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HOLDITCH'S ENVELOPE

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Dedicated to Professor Francis Clarke on the occasion of his 75th birthday

Abstract. An implicit assumption in the original version of Holditch's theorem is C^1-regularity and strict convexity of the envelope generated by a chord traveling around a convex curve $\mathbb{C}$. We establish that this holds when $\mathbb{C}$ is C^2-regular with positive curvature and the chordlength is sufficiently small. We also consider the case where $\mathbb{C}$ is polyhedral. Then, strict convexity of the envelope may not hold, but, for sufficiently small chordlength, it is nevertheless C^1-regular. The case of a general convex curve $\mathbb{C}$ remains an open problem.

Keywords. Convexity; Envelope; Holditch's theorem; Regularity; Tangency curve.

1. Introduction

In 1858, Reverend Hamnet Holditch, president of Gonville and Caius College, Cambridge, published the following rather surprising result [3], which is listed as one of C. Pickover's 250 milestones in the history of mathematics [5]:

Theorem 1.1. (Holditch's original statement) *If a chord of a closed curve, of constant length $a+b$, be divided into two parts of lengths a and b respectively, the difference between the areas of the closed curve, and of the locus of the dividing point, will be πab.*

Some clarification is required, including a discussion of Holditch's unstated assumptions.

- Holditch undoubtedly intended the curve, which we denote by $\mathbb{C}$, to enclose a compact convex region $\mathbb{W}$; that is, $\mathbb{C}$ is a *convex curve*. Since $\mathbb{C}$ is rectifiable, there exists a continuous function $C:\mathbb{R}\to\mathbb{R}^2$ which provides a parametrization with respect to arclength. Here
  - (i) $C(s)$ moves counterclockwise as s increases.
  - (ii) $C(0)=C(S)$, where $S>0$ is the arclength of $\mathbb{C}$.
  - (iii) $C(s+S)=C(s)$ for all $s\in\mathbb{R}$.

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(iv) For s_1 and s_2 in $[0,S)$ we have $s_1 \neq s_2 \implies C(s_1) \neq C(s_2)$.

- As the chord $Q(s) = [C(s), B(s)]$ traverses $\mathbb{C}$, we view $C(s)$ as the tail, and we denote the head by $B(s)$ as s increases. We will assume that the function $B : \mathbb{R} \to \mathbb{C}$ has the following properties (where $\|\cdot\|$ denotes the Euclidean norm).
  (a) $B(\cdot)$ enjoys the same properties (i)-(iv) as $C(\cdot)$ does.
  (b) $\|C(t) - B(t)\| = L = a + b, \ \forall t \in \mathbb{R}$.

  When $C(\cdot)$ and $B(\cdot)$ as above exist for a given L, we say, as in [6], that there is *good chord travel*. In view of our assumptions, the direction angle $\theta(\cdot)$ of the chord, measured counterclockwise from the horizontal, is continuous and nondecreasing. Note that this condition prohibits the so-called retrograde movement of the chord as considered in [4].

  As Holditch, we assume to start with a convex simple curve that bounds a compact convex region. Furthermore, the curve is said to be C^k-*regular* on an open interval if it is k times continuously differentiable with nonvanishing derivative on that interval. It is said to be *strictly convex* if it is locally isomorphic to the graph of a strictly convex function.

- For $s \in \mathbb{R}$, we denote by $\tau(s)$ the minimal value greater than s such that $B(s) = C(\tau(s))$. Then our assumptions imply that $\tau(\cdot)$ is continuous and strictly increasing.

- For the statement of Holditch's theorem to make sense, we require that the locus of the dividing point be a Jordan curve. In Proppe, Stancu and Stern [6] it was proven that all such loci, there called *Holditch curves*, are Jordan if L is small enough. In the same paper, smoothness and convexity properties of Holditch curves were investigated as well.

