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AN ALGORITHM BASED ON A SET-VALUED NONEXPANSIVE MAPPING

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Dedicated to the memory of Professor Hoang Tuy

Abstract. In this paper, we generalize a recent convergence result of Tam (2018) which was proved for iterates of a set-valued operator such that its values can be expressed as a finite union of values of single-valued paracontracting operators. It is shown that this convergence result is true for a general set-valued mapping such that its values are not necessarily finite unions of values of single-valued operators.

Keywords. Convergence analysis, Fixed point; Nonexpansive mapping; Set-valued mapping.

1. INTRODUCTION

For more than sixty years now, there has been considerable research activity regarding the fixed point theory of certain classes of nonlinear mappings. See, for example, [3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 26, 27] and references cited therein. This activity stems from Banach's classical theorem [1] regarding the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, for instance, studies of feasibility, common fixed point problems and variational inequalities, which find significant applications in engineering, medical and the natural sciences [4, 7, 23, 24, 26, 27]. In particular, in [25] it was considered a framework for the analysis of iterative algorithms which can be described in terms of a structured set-valued operator. More precisely, at each point in the ambient space, it is assumed that the value of the operator can be expressed as a finite union of values of single-valued paracontracting operators. For such algorithms a convergence result was proved which generalizes a result obtained in [2]. The work [25] also contains an application of its main result to sparsity constrained minimisation.

In the present paper we generalize the main result of [25] for a general set-valued mapping such that its values are not necessarily finite unions of values of single-valued paracontracting operators.

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2. Preliminaries and the Main Result

Let R^n be the n-dimensional Euclidean space equipped with an inner product

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i,\ x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n)\in R^n$$

which induces the Euclidean norm $\|x\| = \langle x,x\rangle^{1/2}$, $x\in R^n$ and $C\subset R^n$ be a nonempty closed set.

For each $x\in R^n$ and each $r>0$ set

$$B(x,r)=\{y\in R^n:\ \|x-y\|\le r\}.$$

Assume that $T: C\to 2^C\setminus\{\emptyset\}$ is a mappings with the closed graph

$$\mathrm{graph}(T)=\{(x,y)\in R^n\times R^n:\ x\in C,\ y\in T(x)\}$$

such that $T(x)$ is bounded for every $x\in C$.

Define

$$F(T)=\{x\in X:\ x\in T(x)\}$$

and

$$\bar{F}(T)=\{x\in X:\ T(x)=\{x\}\}.$$

In this section, we state our main result which is proved under the following assumptions.

We assume that

$$\bar{F}(T)\neq\emptyset. \tag{2.1}$$

For each $x\in R^n$ and each nonempty set $A\subset R^n$, define

$$\rho(x,A)=\sup\{\|x-y\|:\ y\in A\}. \tag{2.2}$$

Assume that, for each $z\in\bar{F}(T)$, each $x\in F(T)$, and each $y\in C\setminus F(T)$,

$$\rho(z,T(x))\le\|z-x\| \tag{2.3}$$

and

$$\rho(z,T(y))<\|z-y\|. \tag{2.4}$$

Note that (2.3) and (2.4) are known in the literature as quasi-nonexpansive properties [20].

We also assume that the following upper semicontinuity assumption holds.

(A) For each $x\in C$ and each $\varepsilon>0$ there exists $\delta>0$ such that, for each $y\in B(x,\delta)\cap C$,

$$T(y)\subset\cup\{B(\xi,\varepsilon):\ \xi\in T(x)\}.$$

Theorem 2.1. *Assume that $\{x_i\}_{i=0}^{\infty}\subset C$ and that*

$$x_{i+1}\in T(x_i),\ i=0,1,\dots \tag{2.5}$$

Then the sequence $\{x_i\}_{i=0}^{\infty}$ is bounded and if a subsequence $\{x_{i_p}\}_{i=0}^{\infty}$ converges, then

$$\lim_{p\to\infty} x_{i_p}\in T(\lim_{p\to\infty} x_{i_p}).$$

This result is proved in the next section. It should be mentioned that in [25] a particular case of Theorem 2.1 was obtained when the value of the operator T is expressed as a finite union of values of single-valued paracontracting operators.

