \widehat{\Phi}_2(\varepsilon) &= -\varepsilon\big(\widehat{\Gamma}_1 - \varepsilon\widehat{\Gamma}_2\widehat{\Gamma}_3^{-1}(\varepsilon)\widehat{\Gamma}_2^T\big)^{-1}\widehat{\Gamma}_2\widehat{\Gamma}_3^{-1}(\varepsilon),\\ \widehat{\Phi}_3(\varepsilon) &= \varepsilon\widehat{\Gamma}_3^{-1}(\varepsilon) + \varepsilon^2\widehat{\Gamma}_3^{-1}(\varepsilon)\widehat{\Gamma}_2^T\big(\widehat{\Gamma}_1 - \varepsilon\widehat{\Gamma}_2\widehat{\Gamma}_3^{-1}(\varepsilon)\widehat{\Gamma}_2^T\big)^{-1}\widehat{\Gamma}_2\widehat{\Gamma}_3^{-1}(\varepsilon).\end{aligned} \tag{6.47}$$

Further, using the inequality (6.45) and the equation (6.47), we obtain (quite similarly to the inequalities (5.21), (5.23), and (5.24)) the existence of a positive number $\varepsilon_4 \le \varepsilon_3$ such that, for all $\varepsilon \in (0, \varepsilon_4]$, the following inequalities are valid:

$$\left\|\widehat{\Phi}_1(\varepsilon) - \widehat{\Gamma}_1^{-1}\right\| \le \delta_2 \varepsilon, \tag{6.48}$$

$$\begin{aligned} \left\|\varepsilon^{-1}\widehat{\Phi}_2(\varepsilon) + \widehat{\Gamma}_1^{-1}\widehat{\Gamma}_2\Theta_{u,2}^{-1}\right\| \le \delta_2 \varepsilon, \\ \left\|\varepsilon^{-1}\widehat{\Phi}_3(\varepsilon) - \Theta_{u,2}^{-1}\right\| \le \delta_2 \varepsilon, \end{aligned} \tag{6.49}$$

$$\left\|\widehat{\Phi}_2(\varepsilon)\right\| \le \delta_2 \varepsilon, \quad \left\|\widehat{\Phi}_3(\varepsilon)\right\| \le \delta_2 \varepsilon, \tag{6.50}$$

where $\delta_2 > 0$ is some constant independent of ε.

Consider the n-vector

$$\hat{x}_{f,0} \stackrel{\triangle}{=} x_0 + \int_0^{t_f} \mathscr{K}_v(t) v_0^*(t) dt, \tag{6.51}$$

and the $n \times n$-matrix

$$\widehat{\Phi}_0 = \begin{pmatrix} \widehat{\Gamma}_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.52}$$

Now, using the equations (6.42), (6.43), (6.46), (6.47), (6.51), and the inequalities (6.48), (6.50), and that the matrix L is orthogonal, we obtain the inequality

$$\left\|LF\hat{x}(t_f, \varepsilon) - \widehat{\Phi}_0 LF\hat{x}_{f,0}\right\| \le \delta_3 \varepsilon \quad \forall\, \varepsilon \in (0, \varepsilon_4], \tag{6.53}$$

where $\delta_3 > 0$ is some constant independent of ε.

Proceed to the estimate of $\hat{u}(t,\varepsilon)$. Using the equations (4.1), (6.42), (6.43), (6.46), and (6.51), we rewrite the expression (6.39) for $\hat{u}(t,\varepsilon)$ as:

$$\hat{u}(t,\varepsilon) = -R_u^{-1}(t,\varepsilon)\mathscr{K}_u^T(t)(LF)^T\widehat{\Phi}(\varepsilon)LF\hat{x}_{f,0}, \quad t \in [0, t_f]. \tag{6.54}$$

Furthermore, quite similarly to (5.26), (5.27), (5.32), and (5.33), the expression (6.54) is transformed to the following block form:

$$\begin{aligned} \hat{u}(t,\varepsilon) = -\begin{pmatrix} R_{u,1}^{-1}(t) & 0 \\ 0 & \varepsilon^{-1}R_{u,2}^{-1}(t) \end{pmatrix} \begin{pmatrix} \mathscr{K}_{u,1}^T(t) \\ \mathscr{K}_{u,2}^T(t) \end{pmatrix} (LF)^T\widehat{\Phi}(\varepsilon)LF\hat{x}_{f,0} \\ = \begin{pmatrix} \hat{u}_1(t,\varepsilon) \\ \hat{u}_2(t,\varepsilon) \end{pmatrix}, \quad t \in [0, t_f], \end{aligned} \tag{6.55}$$

where

$$\begin{aligned} \hat{u}_1(t,\varepsilon) = -R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)(LF)^T\widehat{\Phi}(\varepsilon)LF\hat{x}_{f,0}, \quad t \in [0, t_f], \\ \hat{u}_2(t,\varepsilon) = -\Big(\varepsilon^{-1}R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^T(t)\widehat{\Phi}_2^T(\varepsilon)\;,\; \varepsilon^{-1}R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^T(t)\widehat{\Phi}_3(\varepsilon)\Big)LF\hat{x}_{f,0}, \quad t \in [0, t_f]. \end{aligned} \tag{6.56}$$