- In [6] it was shown that good chord travel holds when $\mathbb{C}$ is C^1-regular and L is sufficiently small. Also, it is readily noted that if $\mathbb{W}$ is polyhedral with no acute interior angles, a sufficient (but not necessary) condition for good chord travel is that L does not exceed the length of any side.

- Holditch also assumed the existence of an *envelope* $\mathbb{E}$, a curve with the property that the moving chord is tangent to $\mathbb{E}$ at a unique contact point. Hence Holditch required the envelope to be C^1 regular and strictly convex. In his words, this point is where the chord "intersects its consecutive position". Broman [2] suggests that this terminology was influenced by Newton.

Broman's modern proof of Holditch's theorem (and a generalization) manages to avoid the envelope issue entirely, as a line integral technique was employed; see also Monterde and Rochera [4]. Nevertheless, a problem of independent interest in Convex Analysis is establishing the existence of an envelope as Holditch envisioned. Broman's article also includes Holditch's original proof, assuming the existence of a C^1-regular and strictly convex envelope. The method uses "sweeping tangents" as in Mamikon's theorem [1], a technique which Reverend Holditch must have been aware of.

We prove that if $\mathbb{C}$ is C^2-regular with positive curvature, then the smallness of L implies that a C^1-regular envelope $\mathbb{E}$ exists. It is shown that $\mathbb{E}$ is a strictly convex curve in the interior of $\mathbb{W}$. We also consider the case that $\mathbb{W}$ is polyhedral.

2. PRELIMINARIES

We will make reference to the following two results (Propositions 3.3 and 3.5) from [6]. We include the proofs for the benefit of the reader. Here $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^2$.

Proposition 2.1. *Suppose that $C(\cdot)$ is C^1-regular on $\mathbb{R}$ and assume that $\hat{s}$ is such that the chord $Q(\hat{s})$ is perpendicular to the curve $\mathbb{C}$ at $B(\hat{s}) = C(\tau(\hat{s}))$, but not at $C(\hat{s})$. Then $B(\cdot)$ is not differentiable at $\hat{s}$.*

Proof. The geometric hypotheses assert that

$$\langle B(\hat{s}) - C(\hat{s}), \dot{C}(\tau(\hat{s}))\rangle = 0 \tag{2.1}$$

and

$$\langle B(\hat{s}) - C(\hat{s}), \dot{C}(\hat{s})\rangle \neq 0. \tag{2.2}$$

By way of contradiction, suppose that $\tau(\cdot)$ is differentiable at $\hat{s}$. After differentiating the equation $\|C(s) - C(\tau(s))\|^2 = L^2$ and gathering terms, one obtains

$$\dot{\tau}(s)\left[\langle B(s) - C(s), \dot{C}(\tau(s))\rangle\right] = \langle B(s) - C(s), \dot{C}(s)\rangle.$$

Upon taking $s = \hat{s}$ and applying (2.1) - (2.2), the left side is 0 and the right side is not. This contradiction shows that $\tau(\cdot)$ cannot be differentiable at $\hat{s}$. Now, since $B(\cdot) = C(\tau(\cdot))$, $C(\cdot)$ is C^1-regular and $\tau(\cdot)$ is continuous and strictly increasing, the non-differentiability of $B(\cdot)$ at $\hat{s}$ follows. □

Proposition 2.2. *Suppose that $C(\cdot)$ is C^1-regular on $\mathbb{R}$ and assume that $\hat{s}$ is such that the chord $Q(\hat{s})$ is not perpendicular to the curve $\mathbb{C}$ at both $B(\hat{s}) = C(\tau(\hat{s}))$ and $C(\hat{s})$. Then the following hold:*

(1) There exists an open interval containing $\hat{s}$ upon which $B(\cdot)$ is C^1-regular.
(2) If the above non-perpendicularity assumptions hold, but now with $C(\cdot)$ taken to be C^k-regular, then there exists an open interval containing $\hat{s}$ upon which $B(\cdot)$ is C^k-regular.

