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BRANCH-AND-BOUND ALGORITHMS FOR SOLVING A MODIFIED CONSUMER PROBLEM

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Dedicated to the memory of Professor Hoang Tuy

Abstract. We consider an extended version of the consumer problem. The problem is formulated as a nonconvex optimization problem with two bilinear constraints. We propose two brand-and-bound algorithms for finding a global solution of the problem. The first algorithm uses the convex envelope for the bilinear functions for bounding, and a rectangular bisection for branching, whereas the second one employs a decoupling relaxation companied with an adaptive simplicial bisection. Some numerical experiments and results on randomly generated data are reported.

Keywords. Adaptive simplex; Bilinear constraint; Consumer problem; Decoupling relaxation; Envelope function; Rectangular bisection.

1. Introduction and the problem setting

Nowadays, optimization in the manufacturing and consumer sectors is a topic of great interest. However, the optimization operation can only be done when the goods together with their prices and utilities are quantified by the functional relationships. They (producers and consumers) want to choose a packages of goods to producing (or consumption) under the conditions allowed, such that the benefit are greatest possible. In consumption economics, the following two classical problems are of common interest. The first one is maximizing utility subject to consumer budget constraint (see Intriligator [4], p. 149), and the second one is minimizing consumer's expenditure for the utility of a specified level (see Nichoson and Snyder [8], p. 132). Quatitative properties of the problem of maximizing utility subject to consumer budget constraint have been studied by Takayama [10] (pp. 241-242, 253-255], Penot [9], Hadjisavvas and Penot [1], and many other authors. In this paper, we consider the consumer problem in respect of extension the classical consumer problem onto new version and propose some solution

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methods for the suggested problem. For more detail and your convenience to read, we restate the classical problem as follows.

Let us consider a market containing a packages of N types of goods $1,2,...,N$. Denote by $d^T = (d_1,...,d_N) \geq 0$ the price vector and $x^T = (x_1,...,x_N) \geq 0$ the quantity vector of types of goods. Moreover, assume that the utility of a package of goods x is quantified by $U(x)$, where $U : \mathbb{R}^N \to \mathbb{R}$ is a given function (utility function). The feasible domain of x in usually is a convex polyhedral set in $\mathbb{R}^N_+$, denoted by X. When the price vector d is known, with an amount of budget no more a given $M > 0$, the classical consumer problem is as follows.

Find $x \in \mathbb{R}^N$ such that

$$\max_x U(x) \\ \text{s.t.} \begin{cases} x \in X \\ \bar{A}x \leq \bar{b} \\ \langle d,x\rangle \leq M, \end{cases} \tag{CCP}$$

where the constraint $\bar{A}x \leq \bar{b}, \bar{A}$ is a $q \times N$ matrix and $\bar{b} \in \mathbb{R}^q$ represents the structure or the density of types of goods, and it is usually defined by the consumers. Usually, the utility function is concave continuous function such as the Coob-Douglas function [10]. Thus, with a given price d, the classic consumer problem (CCP) is an optimization problem of maximizing a concave function over a convex polyhedral set, which can be solved efficiently by available algorithms.

Derived from the practice that the prices of goods were decided by the markets of the goods. More specifically, with the popular group of goods, which were consumed over the perfectly competitive markets, the prices were decided by both consumers and producers via the law of supply and demand. However, with the special group of goods, which were consumed on the oligopolistic markets, the prices were decided by the producers. From this practice, in this paper, we extend the classical consumer problem in a way that takes into account the requirements of both producers and consumers. Precisely, whereas the consumers want to choose the amount of the goods x such that maximizing the utility, the producers want to choose the level of the prices of some given goods such that the total of revenues obtained from the package of goods x is no less than given $L > 0$. For easily of presentation, we divide a package of $N = n+m$ types of the goods into two groups. The first group is denoted by $x^T = (x_1,...,x_n)$ with the level of prices, respectively, are given $c^T = (c_1,...,c_n)$, whereas the second group, denoted by $y^T = (y_1,...,y_m)$ with the level of prices $p^T = (p_1,..., p_m)$, respectively, are unknowns. In order to show the capacity of the producers, we denote by $\tilde{b}^T = (\tilde{b}_1,...,\tilde{b}_r), \tilde{b}_i > 0, i = 1,...,r$ the vector of the endowment of resources, $\tilde{a}_{ij}$ the amount of the $i-$th resource for producing one unit of the $j-$th commodity, and let $\tilde{A} = (\tilde{a}_{ij})_{r\times(n+m)}$. Moreover, denote by $\tilde{A}_1, \tilde{A}_2$, respectively, the matrices obtained from top n columns and remaining m columns of $\tilde{A}$. Then the requirements of the producers can be written as follows.

Find $(x,y) \in \mathbb{R}^{n+m}, p \in \mathbb{R}^m$ such that

$$\begin{cases} \langle c,x\rangle + \langle p,y\rangle & \geq & L, \\ \tilde{A}_1x + \tilde{A}_2y & \leq & \tilde{b}, \\ p,x,y & \geq & 0. \end{cases}$$

Combining the requirements of both producers and consumers, we suggest an extended version of the consumer problem as follows.

Find $(x,y) \in \mathbb{R}^{n+m}$ and $p \in \mathbb{R}^m$ such that

$$\max_{x,y,p} U(x,y)$$
$$\text{s.t.} \begin{cases} x \in X, y \in Y_0, p \in P_0 & \\ \bar{A}_1 x + \bar{A}_2 y & \leq \ \bar{b}, \\ \tilde{A}_1 x + \tilde{A}_2 y & \leq \ \tilde{b}, \\ \langle c,x\rangle + \langle p,y\rangle & \leq \ M, \\ \langle c,x\rangle + \langle p,y\rangle & \geq \ L, \end{cases} \tag{$ECP0$}$$

where $\bar{A}_1$ and $\bar{A}_2$ are the matrices obtained from top n columns and remaining m columns of the matrix $\bar{A}$, respectively. Obviously, $0 < L < M$ is a necessary condition for this problem has a solution and it is suitable on the practice. For easy on presentation, we define a convex polyhedral by setting

$$D := \{(x,y) \in \mathbb{R}^{n+m} : \bar{A}_1 x + \bar{A}_2 y \leq \bar{b}, \tilde{A}_1 x + \tilde{A}_2 y \leq \tilde{b}\},$$

and $f(x,y) := -U(x,y)$. Then the extended consumer problem $(ECP0)$ is rewritten as the form

$$f_* := \ \min_{x,y,p} f(x,y)$$
$$\text{s.t.} \begin{cases} x \in X, y \in Y_0, (x,y) \in D, p \in P_0, & \\ \langle c,x\rangle + \langle p,y\rangle - M & \leq \ 0, \\ \langle -c,x\rangle + \langle -p,y\rangle + L & \leq \ 0. \end{cases} \tag{$ECP1$}$$

2. A RECTANGULAR BISECTION B-B ALGORITHM

Firstly, we recall from [2] that the convex envelope of a function ψ on a convex set C is the convex function on C denoted by $co_C\psi$ such that $co_C\varphi(x) \leq \psi(x)$ for every $x \in C$. If ζ is any convex function on C satisfying $\zeta(x) \leq \psi(x)$ for every $x \in C$, then $\zeta(x) \leq co_C\psi(x)$ for every $x \in C$. It is well known [2] that the convex envelope function of a concave function is affine, and that if $C = C_1 \times \ldots \times C_N$ is compact and ψ is separable, i.e., $\psi(x_1,\ldots,x_N) = \sum_{j=1}^N \psi_j(x_j)$, then $co\psi(x) = \sum_{j=1}^N co\psi_j(x_j)$, where $co\psi_j$ is the convex envelope function of ψ_j over C_j for every j.

