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A SUBGRADIENT-EXTRAGRADIENT METHOD FOR BILEVEL EQUILIBRIUM PROBLEMS WITH THE CONSTRAINTS OF VARIATIONAL INCLUSION SYSTEMS AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce a subgradient-extragradient method for solving bilevel equilibrium problems with the constraints of variational inclusion systems and fixed point problems. We obtain a strong convergence of solutions in the framework of Hilbert spaces.

Keywords. Equilibrium problem; Fixed point; Subgradient-extragradient method; Varinational Inclusion.

1. Introduction

Let $(\mathscr{H}, \|\cdot\|)$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let C be a nonempty closed convex subset of $\mathscr{H}$. Let the metric projection of $\mathscr{H}$ onto set C be denoted as the symbol P_C. In what follows, we use $\mathrm{Fix}(T)$ to denote the fixed-point set of T, where T is a nonlinear mapping on $\mathscr{H}$. Let the $\mathscr{R}$ denote the set of all real numbers. Let A be a self-mapping on $\mathscr{H}$. The classical variational inequality problem (VIP) is to find $x^* \in C$ satisfying $\langle Ax^*, x-x^*\rangle \geq 0$, $\forall x \in C$. We denote by VI(C,A) the solution set of the VIP. The extragradient method introduced first by Korpelevich [1] in 1976 is one of the most popular methods for solving the VIP. It was shown in [1] that if VI$(C,A) \neq \emptyset$, this method converges weakly to a solution of the VIP. The literature on the VIP is vast and Korpelevich's extragradient method has received much attention; see e.g., [2, 3, 4, 5, 6] and the references therein.

Let $A_1, A_2 : \mathscr{H} \to \mathscr{H}$ be single-valued mappings, and let $B_1, B_2 : C \to 2^{\mathscr{H}}$ be multi-valued mappings with $B_j x \neq \emptyset$, $\forall x \in C$, $j = 1,2$. The system of variational inclusions (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} 0 \in \lambda_1(A_1 y^* + B_1 x^*) + x^* - y^*, \\ 0 \in \lambda_2(A_2 x^* + B_2 y^*) + y^* - x^*, \end{cases} \tag{1.1}$$

with constants $\lambda_1, \lambda_2 > 0$. From [7, Lemma 2], one knows that problem (1.1) can been transformed into a fixed point problem in the following way. For given $x^*, y^* \in C$, (x^*, y^*) is a solution

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of problem (1.1) if and only if $x^* \in \mathrm{Fix}(G)$, where $\mathrm{Fix}(G)$ is the fixed-point set of the mapping $G := J_{\lambda_1}^{B_1}(I-\lambda_1 A_1)J_{\lambda_2}^{B_2}(I-\lambda_2 A_2)$, and $y^* = J_{\lambda_2}^{B_2}(I-\lambda_2 A_2)x^*$.

Let $\Phi : \mathscr{H} \times \mathscr{H} \to \mathscr{R} \cup \{+\infty\}$ be a bifunction such that $\Phi(x,x) = 0$, $\forall x \in C$. The equilibrium problem (shortly, EP(C,Φ)) for the bifunction Φ on the constraint domain C is to find $\hat{x} \in C$ such that

$$\Phi(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The solution set of EP(C,Φ) is denoted by Sol(C,Φ). It is worth mentioning that equilibrium problem (1.2) is a unified model of several important problems, such as, variational inequality problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems and so forth. Many efficient algorithms have been suggested and studied for solving the equilibrium problem (1.2) and its extended versions; see, e.g., [8, 9, 10, 11, 12] and the therein references.

In this paper, we introduce a subgradient-extragradient method for solving a bilevel equilibrium problem with the constraints of variational inclusions systems and fixed point problems. We obtain a strong convergence of solutions in the framework of Hilbert spaces. Our results mainly improve and extend the corresponding results announced in [13, 14, 15].

