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NONTRIVIAL SOLUTIONS OF FOURTH ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH NONLOCAL THREE-POINT BOUNDARY CONDITIONS

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Dedicated to the memory of Professor Ram U. Verma

Abstract. For $0<\eta<1$ fixed, by an application of a Krasnosel'skii-Zabreiko fixed point theorem, nontrivial solutions are established for a nonlinear fourth order differential equation, $u^{(4)}(t)+f(u(t))=0$, $0\le t\le 1$, satisfying nonlocal three-point conditions $u(0)=u'(0)=u''(0)=u''(\eta)-u''(1)=0$, where $f:\mathbb{R}\to\mathbb{R}$ is continuous and almost linear at infinity.

Keywords. Boundary value problem; Fixed point; Fourth order differential equation; Three-point boundary conditions.

1. Introduction

With $0<\eta<1$ fixed throughout, we are concerned with the existence of nontrivial solutions of the fourth order nonlinear ordinary differential equation,

$$u^{(4)}(t)+f(u(t))=0, \quad t\in[0,1], \tag{1.1}$$

satisfying the nonlocal three-point boundary conditions,

$$u(0)=u'(0)=u''(0)=u''(\eta)-u''(1)=0, \tag{1.2}$$

where $f:\mathbb{R}\to\mathbb{R}$ is continuous and $\lim_{|r|\to\infty}\frac{f(r)}{r}$ exists. (By a nontrivial solution, we mean a solution not identically zero; for example, if $f\equiv 0$, the only solution of (1.1), (1.2) is the trivial solution, $u\equiv 0$.)

Fourth order ordinary differential equations have been the subject of much research, both in theory and applications. Common among these studies involve boundary value problems for the deformation of a loaded beam under certain endpoint boundary conditions [1, 3, 9], as well as for the displacement of a cantilever beam under endpoint boundary conditions [26, 27]. In addition, there has also been a good deal of attention given to solutions of fourth order

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ordinary differential equations satisfying nonlocal boundary conditions (such as these three-point boundary conditions); see, e.g., [2, 4, 5, 7, 8, 17, 23, 28]. For other studies devoted to solutions of boundary value problems for fourth order ordinary differential equations, we refer to [18, 19, 20, 21, 24, 25].

In this paper, we apply a Krasnosel'skii and Zabreiko fixed point theorem [16, p.239, Theorem 40.1; first, see the notation on p.77 in equation (17.10)] to obtain nontrivial solutions of (1.1), (1.2). In the context of when the nonlinearity is almost linear at infinity, this fixed point theorem has been used effectively in establishing the existence of solutions of boundary value problems for ordinary differential equations, finite difference equations, and fractional difference equations; see, e.g., [6, 10, 11, 12, 13, 14, 15, 22] and the references therein.

2. Some Preliminaries and the Krasnosel'skii-Zabreiko Fixed Point Theorem

The solution of

$$-u^{(4)}(t)=0, \quad t\in[0,1], \tag{2.1}$$

satisfying the three-point boundary conditions (1.2) is the trivial solution $u(t)\equiv 0$. As a consequence, Sun and Zhu [23] showed by direct computation that the solution of

$$u^{(4)}(t)+p(t)=0, \quad t\in[0,1], \tag{2.2}$$

satisfying the three-point boundary conditions (1.2), has the integral representation,

$$u(t)=\int_0^1 G(t,s)p(s)ds, \tag{2.3}$$

where the kernel of the integral

$$G(t,s):=H(t,s)+\frac{t^3}{6(1-\eta)}K(\eta,s), \tag{2.4}$$

with

$$H(t,s)=\frac{t^3}{6}\begin{cases}(1-s)-(t-s)^3, & 0\le s\le t\le 1,\\ 1-s, & 0\le t\le s\le 1,\end{cases} \tag{2.5}$$

and

$$K(t,s)=\frac{\partial^2}{\partial t^2}H(t,s)=\begin{cases}s(1-t), & 0\le s\le t\le 1,\\ t(1-s), & 0\le t\le s\le 1.\end{cases} \tag{2.6}$$

Sun and Zhu also established the inequalities [23, Lemma 2.2] that, for $(t,s)\in[0,1]\times[0,1]$,

$$\frac{t^3}{6}s(1-s)\le H(t,s)\le s(1-s), \tag{2.7}$$

and it is straightforward that, for $(t,s)\in[0,1]\times[0,1]$,

$$0\le K(t,s)\le s(1-s). \tag{2.8}$$

Properties of $G(t,s)$ which are of importance to us include (see [23]):

(a) For each $s\in[0,1]$,

$$\frac{\partial^i}{\partial t^i}G(0,s)=0,\ i=0,1,2,\ \text{and}\ \frac{\partial^2}{\partial t^2}G(\eta,s)-\frac{\partial^2}{\partial t^2}G(1,s)=0.$$

(b) $G(t,s)>0$, for $(t,s)\in(0,1)\times(0,1)$.