As usual, we assume that the utility function U is continuous concave on $\mathbb{R}^{n+m}$. Then f is continuous convex on $\mathbb{R}^{n+m}$. Since X, Y_0, D are convex, Problem (ECP1) is a convex program with two additional bilinear constraints that makes the problem difficult. First we consider the case when the feasible domain of the price p is box P_0 (often in practical models). In order to employ the separability of the inner product and the rectangular structure of Y_0 and P_0, by using the elementary equality $(a-b)^2 = a^2 - 2ab + b^2$ ($a,b \in \mathbb{R}$, we can write Problem (ECP1) in the form

$$\alpha(P_0,Y_0) := \ \min_{x,y,p} f(x,y)$$
$$\text{s.t.} \begin{cases} x \in X, y \in Y_0, (x,y) \in D, p \in P_0 & \\ \langle c,x\rangle + \frac{1}{2}\sum_{j=1}^m (p_j + y_j)^2 - p_j^2 - y_j^2 - M & \leq \ 0, \\ \langle -c,x\rangle + \frac{1}{2}\sum_{j=1}^m (p_j - y_j)^2 - p_j^2 - y_j^2 + L & \leq \ 0. \end{cases} \tag{ECP}$$

The separability of the last two constraints and rectangular structure of Y_0 and P_0 in Problem (ECP) suggest a branch-and-bound algorithm with rectangular bisection for solving it.

We will use the following adaptive rectangular bisection for a branch-and-bound algorithm to be described below.

An adaptive rectangular bisection (Rule 1). Let I be a given q- dimensional box given as $I := I_1 \times ... \times I_q$, and ψ_j be the concave functions over $I_j, j = 1,...,q$. Denote by $\mathrm{co}_{I_j}\psi_j$ the convex envelope function of ψ_j over I_j. For each $x^I \in I$, denote by $j_{max}(I)$ the index that belongs to the set

$$\mathrm{argmax}_{1\le j\le q}\{\psi_j(x^I_j) - \mathrm{co}_{I_j}\psi_j(x^I_j)\}.$$

Then, we bisect I into two boxes via the middle point of edge $j_{max}(I)$. We call this middle point the *bisection point* and $j_{max}(I)$ the *bisection index*.

For this bisection, we have the following lemma. Its proof can be found, e.g. in [5, 6]

Lemma 2.1. *Let $\{I^k\}$ be an infinite sequence of subboxes generated by the adaptive rectangular bisection Rule 1 such that $I^{k+1} \subset I^k$ for every k. Let b_k be the bisection point and j_k be the bisection index for I^k. Then*

$$\lim_{k\to\infty}(\psi_{j_k}(b^k) - co_{I^k_{j_k}}\psi_{j_k}(b^k)) = 0.$$

Consequently, $\{I^k\}$ tends to a singleton provided that ψ_{j_k} is concave, but not affine on I_{j_k} for every j_k.

By using rectangular bisection (Rule 1), the bounding branching operations now be done as follows.

Bounding by the convex envelope. Let $Y \subseteq Y_0$, and $P \subseteq P_0$ be rectangles. Consider Problem (ECP) with Y_0 and P_0 are replaced by Y and P respectively. Let $B := P \times Y$ and define the relaxed problem

$$\begin{array}{ll} \beta(B) := & \min_{x,y,p} f(x,y) \\ \text{s.t.} & \left\{\begin{array}{lll} x\in X, (x,y)\in D, (p,y)\in B & & \\ \langle c,x\rangle + \frac{1}{2}\sum_{j=1}^m (p_j+y_j)^2 + co_h(-p_j^2) + co_h(-y_j^2) - M & \le & 0, \\ \langle -c,x\rangle + \frac{1}{2}\sum_{j=1}^m (p_j-y_j)^2 + co_h(-p_j^2) + co_h(-y_j^2) + L & \le & 0. \end{array}\right. \end{array} \qquad (RECP)$$

Clearly, $\beta(B)$ is a lower bound for $\alpha(B)$. Let (x^B, y^B, p^B) be the obtained solution to (RECP). We use Rule 1 to bisect B. Let $\sigma(B^Y)$ and $\sigma(B^P)$ be bisection- indeces for P and Y respectively. We first calculate

$$\begin{aligned} \sigma(B^Y) &:= \max_{1\le j\le q}\{h^Y_q(y^B_q) - \mathrm{co}_{B^Y_q} h^Y_q(y^B_q)\}, \\ \sigma(B^P) &:= \max_{1\le l\le m}\{h^P_l(p^B_l) - \mathrm{co}_{B^P_l} h^P_l(p^B_l)\}, \end{aligned}$$

and suppose that the maximums $\sigma(B^Y)$, $\sigma(B^P)$ attained at the indexes denote by $q_{max(B^Y)}$ and $l_{max(B^P)}$, respectively. Then the bisection index $j_{max(B)}$ of the box B defined as

$$j_{max(B)} := \left\{\begin{array}{ll} q_{max(B^Y)} & \text{if} \quad \sigma(B^Y) \ge \sigma(B^P) \\ m + l_{max(B^P)} & \text{if} \quad \text{otherwise}, \end{array}\right.$$

and the bisection point z^B is the middle point of the edge $j_{max(B)}$ of the box B. For the proof of the convergence of the algorithm to be convenient, we denote z^B by u^{B^Y} if $j_{max(B)} \le m$, and by v^{B^P} if otherwise.

