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CONVERGENCE OF A $k+1$-STEP ITERATIVE SCHEME WITH ERRORS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN GENERALIZED CONVEX METRIC SPACES

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Abstract. In this paper, we introduce the concept of a generalized convex metric space as a generalization of a convex metric space, which is due to Takahashi [A convexity in metric space and nonexpansive mappings, Kodai Math. Seminar Reports 22 (1970), 142-149], and give the iterative scheme due to Xiao, Sun and Huang [Approximating common fixed points of asymptotically quasi-nonexpansive mappings by a $k+1$-step iterative scheme with error terms, J. Comput. Appl. Math. 233 (2010), 2062-2070]. We also establish strong convergence of this scheme to a unique common fixed point of a finite family of asymptotically quasi-nonexpansive mappings. Our results generalize/extend several existing results

Keywords. Asymptotically quasi-nonexpansive mapping; Common fixed point; Generalized convex metric space.

1. Introduction

Recently, several convergence theorems of a fixed point for a nonexpansive mapping, a quasi-nonexpansive mapping, and their extensions in metric and Banach spaces were obtained, see, e.g., [1, 2, 3, 4, 5, 6, 7, 8]). On the other hand, the common fixed point problem, which is a certain convex feasibility problem, finds a lot of important applications in the real world. Recently, various new convergence theorems of common fixed points for a finite family of asymptotically quasi-nonexpansive mappings in Banach and convex metric spaces were established; see, e.g., [9, 10, 11, 12, 13, 14] and the references therein.

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Let C be a convex subset of a Banach space X. In 2008, Khan and Ahmed [12] introduced a general iterative scheme for a finite family of mappings $\{T_i : i = 1,2,...,k\}$ as follows:

$$\begin{aligned} & x_0 \in C, \\ & x_{n+1} = (1-\alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n}, \\ & y_{(k-1)n} = (1-\alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ & y_{(k-2)n} = (1-\alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n}, \\ & \ldots \\ & y_{2n} = (1-\alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n}, \\ & y_{1n} = (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \tag{1.1}$$

where $y_{0n} = x_n$ for all $n \in N \cup \{0\}$, and $T_1, T_2, \ldots, T_k$ are asymptotically quasi-nonexpsive mappings. They established a convergence theorem in the framework of convex metric spaces under mild conditions. Applications were also considered.

In 2010, Xiao, Sun and Huang [15] gave some necessary and sufficient conditions for the convergence of the following scheme involving $k+1$ asymptotically quasi-nonexpansive mappings $\{T_i : i = 0,1,2,...,k\}$ to a common fixed point of these mappings:

$$\begin{aligned} & x_1 \in C, \\ & x_{n+1} = (1-\alpha_{kn}-\beta_{kn})x_n + \alpha_{kn}T_k^n y_{kn} + \beta_{kn}u_{kn}, \\ & y_{kn} = (1-\alpha_{(k-1)n}-\beta_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-1)n} + \beta_{kn}u_{(k-1)n}, \\ & y_{(k-1)n} = (1-\alpha_{(k-2)n}-\beta_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-2)n} + \beta_{(k-2)n}u_{(k-2)n}, \\ & \ldots \\ & y_{2n} = (1-\alpha_{1n}-\beta_{1n})x_n + \alpha_{1n}T_1^n y_{1n} + \beta_{1n}u_{1n}, \\ & y_{1n} = (1-\alpha_{0n}-\beta_{0n})x_n + \alpha_{0n}T_0^n y_n + \beta_{0n}u_{0n}. \end{aligned} \tag{1.2}$$

They obtained several convergence theorems of common fixed points in uniformly convex Banach spaces under various assumptions.

Motivated by their results, in this paper, we introduce the concept of a generalized convex metric space as a generalization of a convex metric space, which is due to Takahashi [16], and introduce a new iterative process below in generalized convex metric spaces. We also establish the strong convergence of this new iterative scheme to a unique common fixed point of a finite family of asymptotically quasi-nonexpansive mappings.