Consider the vector-valued function

$$\hat{u}_0(t) = \mathrm{col}\Big(\hat{u}_{10}(t)\ ,\ \hat{u}_{10}(t)\Big), \quad t \in [0, t_f], \tag{6.57}$$

where

$$\hat{u}_{10}(t) \triangleq -R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)(LF)^T\widehat{\Phi}_0 LF\hat{x}_{f,0}, \quad t \in [0, t_f],$$

$$\hat{u}_{20}(t) \triangleq \Big(R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^T(t)\Theta_{u,2}^{-1}\widehat{\Gamma}_2^T\widehat{\Gamma}_1^{-1}\ ,\ -R_{u,2}^{-1}(t)\mathscr{E}_{u,2}^T(t)\Theta_{u,2}^{-1}\Big)LF\hat{x}_{f,0}, \quad t \in [0, t_f]. \tag{6.58}$$

Using the equations (6.52), (6.55)-(6.58) and the inequalities (6.48)-(6.50), we have (quite similarly to the inequalities (5.30), (5.35), and (5.38)) the following inequality:

$$\big\|\hat{u}(t,\varepsilon) - \hat{u}_0(t)\big\| \le \delta_4\varepsilon \quad \forall\, \varepsilon \in (0, \varepsilon_4], \tag{6.59}$$

where $\delta_4 > 0$ is some constant independent of ε.

Proceed to the estimate of $\widehat{J}_\varepsilon^*$. Using the equations (2.6), (4.1), (6.42), (6.43), (6.46), (6.51), (6.55), and taking into account that L is an orthogonal matrix, we can rewrite the expression (6.35) for $\widehat{J}_\varepsilon^*$ in the form

$$\widehat{J}_\varepsilon^* = \frac{1}{2}\hat{x}_{f,0}^T(LF)^T\big(\widehat{\Phi}(\varepsilon)\big)^2 LF\hat{x}_{f,0}$$
$$+\frac{1}{2}\int_0^{t_f}\Big[\hat{u}_1^T(t,\varepsilon)R_{u,1}(t)\hat{u}_1(t,\varepsilon) + \varepsilon\hat{u}_2^T(t,\varepsilon)R_{u,2}(t)\hat{u}_2(t,\varepsilon) - \big(v_0^*(t)\big)^T R_v(t)v_0^*(t)\Big]dt. \tag{6.60}$$

Consider the value

$$\widehat{J}_0^* \triangleq \frac{1}{2}\hat{x}_{f,0}^T(LF)^T\big(\widehat{\Phi}_0\big)^2 LF\hat{x}_{f,0}$$
$$+\frac{1}{2}\int_0^{t_f}\Big[\hat{u}_{10}^T(t)R_{u,1}(t)\hat{u}_{10}(t) - \big(v_0^*(t)\big)^T R_v(t)v_0^*(t)\Big]dt. \tag{6.61}$$

Using the equations (6.52), (6.57), (6.60), (6.61) and the inequalities (6.48), (6.50), and (6.59), we directly obtain the inequality

$$\big|\widehat{J}_\varepsilon^* - \widehat{J}_0^*\big| \le \delta_5\varepsilon \quad \forall\, \varepsilon \in (0, \varepsilon_4], \tag{6.62}$$

where $\delta_5 > 0$ is some constant independent of ε.

Let us transform the expression (6.61) for $\widehat{J}_0^*$. Substitution of the expression for $\hat{u}_{10}(t)$ (see the equation (6.58)) into (6.61) yields after a routine algebra

$$\widehat{J}_0^* = \frac{1}{2}\hat{x}_{f,0}^T(LF)^T\widehat{\Phi}_0\left(I_n + LF\int_0^{t_f}\mathscr{K}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)dt(LF)^T\right)\widehat{\Phi}_0 LF\hat{x}_{f,0}$$
$$-\frac{1}{2}\int_0^{t_f}\big(v_0^*(t)\big)^T R_v(t)v_0^*(t)dt. \tag{6.63}$$

Using the equations (5.9) and (6.44), we directly have

$$I_n + LF\int_0^{t_f}\mathscr{K}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)dt(LF)^T = \begin{pmatrix}\widehat{\Gamma}_1 & \widehat{\Gamma}_2\\ \widehat{\Gamma}_2^T & I_{n-k} + \Omega_{u,13}\end{pmatrix}. \tag{6.64}$$

Substituting (6.64) into (6.63), we obtain after the corresponding matrices' multiplication the following expression for $\widehat{J}_0^*$:

$$\widehat{J}_0^* = \frac{1}{2}\hat{x}_{f,0}^T(LF)^T\widehat{\Phi}_0 LF\hat{x}_{f,0} - \frac{1}{2}\int_0^{t_f}\big(v_0^*(t)\big)^T R_v(t)v_0^*(t)dt. \tag{6.65}$$