The algorithm now can be described as follows. For simplicity of notation, let

$$h(y,p) := -\sum_{j=1,m}(y_j^2 + p_j^2).$$

Algorithm 2.2. *Initial Step. Choose a tolerance* $\varepsilon \geq 0$*, take the initial boxes* Y_0*,* P_0*, and* $B_0 := Y_0 \times P_0$*, and set* $\Gamma_0 := \{B_0\}$*. Calculate the envelope function* $co_{B_0}h$*, and solve the relaxed convex programming problem to obtain the lower bound* $\beta_0 := \beta(B_0)$ *and an optimal solution* (x^0, y^0, p^0)*. If* $|h(y^0, p^0) - co_{B_0}h(y^0, p^0)| \leq \varepsilon$*, let* $(\bar{x}^0, \bar{y}^0, \bar{p}^0) := (x^0, y^0, p^0)$ *and terminate the algorithm with an* $\varepsilon-$ *optimal solution* $(\bar{x}^0, \bar{y}^0, \bar{p}^0)$ *of Problem (ECP1).*

Iteration k $(k = 0, 1, \ldots)$*. At each iteration k, we have* Γ_k *containing a finite number of subboxes of* B_0*. For each subbox* $B \in \Gamma_k$*, an optimal solution* (x^B, y^B, p^B) *of the relaxed problem has been computed. If the condition*

$$|h(y^B, p^B) - co_B h(y^B, p^B)| \leq \varepsilon \tag{2.1}$$

is satisfied, we let $(\bar{x}^B, \bar{y}^B, \bar{p}^B) := (x^B, y^B, p^B)$*, an* ε*- feasible solution of Problem (ECP1), and let* $\alpha(B) = f(\bar{x}^B, \bar{y}^B)$*. If the condition (2.1) is not satisfied, then let* $\alpha(B) = +\infty$*.*

Step 1. If $\Gamma_k = \emptyset$*, terminate. Otherwise, choose* $B^k \in \Gamma_k$ *such that*

$$\beta_k := \beta(B^k) = \min\{\beta(B) : B \in \Gamma_k\}.$$

Rename the objects according to the box B^k*:*

$$\begin{array}{lcll}
(x^k, y^k, p^k) & := & (x^{B^k}, y^{B^k}, p^{B^k}) & \text{– optimal solution of the relaxed problem defined by } B^k, \\
i_k & := & j_{max(B^k)} & \text{– bisection index of } B^k, \\
q_k & := & q_{max(B^{Y^k})} & \text{– bisection index of } B^{Y^k}, \\
l_k & := & l_{max(B^{Y^k})} & \text{– bisection index of } B^{P^k}, \\
z^k & := & z^{B^k} & \text{– bisection point of } B^k, \\
u^k & := & z^{B^k} & \text{– bisection point of } B^{Y^k}, \\
v^k & := & z^{B^k} & \text{– bisection point of } B^{P^k},
\end{array}$$

and

$$(\bar{x}^k, \bar{y}^k, \bar{p}^k) \quad := \quad (\bar{x}^{B^k}, \bar{y}^{B^k}, \bar{p}^{B^k})$$

if (2.1) is satisfied for $(x^{B^k}, y^{B^k}, p^{B^k})$*.*

Step 2. Bisect B^k *into two subboxes* B^{k^+} *and* B^{k^-} *by using Rule 1.*

Step 3. Calculate the lower bound, upper bound, $\varepsilon-$ *feasible solution for Problem (ECP1), and then define the bisection index and bisection point for each newly generated subbox* B^{k^+} *and* B^{k^-}*.*

Step 4. Update the currently best upper bound $\alpha_{k+1} := \min\{\alpha_k, \alpha(B^{k^+}), \alpha(B^{k^-})\}$*. If* $\alpha_{k+1} < +\infty$*, denote by* $(\bar{x}^{k+1}, \bar{y}^{k+1}, \bar{p}^{k+1})$ *the best currently feasible* $\varepsilon-$*solution if* $\alpha_{k+1} = f(\bar{x}^{k+1}, \bar{y}^{k+1})$*.*

Step 5. Set

$$\Gamma_{k+1/2} := (\Gamma_k \setminus B^k) \cup \{B^{k^+}, B^{k^-}\},$$

and update

$$\Gamma_{k+1} := \{B \in \Gamma_{k+1/2} : \beta(B) < \alpha_{k+1}\}.$$

Let $k := k+1$ *and go to Step 1.*

Theorem 2.3. *(i) If the algorithm terminates at iteration k, then $(\bar{x}^k, \bar{y}^k, \bar{p}^k)$ is an $\varepsilon-$optimal solution of Problem (ECP1).*
(ii) Otherwise, if the algorithm never terminates, then any cluster point of the sequence $\{(x^k, y^k, p^k)\}$ is an ε golbal optimal solution of Problem (ECP1). Furthermore, $\beta_k \nearrow f_*$.

Proof. (i) It is obvious from the definitions.

(ii) By taking a subsequence if necessary, without loss of generality, we assume that

$$\lim_{k\to\infty}(x^k, y^k, p^k) = (x^*, y^*, p^*).$$

By the branching operation, there exists a subsequence $\{B^{k_j}\}$ of the sequence $\{B^k\}$ such that $B^{k_{j+1}} \subset B^{k_j}$ for every j. By Lemma 2.1, one has

$$\lim_{j\to\infty}[h_{i_{k_j}}(z^{k_j}) - \mathrm{co}_{B^{k_j}_{i_{k_j}}} h_{i_{k_j}}(z^{k_j})] = 0, \tag{2.2}$$

and $B^{k_j}_{i_{k_j}}$ tends to singleton as $j\to\infty$. This result and (2.2) imply that

$$\lim_{j\to\infty}[h_{i_{k_j}}([y^{k_j}, p^{k_j}]_{i_{k_j}}) - \mathrm{co}_{B^{k_j}_{i_{k_j}}} h_{i_{k_j}}([y^{k_j}, p^{k_j}]_{i_{k_j}})] = 0. \tag{2.3}$$

Furthermore, by the rule for selecting the bisection index i_{k_j}, from (2.3), it follows that

$$\lim_{j\to\infty}[h_r([y^{k_j}, p^{k_j}]_r) - \mathrm{co}_{B^{k_j}_r} h_r([y^{k_j}, p^{k_j}]_r)] = 0 \ \forall 1 \le r \le 2m. \tag{2.4}$$

From (2.4) and $h(y^k, p^k) = \sum_{r=1}^{2m} h_r([y^k, p^k]_r)$, we have

$$\lim_{j\to\infty}[h(y^{k_j}, p^{k_j}) - \mathrm{co}_{B^{k_j}} h(y^{k_j}, p^{k_j})] = 0. \tag{2.5}$$

On the other hand, since $(x^{k_j}, y^{k_j}, p^{k_j})$ is an optimal solution to problem for estimating the value $\beta(B^{k_j})$, for every j, we have

$$\begin{cases} x^{k_j} \in X, (x^{k_j}, y^{k_j}) \in D, (y^{k_j}, p^{k_j}) \in B^{k_j} \\ g_1(x^{k_j}, y^{k_j}, p^{k_j}) + h(y^{k_j}, p^{k_j}) - (h(y^{k_j}, p^{k_j}) - \mathrm{co}_{B^{k_j}} h(y^{k_j}, p^{k_j}) \le 0. \end{cases} \tag{2.6}$$

Passing the limit as $j\to\infty$, combining (2.5) with (2.6), and using the closedness of the sets X, D, Y, P, we can write

$$\begin{cases} x^* \in X, (x^*, y^*) \in D, (y^*, p^*) \in Y \times P, \\ g_1(x^*, y^*, p^*) + h(y^*, p^*) \le 0. \end{cases}$$

which shows that (x^*, y^*, p^*) is a feasible solution of Problem (ECP1).