3. PROOF OF THEOREM 2.1

By (2.1), there exists $z \in C$ such that

$$T(z) = \{z\}. \tag{3.1}$$

It follows from (2.3)-(2.5) and (3.1) that, for each integer $i \geq 0$,

$$\|z - x_i\| \geq \|z - x_{i+1}\|. \tag{3.2}$$

Thus $\{x_i\}_{i=0}^{\infty}$ is bounded. Assume that a subsequence $\{x_{i_p}\}_{i=0}^{\infty}$ converges and set

$$x_* = \lim_{p \to \infty} x_{i_p}. \tag{3.3}$$

We show that $x_* \in T(x_*)$. Assume the contrary. Then

$$x_* \notin F(T). \tag{3.4}$$

By (2.4), (3.1), and (3.4), there exists $\varepsilon > 0$ such that

$$\rho(z, T(x_*)) < \|z - x_*\| - 4\varepsilon. \tag{3.5}$$

Assumption (A) implies that there exists $\delta \in (0, \varepsilon/2)$ such that

$$T(y) \subset \cup\{B(\xi, \varepsilon/2) : \ \xi \in T(x_*)\}, \ y \in B(x_*, \delta) \cap C. \tag{3.6}$$

In view of (3.3), there exists an integer $p_0 \geq 1$ such that

$$\|x_{i_p} - x_*\| \leq \delta \text{ for all integers } p \geq p_0. \tag{3.7}$$

Let $p \geq p_0$ be an integer. By (3.7),

$$\|x_{i_p} - x_*\| \leq \delta. \tag{3.8}$$

Equations (3.6) and (3.8) imply that

$$T(x_{i_p}) \subset \{B(\xi, \varepsilon/2) : \ \xi \in T(x_*)\}. \tag{3.9}$$

It follows from (2.2), (3.5), and (3.9) that

$$\begin{aligned} \rho(z, T(x_{i_p})) &\leq \varepsilon/2 + \rho(z, T(x_*)) \\ &\leq \varepsilon/2 + \|z - x_*\| - 4\varepsilon. \end{aligned} \tag{3.10}$$

By (2.2), (2.5), (3.8), and (3.10),

$$\begin{aligned} \|x_{i_p+1} - z\| \leq \rho(z, T(x_{i_p})) &\leq \|z - x_*\| - 3\varepsilon \\ &\leq \|z - x_{i_p}\| + \|x_{i_p} - x_*\| - 3\varepsilon \\ &\leq \|x_{i_p} - z\| - 2\varepsilon. \end{aligned} \tag{3.11}$$

Since ε depends only on z and x_* (see (3.5)) equations (3.2) and (3.11) imply that, for every integer $m > p_0$,

$$\begin{aligned} \|z - x_0\| &\geq \|z - x_0\| - \|z - x_{i_m+1}\| \\ &= \sum_{j=0}^{i_m} (\|z - x_j\| - \|z - x_{j+1}\|) \\ &\geq \sum_{p=p_0}^{m} (\|z - x_{i_p}\| - \|z - x_{i_p+1}\|) \\ &\geq 2(m - p_0)\varepsilon \to \infty \end{aligned}$$

as $m \to \infty$. The contradiction we have reached completes the proof of this theorem.

4. Conclusions

In our paper, we establish a convergence of iterates of a set-valued operator acting in a finite-dimensional Euclidean space. It generalizes its prototype obtained in a particular case when the values of the set-valued operator can be expressed as a finite union of values of single-valued paracontracting operators. We believe that this result can be useful in the studies of feasibility and common fixed point problems and their applications in optimization theory, engineering, medical and the natural sciences.

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