(c) $\max_{0\le t\le 1} G(t,s) \le s(1-s) + \frac{1}{6(1-\eta)}s(1-s)$.

Remark 2.1. We recall here that, for normed spaces C and D, an operator $T : C \to D$, which is continuous and maps bounded subsets of C into precompact subsets of D, is said to be *completely continuous*.

We will apply the following Krasnosel'skii-Zabreiko fixed point theorem to obtain nontrivial solutions of (1.1), (1.2).

Theorem 2.2. *Let X be a Banach space and $F : X \to X$ be a completely continuous operator. If there exists a bounded linear operator $A : X \to X$ such that 1 is not an eigenvalue and*

$$\lim_{||u||\to\infty} \frac{||F(u)-A(u)||}{||u||} = 0,$$

then F has a fixed point in X.

We will apply Theorem 2.2 to a nonlinear integral operator whose kernel is $G(t,s)$. In that context, let the Banach space $(X, ||\cdot||)$ be defined by

$$X := C[0,1] = \{h : [0,1] \to \mathbb{R} \mid h \text{ is continuous}\}, \tag{2.9}$$

with norm

$$||h|| := \sup_{0\le t\le 1} |h(t)|. \tag{2.10}$$

It is standard that $u \in X$ is a fixed point of the completely continuous operator $F : X \to X$ defined by

$$(Fu)(t) := \int_0^1 G(t,s)f(u(s))ds = \int_0^1 \left[H(t,s) + \frac{t^3}{6(1-\eta)}K(\eta,s)\right] f(u(s))ds, \ \ 0\le t\le 1, \tag{2.11}$$

if and only if u is a solution of (1.1), (1.2) (for detailed proofs, see the papers [5] and [23]).

3. Existence Results

We now apply Theorem 2.2 to the operator F defined in (2.11) and to an associated linear operator in establishing solutions of (1.1), (1.2).

Theorem 3.1. *Assume $f : \mathbb{R} \to \mathbb{R}$ and $\lim_{|r|\to\infty} \frac{f(r)}{r} = m$. If*

$$|m| < d := \frac{1}{\sup_{0\le t\le 1} \int_0^1 \left[H(t,s) + \frac{t^3}{6(1-\eta)}K(\eta,s)\right] ds},$$

then the boundary value problem (1.1), (1.2) has a solution u, and moreover, $u \ne 0$, when $f(0) \ne 0$.

Proof. Let the Banach space $(X, ||\cdot||)$ and the completely continuous operator $F : X \to X$ be as defined in the previous section in (2.9), (2.10), and (2.11), respectively.

Corresponding to (1.1), (1.2), we consider a linear fourth order differential equation,

$$u^{(4)}(t) + mu(t) = 0, \ \ 0\le t\le 1, \tag{3.1}$$

satisfying the boundary conditions (1.2), and we define a completely continuous linear operator $A : X \to X$ by

$$(Au)(t) := m\int_0^1 G(t,s)u(s)ds = m\int_0^1 \left[H(t,s) + \frac{t^3}{6(1-\eta)}K(\eta,s)\right]u(s)ds, \;\; 0 \le t \le 1. \quad (3.2)$$

Of course, solutions of (3.1), (1.2) are fixed points of A, and conversely.

First, we exhibit that 1 is not an eigenvalue of A. To that end, we consider two cases: (a) $m = 0$ and (b) $m \neq 0$.

For (a), if $m = 0$, since the boundary value problem (2.1), (1.2) has only the trivial solution, it is immediate that 1 is not an eigenvalue of A.

For (b), if $m \neq 0$ and (3.1), (1.2) has a nontrivial solution, u, then $||u|| > 0$. And so, we have

$$\begin{aligned}
||u|| &= ||Au|| \\
&= \sup_{0\le t\le 1}\left|m\int_0^1 G(t,s)u(s)\right| \\
&= |m|\sup_{0\le t\le 1}\left|\int_0^1 \left[H(t,s) + \frac{t^3}{6(1-\eta)}K(\eta,s)\right]u(s)ds\right| \\
&\le |m|||u||\sup_{0\le t\le 1}\int_0^1 \left[H(t,s) + \frac{t^3}{6(1-\eta)}K(\eta,s)\right]ds \\
&< d||u||\frac{1}{d} \\
&= ||u||,
\end{aligned}$$

a contradiction. Again, 1 is not an eigenvalue of A.

