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COMMON SOLUTIONS OF A FINITE FAMILY OF MINIMIZATION PROBLEMS AND FIXED POINT PROBLEMS IN HADAMARD MANIFOLDS

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Dedicated to the memory of Professor Ram U. Verma

Abstract. In this paper, an iterative algorithm to approximate a common solution of a finite family of minimization problems and fixed point problems of a finite family of demicontractive mappings in Hadamard manifolds is proposed. Under suitable conditions, some convergence theorems of the sequence generated by the algorithm to the common solution of the two problems in Hardmard manifolds are proved.

Keywords. Demi-contractive mapping; Fejér monotone; Hadamard manifold; Proximal point algorithm; Nonexpansive mapping.

1. Introduction

In 1970, Martinet [14] proposed and analyzed the proximal point algorithms (PPA) as a tool to solve the convex minimization problem. Later on, Rockafellar [18] modified the PPA and studied the convergence analysis of the PPA in Hilbert spaces. In this connection, see also the early papers by [6, 16].

Recently, many convergence results by the proximal point algorithm have been extended from the classical linear spaces to the setting of manifolds; see, e.g., [1, 2, 3, 4, 7, 8, 9, 10, 11, 13, 15, 20, 21, 22] and the references therein.

In 2002, Ferreira and Oliveira [11] considered the proximal point method to solve convex optimization problem in the setting of Hadamard manifolds. While, Li et al. [13] extended the proximal point method for finding a solution of the following problem $x^* \in B^{-1}(\mathbf{0})$, in the setting of Hadamard manifolds.

In 2020, Ansari and Babu [1] extended the proximal point method for solving the following inclusion problem $x^* \in (A+B)^{-1}(\mathbf{0})$ in the setting of Hadamard manifolds with A being a

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continuous monotone vector field and B being a maximal monotone vector field defined on a closed bounded geodesic convex subset of a Hadamard manifold.

Recently, Chang et al. [8] proposed a new algorithm and proved that the sequence generalized by the algorithm converges strongly to a common element of the set of fixed points of a quasi-pseudo-contractive mapping and a demi-contraction mapping and the set of zeros of monotone inclusion problems on Hadamard manifolds. At the same time, Chang et al. [10] considered the inertial proximal point algorithm for finding a zero point of variational inclusions on Hadamard manifolds.

Inspired and motivated by these results, the purpose of this article is to propose an efficient iterative algorithm. Under suitable conditions, we prove that the sequence generated by our algorithm can approximate an element x^* such that

$$x^* \in \bigcap_{i=1}^{m} \{\arg\min_{y \in C} f_i(y)\} \bigcap \bigcap_{i=1}^{m} Fix(T_i),$$

where $f_i,\ i = 1,2,\cdots,m$ is a finite family of proper geodesic convex and lower semi-continuous functions and $T_i,\ i = 1,2,\cdots,m$ is a finite family of demicontractive mappings in Hadamard manifolds.

2. Preliminaries

In this section, we recall some notations, terminologies, and basic results from Riemannian manifold which can be found in any textbook on Riemannian geometry (see, for example, [20]).

Let M be a finite dimensional differentiable manifold, and let T_pM be the tangent space of M at $p \in M$. We denote by $TM = \bigcup_{p \in M} T_pM$ the tangent bundle of M. An inner product $\langle \cdot, \cdot \rangle_p$ on T_pM is called a Riemannian metric on T_pM. A tensor field $\langle \cdot, \cdot \rangle$ is said to be a Riemannian metric on M if, for every $p \in M$, the tensor $\langle \cdot, \cdot \rangle_p$ is a Riemannian metric on T_pM. The corresponding norm to the inner product $\langle \cdot, \cdot \rangle_p$ on T_pM is denoted by $||\cdot||_p$. We omit the subscript p if there is no confusion occurs.

A differentiable manifold M endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$ is called a Riemannian manifold. The length of a piecewise smooth curve $\gamma : [0,1] \to M$ joining p to q (i.e., $\gamma(0) = p$ and $\gamma(1) = q$) is defined as $L(\gamma) = \int_0^1 ||\gamma'(t)||dt$. The Riemannian distance $d(p,q)$ is the minimal length over the set of all such curves joining p to q, which induces the original topology on M.

A Riemannian manifold M is complete if, for any $p \in M$, all geodesics emanating from p are defined for all $t \in \mathbb{R}$. A geodesic joining p to q in M is said to be a minimal geodesic if its length is equal to $d(p,q)$. A Riemannian manifold M equipped with Riemannian distance d is a metric space (M,d). By Hopf-Rinow Theorem [20] if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M,d) is a complete metric space and bounded closed subsets are compact. If M is a complete Riemannian manifold, then the exponential map $exp_p : T_pM \to M$ at $p \in M$ is defined by $exp_p v = \gamma_v(1,p)$ for all $v \in T_pM$, where $\gamma_v(\cdot,p)$ is the geodesic starting from p with velocity v, that is, $\gamma_v(0,p) = p$ and $\gamma_v^{'}(0,p) = v$. It is known that $exp_p tv = \gamma_v(t,p)$ for each real number t. It is easy to see that $exp_p \mathbf{0} = \gamma_v(0,p) = p$, where $\mathbf{0}$ is the zero tangent vector. Note that the exponential map exp_p is differentiable on T_pM for any $p \in M$. A complete simply connected Riemannian manifold of non-positive sectional curvature is called a Hadamard Manifold.