Moreover, from the definition of the bounding operation, it follows that

$$\beta(B^{k_j}) = f(x^{k_j}, y^{k_j}) \le f_* \le f(x^*, y^*). \tag{2.7}$$

Passing the limit as $j\to\infty$, by continuity of f, we obtain

$$\lim_{j\to\infty} f(x^{k_j}, y^{k_j}) = f(x^*, y^*) \le f_* \le f(x^*, y^*). \tag{2.8}$$

Thus $f(x^*, y^*) = f_*$ and (x^*, y^*, p^*) is an optimal solution of the Problem (ECP1). Finally, since the sequence $\{\beta_k\}$ is increasing, by combining with (2.7) and (2.8), we obtain

$$\lim_{k\to\infty}\beta_k = \lim_{j\to\infty}\beta_{k_j} = f(x^*, y^*) = f_*.$$

The proof is completed. □

3. A DECOUPLING BARANCH-AND-BOUND ALGORITHM

Under the assumption that the utility function U is concave and continuous on $\mathbb{R}^{n+m}$, the function f is convex and continuous on $\mathbb{R}^{n+m}$. However, the extended consumer problem (ECP) is still a convex-concave (non convex) optimization problem with two bilinear constraints. There are some algorithms for solving global solution of convex-concave programming problems [3, 5, 6, 7]. However, to our knowledge, an algorithm for finding a global solution of the problem of minimizing a convex function with two bilinear constraints is yet absent in the literature.

In the case that P_0 is a simplex (often in practical consumer models) rather than a rectangle, since a n-dimensional simplex has $n+1$- vertices whereas a $n-$ dimensional rectange has 2^n-vertices, a decoupling algorithm for convex-concave problem (ECP) seems a suitable choice. Bounding operation by a coupling technique for globally minimizing a convex-concave function was introduced in [5].

Before describing a new algorithm to solve the extended consumer problem (ECP), we now emphasize some properties of Problem (ECP).

Let C, X, and Y be convex closed nonempty sets in the spaces $\mathbb{R}^{n+m}$, $\mathbb{R}^n$, and $\mathbb{R}^m$, respectively. A given function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is said to be convex-concave (saddle) on $X \times Y$ if, for each fixed $y \in Y$, the function $f(.,y)$ is convex on X, and for each fixed $x \in X$, the function $f(x,.)$ is concave on Y. A general convex - concave programming problem can be stated as follows

$$\min_{x,y} g(x,y) \quad \text{s.t.} \begin{cases} (x,y) \in C, x \in X, y \in Y, \\ h_j(x,y) \le 0, j = 1,...,l, \end{cases} \tag{CCP}$$

where at least one, either g or $h_j, j = 1,...,l$ is convex-concave. Some algorithms have been proposed for some special cases for Problem (CCP), see, e.g., [3, 5]

One of special case Problem (CCP), when $C = \mathbb{R}^{m+n}$, the objective function g does not depend on variable y and there is only one constraint function $h(x,y)$, which is the problem

$$\min_x g(x) \quad \text{s.t.} \begin{cases} x \in X, y \in Y, \\ h(x,y) \le 0. \end{cases} \tag{CCP1}$$

Let us define the function θ by taking, for each $v \in Y$

$$\theta(v) := \min_x g(x) \quad \text{s.t.} \begin{cases} x \in X, \\ h(x,v) \le 0. \end{cases} \tag{3.1}$$

Then the problem for computing $\theta(v)$ is a convex program, for which we have the following lemma.

Lemma 3.1. *Assume that the Problem (CCP1) has an optimal solution, and let $V(Y)$ denote the set of vertices of the polyhedral convex set Y. If (x^*,y^*) is an optimal solution to Problem (CCP1), then*

$$f_* = f(x^*) := \min\{\theta(v) : v \in V(Y)\}.$$

Proof. As usual, we set $\theta(v) = +\infty$ if the set $\{x \in X : h(x,v) \le 0\}$ is empty. Since (x^*, y^*) is an optimal solution to Problem (CCP1), we obtain from the definition that $y^* \in Y$ and $h(x^*, v^*) \le 0$, which implies that $\min\{h(x^*, v) : v \in Y\} \le 0$. From the concavity of the function $h(x^*, .)$ over Y, it follows that there exists a vertex $\bar{v} \in V(S)$ such that $h(x^*, \bar{v}) \le 0$, which implies that $(x^*, \bar{v})$ is also an optimal solution to Problem (CCP1) and $f(x^*) = \theta(\bar{v})$. Let $\hat{v}$ be any vertex of Y such that $\theta(\hat{v}) < +\infty$ and $\hat{x}$ be an optimal solution of the problem defining the value $\theta(\hat{v})$. Obviously, $(\hat{x}, \hat{v})$ is also a feasible solution to Problem (CCP1). Moreover one has

$$\theta(\hat{v}) = f(\hat{x}) \ge f(x^*) = \theta(v^*).$$

Thus

$$\theta(v^*) \le \theta(v) \; \forall v \in V(Y).$$

□

The following example shows that the assertion of Lemma 3.1 does not longer hold true when Problem (CCP1) has two convex-concave constraints. Let us consider the problem

$$\begin{array}{ll} & \min_{x,y}\{g(x) := x_1 - 10x_2\} \\ \text{s.t.} & \left\{ \begin{array}{lcl} x_1 + x_2 & \le & 18, \\ x_1 & \le & 10, \\ x_2 & \le & 11, \\ x_1, x_2 & \ge & 0, \\ y_1 + y_2 & = & 1, \\ y_1, y_2 & \ge & 0, \\ x_1y_1 + x_2y_2 - 10 & \le & 0, \\ -x_1y_1 - x_2y_2 + 8 & \le & 0. \end{array} \right. \end{array} \tag{3.2}$$

In this problem, the last two constraints are bilinear, whereas $X = [0,10] \times [0,11]$ is a box in $\mathbb{R}^2$, $Y = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 = 1, y_1 \ge 0, y_2 \ge 0\}$ is a normal simplex in $\mathbb{R}^2$ with two vertices $(1,0)$ and $(0,5)$. Moreover, from the definition of the function θ in (3.1), by solving the linear program, we obtain $\theta(1,0) = -102$ and $\theta(0,1) = -100$, whereas, at $(0.5, 0.5) \in Y$, we have $\theta(0.5, 0.5) = -103$. These results show that Problem (3.2) has an optimal solution, however there is no any optimal solution (x,y) such that $v \in V(Y)$.