Let us treat each of the addends in the right-hand side of (6.65) separately. We start with the first one. Using (6.51) yields

$$\frac{1}{2}\hat{x}_{f,0}^T(LF)^T\widehat{\Phi}_0 LF\hat{x}_{f,0} = \frac{1}{2}\Bigg(x_0^T(LF)^T\widehat{\Phi}_0 LFx_0 + 2x_0^T(LF)^T\widehat{\Phi}_0 LF\int_0^{t_f}\mathscr{K}_v(t)v_0^*(t)dt + \int_0^{t_f}\big(v_0^*(t)\big)^T\mathscr{K}_v^T(t)dt(LF)^T\widehat{\Phi}_0 LF\int_0^{t_f}\mathscr{K}_v(t)v_0^*(t)dt\Bigg). \tag{6.66}$$

Using the equation (5.40), as well as the equations (4.2) and (5.9), we can transform the vector $LF\int_0^{t_f}\mathscr{K}_v(t)v_0^*(t)dt$, appearing in the right-hand side of (6.66), as follows:

$$LF\int_0^{t_f}\mathscr{K}_v(t)v_0^*(t)dt = \left(LF\int_0^{t_f}\mathscr{K}_v(t)R_v^{-1}(t)\mathscr{K}_v^T(t)dt(LF)^T\right)\Phi_0 LFx_0 = \left(LF\int_0^{t_f}S_v(t)dt(LF)^T\right)\Phi_0 LFx_0 = \begin{pmatrix}\Lambda_{v,1} & \Lambda_{v,2}\\ \Lambda_{v,2}^T & \Lambda_{v,3}\end{pmatrix}\Phi_0 LFx_0. \tag{6.67}$$

Substitution of (6.67) into the right-hand side of (6.66), and use of the equations (5.28) and (6.52) yield

$$\frac{1}{2}\hat{x}_{f,0}^T(LF)^T\widehat{\Phi}_0 LF\hat{x}_{f,0} = \frac{1}{2}x_0^T(LF)^T\Bigg[\widehat{\Phi}_0 + \widehat{\Phi}_0\begin{pmatrix}\Lambda_{v,1} & \Lambda_{v,2}\\ \Lambda_{v,2}^T & \Lambda_{v,3}\end{pmatrix}\Phi_0 + \Phi_0\begin{pmatrix}\Lambda_{v,1} & \Lambda_{v,2}\\ \Lambda_{v,2}^T & \Lambda_{v,3}\end{pmatrix}\widehat{\Phi}_0 + \Phi_0\begin{pmatrix}\Lambda_{v,1} & \Lambda_{v,2}\\ \Lambda_{v,2}^T & \Lambda_{v,3}\end{pmatrix}\widehat{\Phi}_0\begin{pmatrix}\Lambda_{v,1} & \Lambda_{v,2}\\ \Lambda_{v,2}^T & \Lambda_{v,3}\end{pmatrix}\Phi_0\Bigg]LFx_0$$
$$= \frac{1}{2}x_0^T(LF)^T\begin{pmatrix}\widehat{\Gamma}_1^{-1} + \widehat{\Gamma}_1^{-1}\Lambda_{v,1}\Gamma_1^{-1} + \Gamma_1^{-1}\Lambda_{v,1}\widehat{\Gamma}_1^{-1} + \Gamma_1^{-1}\Lambda_{v,1}\widehat{\Gamma}_1^{-1}\Lambda_{v,1}\Gamma_1^{-1} & 0\\ 0 & 0\end{pmatrix}LFx_0. \tag{6.68}$$

Proceed to the second addend in the right-hand side of (6.65). Using the equation (5.40), as well as the equations (4.2), (5.9), and (5.28), we obtain

$$\frac{1}{2}\int_0^{t_f}\big(v_0^*(t)\big)^T R_v(t)v_0^*(t)dt = \frac{1}{2}x_0^T(LF)^T\Phi_0\left(LF\int_0^{t_f}S_v(t)dt(LF)^T\right)\Phi_0 LFx_0 = \frac{1}{2}x_0^T(LF)^T\begin{pmatrix}\Gamma_1^{-1}\Lambda_{v,1}\Gamma_1^{-1} & 0\\ 0 & 0\end{pmatrix}LFx_0. \tag{6.69}$$