Let M be a Hadamard manifold. Then, for any two points $x,y \in M$, there exists a unique normalized geodesic $\gamma : [0,1] \to M$ joining $x = \gamma(0)$ to $y = \gamma(1)$ which is in fact a minimal geodesic denoted by $\gamma(t) = exp_x texp_x^{-1} y$, $\forall t \in [0,1]$, and, for any sequence $\{x_n\} \subset M$ with $x_n \to x_0 \in M$, $exp_{x_n}^{-1} y \to exp_{x_0}^{-1} y$ and $exp_y^{-1} x_n \to exp_y^{-1} x_0$ for any $y \in M$; see [3].

The following inequalities can be proved easily.

Lemma 2.1. *Let M be a finite dimensional Hadamard manifold.*

(i) Let $\gamma : [0,1] \to M$ be a geodesic joining x to y. Then

$$d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2| d(x,y), \ \ \forall t_1, t_2 \in [0,1];$$

(From now on $d(x,y)$ denotes the Riemannian distance).

(ii) for any $x,y,z,u,w \in M$ and $t \in [0,1]$, the following inequalities hold:

$$d(exp_x texp_x^{-1} y,\ z) \le (1-t) d(x,z) + t d(y,z);$$

$$d^2(exp_x texp_x^{-1} y,\ z) \le (1-t) d^2(x,z) + t d^2(y,z) - t(1-t) d^2(x,y), \tag{2.1}$$

and

$$d(exp_x texp_x^{-1} y,\ exp_u(texp_u^{-1} w) \le (1-t) d(x,u) \ + t d(y,w).$$

Let M be a Hadamard manifold. A subset $C \subset M$ is said to be geodesic convex if, for any two points x and y in C, the geodesic joining x to y is contained in C. In the sequel, unless otherwise specified, we always assume that M is a finite dimensional Hadamard manifold, C is a nonempty, closed, and geodesic convex set in M, and $Fix(S)$ is the fixed point set of a mapping S.

A function $f : C \to (-\infty, \infty]$ is said to be geodesic convex if, for any geodesic $\gamma(\lambda) (0 \le \lambda \le 1)$ joining $x,y \in C$, the function $f \circ \gamma$ is convex, i.e.,

$$f(\gamma(\lambda)) \le \lambda f(\gamma(0)) + (1-\lambda) f(\gamma(1)) = \lambda f(x) + (1-\lambda) f(y).$$

Let X be a complete metric space and $Q \subset X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called Fejér monotone with respect to Q if for any $y \in Q$ and $n \ge 0$, $d(x_{n+1}, y) \le d(x_n, y)$.

Lemma 2.2. [3, 22] *Let X be a complete metric space, and let $Q \subset X$ be a nonempty set. If $\{x_n\} \subset X$ is Fejér monotone with respect to Q, then $\{x_n\}$ is bounded. Moreover, if a cluster point x of $\{x_n\}$ belongs to Q, then $\{x_n\}$ converges to x.*

Definition 2.3. A mapping $S : C \to C$ is said to be

(1) contractive if there exists a constant $k \in (0,1)$ such that

$$d(Sx, Sy) \le k d(x,y), \ \ \forall x,y \in C.$$

If $k = 1$, then S is said to be nonexpansive;

(2) quasinonexpansive if $Fix(S) \ne \emptyset$ and

$$d(Sx, p) \le d(x,p), \forall p \in Fix(S),\ x \in C;$$

(3) firmly nonexpansive [2] if, for all $x,y \in C$, the function $\phi : [0,1] \to [0,\infty]$ defined by

$$\phi(t) := d(exp_x texp_x^{-1} Sx, exp_y texp_y^{-1} Sy),\ \forall t \in [0,1]$$

is nonincreasing.

(4) k-demicontractive [8] if $Fix(T) \neq \emptyset$ and there exists a constant $k \in [0;1)$ such that

$$d^2(Sx,p) \leq d^2(x,p) + kd^2(x,Sx),\ \forall x \in X,\ p \in Fix(S);$$

Let $S : C \to C$ be a mapping. Then the following statements are equivalent; see [2].

(i) S is firmly nonexpansive;

(ii) for any $x,y \in C$ and $t \in [0,1]$, $d(S(x),S(y)) \leq d(exp_x texp_x^{-1} Sx, exp_y texp_y^{-1} Sy)$;

(iii) for any $x,y \in C$, $\langle exp_{S(x)}^{-1} S(y), exp_{S(x)}^{-1} x\rangle + \langle exp_{S(y)}^{-1} S(x), exp_{S(y)}^{-1} y\rangle \leq 0$.