2. Preliminaries

First we recall some basic definitions, lemmas and an iterative schemes from the existing literature for subsequent use.

Definition 2.1. [9, 16] Let (X,d) be a metric space. A convex structure in X is a mapping $w : X \times X \times [0;1] \to X$ satisfying, for all $x,y,u \in X$ and all $\lambda \in [0,1]$,

$$d(u,w(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y).$$

A metric space X together with a convex structure is called a convex metric space. A nonempty subset C of X is said to be convex if $w(x,y,\lambda) \in C$ for all $(x,y,\lambda) \in C \times C \times [0,1]$.

Definition 2.2. Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be:

(1) nonexpansive if $d(Tx,Ty)\le d(x,y)$, $\forall x,y\in X$;

(2) asymptotically nonexpansive if there exists $k_n\in[0,\infty)$ for all $n\in N(N :=$ the set of all positive integers) with $\lim\limits_{n\to\infty}k_n=0$ such that $d(T^nx,T^ny)\le(1+k_n)d(x,y)$, $\forall x,y\in X$;

(3) quasi-nonexpansive if $d(Tx,p)\le d(x,p)$ $\forall x\in X$, $\forall p\in F(T)$, where $F(T)$ is the set of fixed points of T;

(4) asymptotically quasi-nonexpansive if there exists $k_n\in[0,\infty)$ for all $n\in N$ with $\lim\limits_{n\to\infty}k_n=0$ such that $d(T^nx,p)\le(1+k_n)d(x,p)$, $\forall x\in X$, $\forall p\in F(T)$.

Remark 2.3. From Defnition 2.2, if $F(T)\neq\phi$, then nonexpansiveness mappings, quasi-nonexpansiveness mappings, and asymptotically nonexpansiveness mappings are all special cases of asymptotically quasi-nonexpansiveness mappings.

Lemma 2.4. [6] *Let (a_n), (b_n) and (c_n) be three sequences satisfying $a_n\ge0,b_n\ge0,c_n\ge0$ $a_{n+1}\le(1+c_n)a_n+b_n$, $\forall n\in N$, $\sum\limits_{n=1}^{\infty}b_n<\infty$, and $\sum\limits_{n=1}^{\infty}c_n<\infty$. Then*

(i) $\lim_{n\to\infty}a_n$ *exists;*

(ii) if $\liminf_{n\to\infty}a_n=0$, *then* $\lim_{n\to\infty}a_n=0$

3. Main Results

Now we introduce the concept of a generalized convex metric space as a generalization of a convex metric space as follows:

Definition 3.1. Let (X,d) be a metric space. A generalized convex structure in X is a mapping $W:X\times X\times X\times[0,1]\times[0,1]\to X$ satisfying, for all $x,y,z,u\in X$, and for all λ,μ in $[0,1]$,

$$d(u,W(x,y,z,\lambda,\mu))\le(1-\lambda-\mu)d(u,x)+\lambda d(u,y)+\mu d(u,z).$$

A metric space together with a generalized convex structure is called a generalized convex metric space.

A nonempty subset C of X is said to be generalized convex if $W(x,y,z,\lambda,\mu)\in C$ for all $(x,y,z,\lambda,\mu)\in C\times C\times C\times[0,1]\times[0,1]$.

Remark 3.2. If $\mu=0$ and $z=0$, then the concept of a generalized convex metric space is reduced to a convex metric space.

Now, we introduce our iterative scheme in generalized convex metric spaces as follows. Let C be a generalized convex subset of a generalized convex metric space (X,d) and $x_1\in C$. Suppose that $\alpha_{in},\beta_{in}\in[0,1]$ for all $n=1,2,3,...,$ and all $i=0,1,2,...,k$. Let $\{T_i:i=0,1,2,...,k\}$ be a finite family of self-mappings of C. The iterative scheme is