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$. Given a sequence $\{x^k\} \subset \mathscr{H}$, we denote by $x^k \to x$ (resp., $x^k \rightharpoonup x$) the strong (resp., weak) convergence of $\{x^k\}$ to x. A bifunction $\Psi : C \times C \to \mathscr{R}$ is said to be

(i) η-strongly monotone if $\Psi(x,y) + \Psi(y,x) \leq -\eta\|x-y\|^2$ $\forall x,y \in C$;

(ii) monotone if $\Psi(x,y) + \Psi(y,x) \leq 0$ $\forall x,y \in C$;

(iii) Lipschitz-type continuous with constants $c_1, c_2 > 0$ (see [16]) if $\Psi(x,y) + \Psi(y,z) \geq \Psi(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2$, $\forall x,y,z \in C$.

Recall that a mapping $T : C \to C$ is ξ-strictly pseudocontractive for some $\xi \in [0,1)$ if $\langle Tx - Ty, x-y\rangle \leq \|x-y\|^2 - \frac{1-\xi}{2}\|(I-T)x-(I-T)y\|^2$ $\forall x,y \in C$. We know that if T is a ξ-strict pseudocontraction, then T satisfies Lipschitz condition $\|Tx-Ty\| \leq \frac{1+\xi}{1-\xi}\|x-y\|$ $\forall x,y \in C$. If $\xi = 0$, then T is said to be nonexpansive.

Lemma 2.1. [13] *Let $T : C \to C$ be a ξ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers. If $(\gamma+\delta)\xi \leq \gamma$, then $\|\gamma(x-y) + \delta(Tx-Ty)\| \leq (\gamma+\delta)\|x-y\|$, $\forall x,y \in C$.*

Also, recall that a mapping $F : C \to \mathscr{H}$ is said to be

(i) L-Lipschitz continuous or L-Lipschitzian if $\exists L > 0$ such that $\|Fx - Fy\| \leq L\|x-y\|$, $\forall x,y \in C$;

(ii) ζ-contractive if $\exists \zeta \in [0,1)$ such that $\|Fx-Fy\| \leq \zeta\|x-y\|$, $\forall x,y \in C$;

(iii) monotone if $\langle Fx-Fy, x-y\rangle \geq 0$, $\forall x,y \in C$;

(iv) pseudomonotone if $\langle Fx, y-x\rangle \geq 0 \Rightarrow \langle Fy, y-x\rangle \geq 0$, $\forall x,y \in C$;

(v) η-strongly monotone if $\exists \eta > 0$ s.t. $\langle Fx-Fy, x-y\rangle \geq \eta\|x-y\|^2$, $\forall x,y \in C$;

(vi) α-inverse-strongly monotone if $\exists \alpha > 0$ s.t. $\langle Fx-Fy, x-y\rangle \geq \alpha\|Fx-Fy\|^2$, $\forall x,y \in C$.

It is clear that every inverse-strongly monotone mapping is monotone and Lipschitz continuous but the converse is not true. For each point $x \in \mathscr{H}$, we know that there exists a unique nearest point in C, denoted by P_Cx, such that $\|x-P_Cx\| \le \|x-y\|$ $\forall y \in C$. The mapping P_C is said to be the metric projection of $\mathscr{H}$ onto C.

The following facts hold:

(i) $\langle x-y, P_Cx-P_Cy\rangle \ge \|P_Cx-P_Cy\|^2$, $\forall x,y \in \mathscr{H}$;

(ii) $\langle x-P_Cx, y-P_Cx\rangle \le 0$, $\forall x \in \mathscr{H}, y \in C$;

(iii) $\|x-y\|^2 \ge \|x-P_Cx\|^2 + \|y-P_Cx\|^2$, $\forall x \in \mathscr{H}, y \in C$;

(iv) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$, $\forall x,y \in \mathscr{H}$;

(v) $\|sx+(1-s)y\|^2 = s\|x\|^2 + (1-s)\|y\|^2 - s(1-s)\|x-y\|^2$, $\forall x,y \in \mathscr{H}$, $s \in [0,1]$.

The following inequality is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$: $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$, $\forall x,y \in \mathscr{H}$.