Substitution of (6.68) and (6.69) into (6.65) directly yields

$$\widehat{J}_0^* = \frac{1}{2}x_0^T(LF)^T\begin{pmatrix}\Delta_1 & 0\\ 0 & 0\end{pmatrix}LFx_0, \tag{6.70}$$

where

$$\Delta_1 = \widehat{\Gamma}_1^{-1} + \widehat{\Gamma}_1^{-1}\Lambda_{v,1}\Gamma_1^{-1} + \Gamma_1^{-1}\Lambda_{v,1}\widehat{\Gamma}_1^{-1} + \Gamma_1^{-1}\Lambda_{v,1}\widehat{\Gamma}_1^{-1}\Lambda_{v,1}\Gamma_1^{-1} - \Gamma_1^{-1}\Lambda_{v,1}\Gamma_1^{-1}. \qquad (6.71)$$

Treating the expression (6.71) for the matrix Δ_1 and taking into account that $\widehat{\Gamma}_1 = \Gamma_1 + \Lambda_{v,1}$ (see the equations (5.11) and (6.44)), we obtain

$$\Delta_1 = \Gamma_1^{-1}\Big(\Gamma_1\widehat{\Gamma}_1^{-1}\Gamma_1 + \Gamma_1\widehat{\Gamma}_1^{-1}\Lambda_{v,1} + \Lambda_{v,1}\widehat{\Gamma}_1^{-1}\Gamma_1 + \Lambda_{v,1}\widehat{\Gamma}_1^{-1}\Lambda_{v,1} - \Lambda_{v,1}\Big)\Gamma_1^{-1}$$
$$= \Gamma_1^{-1}\Big(\Gamma_1\widehat{\Gamma}_1^{-1}\big(\Gamma_1 + \Lambda_{v,1}\big) + \Lambda_{v,1}\widehat{\Gamma}_1^{-1}\big(\Gamma_1 + \Lambda_{v,1}\big) - \Lambda_{v,1}\Big)\Gamma_1^{-1} = \Gamma_1^{-1}. \qquad (6.72)$$

Furthermore, substituting (6.72) into (6.70) and using the equations (5.28) and (5.43), we have immediately

$$\widehat{J}_0^* = J_0^*. \qquad (6.73)$$

This equality and the inequality (6.62) imply

$$\big|\widehat{J}_\varepsilon^* - J_0^*\big| \le \delta_5\varepsilon \quad \forall\, \varepsilon \in (0, \varepsilon_4]. \qquad (6.74)$$

Now, based on the inequality (6.74), we are going to prove the equality (6.29). Namely, due to (6.28), (6.35), and (6.74), we have

$$\widehat{J}^* = \inf_{u(t)\in L^2[0,t_f;\mathbb{R}^r]} \widehat{J}\big(u(t)\big) \le \widehat{J}\big(\hat{u}(t,\varepsilon)\big) \le \widehat{J}_\varepsilon\big(\hat{u}(t,\varepsilon)\big) = \widehat{J}_\varepsilon^* \le J_0^* + \delta_5\varepsilon \quad \forall\, \varepsilon \in (0,\varepsilon_4], \qquad (6.75)$$

yielding

$$\widehat{J}^* \le J_0^*. \qquad (6.76)$$

Let us show that the equality (6.29) is valid. For this purpose, we assume opposite which, by virtue of (6.76), is

$$\widehat{J}^* < J_0^*. \qquad (6.77)$$

Due to this inequality, there exists $\widetilde{u}(t) \in L^2[0,t_f;\mathbb{R}^r]$ such that

$$\widehat{J}^* < \widehat{J}\big(\widetilde{u}(t)\big) < J_0^*. \qquad (6.78)$$

Using the inequality (6.74) and that $\hat{u}(t,\varepsilon)$ is the optimal control in the problem (6.27) and (6.30), we obtain

$$J_0^* - \delta_5\varepsilon \le \widehat{J}_\varepsilon^* = \widehat{J}_\varepsilon\big(\hat{u}(t,\varepsilon)\big) \le \widehat{J}_\varepsilon\big(\widetilde{u}(t)\big) = \widehat{J}\big(\widetilde{u}(t)\big) + b\varepsilon \quad \forall\, \varepsilon \in (0,\varepsilon_4], \qquad (6.79)$$

where

$$b = \int_0^{t_f} \widetilde{u}_{\mathrm{low}}^T(t) R_2(t) \widetilde{u}_{\mathrm{low}}(t)\, dt$$

and $\widetilde{u}_{\mathrm{low}}(t)$ is the lower block of the vector $\widetilde{u}(t)$ of the dimension $r-q$.

From the chain of the equalities and the inequalities (6.79), we have the validity of the inequality $J_0^* \le \widehat{J}\big(\widetilde{u}(t)\big) + (b+\delta_5)\varepsilon$ for all $\varepsilon \in (0,\varepsilon_4]$, which yields the inequality $J_0^* \le \widehat{J}\big(\widetilde{u}(t)\big)$. The latter contradicts the right-hand side inequality in (6.78). This contradiction means that the inequality (6.77) is wrong, which implies the validity of the equality (6.29). Thus, the lemma is proven. □

Consider the pair of vector-valued functions $\big(u_0^*(t)\ ,\ v_0^*(t)\big)$, $t\in[0,t_f]$, where $u_0^*(t)$ is given by the equations (5.29), (5.34), and (5.37), while $v_0^*(t)$ is given by the equation (5.40).