Lemma 2.4. [8] *If $S : C \to C$ is a firmly nonexpansive mapping and $Fix(S) \neq \emptyset$, then, for any $x \in C$ and $p \in Fix(S)$, the following conclusion holds:*

$$d^2(Sx,p) \leq d^2(x,p) - d^2(Sx,x) \tag{2.2}$$

More information on firmly nonexpansive mappings can be found, for example, in [5, 19].

Remark 2.5. From Definition 2.3 and Lemma 2.4, it is easy to see that if $Fix(S) \neq \emptyset$, then the following implications hold:

$$S\ is\ firmly\ nonexpansive \implies S\ is\ nonexpansive$$
$$\implies S\ is\ quasinonexpansive \implies S\ is\ demicontractive,$$

but the converse is not true. Moreover, the class of demicontractive mappings have more powerful applications in solving mean geodesic problems; see, e.g., [12, 17].

Now we collect some basic concepts related to theory of geodesic convex optimization in Hadamard manifolds.

A function f defined on C is said to be lower semi-continuous (lsc) at a point $x \in C$ if $f(x) \leq \liminf_{n\to\infty} f(x_n)$ for each sequence $x_n \to x$. A function f is said to be lower semi-continuous on C if it is lsc at any point in C. A geodesic convex minimization problem together with the fixed point problem of a k-demicontractive mapping is defined as follows:

$$find\ x \in C\ such\ that\ f(x) = \min_{y\in C} f(y),\ \ and\ x = Tx.$$

The solution set of the geodesic convex minimization problem is denoted by

$$\arg\min_{y\in C} f(y) = \{x \in C:\ f(x) = \min_{y\in C} f(y)\}$$

Lemma 2.6. [3] *Let $f : C \to (-\infty,\infty]$ be a proper geodesic convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in Hadamard manifold M as*

$$J_\lambda^f(x) = \arg\min_{y\in C}[f(y) + \frac{1}{2\lambda} d^2(y,x)],\ \ \forall x \in C.$$

Then (i) the set $Fix(J_\lambda^f)$ of fixed points of the resolvent of f coincides with the set $argmin_{y\in C} f(y)$ of minimizers of f, and, for any $\lambda > 0$, the resolvent J_λ^f of f is a firmly nonexpansive mapping. Hence it is nonexpansive;

(ii) In addition, if $Fix(J_\lambda^f) \neq \emptyset$, then we have from (2.2) that

$$d^2(J_\lambda^f x,p) \leq d^2(x,p) - d^2(J_\lambda^f x,x),\ \forall x \in C,\ p \in Fix(J_\lambda^f).$$

Recall that a mapping $S : C \to C$ is said to be demiclosed at $\mathbf{0}$ if for any bounded sequence $\{x_n\} \subset C$ with $x_n \to x^* \in C$ and $d(x_n, Sx_n) \to 0$, then $x^* \in Fix(S)$). One remarks that it is easy to see that each nonexpansive mapping from C to C is demiclosed at $\mathbf{0}$.

Lemma 2.7. *Let C be a nonempty closed and convex subset of a Hadamard manifold M.*

(1) If $T : C \to C$ is a k-demicontractive mappings with $0 \le k \le \delta < 1$, then the mapping $K : C \to C$ defined by, $K(x) := exp_x(1-\delta)exp_x^{-1}Tx$, $\forall x \in C$ is a quasinonexpansive mapping and $Fix(K) = Fix(T)$.

(2) In addition, if T is demiclosed at $\mathbf{0}$, then K is also demiclosed at $\mathbf{0}$.

Proof. (1) It is easy to prove that $Fix(T) = Fix(K)$. By the assumption, T is a k-demicontractive mappings. Hence $Fix(T) \neq \emptyset$, and $Fix(K) \neq \emptyset$. Now we prove that $K : C \to C$ is a quasi-nonexpansive mapping. Indeed, for any $p \in Fix(K)$ and $x \in C$, it follows from (2.1) that

$$\begin{aligned} d^2(Kx,p) &\le \delta d^2(p,x) + (1-\delta)d^2(Tx,p) - \delta(1-\delta)d^2(x,Tx) \\ &\le \delta d^2(p,x) + (1-\delta)\{d^2(x,p) + kd^2(x,Tx)\} - \delta(1-\delta)d^2(x,Tx) \\ &\le d^2(p,x) \ (\textit{due to } k \le \delta), \end{aligned}$$

i.e., $K : C \to C$ is a quasinonexpansive mapping.

(2) Now we prove that the mapping K is demiclosed at zero.