Proof. Consider the continuously differentiable function given by

$$g(s,\tau) := \|C(s) - C(\tau)\|^2 - L^2.$$

The non-perpendicularity condition at $B(\hat{t})$ is

$$\langle \dot{C}(\tau(\hat{s})), [C(\hat{s}) - C(\tau(\hat{s}))\rangle] \neq 0,$$

which is equivalent to

$$\frac{\partial g}{\partial \tau}(\hat{s}, \tau(\hat{s})) \neq 0.$$

Since $g(\hat{s}, \tau(\hat{s})) = 0$, the Implicit Function Theorem asserts that on a neighborhood of $\hat{s}$ there is defined a unique continuously differentiable function $h(\cdot)$, such that $g(s, h(s)) = 0$. But, since $g(s, \tau(s)) = 0$ for all s, we have that $h(\cdot) = \tau(\cdot)$ near $\hat{s}$. The Implicit Function Theorem also asserts that for s near $\hat{s}$ one has

$$\dot{\tau}(s) = -\frac{\frac{\partial g}{\partial s}(s,\tau(s))}{\frac{\partial g}{\partial \tau}(s,\tau(s))}.$$

The non-perpendicularity condition at $C(\hat{s})$ is

$$\langle \dot{C}(\hat{s}), C(\hat{s}) - C(\tau(\hat{s}))\rangle \neq 0,$$

which is equivalent to the numerator in the above expression being nonzero. Hence $\dot{\tau}(\cdot)$ is nonzero near $\hat{s}$. (In fact, since $\tau(\cdot)$ is strictly increasing, we have positivity of this derivative near $\hat{s}$.) Since $B(\cdot) = C(\tau(\cdot))$, we have $\dot{B}(s) = \dot{C}(\tau(s))\dot{\tau}(s)$, and therefore $\dot{B}(s)$ is continuously differentiable and nonzero near $\hat{s}$. This proves (1).

In order to prove (2), note that $g(\cdot,\cdot)$ is now a k times continuously differentiable function of (s,τ). Then (2) follows from the proof of (1), upon noting that the Implicit Function Theorem provides for $\tau(\cdot)$ to be k times continuously differentiable. □

Remark 2.3. The case where one has perpendicularity at *both* endpoints of the chord is indeterminate as far as differentiability of $B(\cdot)$ is concerned; see [6].

3. The case of C^2-regular $C(\cdot)$

We now assume that $C(\cdot)$ is C^2-regular.

3.1. **Special case: horizontal chord.** By Lemma 2.4 in [6], we can take $L > 0$ sufficiently small to guarantee that the traveling chord is non-perpendicular to $\mathbb{C}$ at both its endpoints.

Given a parameter value s_0, we first consider the case where the chord $Q(s_0)$ is horizontal with $C(s_0)$ to the left of $B(s_0)$. Then, there exists an open interval $I(s_0)$ around s_0 for which $s \in I(s_0)$ implies that $Q(s)$ is non-vertical, in which case the slope

$$m(s) := \frac{B_2(s) - C_2(s)}{B_1(s) - C_1(s)}$$

is well defined. (Here the subscripts denote x and y components.) Also define $b(s) = C_2(s) - m(s)C_1(s)$. Then, for $s \in I(s_0)$, we have that

$$y = m(s)x + b(s) \tag{3.1}$$

is the equation of $L(s)$, the line generated by $Q(s)$.

Applying the classical definition, the *envelope* of the family of lines $\{L(s) : s \in I(s_0)\}$ is the set of all (x,y) satisfying (3.1), now written as

$$m(s)x - y = -b(s) \tag{3.2}$$

and

$$m'(s)x = -b'(s), \tag{3.3}$$

as the parameter s varies in $I(s_0)$. Note that $b(\cdot)$ and $m(\cdot)$ are C^2 on that interval by virtue of Proposition 2.2.