$$\begin{aligned}
x_{n+1}&=W(T_k^ny_{kn},x_n,u_{kn},\alpha_{kn},\beta_{kn}),\\
y_{kn}&=W(T_{k-1}^ny_{(k-1)n},x_n,u_{(k-1)n},\alpha_{(k-1)n},\beta_{(k-1)n}),\\
y_{(k-1)n}&=W(T_{k-2}^ny_{(k-2)n},x_n,u_{(k-2)n},\alpha_{(k-2)n},\beta_{(k-2)n}),\\
&\dots\\
y_{2n}&=W(T_1^ny_{1n},x_n,u_{1n},\alpha_{1n},\beta_{1n}),\\
y_{1n}&=W(T_0^ny_{0n},x_n,u_{0n},\alpha_{0n},\beta_{0n}).
\end{aligned}\tag{3.1}$$

Remark 3.3. (1) If $W(x,y,z,\lambda,\mu)=(1-\lambda-\mu)x+\lambda y+\mu z$ for all $(x,y,z,\lambda,\mu)\in X\times X\times X\times[0,1]\times[0,1]$ then iterative process (3.1) reduces to (1.2).

(2) Khan and Ahmed [12] noted that Lemma 2.4 (ii) holds under the hypothesis $\limsup_{n\to\infty}a_n=0$ as well. Therefore, the condition (ii) in Lemma 2.4 can be reformulated as follows:

$(ii)'$ if $\liminf_{n\to\infty}a_n=0$ or $\limsup_{n\to\infty}a_n=0$, then $\lim_{n\to\infty}a_n=0$.

Proposition 3.4. *Let C be a nonempty generalized convex subset of a generalized convex metric space X. Let $\{T_i:i=0,1,2,...,k\}$ be $k+1$ asymptotically quasi-nonexpansive self-mappings of C with $F:=\cap_{i=0}^{k}(Ti)\neq\phi$. Then, there exist a point $p\in F$ and a sequence $(u_n)\subset[0,\infty)$ with $\lim_{n\to\infty}v_n=0$ such that*

$$d(T_i^n x,p)\leq(1+v_n)d(x,p),$$

for all $x\in C$ and for each $i=0,1,2,...,k$.

Proof. The proof is similar to the proof of [12, Proposition 2.1] and is omitted. □

We need the following lemmas to state our main theorem.

Lemma 3.5. *Let C be a nonempty generalized convex subset of a generalized convex metric space X, and let $\{T_i:i=0,1,2,...,k\}$ be $k+1$ asymptotically quasi-nonexpansive mappings from C into X with $F\neq\phi$. Let (x_n) be a sequence defined in (3.1). Then*

(a) $d(x_{n+1},p)\leq(1+v_n)^{k+1}d(x_n,p)+\beta_{kn}d(u_{kn},p)$,
for all $p\in F$ and for each $n\in N$,

(b) there exists a constant $M>0$ such that, for all $n,m\in N$ and for every $p\in F$,

$$d(x_{n+m},p)\leq Md(x_n,p)+M\sum_{i=n}^{n+m-1}\beta_{ki}d(u_{ki},p).$$

Proof. (a) For all $p\in F$, we have from Proposition 3.4 that

$$\begin{aligned}
d(y_{1n},p) &= d(W(T_0^n x_n,x_n,u_{0n},\alpha_{0n},\beta_{0n}),p)\\
&\leq (1-\alpha_{0n}-\beta_{0n})d(T_0^n x_n,p)+\alpha_{0n}d(x_n,p)+\beta_{0n}d(u_{0n},p)\\
&\leq (1-\alpha_{0n}-\beta_{0n})(1+v_n)d(x_n,p)+\alpha_{0n}d(x_n,p)+\beta_{0n}d(u_{0n},p)\\
&= [1-\alpha_{0n}-\beta_{0n}+v_n-\alpha_{0n}v_n-\beta_{0n}v_n+\alpha_{0n}]d(x_n,p)+\beta_{0n}d(u_{0n},p)\\
&\leq (1+v_n)d(x_n,p)+\beta_{0n}d(u_{0n},p).
\end{aligned}$$