In fact, for any bounded sequence $\{x_n\}$ in C such that $\lim x_n = p$ and $\lim_{n\to\infty} d(x_n, Kx_n) = 0$, we have

$$d(x_n, Kx_n) = d(x_n, exp_{x_n}(1-\delta)exp_{x_n}^{-1}Tx_n) = (1-\delta)d(x_n, Tx_n) \to 0.$$

Since T is demiclosed at zero. Thus $Tp = p$. Since $Fix(T) = Fix(K)$, this implies that $Kp = p$. Hence K is demiclosed at zero. The conclusion is proved. □

Lemma 2.8. [4] *Let M be a Hadamard manifold and $f : M \to (-\infty, +\infty]$ be a proper geodesic convex and lsc function. Then, the following inequality holds:*

$$d(J_\lambda^f(x), J_\mu^f(x) \le \frac{|\lambda - \mu|}{\lambda} d(u, J_\lambda^f(x)), \ \forall \lambda > 0, \ \mu > 0.$$

3. Main Results

Throughout this section, we assume that

(1) M is a finite dimensional Hadamard manifold, and C is a nonempty closed and geodesic convex subset of M.

(2) $f_i : C \to \mathbb{R}$, $i = 1, 2, \cdots, m$ is a proper geodesic convex and lower semi-continuous function. For given sequence $\{\lambda_n\}$, $\lambda_n \ge \lambda > 0$, define the Moreau-Yosida resolvent of f_i in C by

$$J_{\lambda_n}^{f_i}(x) = \arg\min_{y \in C}(f_i(y) + \frac{1}{2\lambda_n} d^2(y,x)), \ i == 1, 2, \cdots, m;$$

Denote by

$$S_{\lambda_n}^i := J_{\lambda_n}^{f_i} \circ J_{\lambda_n}^{f_{i-1}} \circ \cdots \circ J_{\lambda_n}^{f_2} \circ J_{\lambda_n}^{f_1}, \ i = 1, 2, \cdots, m;$$

(3) $T_i : C \to C$, $i = 1, 2, \cdots, m$ is a k-demicontractive mapping with $0 \le k \le \delta < 1$, $i = 1, 2, \cdots, m$, and T_i is demiclosed at zero;

Denote by

$$K_i(x) := exp_x(1-\delta)exp_x^{-1}T_i x,\ x \in C,\ i = 1,2,\cdots,m.$$

We are now in a position to give the following main result of this article.

Theorem 3.1. *Let M, C, $\{f_i\}_{i=1}^m$, $\{T_i\}_{i=1}^m$, $\{J_{\lambda_n}^{f_i}\}_{i=1}^m$, $\{S_{\lambda_n}^i\}_{i=1}^m$, $\{K_i\}_{i=1}^m$ be the same as above. For any given $x_0 \in C$, let $\{x_n\}$, $\{u_n\}$, and $\{y_n^{(i)}\}$, $i = 1,2,\cdots,m-1$ be the sequences generated by*

$$\begin{cases} (a)\ u_n = S_{\lambda_n}^m(x_n), \\ (b)\ y_n^{(1)} = exp_{u_n}\beta_n^{(1)}exp_{u_n}^{-1}K_1 u_n, \\ (c)\ y_n^{(i)} = exp_{u_n}\beta_n^{(i)}exp_{u_n}^{-1}K_i y_n^{(i-1)},\ i = 2,3,\cdots,m-1, \\ (d)\ x_{n+1} = exp_{u_n}\alpha_n exp_{u_n}^{-1}K_m y_n^{(m-1)}, \end{cases} \quad n \geq 0, \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n^{(i)}\}$, $i = 1,2,\cdots,m-1$, are sequences in $(0,1)$. If the set

$$\Omega := \bigcap_{i=1}^m \{\arg\min_{y\in C} f_i(y)\} \bigcap \bigcap_{i=1}^m Fix(T_i) \neq \emptyset,$$

and there exists $a,b \in (0,1)$ such that

$$(A) \qquad 0 < a \leq \alpha_n, \beta_n^{(i)} < b < 1,\ \forall n \geq 0 \text{ and } i = 1,2,\cdots,m-1,$$

then there exists $x^* \in \Omega$ such that $\{x_n\}$ converges to x^* which is a common minimization of $\{f_i\}_{i=1}^m$, as well as it also a common fixed point of $\{T_i\}_{i=1}^m$ in C.

Proof. (I) From Lemma 2.6 and Lemma 2.7, we know that

(1) if $p \in \Omega$, then $p \in \bigcap_{i=i}^m Fix(T_i)$, p is a common minimizer of $\{f_i\}_{i=1}^m$, and $p \in \bigcap_{i=1}^m Fix(J_{\lambda_n}^{f_i})$;

(2) for each $i = 1,2,\cdots,m$, $Fix(K_i) = Fix(T_i)$, K_i is a quasinonexpansive mapping, and demiclosed at zero. Moreover, $Fix(T_i)$ is a closed convex subset of C.

(II) Prove that $\{x_n\}$ is Fejér monotone with respect to Ω.