As we have known, the main difficulty for finding an optimal solution of the extended consumer problem (ECP) is caused by the two bilinear constraints

$$\begin{array}{lcl} \langle c,x\rangle + \langle p,y\rangle & \le & M, \\ -\langle c,x\rangle - \langle p,y\rangle & \le & -L. \end{array}$$

A class of branch-and-bound algorithms were proposed in [3] for minimizing a quasiconvex-concave function subject to convex and one quasiconvex-concave inequality constraints. It should be noticed, since the extended consumer problem (ECP) has two convex-concave constraints, that the algorithms proposed in [3] fail to apply. In this section, also by using the technique of decoupling relaxation, we propose a new branch-and-bound for approximating an globally optimal solution of the extended consumer problem (ECP). Precisely, we assume that the feasible domain of the price p is a normal simplex of $m-1$ dimension in $\mathbb{R}^m$ and is denoted by S, namely,

$$S = \{p^T = (p_1, ..., p_m) \in \mathbb{R}^m : \sum_{i=1}^{m} p_i = 1, p_i \geq 0, i = 1, ..., m\}.$$

This assumption is appropriate when the prices $p_i, i = 1, ..., m$, amounts budget M and L are expressed by the relative numbers. With this assumption, for convenience of following, we rewrite the Problem (ECP) as follows

$$\begin{array}{rl} f_* := & \min_{x,y,p} f(x,y) \\ \text{s.t.} & \left\{ \begin{array}{lll} x \in X, y \in Y, (x,y) \in D, p \in S & & \\ \langle c,x\rangle + \langle p,y\rangle - M & \leq & 0, \\ \langle -c,x\rangle + \langle -p,y\rangle + L & \leq & 0. \end{array} \right. \end{array} \tag{ECP2}$$

Let Δ be a subsimplex of the initial simplex S. Let us consider the extended consumer problem (ECP2) restricted on Δ that is defined as

$$\begin{array}{rl} \alpha(\Delta) := & \min_{x,y,p} f(x,y) \\ \text{s.t.} & \left\{ \begin{array}{lll} x \in X, y \in Y, (x,y) \in D, p \in \Delta & & \\ \langle c,x\rangle + \langle p,y\rangle - M & \leq & 0, \\ \langle -c,x\rangle + \langle -p,y\rangle + L & \leq & 0. \end{array} \right. \end{array} \tag{3.3}$$

Obviously, $\alpha(\Delta)$ is an upper bound of f_*. In order to obtain a lower bound of $\alpha(\Delta)$, we decouple the two bilinear constraints by adding a new variable q and define a relaxed problem as follows

$$\begin{array}{rl} \beta(\Delta) := & \min_{x,y,p,q} f(x,y) \\ \text{s.t.} & \left\{ \begin{array}{lll} x \in X, y \in Y, (x,y) \in D, p \in \Delta, q \in \Delta & & \\ \langle c,x\rangle + \langle p,y\rangle - M & \leq & 0, \\ \langle -c,x\rangle + \langle -q,y\rangle + L & \leq & 0. \end{array} \right. \end{array} \tag{3.4}$$

As usual, we set $\alpha(\Delta) = +\infty$ and $\beta(\Delta) = +\infty$ if the feasible set of this problem is empty. The following lemma will show a relationship between $\alpha(\Delta)$ and $\beta(\Delta)$.

Lemma 3.2. *For each subsimplex $\Delta \subseteq S$, one has*
(i) if $\beta(\Delta) = +\infty$, then the feasible set of Problem (3.3) is empty. Otherwise, we have $\beta(\Delta) \leq \alpha(\Delta)$;
(ii) if $(\bar{x}, \bar{y}, \bar{p}, \bar{q})$ is an optimal solution of the relaxed problem (3.4) satisfying $\bar{p} = \bar{q}$, then $(\bar{x}, \bar{y}, \bar{p})$ is an optimal solution to Problem (3.3), and $\alpha(\Delta) = \beta(\Delta)$.

Proof. (i) It is obvious, by if (x, y, p) is a feasible solution to problem (3.3), then (x, y, p, p) is a feasible solution to the relaxed problem (3.4).
(ii) From the assumption of the theorem and the assertion (i), it follows that $(\bar{x}, \bar{y}, \bar{p})$ is a feasible solution of Problem (3.3). On the other hand, for every feasible solution $(\hat{x}, \hat{y}, \hat{p})$ of Problem (3.3), the point $(\hat{x}, \hat{y}, \hat{p}, \hat{p})$ is a feasible solution of Problem (3.4) satisfying $f(\bar{x}, \bar{y}) \leq f(\hat{x}, \hat{y})$, which means that $(\bar{x}, \bar{y}, \bar{p})$ is an optimal solution to Problem (3.3) and $\alpha(\Delta) = \beta(\Delta)$. The proof is completed. □

Obviously, the relaxed problem (3.4) is still a nonconvex optimization problem with two bilinear constraints. However, by its objective function, f does not depend on the two variables p, q, and by the special structure of the two bilinear constraints, solving this problem can be attributed by solving a finite number the convex optimization problems, one for each vertex of the simplex. Indeed, we first define the function $\varphi : S \times S \to \mathbb{R}$ by taking

$$\varphi(u,v) := \min_{x,y} f(x,y) \quad \text{s.t.} \quad \begin{cases} x \in X, y \in Y, (x,y) \in D, \\ \langle c,x\rangle + \langle u,y\rangle - M \leq 0, \\ -\langle c,x\rangle - \langle v,y\rangle + L \leq 0. \end{cases} \tag{3.5}$$

Then we have the following lemma.

Lemma 3.3. *Let Δ be a subsimplex of the initial simplex S and assume that the relaxed problem (3.4) has an optimal solution. Then $\beta(\Delta) = \min\{\varphi(u,v) : u,v \in V(\Delta)\}$. Moreover, let $\beta(\Delta) = \varphi(\hat{u},\hat{v})$, and denote by $(\hat{x},\hat{y})$ the optimal solution of Problem (3.5) of estimating the value $\varphi(\hat{u},\hat{v})$. Then $(\hat{x},\hat{y},\hat{u},\hat{v})$ is an optimal solution to Problem (3.4).*

Proof. Denote by $V(\Delta)$ the set of the vertices of the simplex Δ. We first show that there exists an optimal solution (x,y,p,q) of Problem (3.4) such that $p,q \in V(\Delta)$. Indeed, denote by $(\bar{x},\bar{y},\bar{p},\bar{q})$ the optimal solution of Problem (3.4), from the definition, one has

$$\begin{cases} \langle c,\bar{x}\rangle + \langle \bar{p},\bar{y}\rangle - M \leq 0, \\ -\langle c,\bar{x}\rangle - \langle \bar{q},\bar{y}\rangle + L \leq 0. \end{cases} \tag{3.6}$$

This results in

$$\begin{cases} \min\{\langle c,\bar{x}\rangle + \langle p,\bar{y}\rangle - M \ : \ p \in \Delta\} \leq 0, \\ \min\{-\langle c,\bar{x}\rangle - \langle q,\bar{y}\rangle + L \ : \ q \in \Delta\} \leq 0. \end{cases} \tag{3.7}$$