Assume that $d(y_{jn},p)\leq(1+v_n)^j d(x_n,p)+\beta_{jn}d(u_{jn},p)$ holds for some $1\leq j\leq k$. Then

$$\begin{aligned}
d(y_{(j+1)n},p) &= d(w(T_j^n y_{jn},x_n,u_{jn},\alpha_{jn},\beta_{jn}),p)\\
&\leq (1-\alpha_{jn}-\beta_{jn})d(T_j^n y_{jn},p)+\alpha_{jn}d(x_n,p)+\beta_{jn}d(u_{jn},p)\\
&\leq (1-\alpha_{jn}-\beta_{jn})(1+v_n)d(y_{jn},p)+\alpha_{jn}d(x_n,p)+\beta_{jn}d(u_{jn},p)\\
&\leq [1-\alpha_{jn}-\beta_{jn}](1+v_n)^{j+1}d(x_n,p)+\alpha_{jn}d(x_n,p)+\beta_{jn}d(u_{jn},p),\\
&= [(1+v_n)^{j+1}-\alpha_{jn}\{1+\frac{j+1}{1!}v_n+\frac{(j+1).j}{2!}v_n^2+...\}\\
&\quad -\beta_{jn}(1+v_n)^{j+1}+\alpha_{jn}]d(x_n,p)+\beta_{jn}d(u_{jn},p)\\
&\leq (1+v_n)^{j+1}d(x_n,p)+\beta_{jn}d(u_{jn},p).
\end{aligned}$$

By mathematical induction, we obtain that

$$(2.2)\ d(y_{in},p) \leq (1+\nu_n)^i d(x_n,p) + \beta_{in} d(u_{in},p),$$

for all $i = 1,2,...,k$.

Now, it follows from (3.1) that

$$\begin{aligned}
d(x_{(n+1)n},p) &= d(W(T_k^n y_{kn}, x_n, u_{kn}, \alpha_{kn}, \beta_{kn}), p) \\
&\leq (1-\alpha_{kn}-\beta_{kn}) d(T_k^n y_{kn}, p) + \alpha_{kn} d(x_n,p) + \beta_{kn} d(u_{kn},p) \\
&\leq (1-\alpha_{kn}-\beta_{kn})(1+\nu_n) d(y_{kn},p) + \alpha_{kn} d(x_n,p) + \beta_{kn} d(u_{kn},p) \\
&\leq [1-\alpha_{kn}-\beta_{kn}](1+\nu_n)^{k+1} d(x_n,p) + \alpha_{kn} d(x_n,p) + \beta_{kn} d(u_{kn},p), \\
&= [(1+\nu_n)^{k+1} - \alpha_{kn}\{1 + \frac{k+1}{1!}\nu_n + \frac{(k+1).k}{2!}\nu_n^2 + ...\} \\
&\quad -\beta_{kn}(1+\nu_n)^{k+1} + \alpha_{kn}] d(x_n,p) + \beta_{kn} d(u_{kn},p) \\
&\leq (1+\nu_n)^{k+1} d(x_n,p) + \beta_{kn} d(u_{kn},p).
\end{aligned}$$

(b) From (a), one sees that

$$\begin{aligned}
d(x_{n+m},p) &\leq (1+\nu_{n+m-1})^{k+1} d(x_{n+m-1},p) + \beta_{k(n+m-1)} d(u_{k(n+m-1)},p) \\
&\leq e^{(k+1)\nu_{n+m-1}} d(x_{n+m-1},p) + \beta_{k(n+m-1)} d(u_{k(n+m-1)},p) \\
&\leq e^{(k+1)\nu_{n+m-1}} [e^{(k+1)\nu_{n+m-2}} d(x_{n+m-2},p) + \beta_{k(n+m-2)} d(u_{k(n+m-2)},p)] \\
&\quad + \beta_{k(n+m-1)} d(u_{k(n+m-1)},p) \\
&\leq e^{(k+1)[\nu_{n+m-1}+\nu_{n+m-2}]} d(x_{n+m-2},p) \\
&\quad + e^{(k+1)\nu_{n+m-1}} [\beta_{k(n+m-2)} d(u_{k(n+m-2)},p) + \beta_{k(n+m-1)} d(u_{k(n+m-1)},p)] \\
&\leq ... \\
&\leq e^{(k+1)\sum_{i=n}^{n+m-1}\nu_i} d(x_n,p) + e^{(k+1)\sum_{i=n}^{n+m-1}\nu_i} \sum_{i=n}^{n+m-1} \beta_{ki} d(u_{ki},p) \\
&\leq M d(x_n,p) + M \sum_{i=n}^{n+m-1} \beta_{ki} d(u_{ki},p),
\end{aligned}$$