In fact, for each $i = 1,2,\cdots,m$, $J_{\lambda_n}^{f_i}$ is nonexpansive. Thus S_λ^m is also nonexpansive. Letting $p \in \Omega$, we have

$$d(u_n,p) = d(S_{\lambda_n}^m(x_n), S_{\lambda_n}^m(p)) \leq d(x_n,q). \tag{3.2}$$

By Lemma 2.7, for each $i = 1,,\cdots,m$, K_i is quasinonexpansive. From (3.1), (3.2) and Lemma 2.1, we have

$$\begin{aligned} d(y_n^{(1)},p) &\leq (1-\beta_n^{(1)})d(u_n,p) + \beta_n^{(1)}d(K_1 u_n,p) \\ &\leq (1-\beta_n^{(1)})d(u_n,p) + \beta_n^{(1)}d(u_n,p) \\ &= d(u_n,p) \leq d(x_n,p), \end{aligned}$$

which in turn implies that

$$\begin{aligned} d(y_n^{(2)},p) &\leq (1-\beta_n^{(2)})d(u_n,p) + \beta_n^{(2)}d(K_2 y_n^{(1)},p) \\ &\leq (1-\beta_n^{(2)})d(u_n,p) + \beta_n^{(2)}d(y_n^{(1)},p) \\ &\leq d(u_n,p) \leq d(x_n,p). \end{aligned}$$

Similarly, for each $i = 3, \cdots, m-1$ we can prove that

$$\begin{aligned} d(y_n^{(i)}, p) &\le (1-\beta_n^{(i)}) d(u_n, p) + \beta_n^{(i)} d(K_i y_n^{(i-1)}, p) \\ &\le (1-\beta_n^{(i)}) d(u_n, p) + \beta_n^{(i)} d(y_n^{(i-1)}, p) \\ &= d(u_n, p) \le d(x_n, p). \end{aligned}$$

On the other hand, it follows from (3.1), (3.2), and Lemma 2.1 that

$$\begin{aligned} d(x_{n+1}, p) &\le (1-\alpha_n) d(u_n, p) + \alpha_n d(K_m y_n^{(m-1)}, p)\} \\ &\le (1-\alpha_n) d(u_n, p) + \alpha_n d(y_n^{(m-1)}, p) \\ &\le d(u_n, p) \le d(x_n, p), \ \forall n \ge 0. \end{aligned} \tag{3.3}$$

This shows that $\{d(x_n, p)\}$ is decreasing and bounded below, and then the limit $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \Omega$. This indicates that $\{x_n\}$ is Fejér monotone with respect to Ω. Hence the sequence $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n^{(i)}\}$, $i = 1, 2, \cdots, m-1$.

(III) Prove that

$$\lim_{n\to\infty} d(x_n, u_n) = \lim_{n\to\infty} d(x_n, S_{\lambda_n}^m x_n) = 0. \tag{3.4}$$

In fact, it follows from (3.3) and Lemma 2.6 (ii) that, for any given $p \in \Omega$,

$$\begin{aligned} d^2(u_n, S_{\lambda_n}^{m-1} x_n) &\le d^2(S_{\lambda_n}^{m-1} x_n, p) - d^2(u_n, p) \\ &\le d^2(x_n, p) - d^2(u_n, p) \\ &\le d^2(x_n, p) - d^2(x_{n+1}, p). \end{aligned}$$

Thus $\lim_{n\to\infty} d(u_n, S_{\lambda_n}^{m-1} x_n) = 0$. Similarly, by using the same method, we can prove that

$$\lim_{n\to\infty} d(S_{\lambda_n}^{m-i} x_n, S_{\lambda_n}^{m-(i+1)} x_n) = 0, \ \ i = 0, 1, 2, \cdots m-1.$$

Hence,

$$\begin{aligned} d(u_n, x_n) &= d(S_{\lambda_n}^m x_n, x_n) \\ &\le d(S_{\lambda_n}^m x_n, S_{\lambda_n}^{m-1} x_n) + d(S_{\lambda_n}^{m-1} x_n, S_{\lambda_n}^{m-2} x_n) \\ &\quad + \cdots + d(S_{\lambda_n}^2 x_n, S_{\lambda_n}^1 x_n) + d(S_{\lambda_n}^1 x_n, x_n) \to 0 \ (n \to \infty). \end{aligned}$$

The conclusion (3.4) is proved.