Theorem 6.3. *Let the assumptions A1-A7 be satisfied. Then, the pair $\big(u_0^*(t)\ ,\ v_0^*(t)\big)$, $t\in[0,t_f]$ is the open-loop saddle point solution of the ODG. The value J_0^*, given by (5.43), is the ODG value.*

Proof. The statements of the theorem directly follow from the definitions of the open-loop saddle point solution to the ODG and the ODG value (see Remark 3.2 and Definition 2.4), as well as from the equation (6.2), Lemma 6.1 and Lemma 6.2. □

Corollary 6.4. *Let the assumptions A1-A7 be satisfied. Then, for $\varepsilon\to+0$, the open-loop saddle point solution of the OPCCG tends to the open-loop saddle point solution of the ODG uniformly in $t\in[0,t_f]$, and the OPCCG value tends to the ODG value, i.e.,*

$$\lim_{\varepsilon\to+0}\big(u^*(t,\varepsilon)\ ,\ v^*(t,\varepsilon)\big)=\big(u_0^*(t)\ ,\ v_0^*(t)\big),\quad t\in[0,t_f], \tag{6.80}$$

$$\lim_{\varepsilon\to+0}J_\varepsilon^*=J_0^*. \tag{6.81}$$

Proof. The statements of the corollary directly follow from Lemma 4.2, Theorem 6.3 and from the inequalities (5.38), (5.41), and (5.44). □

Corollary 6.5. *Let the assumptions A1-A7 be satisfied. Then, for $\varepsilon\to+0$, the open-loop saddle point solution of the game (2.1)-(2.5) tends to the open-loop saddle point solution of the game (2.1)-(2.4), (2.8) uniformly in $t\in[0,t_f]$, and the value of the game (2.1)-(2.5) tends to the value of the game (2.1)-(2.4), and (2.8).*

Proof. The statements of the corollary are an immediate consequence of Definition 3.3, Remark 3.6 and Corollary 6.4. □

7. Example

Consider the following particular case of the system (2.1):

$$\begin{aligned}\frac{dz_1(t)}{dt}&=z_1(t-1)+u_1(t)-u_2(t)-2u_1(t-1.5)+2u_2(t-1.5)\\&\quad -v_1(t)+v_2(t)+v_1(t-1)-v_2(t-1),\quad t\in[0,2],\end{aligned} \tag{7.1}$$

$$\begin{aligned}\frac{dz_2(t)}{dt}&=z_2(t-1)+2u_1(t)-u_2(t)+u_1(t-1.5)+u_2(t-1.5)\\&\quad +v_1(t)+v_2(t)-v_1(t-1)-v_2(t-1),\quad t\in[0,2].\end{aligned} \tag{7.2}$$

where $z_1(t)$, $z_2(t)$, $u_1(t)$, $u_2(t)$, $v_1(t)$, $v_2(t)$ are scalar variables.

The system (7.1)-(7.2) is subject to the initial conditions

$$z_1(\tau)=0,\quad \tau\in[-1,0),\quad z_1(0)=1, \tag{7.3}$$

$$z_2(\tau)=0,\quad \tau\in[-1,0),\quad z_2(0)=2, \tag{7.4}$$

$$u_1(\eta)=0,\quad u_2(\eta)=0,\quad \eta\in[-1.5,0), \tag{7.5}$$

$$v_1(\zeta)=0,\quad v_2(\zeta)=0,\quad \zeta\in[-1,0). \tag{7.6}$$

Comparing the system (7.1)-(7.2) with the system (2.1), we can conclude that in (7.1)-(7.2) $n = 2$, $r = 2$, $s = 2$,

$$N_z = 1, \quad N_u = 1, \quad N_v = 1, \quad h_{z,1} = 1, \quad h_{u,1} = 1.5, \quad h_{v,1} = 1, \quad t_f = 2; \quad f(t) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in [0,2], \tag{7.7}$$

and the matrices of the coefficients have the form

$$\mathscr{A}_0(t) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathscr{A}_1(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in [0,2], \tag{7.8}$$

$$\mathscr{G}(t,\tau) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (t,\tau) \in [0,2] \times [-1,0], \tag{7.9}$$

$$\mathscr{B}_0(t) \equiv \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad \mathscr{B}_1(t) \equiv \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}, \quad t \in [0,2], \tag{7.10}$$

$$\mathscr{P}(t,\eta) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (t,\eta) \in [0,2] \times [-1.5,0], \tag{7.11}$$

$$\mathscr{C}_0(t) \equiv \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathscr{C}_1(t) \equiv \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad t \in [0,2], \tag{7.12}$$

$$\mathscr{Q}(t,\zeta) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (t,\zeta) \in [0,2] \times [-1,0]. \tag{7.13}$$