Let $B : C \to 2^{\mathscr{H}}$ be a set-valued operator with $Bx \ne \emptyset$, $\forall x \in C$. B is said to be monotone if, for each $x,y \in C$, $\langle u-v, x-y\rangle \ge 0$, $\forall u \in Bx, v \in By$. Also, B is said to be maximal monotone if $(I+\lambda B)C = \mathscr{H}$ for all $\lambda > 0$. For a monotone operator B, we define the mapping $J_\lambda^B : (I+\lambda B)C \to C$ by $J_\lambda^B = (I+\lambda B)^{-1}$ for each $\lambda > 0$. Such J_λ^B is called the resolvent of B for $\lambda > 0$.

Proposition 2.2. [17] *Let $B : C \to 2^{\mathscr{H}}$ be a maximal monotone operator. Then the following statements hold: (i) the resolvent identity: $J_\lambda^B x = J_\mu^B(\frac{\mu}{\lambda}x + (1-\frac{\mu}{\lambda})J_\lambda^B x)$, $\forall \lambda, \mu > 0, x \in \mathscr{H}$; (ii) if J_λ^B is a resolvent of B for $\lambda > 0$, then J_λ^B is a firmly nonexpansive mapping with* $\mathrm{Fix}(J_\lambda^B) = B^{-1}0$, *where $B^{-1}0 = \{x \in C : 0 \in Bx\}$.*

Let $A : \mathscr{H} \to \mathscr{H}$ be an α-inverse-strongly monotone mapping, and let $B : C \to 2^{\mathscr{H}}$ be a maximal monotone operator. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I-\lambda A) = (I+\lambda B)^{-1}(I-\lambda A)$, $\forall \lambda > 0$. For a given $\lambda \ge 0$, one has $\|(I-\lambda A)x - (I-\lambda A)y\|^2 \le \|x-y\|^2 - \lambda(2\alpha-\lambda)\|Ax-Ay\|^2$. In particular, if $0 \le \lambda \le 2\alpha$, then $I-\lambda A$ is nonexpansive.

Proposition 2.3. [18] *The following statements hold: (i)* $\mathrm{Fix}(T_\lambda) = (A+B)^{-1}0$ $\forall \lambda > 0$; *(ii)* $\|y-T_\lambda y\| \le 2\|y-T_r y\|$ *for $0 < \lambda \le r$ and $y \in C$.*

Utilizing Proposition 2.2 (ii), we immediately obtain the following result.

Lemma 2.4. *Let $B_1, B_2 : C \to 2^{\mathscr{H}}$ be two maximal monotone operators. Let the mappings $A_1, A_2 : \mathscr{H} \to \mathscr{H}$ be α-inverse-strongly monotone and β-inverse-strongly monotone, respectively. Let the mapping $G : \mathscr{H} \to C$ be defined as $G := J_{\lambda_1}^{B_1}(I-\lambda_1 A_1)J_{\lambda_2}^{B_2}(I-\lambda_2 A_2)$. If $0 \le \lambda_1 \le 2\alpha$ and $0 \le \lambda_2 \le 2\beta$, then $G : \mathscr{H} \to C$ is nonexpansive.*

Lemma 2.5. [19] *Let $A : C \to \mathscr{H}$ be pseudomonotone and continuous. Given a point $x \in C$. Then $\langle Ax, y-x\rangle \ge 0$, $\forall y \in C \Leftrightarrow \langle Ay, y-x\rangle \ge 0$, $\forall y \in C$.*

Lemma 2.6. [20] *Let $T : C \to C$ be a ξ-strict pseudocontraction. Then $I-T$ is demiclosed at zero, i.e., if $\{x^k\}$ is a sequence in C such that $x^k \rightharpoonup x \in C$ and $(I-T)x^k \to 0$, then $(I-T)x = 0$, where I is the identity mapping of $\mathscr{H}$.*

The following lemma is useful to analyze the convergence of the proposed algorithms in this paper.

Lemma 2.7. [21] *Let $\{\Gamma_k\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{k_j}\}$ of $\{\Gamma_k\}$ which satisfies $\Gamma_{k_j} < \Gamma_{k_j+1}$ for each* *integer* $j \geq 1$. *Define the sequence* $\{\tau(k)\}_{k \geq k_0}$ *of integers as follows:*

$$\tau(k) = \max\{j \leq k : \Gamma_j < \Gamma_{j+1}\},$$

where integer $k_0 \geq 1$ *such that* $\{j \leq k_0 : \Gamma_j < \Gamma_{j+1}\} \neq \emptyset$. *Then, the following hold:*

(i) $\tau(k_0) \leq \tau(k_0+1) \leq \cdots$ *and* $\tau(k) \to \infty$*;*

(ii) $\Gamma_{\tau(k)} \leq \Gamma_{\tau(k)+1}$ *and* $\Gamma_k \leq \Gamma_{\tau(k)+1}$, $\forall k \geq k_0$.