where $M = e^{(k+1)\sum_{i=n}^{\infty}\nu_i}$. □

Lemma 3.6. *Let $\{T_i : i = 0,1,2,...,k\}$ be $k+1$ asymptotically quasi-nonexpansive mappings from a nonempty generalized convex subset C of a generalized convex metric space X into C with $F(T) \neq \phi$ (T_i, $\forall\, i \in \{0,1,2,...,k\}$ need not be continuous). Let (x_n) be a sequence defined by (3.1). If $\lim_{n\to\infty} d(x_n,F) = 0$, and $(\sum_{i=n}^{n+m-1} \beta_{ki} d(u_{ki},p))$ is a bounded sequence for each $p \in F$, then (x_n) is a Cauchy sequence ,where $d(y,E) = \inf\{d(y,e) : e \in E\}$.*

Proof. From Lemma 3.5 (b), there exists a constant $M > 0$ such that

$$(2.3)\ d(x_{n+m},p) \leq M d(x_n,p) + M \sum_{i=n}^{n+m-1} \beta_{ki} d(u_{ki},p)$$

for all $n,m \in N$ and for each $p \in F$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(x_n,F) = 0$ and $(\sum_{i=n}^{n+m-1} \beta_{ki} d(u_{ki},p))$ is a bounded sequence for each $p \in F$, there exists a constant $N_1 \in N$ such that, for all $n \geq N_1$,

$$d(x_n,F) < \frac{\varepsilon}{2(M+1)} \text{ and } \sum_{i=n}^{n+m-1} \beta_{ki} d(u_{ki},p) < \frac{\varepsilon}{2M}.$$

Thus, there exists $q \in F$ such that

$$(x_n,q) < \frac{\varepsilon}{M+1}, \forall n \geq N_1. \tag{3.2}$$

From (3.1), we obtain that

$$\begin{aligned} d(x_{n+m},x_n) &\leq d(x_{n+m},q) + d(x_n,q) \\ &\leq (M+1)d(x_n,q) + M \sum_{i=n}^{n+m-1} \beta_{ki} d(u_{ki},p) \\ &< (M+1)\frac{\varepsilon}{2(M+1)} + M\frac{\varepsilon}{2M} \\ &= 0. \end{aligned}$$

for all $n,m \geq N_1$. Therefore, (x_n) is a Cauchy sequence in C. □

Now, we state and prove the main result of this section.

Theorem 3.7. *Let $\{T_i : i = 0,1,2,\ldots,k\}$ be $k+1$ mappings from a nonempty generalized convex subset C of a generalized convex metric space (X,d) into C with $F \neq \phi$ ($T_i, i = 1,2,\ldots,k$, need not to be continuous). Let (x_n) be a sequence defined by (3.1). Then*

(A) $\liminf_{n\to\infty} d(x_n,F) = \limsup_{n\to\infty} d(x_n,F) = 0$ if (x_n) converges to a unique point in F;

(B) (x_n) converges to a unique point in F if C is complete, $\{T_i : i = 0,1,2,\ldots,k\}$ are $k+1$ asymptotically quasi-nonexpansive mappings with $x_0 \in C$, and

$$\liminf_{n\to\infty} d(x_n,F) = 0 \text{ or } \limsup_{n\to\infty} d(x_n,F) = 0.$$

Proof. (A) Let $p \in F$. Since (x_n) converges to p, $\lim_{n\to\infty}(x_n,p) = 0$. So, for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$d(x_n,p) < \varepsilon,\ \forall n \geq n_0.$$

Taking the infimum over $p \in F$, we find that

$$d(x_n,F) < \varepsilon,\ \forall n \geq n_0.$$

This means that $\lim_{n\to\infty}(x_n,F) = 0$. We obtain that $\liminf_{n\to\infty} d(x_n,F) = \limsup_{n\to\infty} d(x_n,F) = 0$.