Comparing the initial conditions (7.3)-(7.6) with the initial conditions (2.2)-(2.4), we can see that in (7.3)-(7.6)

$$\varphi_z(\tau) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \tau \in [-1,0); \quad \varphi_{0,z} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \tag{7.14}$$

$$\varphi_u(\eta) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \eta \in [-1.5,0); \quad \varphi_v(\zeta) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \zeta \in [-1,0). \tag{7.15}$$

In this example, the cost functional, to be minimized by the control $u(t) = \mathrm{col}\big(u_1(t), u_2(t)\big)$ and maximized by the control $v(t) = \mathrm{col}\big(v_1(t), v_2(t)\big)$, is

$$\begin{aligned} \mathscr{J}_\varepsilon\big(u(t), v(t)\big) = {} & \frac{1}{2}\big(0.1 z_1^2(2) + 0.2 z_1(2) z_2(2) + 0.1 z_2^2(2)\big) \\ & + \frac{1}{2}\int_0^2 \big[u_1^2(t) + \varepsilon u_2^2(t) - v_1^2(t) - v_2^2(t)\big]\, dt, \end{aligned} \tag{7.16}$$

where $\varepsilon > 0$ is a small parameter.

Comparing the cost functional (7.16) with the cost functional (2.5), we have that $q = 1$ and

$$\mathscr{F} = \begin{pmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}; \quad R_{u,1}(t) \equiv 1, \quad R_{u,2}(t) \equiv 1, \quad R_v(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in [0,2]. \tag{7.17}$$

Along with the cost functional (7.16), we consider the following cost functional:

$$\mathscr{J}_0\big(u(t),v(t)\big) = \frac{1}{2}\big(0.1z_1^2(2)+0.2z_1(2)z_2(2)+0.1z_2^2(2)\big) + \frac{1}{2}\int_0^2 \big[u_1^2(t)-v_1^2(t)-v_2^2(t)\big]dt. \tag{7.18}$$

Thus, the zero-sum differential game (7.1)-(7.6), (7.16) is a particular case of the zero-sum differential game (2.1)-(2.5), while the zero-sum differential game (7.1)-(7.6), (7.18) is a particular case of the zero-sum differential game (2.1)-(2.4), and (2.8).

In this example, the matrix-valued function $\Psi(t)$, defined by the terminal-value problem (3.1), is

$$\Psi(t) = \psi(t)I_2, \quad t\in[0,2], \tag{7.19}$$

where the scalar function $\psi(t)$ has the form

$$\psi(t) = \begin{cases} 2-t, & t\in[0,1], \\ 1, & t\in(1,2]. \end{cases} \tag{7.20}$$

Using the equations (7.7), (7.10)-(7.13), and (7.19)-(7.20), we obtain the matrix-valued functions $\mathscr{K}_u(t)$ and $\mathscr{K}_v(t)$, defined by (3.6)-(3.7), as follows:

$$\mathscr{K}_u(t) = \begin{cases} \begin{pmatrix} -t & t \\ 5-2t & t-1 \end{pmatrix}, & 0\le t\le 0.5, \\[2ex] \begin{pmatrix} 2-t & t-2 \\ 4-2t & t-2 \end{pmatrix}, & 0.5 < t \le 1, \\[2ex] \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, & 1 < t \le 2, \end{cases} \tag{7.21}$$

$$\mathscr{K}_v(t) = \begin{cases} \begin{pmatrix} t-1 & 1-t \\ 1-t & 1-t \end{pmatrix}, & 0\le t\le 1, \\[2ex] \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, & 1 < t \le 2. \end{cases} \tag{7.22}$$

Furthermore, using the equations (3.12), (3.13), (7.14)-(7.15), and (7.19)-(7.20), we can directly calculate the vector x_0

$$x_0 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}. \tag{7.23}$$

Thus, the zero-sum differential game (7.1)-(7.6), (7.16) is transformed to the equivalent zero-sum differential game (3.10)-(3.11) and (3.15), where the coefficients $\mathscr{K}_u(t)$ and $\mathscr{K}_v(t)$ in the equation of dynamics are given by (7.21) and (7.22), the initial position of the game is given by (7.23), the final time instant t_f is given in (7.7), the coefficients in the cost functional are given by (7.17). Due to Remark 3.6, we call this game the Original Partial Cheap Control Game (OPCCG). Similarly, the zero-sum differential game (7.1)-(7.6), (7.18) is transformed to the equivalent zero-sum differential game (3.10)-(3.11), (3.16) with the data given in (7.7), (7.17), (7.21), (7.22), and (7.23). Due to Remark 3.6, we call this game the Original Degenerate Game (ODG).