On the other hand, the normal cone $N_C(x)$ of C at $x \in C$ is defined as $N_C(x) = \{z \in \mathscr{H} : \langle z, y-x\rangle \leq 0 \ \forall y \in C\}$. The subdifferential of a convex function $g : C \to \mathscr{R} \cup \{+\infty\}$ at $x \in C$ is defined by

$$\partial g(x) = \{z \in \mathscr{H} : g(y) - g(x) \geq \langle z, y-x\rangle, \ \forall y \in C\}.$$

In this paper, we are devoted to finding a solution $x^* \in \mathrm{Sol}(\Omega, \Psi)$ of the problem $\mathrm{EP}(\Omega, \Psi)$, where $\Omega = \bigcap_{i=0}^N \mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi)$ with $T_0 := T$. We assume always that the following hold:

T_i is a nonexpansive self-mapping on $\mathscr{H}$ for $i = 1, ..., N$, and T is a ξ-strictly pseudocontractive self-mapping on $\mathscr{H}$.

$B_1, B_2 : C \to 2^{\mathscr{H}}$ are two maximal monotone operators, and $A_1, A_2 : \mathscr{H} \to \mathscr{H}$ are α-inverse-strongly monotone and β-inverse-strongly monotone, respectively.

$G : \mathscr{H} \to C$ is defined as $G := J_{\lambda_1}^{B_1}(I - \lambda_1 A_1) J_{\lambda_2}^{B_2}(I - \lambda_2 A_2)$ where $0 < \lambda_1 < 2\alpha$ and $0 < \lambda_2 < 2\beta$.

Choose the sequences $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}, \{\delta_k\}$ in $(0,1)$, and positive sequences $\{\alpha_k\}$, $\{s_k\}$ such that

(C1) $\beta_k + \gamma_k + \delta_k = 1$, $\forall k \geq 1$ and $(\gamma_k + \delta_k)\xi \leq \gamma_k$, $\forall k \geq 1$;

(C2) $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$, $s_k \downarrow 0$ and $\sum_{k=1}^\infty s_k = \infty$;

(C3) $\liminf_{k\to\infty} \delta_k > 0$ and $0 < \liminf_{k\to\infty} \varepsilon_k \leq \limsup_{k\to\infty} \varepsilon_k < 1$;

(C4) $\{\alpha_k\} \subset (a,b) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$ and $\lim_{k\to\infty} \alpha_k = \tilde{\alpha}$;

(C5) $2s_k\nu - s_k^2 S^2 < 1$, $0 < \lambda < \min\{\nu, S\}$ and $0 < s_k < \min\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{S^2 - \lambda^2}, \frac{2\nu}{S^2}\}$.

In terms of Xu and Kim [22], we write $T_k := T_{k \bmod N}$ for integer $k \geq 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$, i.e., if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_k = T_N$ if $q = 0$ and $T_k = T_q$ if $0 < q < N$.

Algorithm 2.8. *Initial Step:* Given $x^1 \in \mathscr{H}$ arbitrarily. The sequences $\{\varepsilon_k\}, \{\beta_k\}, \{\gamma_k\}$, $\{\delta_k\}$ in $(0,1)$, and positive sequences $\{\alpha_k\}, \{s_k\}$ satisfy the conditions (C1)-(C5).