(IV) Prove that

$$\begin{cases} (a) \ \ \lim_{n\to\infty} d(u_n, K_1 u_n) = 0; \\ (b) \ \ \lim_{n\to\infty} d(y_n^{(i-1)}, K_i y_n^{(i-1)}) = 0, \ i = 2, 3, \cdots, m; \\ (c) \ \ \lim_{n\to\infty} d(S_{\lambda_n}^m (x_n), x_n) = 0. \end{cases} \tag{3.5}$$

Indeed, it follows from (2.1), (3.1), and condition (A) that

$$\begin{aligned} d^2(y_n^{(1)}, p) &\le (1-\beta_n^{(1)}) d^2(u_n, p) + \beta_n^{(1)} d^2(K_1 u_n, p) - \beta_n^{(1)} (1-\beta_n^{(1)}) d^2(u_n, K_1 u_n) \\ &\le (1-\beta_n^{(1)}) d^2(u_n, p) + \beta_n^{(1)} d^2(u_n, p) - \beta_n^{(1)} (1-\beta_n^{(1)}) d^2(u_n, K_1 u_n) \\ &= d^2(u_n, p) - \beta_n^{(1)} (1-\beta_n^{(1)}) d^2(u_n, K_1 u_n) \\ &\le d^2(u_n, p) - a(1-b) d^2(u_n, K_1 u_n). \end{aligned}$$

This together with (2.1) and (3.1) yields

$$\begin{aligned}
d^2(y_n^{(2)},p) &\le (1-\beta_n^{(2)})d^2(u_n,p)+\beta_n^{(2)}d^2(K_2y_n^{(1)},p)-\beta_n^{(2)}(1-\beta_n^{(2)})d^2(u_n,K_2y_n^{(1)})\\
&\le (1-\beta_n^{(2)})d^2(u_n,p)+\beta_n^{(2)}d^2(y_n^{(1)},p)-\beta_n^{(2)}(1-\beta_n^{(2)})d^2(u_n,K_2y_n^{(1)})\\
&\le (1-\beta_n^{(2)})d^2(u_n,p)+\beta_n^{(2)}\{d^2(u_n,p)-a(1-b)d^2(u_n,K_1u_n)\}\\
&\quad -\beta_n^{(2)}(1-\beta_n^{(2)})d^2(u_n,K_2y_n^{(1)})\\
&\le d^2(u_n,p)-a^2(1-b)d^2(u_n,K_1u_n)-\beta_n^{(2)}(1-\beta_n^{(2)})d^2(u_n,K_2y_n^{(1)})\\
&\le d^2(u_n,p)-a^2(1-b)d^2(u_n,K_1u_n)-a(1-b)d^2(u_n,K_2y_n^{(1)}).
\end{aligned}$$

Similarly, by using the same method, we can prove that, for each $i=3,4,\cdots,m-1$,

$$\begin{aligned}
d^2(y_n^{(i-1)},p) &\le (1-\beta_n^{(i-1)})d^2(u_n,p)+\beta_n^{(i-1)}d^2(K_{i-1}y_n^{(i-2)},p)\\
&\quad -\beta_n^{(i-1)}(1-\beta_n^{(i-1)})d^2(u_n,K_{i-1}y_n^{(i-2)})\\
&\le (1-\beta_n^{(i-1)})d^2(u_n,p)+\beta_n^{(i-1)}d^2(y_n^{(i-2)},p)\\
&\quad -\beta_n^{(i-1)}(1-\beta_n^{(i-1)})d^2(u_n,K_{i-1}y_n^{(i-2)})\\
&\le d^2(u_n,p)-a^{(i-1)}(1-b)d^2(u_n,K_1u_n)\\
&\quad -a^{i-2}(1-b)d^2(u_n,K_2y_n^{(1)})-a^{i-3}(1-b)d^2(u_n,K_3y_n^{(2)})-\cdots\\
&\quad -a^1(1-b)d^2(u_n,K_{(i-1)}y_n^{(i-2)}),
\end{aligned}$$

and

$$\begin{aligned}
d^2(y_n^{(i)},p) &\le (1-\beta_n^{(i)})d^2(u_n,p)+\beta_n^{(i)}d^2(K_iy_n^{(i-1)},p)-\beta_n^{(i)}(1-\beta_n^{(i)})d^2(u_n,K_iy_n^{(i-1)})\\
&\le (1-\beta_n^{(i)})d^2(u_n,p)+\beta_n^{(i)}d^2(y_n^{(i-1)},p)-\beta_n^{(i)}(1-\beta_n^{(i)})d^2(u_n,K_iy_n^{(i-1)})\\
&\le (1-\beta_n^{(i)})d^2(u_n,p)+\beta_n^{(i)}\{d^2(u_n,p)-a^{(i-1)}(1-b)d^2(u_n,K_1u_n)\\
&\quad -a^{i-2}(1-b)d^2(u_n,K_2y_n^{(1)})-a^{i-3}(1-b)d^2(u_n,K_3y_n^{(2)})-\cdots\\
&\quad -a^2(1-b)d^2(u_n,K_{(i-1)}y_n^{(i-2)})\}-a(1-b)d^2(u_n,K_iy_n^{(i-1)})\\
&\le d^2(u_n,p)-a^{(i)}(1-b)d^2(u_n,K_1u_n)-a^{i-1}(1-b)d^2(u_n,K_2y_n^{(1)})\\
&\quad -a^{i-2}(1-b)d^2(u_n,K_3y_n^{(2)})-\cdots-a^2(1-b)d^2(u_n,K_{(i-1)}y_n^{(i-2)})\\
&\quad -a(1-b)d^2(u_n,K_iy_n^{(i-1)})
\end{aligned}$$