Let us choose the matrix F as:

$$F = \begin{pmatrix} \sqrt{0.05} & \sqrt{0.05} \\ \sqrt{0.05} & \sqrt{0.05} \end{pmatrix}. \tag{7.24}$$

For this matrix and the matrix $\mathscr{F}$, given in (7.17), the equality (4.1) is fulfilled. Moreover, using the equations (4.2), (7.7), (7.17), (7.22), and (7.24), we obtain by a routine matrix algebra

$$W_v(t) = \begin{cases} \begin{pmatrix} 1+0.2g_v(t) & 0.2g_v(t) \\ 0.2g_v(t) & 1+0.2g_v(t) \end{pmatrix}, & 0 \le t \le 1, \\ \\ \begin{pmatrix} 1+0.2(t-2) & 0.2(t-2) \\ 0.2(t-2) & 1+0.2(t-2) \end{pmatrix}, & 1 < t \le 2, \end{cases} \tag{7.25}$$

where

$$g_v(t) = \frac{(t-1)^3}{3} - 1, \quad t \in [0,1]. \tag{7.26}$$

The equations (7.25)-(7.26) yield the positive definite matrix

$$W_v(0) = \begin{pmatrix} \frac{22}{30} & -\frac{8}{30} \\ \\ -\frac{8}{30} & \frac{22}{30} \end{pmatrix}, \tag{7.27}$$

meaning the fulfilment of the assumption A5. Hence, by virtue of Lemma 4.2, the OPCCG is open-loop solvable. Let us derive the open-loop saddle point solution to this game and the OPCCG value. For this purpose, due to Lemma 4.2, we should calculate the matrix $M(0,\varepsilon)$. To make this calculation, we use the procedure proposed in Subsection 5.1. Namely, due to the equation (5.1), we partition the matrix $\mathscr{K}_u(t)$ into two blocks $\mathscr{K}_{u,1}(t)$ and $\mathscr{K}_{u,2}(t)$. Using (7.21), we have

$$\mathscr{K}_{u,1}(t) = \begin{cases} \begin{pmatrix} -t \\ 5-2t \end{pmatrix}, & 0 \le t \le 0.5 \\ \\ \begin{pmatrix} 2-t \\ 4-2t \end{pmatrix}, & 0.5 < t \le 1, \\ \\ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, & 1 < t \le 2, \end{cases} \tag{7.28}$$

$$\mathscr{K}_{u,2}(t) = \begin{cases} \begin{pmatrix} t \\ t-1 \end{pmatrix}, & 0 \le t \le 0.5 \\ \\ \begin{pmatrix} t-2 \\ t-2 \end{pmatrix}, & 0.5 < t \le 1, \\ \\ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, & 1 < t \le 2, \end{cases} \tag{7.29}$$

Using the equation (5.3), as well as the final instant $t_f = 2$, the scalar function $R_{u,2}(t)$ (see the equation (7.17)), the matrix F (see the equation (7.24)) and the vector-valued function $\mathscr{K}_{u,2}(t)$ (see the equation (7.29)), we directly calculate the matrix $K_{u,2}$

$$K_{u,2} = \begin{pmatrix} \frac{11}{30} & \frac{11}{30} \\ \\ \frac{11}{30} & \frac{11}{30} \end{pmatrix}. \tag{7.30}$$

This matrix has a simple zero eigenvalue, i.e., the assumption A6 is fulfilled. The orthogonal matrix L, appearing in the equation (5.4), can be chosen as

$$L = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{7.31}$$

Thus, using the matrices $K_{u,2}$, L, given by the equations (7.30) and (7.31), and calculating the matrix $D_{u,2}$, defined by the equation (5.4), we obtain

$$D_{u,2} = \begin{pmatrix} 0 & 0 \\ \\ 0 & \frac{22}{30} \end{pmatrix}. \tag{7.32}$$

Moreover, from (5.4) and (7.32), we have

$$\Theta_{u,2} = \frac{22}{30}. \tag{7.33}$$

Proceed to the calculation of the matrices defined in (5.9). Using the data of the OPCCG (see the equations (7.7) and (7.17)), as well as the matrix-valued function $\mathscr{K}_v(t)$, the matrix F, the vector-valued function $\mathscr{K}_{u,1}(t)$ and the matrix L (see the equations (7.22), (7.24), (7.28), and (7.31)), we obtain

$$\begin{aligned} LF\int_0^2 \mathscr{K}_{u,1}(t)R_{u,1}^{-1}(t)\mathscr{K}_{u,1}^T(t)dt(LF)^T = \begin{pmatrix} \Omega_{u,11} & \Omega_{u,12} \\ \Omega_{u,12}^T & \Omega_{u,13} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2.525 \end{pmatrix}, \\ LF\int_0^2 \mathscr{K}_v(t)R_v^{-1}(t)\mathscr{K}_v^T(t)dt(LF)^T = LF\int_0^2 S_v(t)dt(LF)^T \\ = \begin{pmatrix} \Lambda_{v,1} & \Lambda_{v,2} \\ \Lambda_{v,2}^T & \Lambda_{v,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \\ 0 & \frac{16}{30} \end{pmatrix}. \end{aligned} \tag{7.34}$$