Iterative Steps: Calculate x^{k+1} as follows:

Step 1. Compute

$$u^k = \varepsilon_k x^k + (1 - \varepsilon_k) T_k x^k,$$
$$v^k = J_{\lambda_2}^{B_2}(u^k - \lambda_2 A_2 u^k).$$

Step 2. Compute

$$p^k = J_{\lambda_1}^{B_1}(v^k - \lambda_1 A_1 v^k),$$
$$y^k = \mathrm{argmin}\{\alpha_k \Phi(p^k, y) + \tfrac{1}{2}\|y - p^k\|^2 : y \in C\}.$$

Step 3. Choose $\bar{w}^k \in \partial_2 \Phi(p^k, y^k)$, and compute

$$C_k = \{v \in \mathscr{H} : \langle p^k - \alpha_k \bar{w}^k - y^k, v - y^k\rangle \leq 0\},$$
$$z^k = \mathrm{argmin}\{\alpha_k \Phi(y^k, z) + \tfrac{1}{2}\|z - p^k\|^2 : z \in C_k\}.$$

Step 4. Compute

$$t^k = \beta_k x^k + \gamma_k z^k + \delta_k T z^k,$$
$$x^{k+1} = \operatorname{argmin}\{s_k \Psi(t^k,t) + \tfrac{1}{2}\|t-t^k\|^2 : t \in C\}.$$

Set $k := k+1$ and return to Step 1.

We also need the following technical propositions.

Proposition 2.9. [23] *Let C be a convex subset of a real Hilbert space $\mathscr{H}$ and $g : C \to \mathscr{R} \cup \{+\infty\}$ be subdifferentiable. Then, $\bar{x}$ is a solution to the following convex minimization problem $\min\{g(x) : x \in C\}$ if and only if $0 \in \partial g(\bar{x}) + N_C(\bar{x})$, where ∂g denotes the subdifferential of g.*

Proposition 2.10. [24] *Let X and Y be two sets, $\mathscr{G}$ be a set-valued map from Y to X, and W be a real valued function defined on $X \times Y$. The marginal function M is defined by*

$$M(y) = \{x^* \in \mathscr{G}(y) : W(x^*,y) = \sup\{W(x,y) : x \in \mathscr{G}(y)\}\}.$$

If W and $\mathscr{G}$ are continuous, then M is upper semicontinuous.

Next, we assume that two bifunctions $\Psi : C \times C \to \mathscr{R} \cup \{+\infty\}$ and $\Phi : \mathscr{H} \times \mathscr{H} \to \mathscr{R} \cup \{+\infty\}$ satisfy the following conditions:

Ass$_\Phi$:

(Φ_1) $\Omega = \bigcap_{i=0}^N \operatorname{Fix}(T_i) \cap \operatorname{Fix}(G) \cap \operatorname{Sol}(C,\Phi) \neq \emptyset$.

(Φ_2) Φ is monotone and Lipschitz-type continuous with constants $c_1, c_2 > 0$, and Φ is weakly continuous, i.e., $\{x^k \rightharpoonup \hat{x}$ and $y^k \rightharpoonup \hat{y}\} \Rightarrow \{\Phi(x^k,y^k) \to \Phi(\hat{x},\hat{y})\}$.

Ass$_\Psi$:

(Ψ_1) Ψ is ν-strongly monotone and weakly continuous.

(Ψ_2) There exist the mappings $\bar{\Psi}_i : C \times C \to \mathscr{H}$ and $\hat{\psi}_i : C \to \mathscr{H}$ for each $i \in \{1,\ldots,m\}$ such that $\bar{\Psi}_i(x,y) + \bar{\Psi}_i(y,x) = 0$, $\|\bar{\Psi}_i(x,y)\| \le \bar{L}_i\|x-y\|$ and $\|\hat{\psi}_i(x) - \hat{\psi}_i(y)\| \le \hat{L}_i\|x-y\|$ for all $x,y \in C$, and

$$\Psi(x,y) + \Psi(y,z) \ge \Psi(x,z) + \sum_{i=1}^m \langle \bar{\Psi}_i(x,y), \hat{\psi}_i(y-z)\rangle, \quad \forall x,y,z \in C.$$

(Ψ_3) For any sequence $\{y^k\} \subset C$ such that $y^k \to d$, we have $\limsup_{k\to\infty} \frac{|\Psi(d,y^k)|}{\|y^k-d\|} < +\infty$.