On the other hand, one obtains from (2.1) and (3.1) that

$$\begin{aligned} d^2(x_{n+1},p) &\le (1-\alpha_n)d^2(u_n,p)+\alpha_n d^2(K_m y_n^{(m-1)},p)-\alpha_n(1-\alpha_n)d^2(u_n,K_m y_n^{(m-1)}) \\ &\le (1-\alpha_n)d^2(u_n,p)+\alpha_n d^2(y_n^{(m-1)},p)-\alpha_n(1-\alpha_n)d^2(u_n,K_m y_n^{(m-1)}) \\ &\le (1-\alpha_n)d^2(u_n,p)+\alpha_n\{d^2(u_n,p)-a^{(m-1)}(1-b)d^2(u_n,K_1u_n) \\ &\quad -a^{m-2}(1-b)d^2(u_n,K_2y_n^{(1)}) \\ &\quad -a^{m-3}(1-b)d^2(u_n,K_3y_n^{(2)})-\cdots-a^2(1-b)d^2(u_n,K_{(m-2)}y_n^{(m-3)}) \\ &\quad -a(1-b)d^2(u_n,K_m y_n^{(m-2)})\}-\alpha_n(1-\alpha_n)d^2(u_n,K_m y_n^{(m-1)}) \\ &\le d^2(u_n,p)-a^{(m)}(1-b)d^2(u_n,K_1u_n)-a^{m-1}(1-b)d^2(u_n,K_2y_n^{(1)}) \\ &\quad -a^{m-2}(1-b)d^2(u_n,K_3y_n^{(2)})-\cdots-a^3(1-b)d^2(u_n,K_{(m-2)}y_n^{(m-3)}) \\ &\quad -a^2(1-b)d^2(u_n,K_{m-1}y_n^{(m-2)})\}-a(1-b)d^2(u_n,K_m y_n^{(m-1)}). \end{aligned}$$

This implies that

$$\begin{aligned} & a^{(m)}(1-b)d^2(u_n,K_1u_n)+a^{m-1}(1-b)d^2(u_n,K_2y_n^{(1)})+a^{m-2}(1-b)d^2(u_n,K_3y_n^{(2)}) \\ & +\cdots+a^3(1-b)d^2(u_n,K_{(m-2)}y_n^{(m-3)})+a^2(1-b)d^2(u_n,K_m y_n^{(m-2)}) \\ & \quad +a(1-b)d^2(u_n,K_m y_n^{(m-1)}) \\ & \le d^2(u_n,p)-d^2(x_{n+1},p)\le d^2(x_n,p)-d^2(x_{n+1},p)\to 0\ (as\ n\to\infty), \end{aligned}$$

which further yields that

$$\lim_{n\to\infty} d^2(u_n,K_1u_n)=0;\quad \lim_{n\to\infty} d^2(u_n,K_i y_n^{(i-1)})=0,\ i=2,3,\cdots,m. \tag{3.6}$$

In view of (3.1), (3.6), and Lemma 2.1, for each $i=2,3,\cdots,m-1$, we have that

$$\begin{aligned} & d(y_n^{(1)},u_n)=d(exp_{u_n}\beta_n^{(1)}exp_{u_n}^{-1}K_1u_n,u_n)\le \beta_n^{(1)}d(K_1u_n,u_n)\le ad(K_1u_n,u_n)\to 0; \\ & d(y_n^{(i)},u_n)=d(exp_{u_n}\beta_n^{(i)}exp_{u_n}^{-1}K_i y_n^{(i-1)},u_n)\le \beta_n^{(i)}d(K_i y_n^{(i-1)},u_n)\le ad(K_i y_n^{(i-1)},u_n)\to 0. \end{aligned} \tag{3.7}$$

Thus it follows from (3.4), (3.6), and (3.7) that

$$\begin{cases} (a)\ \ \lim_{n\to\infty} d(u_n,K_1u_n)=0; \\ (b)\ \ \lim_{n\to\infty} d(y_n^{(i-1)},K_i y_n^{(i-1)})=0,\ i=2,3,\cdots,m; \\ (c)\ \ \lim_{n\to\infty} d(S_{\lambda_n}^m(x_n),x_n)=0. \end{cases}$$

The conclusion (3.5) is proved.

(V) Prove that

$$\lim_{n\to\infty} d(x_n,S_\lambda^m(x_n))=0,\ \lambda_n\ge\lambda. \tag{3.8}$$

In fact, by the assumption that $\lambda_n \geq \lambda > 0$. Thanks to Lemma 2.8, we obtain from (3.5) (c) that

$$\begin{aligned} d(x_n, S_\lambda^m(x_n)) &\leq d(x_n, S_{\lambda_n}^m(x_n)) + d(S_{\lambda_n}^m(x_n), S_\lambda^m(x_n)) \\ &= (1 + \frac{\lambda_n - \lambda}{\lambda_n}) d(x_n, S_{\lambda_n}^m(x_n)) \\ &\leq 2d(x_n, S_{\lambda_n}^m x_n) \to 0 \ (as\ n \to \infty). \end{aligned}$$

The conclusion (3.8) is proved.