(B) Since $\{T_i : i = 0,1,2,\ldots,k\}$ is a finite family of asymptotically quasi-nonexpansive self-mappings of C, Proposition 3.4 yields that there exists a sequence (ν_n) in $[0,\infty)$ with $\lim_{n\to\infty} \nu_n = 0$. From the completeness of C and Lemma 3.6, we get that $\lim_{n\to\infty} x_n$ exists and equals $q \in C$. Therefore, for all $\varepsilon_1 > 0$, there exists a $n_1 \in N$ such that, for all $n \geq n_1$,

$$d(x_n,q) < \frac{\varepsilon_1}{2(2+\nu_1)}. \tag{3.3}$$

Suppose that $\liminf_{n\to\infty} d(x_n,F)=0$ or $\limsup_{n\to\infty} d(x_n,F)=0$. Then, we have from Lemma 2.4 (ii) and Remark 3.2 (2) that $\lim_{n\to\infty} d(x_n,F)=0$. So, there exists $n_2\in N$ with $n_2\geq n_1$ such that, for all $n\geq n_2$,

$$d(x_n,F)<\frac{\varepsilon_1}{2(4+3\nu_1)}.$$

Thus, there exists $r\in F$ such that

$$d(x_{n_2},r)<\frac{\varepsilon_1}{2(4+3u_1)}. \tag{3.4}$$

For any $T_i, i=0,1,...,k$, we obtain from (3.2) and (3.4) that

$$\begin{aligned}
d(T_iq,q) &\leq d(T_iq,r)+d(r,T_ix_{n_2})+d(T_ix_{n_2},r)+d(r,x_{n_2})+d(x_{n_2},q)\\
&= d(T_iq,r)+2d(T_ix_{n_2},r)+d(r,x_{n_2})+d(x_{n_2},q)\\
&\leq (1+\nu_1)d(q,r)+2(1+\nu_1)d(x_{n_2},r)+d(r,x_{n_2})+d(x_{n_2},q)\\
&\leq (2+\nu_1)d(x_{n_2},q)+(4+3\nu_1)d(x_{n_2},r)\\
&< (2+\nu_1)\frac{\varepsilon_1}{2+\nu_1}+(4+3\nu_1)\frac{\varepsilon_1}{2(4+3\nu_1)}\\
&= \varepsilon_1
\end{aligned}$$

Since ε_1 is arbitrary, so $d(T_iq,q)=0$ for all $i=1,2,...,k$, i.e., $T_iq=q$. Therefore, $q\in F$. □

Remark 3.8. Our Theorem 3.7 generalizes/extends the results in [12, Theorem 2.2], [7, Theorem 3.2] and [6, Theorem 1, and Corollaries 1 and 2].

Since an asymptotically nonexpansive mapping with $F(T)\neq\phi$ is asymptotically quasi-nonexpansive mapping, we obtain the following theorem immediately.

Theorem 3.9. *Let $\{T_i : i=0,1,2,...,k\}$ be $k+1$ mappings from a nonempty generalized convex subset C of a generalized convex metric space (X,d) into C with $F\neq\phi$ ($T_i, i=1,2,...,k$, need not to be continuous). Let (x_n) be a sequence defined by (3.1). Then*

(A) $\liminf_{n\to\infty} d(x_n,F)=\limsup_{n\to\infty} d(x_n,F)=0$ if (x_n) converges to a unique point in F;

(B) (x_n) converges to a unique point in F if C is complete, $\{T_i : i=0,1,2,...,k\}$ are $k+1$ asymptotically quasi-nonexpansive mappings with $x_0\in C$, and

$$\liminf_{n\to\infty} d(x_n,F)=0 \text{ or } \limsup_{n\to\infty} d(x_n,F)=0.$$

Now, we give an application of Theorem 3.7 as follows.