By the properties of the linear programming, there exist two vertices $\hat{p},\hat{q} \in V(\Delta)$ such that

$$\begin{cases} \langle c,\bar{x}\rangle + \langle \hat{p},\bar{y}\rangle - M \leq 0, \\ -\langle c,\bar{x}\rangle - \langle \hat{q},\bar{y}\rangle + L \leq 0. \end{cases} \tag{3.8}$$

Thus $(\bar{x},\bar{y},\hat{p},\hat{q})$ is also an optimal solution to the relaxed problem (3.4) of estimating the value $\beta(\Delta)$. On the other hand, for every feasible solution (x,y) of the problem (3.5) of estimating the value $\varphi(\hat{u},\hat{v})$, the point $(x,y,\hat{u},\hat{v})$ is a feasible solution of problem (3.4), which implies that $\beta(\Delta) = f(\bar{x},\bar{y}) \leq f(x,y)$. Thus $(\bar{x},\bar{y})$ is an optimal solution to the problem (3.5) satisfying

$$\beta(\Delta) = f(\bar{x},\bar{y}) = \varphi(\hat{u},\hat{v}).$$

Now, take an any pair of two vertices $\tilde{u},\tilde{v} \in V(\Delta)$, and assume that $(\tilde{x},\tilde{y})$ is an optimal solution to Problem (3.5). It follows from the definition that $(\tilde{x},\tilde{y},\tilde{u},\tilde{v})$ is a feasible solution of Problem (3.4), and

$$\varphi(\hat{u},\hat{v}) = \beta(\Delta) = f(\bar{x},\bar{y}) \leq f(\tilde{x},\tilde{y}) = \varphi(\tilde{u},\tilde{v}).$$

Finally, we obtain

$$\beta(\Delta) = \varphi(\hat{u},\hat{v}) \leq \varphi(u,v) \ \forall u,v \in V(\Delta).$$

The remaining assertion of the lemma is obvious. The proof is completed. □

Note that, for each vertex $u \in V(\Delta)$, for which $\varphi(u,u) < +\infty$, let denote by (x,y) the optimal solution of the relaxed problem (3.5). Then (x,y,u,u) is a feasible solution of the extended consumer problem $(ECP2)$. Obviously, if there exists an optimal solution (x,y,p) of Problem (ECP2) such that $p \in \Delta$, then $\beta(\Delta)$ is a lower bound for f_*. In order to reduce the difference between $\beta(\Delta)$ and f_*, we realize a branching operation using the following simplex subdivision.

An adaptive simplex subdivision - Rule 2

The adaptive simplex subdivision rule is often used in branch-and-bound algorithms for solving non-convex optimization problems; see, e.g., [11]. Precisely, let Δ be a $m-$simplex in $\mathbb{R}^n$, take any point $\bar{x} \in \Delta$, and assume that $\bar{v}$ is an vertex of Δ such that $\bar{x} \neq \bar{v}$. Denote by $\bar{\eta}$ the midpoint of the line segment $[\bar{x}, \bar{v}]$. By this rule, the simplex Δ is divided into m subsimplices, which have the same vertex $\bar{\eta}$. In particular, if $\bar{x}$ is also a vertex of Δ, then Δ is bisected into two subsimplices. We call $\{\bar{x}, \bar{\eta}, \bar{v}\}$ a trio of division points. For this simplex subdivision, we have the following lemma. Its proof can be found in [11].

Lemma 3.4. *Let $\{\Delta^k\}_0^{+\infty}$ be an infinite sequence of simplices generated by the adaptive simplex subdivision Rule 2 such that $\Delta^{k+1} \subset \Delta^k$ for all k. Let $\{\bar{x}^k, \bar{\eta}^k, \bar{v}^k\}$ be the trio of division points for the simplex Δ^k. Then $\lim_{k\to\infty} \|\bar{x}^k - \bar{v}^k\| = 0$. Consequently, three sequences $\{\bar{x}^k\}, \{\bar{\eta}^k\}$ and $\{\bar{v}^k\}$ tend to the same point $\hat{x} \in \Delta^0$.*

The algorithm now can be described as follows.

Algorithm 3.5. *Initial iteration. Choose a tolerance $\varepsilon \geq 0$ and take $\Sigma_0 := \{\Delta^0 = S\}$. Calculate the lower bound*

$$\beta_0 = \beta(\Delta^0) = \min\{\varphi(u,v) : u, v \in V(\Delta^0)\},$$

and denote by (x^0, y^0, u^0, v^0) the obtained optimal solution of Problem (3.5) of estimating the value $\varphi(u^0, v^0) = \beta_0$. Moreover, if there exists $u \in V(\Delta^0)$ such that $\varphi(u,u) < +\infty$, then take

$$\alpha_0 := \varphi(u^0, u^0) = \min\{\varphi(u,u) : u \in V(\Delta^0)\},$$

and let α_0 be an upper bound of f_*. On the other hand, $(\bar{x}^0, \bar{y}^0, \bar{u}^0)$ is the obtained optimal solution of Problem (3.5) of estimating the value $\varphi(u^0, u^0)$, which is also a feasible solution of Problem (ECP2). Otherwise, we take the upper bound $\alpha_0 := +\infty$. If $|\alpha_0 - \beta_0| \leq \varepsilon$ or $\beta_0 = +\infty$, then regain $\Sigma_0 := \emptyset$.

Iteration $k, k = 0, 1, \ldots$.
At each iteration k, we have a finite family Σ_k of subsimplices of S. For each $\Delta \in \Sigma_k$ with $\beta(\Delta) < +\infty$, we have a vertex pair $u^\Delta, v^\Delta \in V(\Delta)$ such that

$$\beta(\Delta) = \varphi(u^\Delta, v^\Delta) = f(x^\Delta, y^\Delta),$$

where (x^Δ, y^Δ) is an optimal solution of Problem (3.5) of estimating the value $\varphi(u^\Delta, v^\Delta)$. Beside, if there exists $u \in V(\Delta)$ such that $\varphi(u,u) < +\infty$, we have more $(\bar{x}^\Delta, \bar{y}^\Delta, \bar{u}^\Delta)$ with

$$\varphi(\bar{u}^\Delta, \bar{u}^\Delta) = \min\{\varphi(u,u) : u \in V(\Delta)\},$$

where $(\bar{x}^\Delta, \bar{y}^\Delta)$ is an optimal solution of Problem (3.5) of estimating the value $\varphi(\bar{u}^\Delta, \bar{u}^\Delta)$.

Step 1. If $\Sigma_k = \emptyset$, then terminate. In this case, either $(\bar{x}^k, \bar{y}^k, \bar{p}^k)$ is an optimal $\varepsilon-$solution of Problem (ECP2), or Problem (ECP2) has no feasible solution. Otherwise, take $\Delta^k \in \Sigma_k$ such that

$$\beta(\Delta^k) = \min\{\beta(\Delta) : \Delta \in \Sigma_k\},$$

and for ease of presenting we rename the iterative points on the simplex Δ_k as follows

$$x^k := x^{\Delta_k};\ y^k := y^{\Delta_k};\ u^k := u^{\Delta_k};\ v^k := v^{\Delta_k},$$

and the best feasible solution of the Problem (ECP2) as

$$\bar{x}^k := \bar{x}^{\Delta_k};\ \bar{y}^k := \bar{y}^{\Delta_k};\ \bar{u}^k := u^{\Delta_k}$$

if there exists $u \in V(\Delta_k)$ such that $\varphi(u,u) < +\infty$.