Remark 2.11. Suppose that the bifunction Ψ satisfies the condition **Ass**$_\Psi(\Psi_2)$. Then

$$\begin{aligned}
\Psi(x,y) + \Psi(y,z) &\ge \Psi(x,z) + \sum_{i=1}^m \langle \bar{\Psi}_i(x,y), \hat{\psi}_i(y-z)\rangle \\
&\ge \Psi(x,z) - \sum_{i=1}^m |\langle \bar{\Psi}_i(x,y), \hat{\psi}_i(y-z)\rangle| \\
&\ge \Psi(x,z) - \sum_{i=1}^m \|\bar{\Psi}_i(x,y)\|\|\hat{\psi}_i(y-z)\| \\
&\ge \Psi(x,z) - \sum_{i=1}^m \bar{L}_i\hat{L}_i\|x-y\|\|y-z\| \\
&\ge \Psi(x,z) - (\tfrac{1}{2}\sum_{i=1}^m \bar{L}_i\hat{L}_i)\|x-y\|^2 - (\tfrac{1}{2}\sum_{i=1}^m \bar{L}_i\hat{L}_i)\|y-z\|^2.
\end{aligned}$$

Thus, Ψ is Lipschitz-type continuous with constants $c_1 = c_2 = \frac{1}{2}\sum_{i=1}^m \bar{L}_i\hat{L}_i$.

3. Main Results

We are now in a position to state and prove the first main convergence theorem in this paper.

Theorem 3.1. *Let $\{x^k\}$ be the sequence generated by Algorithm 2.8. Let the bifunctions Ψ and Φ satisfy the assumptions Ass$_\Phi$-Ass$_\Psi$. Then, under the conditions (C1)-(C5), the sequence $\{x^k\}$ converges strongly to the unique solution x^* of the problem EP(Ω,Ψ).*

Proof. Choose a fixed $\hat{p} \in \Omega = \bigcap_{i=0}^{N} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C,\Phi)$ arbitrarily, where $G = J_{\lambda_1}^{B_1}(I - \lambda_1 A_1) J_{\lambda_2}^{B_2}(I - \lambda_2 A_2)$ with $0 < \lambda_1 < 2\alpha$, and $0 < \lambda_2 < 2\beta$. We then divide the proof into several steps as follows:

Step 1. Show that the following inequality holds

$$\|z^k - \hat{p}\|^2 \leq \|p^k - \hat{p}\|^2 - (1 - 2\alpha_k c_1)\|y^k - p^k\|^2 - (1 - 2\alpha_k c_2)\|z^k - y^k\|^2, \quad \forall k \geq 1.$$

From Proposition 2.9, we know that for $y^k = \mathrm{argmin}\{\alpha_k \Phi(p^k, y) + \frac{1}{2}\|y - p^k\|^2 : y \in C\}$ there exists $\bar{w}^k \in \partial_2 \Phi(p^k, y^k)$ such that $\alpha_k \bar{w}^k + y^k - p^k \in -N_C(y^k)$, which hence yields

$$\langle \alpha_k \bar{w}^k + y^k - p^k, x - y^k \rangle \geq 0, \quad \forall x \in C. \tag{3.1}$$

From the definition of $\bar{w}^k \in \partial_2 \Phi(p^k, y^k)$, it follows that

$$\alpha_k [\Phi(p^k, x) - \Phi(p^k, y^k)] \geq \langle \alpha_k \bar{w}^k, x - y^k \rangle, \quad \forall x \in \mathscr{H}. \tag{3.2}$$

Adding (3.1) and (3.2), we get

$$\alpha_k [\Phi(p^k, x) - \Phi(p^k, y^k)] + \langle y^k - p^k, x - y^k \rangle \geq 0, \quad \forall x \in C. \tag{3.3}$$

It follows from $z^k \in C_k$ and the definition of C_k that

$$\langle p^k - \alpha_k \bar{w}^k - y^k, v - y^k \rangle \leq 0,$$

and hence

$$\alpha_k \langle \bar{w}^k, z^k - y^k \rangle \geq \langle p^k - y^k, z^k - y^k \rangle. \tag{3.4}$$

Putting $x = z^k$ in (3.2), we get

$$\alpha_k [\Phi(p^k, z^k) - \Phi(p^k, y^k)] \geq \alpha_k \langle \bar{w}^k, z^k - y^k \rangle.$$