3.2. **Positive curvature case.** We now add the assumption that $\mathbb{C}$ has positive curvature at every point. Then the curvature of $\mathbb{C}$ has a positive lower bound.

The coefficient matrix of the system (3.2) - (3.3) is

$$J(s) := \begin{pmatrix} m(s) & -1 \\ m'(s) & 0 \end{pmatrix}.$$

Note that $m'(s) > 0$ on $I(s_0)$ by convexity of $\mathbb{C}$ and the positive curvature assumption, which implies strict convexity of $\mathbb{C}$ on that interval. Hence $J(s)$ is nonsingular on $I(s_0)$ and the envelope is given by the C^1 curve $(x(\cdot), y(\cdot))$, where

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = J(s)^{-1} \begin{pmatrix} -b(s) \\ -b'(s) \end{pmatrix}.$$

We denote this curve on the interval $I(s_0)$ by $E(s_0)$ and refer to it as the *envelope on* $I(s_0)$.

Remark 3.1.

- From [6], we know of examples of ellipses where we have good chord travel but where L is too large for the non-perpendicularity of Proposition 2.2 to hold (i.e. the conditions of Proposition 2.1 hold). Then $m'(\cdot)$ does not exist at some $\hat{s}$, which leads to $(x(\hat{s}), y(\hat{s}))$ not being defined at that parameter value.
- A related example, but one that is more visual, will now be discussed heuristically, yet can be verified rigorously. First consider the polyhedral example from [2], where $\mathbb{C}$ is a square whose sides match the length of the traveling chord. As Figure 1 shows, as the chord traverses the square from a left-vertical to right-vertical position, the envelope consists of the union of two smooth branches. Note that the line supporting the horizontal chord cannot meet the boundary of the envelope because each point of the horizontal edge of the square is strictly below some (non-horizontal) chord. Now "slightly round" the square so that now $\mathbb{C}$ becomes C^2-regular with positive curvature, and with a chordlength that allows for good chord travel, but which violates the non-perpendicularity condition of Proposition 2.2. Then the same behaviour as in the square example occurs, since at the point where perpendicularity occurs, $m'(\cdot)$ does not exist and the envelope cannot be defined, resulting in a right-to-left jump of the envelope and the "two branch" behaviour shown in Figure 2.

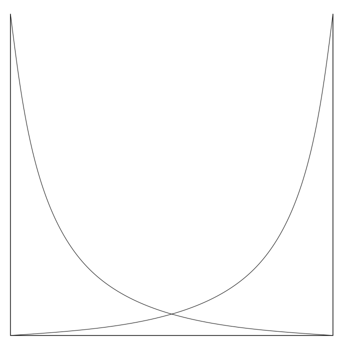

FIGURE 1. Square example

We do not (yet) assert that the curve $E(s_0)$ is regular; that is, has nonvanishing velocity. Nevertheless, we can show that

$$m(s)x'(s) - y'(s) = 0 \tag{3.4}$$

for every $s \in I(s_0)$. In case the curve $E(s_0)$ is regular, this says that it is tangent to $L(s)$ at $(x(s), y(s))$ for each $s \in I(s_0)$, which agrees with the classical notion of the envelope. To verify (3.4), differentiate (3.2) and apply (3.3) in order to obtain

$$m(s)x'(s) - y'(s) = -(m'(s)x(s) + b'(s)) = 0,$$

as claimed.

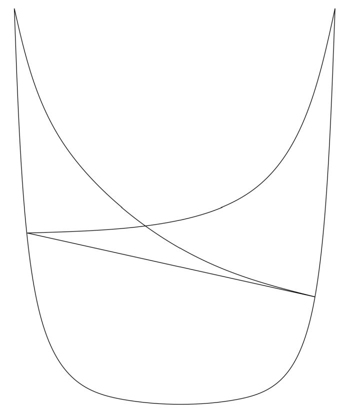

FIGURE 2. Slightly rounded square example

3.2.1. *Regularity and strict convexity of envelope on $I(s_0)$.* Differentiation of (3.2)-(3.3) yields

$$J(s)\begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} -(m'(s)x(s)+b'(s)) \\ -(m''(s)x(s)+b''(s)) \end{pmatrix} = \begin{pmatrix} 0 \\ -(m''(s)x(s)+b''(s)) \end{pmatrix}.$$