Our next claim is that

$$\lim_{||u||\to\infty} \frac{||F(u) - A(u)||}{||u||} = 0.$$

In that direction, let $\varepsilon > 0$ be given. Now $\lim_{|r|\to\infty} \frac{f(r)}{r} = m$ implies there exists an $N_1 > 0$ such that, for $|r| > N_1$,

$$|f(r) - mr| < \varepsilon|r|. \quad (3.3)$$

Let

$$N = \sup_{|r|\le N_1} |f(r)|,$$

and let $L > N_1$ be such that

$$\frac{N + |m|N_1}{L} < \varepsilon.$$

Next, choose $u \in X$ with $||u|| > L$. Then for $0 \le s \le 1$,

(i) if $|u(s)| \le N_1$, we have

$$\begin{aligned}
|f(u(s)) - mu(s)| &\le |f(u(s))| + |m||u(s)| \\
&\le N + |m|N_1 \\
&< \varepsilon L \\
&< \varepsilon||u||,
\end{aligned}$$

and

(ii) if $|u(s)| > N_1$, we have from (3.3) that

$$|f(u(s)) - mu(s)| < \varepsilon|u(s)| \leq \varepsilon||u||.$$

Thus, from (i) and (ii), for all $0 \leq s \leq 1$,

$$|f(u(s)) - mu(s)| \leq \varepsilon||u||. \tag{3.4}$$

It follows from (3.4) that, for $u \in X$ with $||u|| > L$,

$$\begin{aligned}
||F(u) - A(u)|| &= \sup_{0\leq t\leq 1} \left| \int_0^1 G(t,s)[f(u(s)) - mu(s)] \right| \\
&= \sup_{0\leq t\leq 1} \int_0^1 \left[H(t,s) + \frac{t^3}{6(1-\eta)} K(\eta,s) \right] |f(u(s)) - mu(s)|\, ds \\
&\leq \varepsilon||u|| \sup_{0\leq t\leq 1} \int_0^1 \left[H(t,s) + \frac{t^3}{6(1-\eta)} K(\eta,s) \right] ds \\
&= \frac{\varepsilon}{d}||u||.
\end{aligned}$$

Therefore,

$$\lim_{||u||\to\infty} \frac{||F(u) - A(u)||}{||u||} = 0.$$

By Theorem 2.2, F has a fixed point $u \in X$, and u is a desired solution of (1.1), (1.2). Further, if in addition, $f(0) \neq 0$, then immediately $u \neq 0$. □

From inequalities (2.7) and (2.8), for $0 \leq t \leq 1$, as listed in Section 2,

$$\begin{aligned}
H(t,s) + \frac{t^3}{6(1-\eta)} K(\eta,s) &\leq s(1-s) + \frac{1}{6(1-\eta)} s(1-s) \\
&= \left[\frac{7-6\eta}{6-6\eta} \right] s(1-s), \ \ 0 \leq s \leq 1.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
\sup_{0\leq t\leq 1} \int_0^1 \left[H(t,s) + \frac{t^3}{6(1-\eta)} K(\eta,s) \right] ds &\leq \left[\frac{7-6\eta}{6-6\eta} \right] \int_0^1 s(1-s)ds \\
&= \frac{1}{36} \left[\frac{7-6\eta}{1-\eta} \right].
\end{aligned}$$

We can state a corollary to Theorem 3.1.

Corollary 3.2. *Assume $f : \mathbb{R} \to \mathbb{R}$ and $\lim_{|r|\to\infty} \frac{f(r)}{r} = m$. If*

$$|m| < d := 36 \left[\frac{1-\eta}{7-6\eta} \right],$$

then the boundary value problem (1.1), (1.2) has a solution u, and moreover, $u \neq 0$, when $f(0) \neq 0$.

As another corollary to Theorem 3.1, we will show that if $f \geq 0$, then (1.1), (1.2) has a positive solution.

Corollary 3.3. *Assume that $f:[0,\infty)\to[0,\infty)$ is continuous and $\lim_{r\to\infty}\frac{f(r)}{r}=0$. Then the boundary value problem (1.1), (1.2) has a nonnegative solution u. Furthermore, if $f(0)\neq 0$, then u is a positive solution.*

Proof. Let $\overline{f}:\mathbb{R}\to\mathbb{R}$ be defined by

$$\overline{f}(r):=\begin{cases} f(r), & r\geq 0,\\ f(-r), & r<0.\end{cases}$$

Then, $\overline{f}$ is continuous on $\mathbb{R}$ and $\lim_{|r|\to\infty}\frac{\overline{f}(r)}{r}=0$. It follows from Theorem 3.1 that the fourth order differential equation

$$u^{(4)}(t)+\overline{f}(u(t))=0,\ \ 0\leq t\leq 1, \tag{3.5}$$

satisfying the boundary conditions (1.2) has a solution u. In particular, u satisifes

$$u(t)=\int_0^1 G(t,s)\overline{f}(u(s))ds=\int_0^1\left[H(t,s)+\frac{t^3}{6(1-\eta)}K(\eta,s)\overline{f}(u(s))\right]ds,\ \ 0\leq t\leq 1,$$

and hence $u(t)\geq 0$, $0\leq t\leq 1$. In view of $\overline{f}(u(s))=f(u(s))$, $0\leq s\leq 1$, one has that u satisfies (1.1), (1.2). That is, u is a nonnegative solution of (1.1), (1.2). As before, if $f(0)\neq 0$, then $u\neq 0$. □

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