This equation and the equation (5.11) yield that $\Gamma_1 = 1$ meaning the fulfilment of the assumption A7. Moreover, using (5.10)-(5.11), (7.24), (7.31), (7.33), and (7.34), we have

$$\begin{aligned} M(0,\varepsilon) = 0.1\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{22}{30\varepsilon} + \frac{359}{120} \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \\ = \varepsilon g_M(\varepsilon)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad g_M(\varepsilon) = 3(22 + 89.75\varepsilon)^{-1}. \end{aligned} \tag{7.35}$$

Now, using Lemma 4.2, as well as the equations (7.17), (7.21), (7.22), (7.23), and (7.35), we can calculate the components of the open-loop saddle point solution to the OPCCG and the value of this game as follows:

$$u^*(t,\varepsilon) = 6g_M(\varepsilon)\begin{cases} \begin{pmatrix} (3t-5)\varepsilon \\ 1-2t \end{pmatrix}, & 0 \le t \le 0.5, \\ \begin{pmatrix} (3t-6)\varepsilon \\ 4-2t \end{pmatrix}, & 0.5 < t \le 1, \\ \begin{pmatrix} -3\varepsilon \\ 2 \end{pmatrix}, & 1 < t \le 2, \end{cases} \tag{7.36}$$

$$v^*(t,\varepsilon) = 12\varepsilon g_M(\varepsilon)\begin{cases} \begin{pmatrix} 0 \\ 1-t \end{pmatrix}, & 0 \le t \le 1, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & 1 < t \le 2, \end{cases} \tag{7.37}$$

$$J_\varepsilon^* = 18\varepsilon g_M(\varepsilon). \tag{7.38}$$

Due to Corollary 6.4, the following players' controls constitute the open-loop saddle point solution to the ODG:

$$u_0^*(t) = \lim_{\varepsilon\to+0} u^*(t,\varepsilon) = \frac{18}{22}\begin{cases} \begin{pmatrix} 0 \\ 1-2t \end{pmatrix}, & 0 \le t \le 0.5, \\ \begin{pmatrix} 0 \\ 4-2t \end{pmatrix}, & 0.5 < t \le 1, \\ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, & 1 < t \le 2, \end{cases} \tag{7.39}$$

$$v_0^*(t) = \lim_{\varepsilon\to+0} v^*(t,\varepsilon) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad t \in [0,2]. \tag{7.40}$$

Furthermore, the ODG value is

$$J_0^* = \lim_{\varepsilon\to+0} J_\varepsilon^* = 0. \tag{7.41}$$

Also, it should be noted that, due to the results of Subsection 5.3, the controls $u_0^*(t)$ and $v_0^*(t)$, given by (7.39) and (7.40), constitute the open-loop quasi saddle point of the OPCCG.

8. Conclusions

In this paper, the finite-horizon zero-sum linear-quadratic differential game was considered. The dynamics of the game has multiple point-wise and distributed delays in the state variable and in the players' controls. The weight matrix of the control cost of the minimizing player (the minimizer) in the game's cost functional is block-diagonal with the small positive multiplier ε in one of the blocks. Both blocks are positive definite. However when ε is replaced with zero, the weight matrix of the minimizer's control cost becomes singular (but, in general, nonzero). Due to such a structure of this matrix, the considered game is a partial cheap control game. One more feature of the considered game is that the integral part of its cost functional does not contain the cost of the state variable. By the proper linear change of the state variable, the initially formulated differential game was transformed equivalently to a much simpler one. This new game also is a partial cheap control one, while its equation of dynamics does not have delays any more. In the sequel of the paper, this new undelayed game was considered as an original partial cheap control game. The solvability condition of the original game was presented. Due to this condition, the derivation of the open-loop saddle point and the value of this game is reduced to solution of the terminal-value problem for the matrix Riccati differential equation. Asymptotic analysis (with respect to ε) of solution to this terminal-value problem was carried out. Based on this analysis, the zero-order asymptotic expansions with respect to ε of the saddle point and value of the original partial cheap control game were constructed and justified. Using the asymptotic expansion of the open-loop saddle point, its boundedness with respect to ε was established for all sufficiently small values of this parameter. Thus, for the original partial cheap control game (and, therefore, for the initially formulated game), the open-loop saddle point does not have an impulse-like behaviour as $\varepsilon \to +0$. Based on the asymptotic expansion of the open-loop saddle point for the original game, the open-loop quasi saddle point of this game was derived. Along with the original partial cheap control game, its degenerate version was considered. This version is obtained from the partial cheap control game by setting there formally $\varepsilon = 0$, yielding the new zero-sum linear-quadratic differential game. This new game is singular, because it can be solved neither by the Isaacs's MinMax principle nor by the Bellman-Isaacs equation method. The open-loop saddle point and the value of the degenerate game were derived. It was established that the open-loop saddle point and the value of the degenerate game coincide with the limits (for $\varepsilon \to +0$) of the open-loop saddle point and the value, respectively, of the original partial cheap control game.

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