Theorem 3.10. *Let $\{T_i : i=0,1,2,...,k\}$ be $k+1$ mappings from a nonempty generalized convex subset C of a generalized convex metric space (X,d) into C with $F\neq\phi$ ($T_i, i=1,2,...,k$, need not to be continuous). Let (x_n) be a sequence defined by (3.1). Then*

(A) $\liminf_{n\to\infty} d(x_n,F)=\limsup_{n\to\infty} d(x_n,F)=0$ if (x_n) converges to a unique point in F;

(B1) (x_n) converges to a unique point in F if C is complete, $\{T_i : i=0,1,2,...,k\}$ are $k+1$ asymptotically quasi-nonexpansive mappings with $x_0\in C$, and there exists a subsequence (x_{nj}) of (x_n) which converges to a unique point in F.

Proof. (A) The proof is as in Theorem 3.7.

(B1) Since a subsequence (x_{nj}) of (x_n) converges to a unique point in F, then $\liminf_{n\to\infty} d(x_n,F) = 0$ or $\limsup_{n\to\infty} d(x_n,F) = 0$. From Theorem 3.7 (B), we find that (x_n) converges to a unique point in F. □

Remark 3.11. Theorem 3.9 and Theorem 3.10 generalize [15, Theorem 2.4 and Theorem 2.5], respectively.

Next we give another application of our Theorem 3.7.

Theorem 3.12. *Let $\{T_i : i = 0,1,2,...,k\}$ be $k+1$ asymptotically quasi-nonexpansive mappings from a nonempty complete generalized convex subset C of a generalized convex metric space (X,d) into C with $F \neq \phi$ ($T_i, i = 1,2,...,k$, need not to be continuous). Let (x_n) be a sequence defined by (3.1). Assume that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, and if the sequence (z_n) in C satisties $\lim_{n\to\infty} d(z_n,z_{n+1}) = 0$, then*

$$\liminf_{n\to\infty} d(z_n,F) = 0 \text{ or } \limsup_{n\to\infty} d(z_n,F) = 0.$$

Then, (x_n) converges to a unique point in F.

Proof. From the assumptions, we have that

$$\liminf_{n\to\infty} d(x_n,F) = 0 \text{ or } \limsup_{n\to\infty} d(x_n,F) = 0.$$

Therefore, we obtain from Theorem 3.7 (B) that the sequence (x_n) converges to a unique point in F. □

Theorem 3.13. *Let C, $\{T_i : i = 0,1,2,...,k\}$, F and x_n be as in Theorem 3.12. Suppose that there exists a map T_j, which satisfies the following conditions:*

(i) $\lim_{n\to\infty} d(x_n,T_jx_n) = 0$,

(ii) there exists a function $\theta : [0,\infty) \to [0,\infty)$ which is right continuous at $0, \theta(0) = 0$,and $\theta(d(x_n,T_jx_n)) \geq d(x_n,F)$ for all n.

Then the sequence (x_n), defined by (3.1), converges to a unique point in F.

Proof. (i) and (ii) yield that

$$\begin{aligned}
\lim_{n\to\infty} d(x_n,F) \quad & \leq \lim_{n\to\infty} \theta d(x_n,T_jx_n) \\
&= \theta(\lim_{n\to\infty} d(x_n,T_jx_n)) \\
&= \theta(0) \\
&= 0,
\end{aligned}$$

i.e., $\lim_{n\to\infty} d(x_n,F) = 0$. Thus, $\liminf_{n\to\infty} d(x_n,F) = \limsup_{n\to\infty} d(x_n,F) = 0$. By Theorem 3.7 (B), (x_n) converges to a unique point in F. □

Remark 3.14. Theorems 3.12 and 3.13 extend/improve [13, Theorem 3.2], and [15, Theorem 3.4], respectively.

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