Hence regularity of $E(s_0)$ is equivalent to the second order condition

$$m''(s)x(s)+b''(s) \neq 0, \ \forall s \in I(s_0). \tag{3.5}$$

Our goal now is to show that this condition holds if L is sufficiently small a priori. For given s_0, we apply a rotation of coordinates which moves $Q(s_0)$ to a horizontal position with $C(s_0)$ to the left of $B(s_0)$. Let z denote the unique point on $\mathbb{C}$ below $Q(s_0)$, where the tangent line to $\mathbb{C}$ is horizontal, and consider the osculating circle $\overline{\mathbb{C}}$ to $\mathbb{C}$ at z. We parametrize $\overline{\mathbb{C}}$ by arclength with $\overline{C}(s_1) = z = C(s_1)$. From the properties of the osculating circle, we have

$$C'(s_1) = \overline{C}'(s_1),$$

and

$$C''(s_1) = \overline{C}''(s_1)$$

with the curvature of the two curves agreeing at z.

With apparent notation, we denote by $\overline{C}(s_0)$ the point on the osculating circle which is $s_0 - s_1$ units of arclength to the left of z, and we denote by $\overline{B}(s_0)$ the point to the right of z such that $\|\overline{B}(s_0) - \overline{C}(s_0)\| = L$. Figure 3 summarizes this setup.

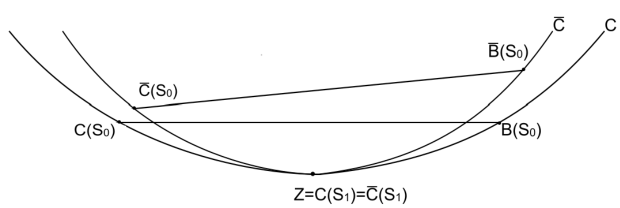

FIGURE 3. Approximation by osculating circle

Note that the approximation between the two curves must be sufficiently tight for $\overline{B}(s_0)$ to be defined, and that this will hold for small enough L. Also, as already pointed out, smallness of L implies non-perpendicularity to both curves at the endpoints of the respective chords.

On $I(s_0)$, $\overline{C}(\cdot)$ provides a second order approximation of $C(\cdot)$ to arbitrary tolerance, for small enough L. Also, from the proof of Proposition 2.2, we can reduce $I(s_0)$ to ensure that $\overline{\tau}(\cdot)$ approximates $\tau(\cdot)$ uniformly up to second order on $I(s_0)$ to any specified tolerance. Then the same holds for $\overline{B}(\cdot)$ vis a vis $B(\cdot)$, $\overline{m}(\cdot)$ vis a vis $m(\cdot)$, $\overline{b}(\cdot)$ vis a vis $b(\cdot)$, $\overline{J}(\cdot)$ vis a vis $J(\cdot)$, and $(\overline{x}(\cdot), \overline{y}(\cdot))$ vis a vis $E(s_0)$.

We now turn to the question of C^1-regularity of $E(s_0)$. We will show that (3.5) holds for small L and suitably reduced $I(s_0)$. Note that on $I(s_0)$

$$\begin{pmatrix} m'(s) & b'(s) \\ m''(s) & b''(s) \end{pmatrix} \begin{pmatrix} x(s) \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ m''(s)x(s) + b''(s) \end{pmatrix}.$$

Hence, if we can show that on $I(s_0)$, the matrix

$$W(s) := \begin{pmatrix} m'(s) & b'(s) \\ m''(s) & b''(s) \end{pmatrix}$$

is nonsingular on $I(s_0)$, then (3.5) holds.