(VI) Finally, we prove that $\{x_n\}$ converges to some point in Ω

In fact, in (II) we have proved that $\{x_n\}$ is a bounded sequence in C, and it is also Fejér monotone with respect to Ω. By Lemma 2.2, in order to prove $\{x_n\}$ converges to some point in Ω, it suffices to prove that there exists a cluster point of $\{x_n\}$ belongs to Ω. Indeed, let x^* be a cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = x^*$. By (3.4) and (3.7), $d(x_n, u_n) \to 0$, and $d(y_n^{(i)}, u_n) \to 0$, $i = 1, 2, \cdots, m-1$. These imply that $\lim_{j\to\infty} u_{n_j} = x^*$ and $\lim_{j\to\infty} y_{n_j}^{(i)} = x^*$, $i = 1, 2, \cdots, m-1$.

On the other hand, by (3.7), $d(u_{n_j}, K_1 u_{n_j}) \to 0$, $d(y_{n_j}^{(i)}, K_i y_{n_j}^{(i)}) \to 0$, and $d(S_\lambda^m(x_{n_j}), x_{n_j}) \to 0$ as $j \to \infty$. Since S_λ^m is a nonexpansive mapping, it is demiclosed at zero. Also in (I) we have proved that K_i, $i = 1, 2, \cdots, m$ all are demiclosed at zero, which implies that

$$x^* \in \bigcap_{i=1}^m Fix(K_i) \bigcap Fix(S_\lambda^m).$$

In order to prove that $x^* \in \Omega$, it should be proved that $Fix(S_\lambda^m) = \bigcap_{i=1}^m Fix(J_\lambda^{f_i})$. It is obvious that $\bigcap_{i=1}^m Fix(J_\lambda^{f_i}) \subseteqq Fix(S_\lambda^m)$.

Next we prove that $Fix(S_\lambda^m) \subset \bigcap_{i=1}^m Fix(J_\lambda^{f_i})$. Let $q \in Fix(S_\lambda^m)$ and $p \in \bigcap_{i=1}^m Fix(J_\lambda^{f_i})$. It follows that

$$\begin{aligned} d(q, p) = d(S_\lambda^m q,\ p) &= d(J_\lambda^{f_m} S_\lambda^{m-1} q, J_\lambda^{f_m} p) \leq d(S_\lambda^{m-1} q, p) \\ &\leq d(S_\lambda^{m-2} q, p) \leq \cdots \leq d(S_\lambda^1 q, p) = d(J_\lambda^{f_1} q, p) \leq d(q, p), \end{aligned}$$

which implies that

$$d(q, p) = d(S_\lambda^m q,\ p) = d(S_\lambda^{m-1} q, p) = d(S_\lambda^{m-2} q, p) = \cdots = d(S_\lambda^1 q, p) = d(J_\lambda^{f_1} q, p).$$

It follows from (2.2) that, for each $i = 1, 2, \cdots, m$,

$$d(S_\lambda^i q, p) + d(S_\lambda^i q, S_\lambda^{i-1} q) \leq d(S_\lambda^{i-1} q, p) = d(q, p).$$

Since $d(S_\lambda^i q, p) = d(q, p)$, this implies that, for each $i = 1, 2, \cdots, m$,

$$d(S_\lambda^i q, S_\lambda^{i-1} q) = 0,\ \ i.e.,\ S_\lambda^{i-1} q \in Fix(J_\lambda^{f_i}) q. \tag{3.9}$$

Taking $i = 1$ in (3.9), we have $q = J_\lambda^{f_1}(q)$. Taking $i = 2$ in (3.19), we have that $q = J_\lambda^{f_1}(q) = J_\lambda^{f_2} q$. Taking $i = 1, 2, \cdots, m$ in (3.9), we can prove that

$$q = J_\lambda^{f_1}(q) = J_\lambda^{f_2} q = \cdots = J_\lambda^{f_{m-1}} q = J_\lambda^{f_m} q,\ i.e.,\ \ q \in \bigcap_{i=1}^m Fix(J_\lambda^{f_i}).$$

This completes the proof. □

4. Conclusion

In this paper, an iterative algorithm was introduced for finding a common solution of a finite family of minimization problems and the fixed point problems of a finite family of demicontractive mappings in Hadamard manifolds. Under suitable conditions, a convergence theorem of the sequence generated by our algorithm was established in Hardmard manifolds. Since the demicontractive maping is more general than nonexpansive mappings, and quasinonexpansive mappings, it has more powerful applications in solving mean ergodic problems. Thus the problem studied in our paper is quite general. It includes many kinds of problems, such as convex optimization problems, fixed point problems, variational inclusion problems, and variational inequality problems as its special cases.

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