Adding (3.4) and the last inequality, we have

$$\alpha_k [\Phi(p^k, z^k) - \Phi(p^k, y^k)] \geq \langle p^k - y^k, z^k - y^k \rangle. \tag{3.5}$$

By Proposition 2.9, we know that for $z^k = \mathrm{argmin}\{\alpha_k \Phi(y^k, y) + \frac{1}{2}\|y - p^k\|^2 : y \in C_k\}$ there exist $q^k \in \partial_2 \Phi(y^k, z^k)$ and $h^k \in N_{C_k}(z^k)$ such that $\alpha_k q^k + z^k - p^k + h^k = 0$. So, we infer that $\alpha_k \langle q^k, y - z^k \rangle \geq \langle p^k - z^k, y - z^k \rangle$, $\forall y \in C_k$, and

$$\Phi(y^k, y) - \Phi(y^k, z^k) \geq \langle q^k, y - z^k \rangle, \quad \forall y \in \mathscr{H}.$$

Putting $y = \hat{p} \in C \subset C_k$ in two last inequalities and adding them, we get

$$\alpha_k [\Phi(y^k, \hat{p}) - \Phi(y^k, z^k)] \geq \langle p^k - z^k, \hat{p} - z^k \rangle.$$

By the monotonicity of Φ, $\hat{p} \in \mathrm{Sol}(C, \Phi)$, and $y^k \in C$, we get $\Phi(y^k, \hat{p}) \leq -\Phi(\hat{p}, y^k) \leq 0$. Therefore,

$$-\alpha_k \Phi(y^k, z^k) \geq \langle p^k - z^k, \hat{p} - z^k \rangle.$$

Combining this and the following Lipschitz-type continuity of Φ

$$\Phi(p^k,y^k)+\Phi(y^k,z^k)\geq \Phi(p^k,z^k)-c_1\|p^k-y^k\|^2-c_2\|y^k-z^k\|^2,$$

we obtain that

$$\begin{aligned}\langle p^k-z^k,z^k-\hat{p}\rangle &\geq \alpha_k\Phi(y^k,z^k)\\ &\geq \alpha_k[\Phi(p^k,z^k)-\Phi(p^k,y^k)]-\alpha_kc_1\|p^k-y^k\|^2-\alpha_kc_2\|y^k-z^k\|^2.\end{aligned}$$

This together with (3.5) implies that

$$\langle p^k-z^k,z^k-\hat{p}\rangle \geq \langle p^k-y^k,z^k-y^k\rangle -\alpha_kc_1\|p^k-y^k\|^2-\alpha_kc_2\|y^k-z^k\|^2. \tag{3.6}$$

Therefore, applying the equality

$$\langle u,v\rangle=\frac{1}{2}(\|u+v\|^2-\|u\|^2-\|v\|^2)\quad \forall u,v\in\mathscr{H}, \tag{3.7}$$

for $\langle p^k-z^k,z^k-\hat{p}\rangle$ and $\langle y^k-p^k,z^k-y^k\rangle$ in (3.6), we obtain the desired result.

Step 2. Show that the following inequality holds

$$\|x^{k+1}-x\|^2\leq \|t^k-x\|^2-\|x^{k+1}-t^k\|^2+2s_k[\Psi(t^k,x)-\Psi(t^k,x^{k+1})]\quad \forall x\in C.$$

Indeed, since $x^{k+1}=\mathrm{argmin}\{s_k\Psi(t^k,t)+\frac{1}{2}\|t-t^k\|^2:t\in C\}$, there exists $m^k\in\partial_2\Psi(t^k,x^{k+1})$ such that

$$0\in s_km^k+x^{k+1}-t^k+N_C(x^{k+1}).$$

By the definition of normal cone N_C and the subgradient m^k, we get

$$\langle s_km^k+x^{k+1}-t^k,x-x^{k+1}\rangle\geq 0,\quad \forall x\in C,$$

and

$$s_k[\Psi(t^k,x)-\Psi(t^k,x^{k+1})]\geq \langle s_km^k,x-x^{k+1}\rangle,\quad \forall x\in C.$$

Adding two last inequalities, we get

$$2s_k[\Psi(t^k,x)-\Psi(t^k,x^{k+1})]+2\langle x^{k+1}-t^k,x-x^{k+1}\rangle\geq 0,\quad \forall x\in C. \tag{3.8}$$

Putting $u=x^{k+1}-t^k$ and $v=x-x^{k+1}$ in (3.7), we get

$$2s_k[\Psi(x^{k+1},x)-\Psi(t^k,x^{k+1})]+\|t^k-x\|^2-\|x^{k+1}-t^k\|^2-\|x^{k+1}-x\|^2\geq 0,\quad \forall x\in C.$$

This attains the desired result.