Note that the deviation of the chord $[\overline{C}(s_0), \overline{B}(s_0)]$ from horizontal can be made arbitrarily small by reduction of L. Combined with the circle symmetry with respect to any line passing through its center, this justifies our assuming that the chord is horizontal in the ensuing analysis. Denote the radius of the osculating circle by a, and for ease of notation, take $z = (0,0)$, so the equation of the circle in polar coordinates is $r = 2a\sin\theta$. The chord subtends an angle $2\phi_L$ at the centre and so ϕ_L is the angle between the y-axis and the radial line to the point where the chord intersects the circle in quadrant 1. See Figure 4.

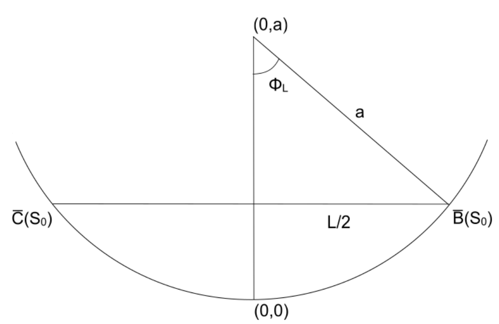

FIGURE 4. Traveling chord along the osculating circle

If the chord is now rotated counterclockwise through a (small) angle γ, its head travels through a distance $\sigma = a\gamma$ along the circle and intersects it at the point $(h(\sigma), k(\sigma))$, where

$$h(\sigma) = a\sin\left(\phi_L + \frac{\sigma}{a}\right)$$

and

$$k(\sigma) = a\left(1-\cos\left(\phi_L+\frac{\sigma}{a}\right)\right).$$

Writing $y=\overline{m}(\sigma)x+\overline{b}(\sigma)$ as the equation of the moving chord on the osculating circle, we obtain

$$\overline{m}(\sigma)=\tan\left(\frac{\sigma}{a}\right)$$

and

$$\overline{b}(\sigma)=k(\sigma)-\tan\left(\frac{\sigma}{a}\right)h(\sigma).$$

Then $\overline{m}'(0)=\frac{1}{a}$, $\overline{m}''(0)=0$, and

$$\overline{b}''(0)=-\frac{1}{a}\cos(\phi_L).$$

Since $\sin\phi_L = L/2a$, we have $\overline{b}''(0)\to -\frac{1}{a}$ as $L\searrow 0$. It follows that for small L, the matrix $W(\cdot)$ is nonsingular on (the suitably reduced interval) $I(s_0)$. We conclude that the curve $E(s_0)$ is C^1-regular. Therefore, as mentioned above, we then have that $E(s_0)$ is tangent to $L(s)$ at $(x(s),y(s))$ for $s\in I(s_0)$.

Since $m(s)$ is bounded on $I(s_0)$, we have that $x'(s)\neq 0$. Then $E(s_0)$ is the graph of a strictly convex function (since its derivative is strictly increasing by virtue of the fact that $m(s)$, the slope of of $L(s)$, is strictly increasing). It follows that $(x(s),y(s))$ is the unique point on $E(s_0)$ which makes contact with $L(s)$.

Finally, we rotate coordinates again, this time reverting to the original coordinates.

3.3. **Existence of a C^1-regular strictly convex envelope on $\mathbb{R}$.** We have that $\{I(s_0): s_0\in[0,S]\}$ is an open cover of the compact interval $[0,S]$. For each s_0, denote by $\Lambda(s_0)$ a chordlength which in conjunction with $I(s_0)$ results in regularity and strict convexity of $E(s_0)$, as established in the previous section. Then there exists a finite subcover $\{I((s_0)_i): i=1,2,\ldots,k\}$. Let $\tilde{\Lambda}$ be the minimal $\Lambda((s_0)_i)$ for this cover. We then do further chordlength reductions so that $\tilde{\Lambda}$ is the common value of all the $\Lambda((s_0)_i)$. Note that this does not necessitate adjusting the intervals $I((s_0)_i)$. Also, using a common chordlength on the intervals of this cover implies that if $s\in I((s_0)_i)\cap I((s_0)_j)$, then the envelopes $E((s_0)_i)$ and $E((s_0)_j)$ agree at s.