Step 2. Bisect the simplex Δ^k by the Rule 2 with the trio of division points $\{u^k, w^k, v^k\}$, where w^k is the middle point of the edge $[u^k, v^k]$. We obtain two new subsimplices, that is denoted by Δ^{k^+} and Δ^{k^-}, with the sets of vertices, respectively, as

$$V(\Delta^{k^+}) = V(\Delta^k) \setminus \{v^{\Delta^k}\} \cup \{w^{\Delta^k}\}$$

and

$$V(\Delta^{k^-}) = V(\Delta^k) \setminus \{u^{\Delta^k}\} \cup \{w^{\Delta^k}\}.$$

Step 3. Calculate the lower bounds and update the currently best feasible solutions for each newly generated subsimplex, that is,

$$\beta(\Delta^{k^+}), x^{\Delta^{k^+}}, y^{\Delta^{k^+}}, u^{\Delta^{k^+}}, v^{\Delta^{k^+}}, \bar{x}^{\Delta^{k^+}}, \bar{y}^{\Delta^{k^+}}, \bar{u}^{\Delta^{k^+}} \text{ for } \Delta^{k^+},$$

and

$$\beta(\Delta^{k^-}), x^{\Delta^{k^-}}, y^{\Delta^{k^-}}, u^{\Delta^{k^-}}, v^{\Delta^{k^-}}, \bar{x}^{\Delta^{k^-}}, \bar{y}^{\Delta^{k^-}}, \bar{u}^{\Delta^{k^-}} \text{ for } \Delta^{k^-}.$$

Step 4. Update the currently best upper bound by taking

$$\alpha_{k+1} := \min\{\alpha_k, f(\bar{x}^{\Delta^{k^+}}, \bar{y}^{\Delta^{k^+}}), f(\bar{x}^{\Delta^{k^-}}, \bar{y}^{\Delta^{k^-}})\},$$

and $(\bar{x}^{k+1}, \bar{y}^{k+1})$, is the currently best feasible solution of Problem (ECP2).
Step 5. Set

$$\Sigma_{k+1/2} := \big(\Sigma_k \setminus \{\Delta^k\}\big) \cup \{\Delta^{k^+}, \Delta^{k^-}\}$$

and update

$$\Sigma_{k+1} := \{\Delta \in \Sigma_{k+1/2} : \beta(\Delta) < \alpha_{k+1}\}.$$

Let $k := k+1$ and go to Step 1.

Remark 3.6. (i) When the simplex Δ^k is bisected into two subsimplices Δ^{k^+} and Δ^{K^-}, on each newly generated simplex there is only one new vertex arisen,and the others are all old. Therefore, in order to estimate the bounds and the iterative points for these new subsimplices, we just have to solve $2m-1$ convex optimization problems (3.5). This greatly reduces the computational cost of the algorithm.

(ii) Thanks to the special structure of the feasible set in Problem (ECP2) restricted on the simplex Δ, that is,

$$\begin{cases} \langle c,x\rangle + \langle p,y\rangle & \leq & M, \\ \langle c,x\rangle + \langle p,y\rangle & \geq & L, \end{cases}$$

we can reduce the number of convex programs, which have to be solved for each simplex Δ. For each vertex $u \in V(\Delta)$, we solve two linear programs

$$\begin{array}{rcl} \bar{M}(u) & := & \max\{\langle c,x\rangle + \langle u,y\rangle : (x,y) \in D\}, \\ \bar{L}(u) & := & \min\{\langle c,x\rangle + \langle u,y\rangle : (x,y) \in D\}. \end{array}$$

Then we divide the vertex set $V(\Delta)$ into three groups as

$$\begin{array}{rcl} V^1(\Delta) & := & \{u \in V(\Delta) : \bar{M}(u) < L \text{ or } \bar{L}(u) > M\}; \\ V^2(\Delta) & := & \{u \in V(\Delta) : L \leq \bar{L}(u) \leq \bar{M}(u) \leq M\}; \\ V^3(\Delta) & := & V(\Delta) \setminus \big(V^1(\Delta) \cup V^2(\Delta)\big). \end{array}$$

Obviously, for every $u,v \in V^1(\Delta)$, we have $\varphi(u,v) = +\infty$, whereas for every $u,v \in V^2(\Delta)$,

$$\varphi(u,v) = \min\{f(x,y) : (x,y) \in D\}.$$

Thus we just have to solve the convex problems (3.5) for estimating the values $\varphi(u,v)$ with $u,v \in V^3(\Delta)$.

Theorem 3.7. *Suppose that f is a continuous function over $X \times Y$ and Problem (ECP2) has an optimal solution. If the algorithm terminates at iteration k, then $(\bar{x}^k, \bar{y}^k, \bar{p}^k)$ is an $\varepsilon-$ global optimal solution to Problem (ECP2). Otherwise, if the algorithm does not terminate after a finite number of iterations, then every cluster point of the sequence $\{(x^k, y^k, u^k)\}$ is an optimal solution to Problem (ECP2). Furthermore, $\beta_k \nearrow f_*$.*

Proof. The first assertion is obvious. Otherwise, let (x^*, y^*, u^*) be any cluster point of the sequence $\{x^k, y^k, u^k\}$. Take a subsequence if necessary, without lost of the generality, we may assume that

$$\lim_{k\to\infty}(x^k, y^k, u^k) = (x^*, y^*, u^*). \tag{3.9}$$

By the rule of simlex-subdivision (Rule 2) and the bounding operation, there exists a sequence of nested simpleciees $\{\Delta^{k_j}\}$ of the sequence $\{\Delta^k\}$ such that $\Delta^{k_{j+1}} \subset \Delta^{k_j}$ and $\beta(\Delta^{k_{j+1}}) \leq \beta(\Delta^{k_j})$. For each subsimplex Δ^{k_j}, we have $f(x^{k_j}, y^{k_j}) = \varphi(u^{k_j}.v^{k_j}) = \beta(\Delta^{k_j})$ and the trio of division points $\{u^{k_j}, w^{k_j}, v^{k_j}\}$. From Lemma 3.4, one has

$$\lim_{j\to\infty}\|u^{k_j} - v^{k_j}\| = 0. \tag{3.10}$$

Combining (3.9) with (3.10), it results

$$\lim_{j\to\infty} u^{k_j} = \lim_{j\to\infty} v^{k_j} = u^*. \tag{3.11}$$

On the other hand, from the definition of Problem (3.5) of estimating the value $\varphi(u^{k_j}, v^{k_j})$, it follows that

$$\left\{\begin{array}{lll} (x^{k_j}, y^{k_j}) \in D; u^{k_j}, v^{k_j} \in S\ \forall j, & & \\ \langle c, x^{k_j}\rangle + \langle u^{k_j}, y^{k_j}\rangle - M & \leq & 0, \\ -\langle c, x^{k_j}\rangle - \langle v^{k_j}, y^{k_j}\rangle + L & \leq & 0. \end{array}\right. \tag{3.12}$$

Passing the limit as $j \to \infty$ and using (3.9), (3.11), and the closedness of the sets D and S we obtain

$$\left\{\begin{array}{lll} (x^*, y^*) \in D, u^* \in S, & & \\ \langle c, x^*\rangle + \langle u^*, y^*\rangle - M & \leq & 0, \\ -\langle c, x^*\rangle - \langle u^*, y^*\rangle + L & \leq & 0, \end{array}\right. \tag{3.13}$$

which shows that (x^*, y^*, u^*) is a feasible solution to Problem (ECP2) and

$$f(x^*, y^*) \geq f_*. \tag{3.14}$$

Moreover, from the bounding operation, one has

$$\beta_{k_j} = \varphi(u^{k_j}, v^{k_j}) = f(x^{k_j}, y^{k_j}) \leq f_*\ \forall j. \tag{3.15}$$

By using the continuity of f, (3.9), and (3.15), and passing the limit as $j \to \infty$, we obtain

$$\lim_{j\to\infty}\beta_{k_j} = \lim_{j\to\infty} f(x^{k_j}, y^{k_j}) = f(x^*, y^*) \leq f_*. \tag{3.16}$$

Combining (3.13) with (3.16), we see that (x^*, y^*, u^*) is an optimal solution to Problem (ECP2). Finally, since the sequence $\{\beta_k\}$ is nondecreasing, by (3.16), we have $\beta_k \nearrow f_*$ as $k \to \infty$. □

4. Computational Results and Experience

We tested two proposed algorithms in MATLAB and executed on a PC Core 2Duo 2*2.0 GHz, RAM 2GB. We tested the program on different groups of problems, each of them contains ten problems of the same sizes, but having randomly generated input data.

With the Algorithm 2.2, for solving the Problem (ECP1) and for each problem, we take $D = \mathbb{R}^{n+m}$; $X = [l_X, u_X]$ to be a box in $\mathbb{R}^n$, whereas $Y = [l_Y, u_Y]$ and $P_0 = [l_P, u_P]$ are boxes in $\mathbb{R}^m$. The lower bounds of the boxes X, Y, P_0 are fixed with $l_X = 0, l_Y = 0, l_P = 0$, whereas the upper bounds u_X, u_Y, u_P contain the numbers that are randomly generated in the interval $[1,5]$. The components of the vector of price c of n -first types of goods are randomly generated in the interval $[1,2]$. We take the utility function $U(x,y)$ to be a negative semidefinit quadratic form in $\mathbb{R}^{n+m}$. Thus the objective function of Problem (ECP1) is a convex one given as

$$f(x,y) = (x,y)^T Q(x,y) + d^T x + e^T y,$$

where Q is a diagonal matrix of $(n+m)-$dimension with the components $q_{ii}, i = 1,...,m+n$ being randomly generated in the interval $[0,0.5]$, and the components of the two vectors $d \in \mathbb{R}^n$ and $e \in \mathbb{R}^m$ are randomly generated in the interval $[-2,2]$. Finally, in order to guarantee that the problem to be solved has a feasible solution, we take $M = 2(m+n)$ and $L = M - t$, where t is randomly generated in the interval $[5,15]$. In addition, we limit the maximum number of iterations for solving each problem to be 2000. The obtained results are reported in Table 4.1 below, where we use the following headings:

- N: number of the tested problem;
- n: number of the goods whose prices are defined (the number of the convex variables);
- m: number of the goods whose prices are unknowns, ($2m$ is the number of the concave variables);
- *Average time*: the average time (in second) needed to solve one problem;
- *Average iter.*: the average number of iterations for one problem;
- N*: number of the problems for which a global solution was obtained.

N	n	m	Average iter.	Average time	N*
10	2	2	22	12.35	10
10	10	2	459	276.02	10
10	100	2	479	287.54	10
10	2	5	296	164.28	10
10	2	8	1484	791.02	9
10	2	10	1793	893.19	3

Table 4.1

The results in Table 4.1 show that the algorithm is efficient when the number of goods whose prices have not been determined does not exceed 8, or the number of concave variables does not exceed 16, while the number of goods whose prices have been determined may be large (up to hundreds).

With the Algorithm 3.5, for solving the Problem (ECP2), in order to consider the relationship between the efficiency of the algorithm with the number of concave variables in Problem (ECP), for each problem, we assume that the prices of all types of goods are unknown. With this assumption, the constraint set D in Problem (ECP2) can be written as $Ay \leq b$. We fix the number of rows of A by 5, whereas the components a_{ij} $(i = 1,...,5; j = 1,...,m)$ are randomly generated in the interval $[-2,2]$. In addition, in order to guarantee that Problem (3.4) has a feasible solution, the components of the matrix A and the vector b are randomly generated in the interval $[M+l, 2(M+L)]$, where M is randomly generated in the interval $[10,15]$, and L is randomly generated in the interval $[7,13]$. The constraint set S of prices is a normal simplex in $\mathbb{R}^m$ given by $S := \{p \in \mathbb{R}^m_+ : \sum_{i=1}^m p_i = 1\}$. Finally, we take the utility function $U(y)$ to be a negative semidefinit quadratic form in $\mathbb{R}^m$. Thus the objective function of Problem (ECP2) is convex quadratic of the form $f(y) = y^T Q y + c^T y$, where $Q = (q_{ij})_{m\times m}$ is a diagonal matrix with q_{ii} being randomly generated in the interval $[0, 0.1]$, and $c \in \mathbb{R}^m$ with the components being randomly generated in the interval $[-20,2]$. The obtained results are reported in Table 4.2 below, with the same headings are in Table 4.1

N	m	Average iter.	Average time	N*
10	2	2	0.36	10
10	5	3	0.76	10
10	10	6	7.91	10
10	20	12	73.43	10
10	30	11	157.24	10
10	50	38	1812.99	10

Table 4.2

The results in Table 4.2 show that the second algorithm can solve the problems with the number of concave variables up to dozens.

5. Conclusion

An extended version of the classical customer problem was considered in this paper. The problem was formulated as a global optimization with two bilinear constraints. Two algorithms were proposed for globally solving the problem. The first algorithm used the convex envelope for bounding and a rectangular subdivision for the branching to handle the rectangular feasible domain. The second algorithm employed decoupling and simplicial operations to handle the two convex-concave constraints and the simplicial structure of the prices. Some computational results for many randomly generated problems were presented to show the efficiency and behavior of the proposed algorithms.

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