We now have the following result on the existence of a C^1-regular envelope.

Theorem 3.2. *Assume that $\mathbb{C}(\cdot)$ is C^2-regular with positive curvature. Then for L sufficiently small we have that $E:=\bigcup\{E((s_0)_i): i=1,2,\ldots,k\}$ is a C^1-regular curve which is tangent to $L(s)$ for every $s\in\mathbb{R}$ at a unique contact point and encloses a strictly convex region $\mathbb{Z}$. This region is the intersection of halfspaces associated with the collection of lines $L(s)$ for $s\in\mathbb{R}$. Furthermore, $\mathbb{Z}$ is compact and contained in the interior of $\mathbb{W}$.*

Proof. Everything follows from the preceding discussion except the "furthermore" part. Now, since $\mathbb{Z}$ is the intersection of halfspaces, it is closed. To see that it is in the interior of $\mathbb{W}$, note that any point not in the interior of $\mathbb{W}$ can be strictly separated from the intersection of halfspaces by one of the lines $L(\cdot)$, by virtue of the strict convexity of $\mathbb{C}$. $\square$

4. POLYHEDRAL CASE

Let us now assume that the convex curve $\mathbb{C}$ is polyhedral; that is, the enclosed region $\mathbb{W}$ is the intersection of finitely many halfspaces. Then $\mathbb{W}$ has a finite number of corners (extreme points). As mentioned earlier, good chord travel is assured if the chordlength L does not exceed the minimum distance between corners, and at the corners, no interior angle is less than 90 degrees.

Our goal now is to show that in the polyhedral case, as in the case where $\mathbb{C}$ is C^2-regular with positive curvature, the intersection of halfspaces associated with linear extensions of the traveling chord forms a convex subset of $\mathbb{W}$ which is enclosed by a C^1-regular curve $\mathbb{E}$ if L is sufficiently small. Note, however, that now $\mathbb{E}$ can have flat segments and on these segments, $\mathbb{E}$ and $\mathbb{C}$ coincide. Hence, unlike the case where $\mathbb{C}$ is C^2-regular with positive curvature, we cannot assert that the moving chord is tangent to $\mathbb{E}$ at a unique contact point, nor can we assert that the enclosed region is in the interior of $\mathbb{W}$. To facilitate our analysis, let us assume that $L < \frac{L'}{2}$, where L' denotes the minimum distance between corners of $\mathbb{C}$. Consider adjacent corners P', P and P'' as shown in Figure 5, where the chord begins its traverse of the corner P with its head at P and tail at A. It finishes its traverse with its tail at P and head at B. Since $L < \frac{L'}{2}$, the entire chord has already traveled along the left leg of the corner across a segment with length at least $L' - 2L$ prior to the head arriving at P, and then again prior to the head arriving at P''.

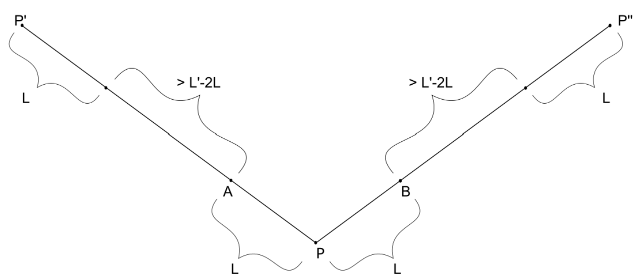

FIGURE 5. Polyhedral corner prior to traverse

We now consider the chord's traverse of the corner P. We will verify the following:

Lemma 4.1. *Consider a polyhedral curve, a chordlength that insures good chord travel along the curve and a traveling chord whose head and tail are on adjacent edges of the polyhedral curve meeting at a corner point P. Then, when the chord crosses any of its prior positions, the crossing point E between the present chord position and a previous chord position cannot be in the intersection of halfspaces that do not contain P determined by the linear extensions of the traveling chord.*

Since the slope of the chord strictly increases during the traverse, this implies that the boundary of the intersection of these halfspaces includes a strictly convex C^1-regular curve between A and B. Upon applying this to every corner of $\mathbb{C}$ results in a C^1-regular envelope $\mathbb{E}$ as illustrated in Figure 6 for a five-sided $\mathbb{C}$.

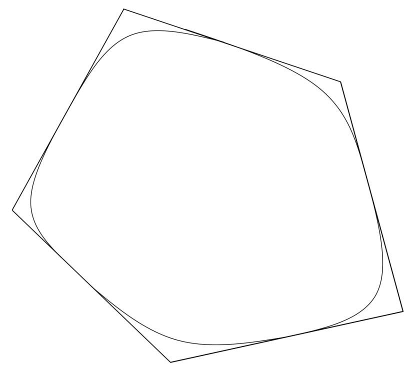

FIGURE 6. Envelope for five-sided polygonal $\mathbb{C}$

It remains to verify the lemma. To this end, we refer to Figure 7.

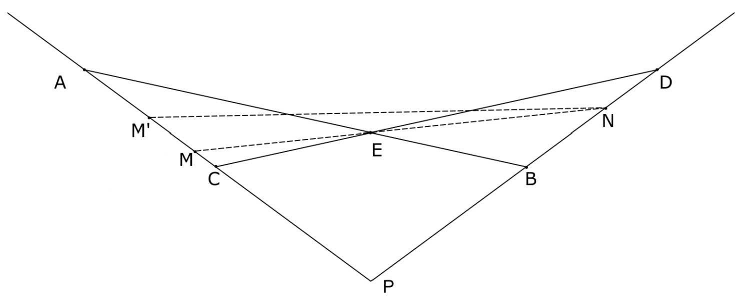

FIGURE 7. For two chords meeting at E, any third chord of intermediate slope cannot pass through E nor separate E from the corner P of the curve.

Proof the lemma: Consider the segment $[M,N]$, which divides the angle at E in some ratio as in Figure 7. We will show that the length of this segment is strictly less than L, the chordlength, as long as $[M,N]$ remains between the two chords meeting at E. Note that any segment in $\mathbb{W}$ which is parallel to and below $[M,N]$ would have even shorter length. It follows that during the traverse, when the chord is parallel to $[M,N]$, it is necessarily above E, which would verify the claim of the lemma.

Consider the quadrilateral $\mathcal{Q}$ with vertices A,C,B,D. It is a standard geometric fact (in any dimension) that the diameter of a polygon (polytope in higher dimension) is equal to length of its maximal diagonal, which is the chordlength L for the quadrilateral $\mathcal{Q}$. Now consider the chord $[M',N]$, crossing above E, as shown. Since its length exceeds that of $[M,N]$ but cannot exceed L, it follows that the length of $[M,N]$ is indeed strictly less than L. $\square$

If $[A,B]$ is a generic traveling chord along a polyhedral curve as sketched in Figure 7, with the standard convention as the chord moves counterclockwise along $\mathbb{C}$, we call the left halfspace associated with the linear extension of the chord $[A,B]$ the halfspace determined by the chord's linear extension that lies to the left of $[A,B]$ as the chord is traversed from A to B.

We can now state the result of this last section as follows.

Theorem 4.2. *Let $\mathbb{C}$ be a planar curve bounding a polyhedral convex set $\mathbb{W}$ with no acute interior angles. Then, if the chordlength is sufficiently small, then the intersection of the left halfspaces associated with the linear extensions of the traveling chord is a convex subset of $\mathbb{W}$ which is bounded by a C^1-regular curve.*

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