Step 3. Show that if x^* is a solution of the MBEP with the GSVI and CFPP constraint, then

$$\|x^{k+1}-y_*^k\|\leq \eta_k\|t^k-x^*\|\leq (1-\lambda s_k)\|t^k-x^*\|,$$

where

$$y_*^k=\mathrm{argmin}\{s_k\Psi(x^*,v)+\frac{1}{2}\|v-x^*\|^2:v\in C\},$$

$$\eta_k=\sqrt{1-2s_k\nu+s_k^2S^2},$$

$$0<\lambda<\min\{\nu,S\},$$

and

$$0<s_k<\min\{\frac{1}{\lambda},\frac{2\nu-2\lambda}{S^2-\lambda^2}\},$$

and $S=\sum_{i=1}^m\bar{L}_i\hat{L}_i$. Put $y_*^k=\mathrm{argmin}\{s_k\Psi(x^*,v)+\frac{1}{2}\|v-x^*\|^2:v\in C\}$. By the similar arguments to those of (3.8), we also have

$$s_k[\Psi(x^*,x)-\Psi(x^*,y_*^k)]+\langle y_*^k-x^*,x-y_*^k\rangle\geq 0,\quad \forall x\in C. \tag{3.9}$$

Setting $x = y_*^k \in C$ in (3.8), and $x = x^{k+1} \in C$ in (3.9), respectively, we obtain that

$$s_k[\Psi(t^k, y_*^k) - \Psi(t^k, x^{k+1})] + \langle x^{k+1} - t^k, y_*^k - x^{k+1}\rangle \geq 0,$$
$$s_k[\Psi(x^*, x^{k+1}) - \Psi(x^*, y_*^k)] + \langle y_*^k - x^*, x^{k+1} - y_*^k\rangle \geq 0.$$

Adding two last inequalities, we have

$$\begin{aligned} 0 \ &\leq 2s_k[\Psi(t^k, y_*^k) - \Psi(t^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, y_*^k)] \\ &+2\langle x^{k+1} - t^k - y_*^k + x^*, y_*^k - x^{k+1}\rangle \\ &= 2s_k[\Psi(t^k, y_*^k) - \Psi(t^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, y_*^k)] + \|t^k - x^*\|^2 \\ &-\|x^{k+1} - t^k - y_*^k + x^*\|^2 - \|x^{k+1} - y_*^k\|^2, \end{aligned} \tag{3.10}$$

where the last equality follows directly from (3.7). Note that, under assumption $\text{Ass}_\Psi(\Psi_2)$, it follows that

$$\Psi(t^k, y_*^k) - \Psi(x^*, y_*^k) \leq \Psi(t^k, x^*) - \sum_{i=1}^m \langle \bar{\Psi}_i(t^k, x^*), \hat{\psi}_i(x^* - y_*^k)\rangle,$$
$$\Psi(x^*, x^{k+1}) - \Psi(t^k, x^{k+1}) \leq \Psi(x^*, t^k) - \sum_{i=1}^m \langle \bar{\Psi}_i(x^*, t^k), \hat{\psi}_i(t^k - x^{k+1})\rangle.$$

Therefore,

$$\begin{aligned} &\Psi(t^k, y_*^k) - \Psi(t^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, y_*^k) \\ &\leq \Psi(t^k, x^*) + \Psi(x^*, t^k) - \sum_{i=1}^m \langle \bar{\Psi}_i(t^k, x^*), \hat{\psi}_i(x^* - y_*^k)\rangle - \sum_{i=1}^m \langle \bar{\Psi}_i(x^*, t^k), \hat{\psi}_i(t^k - x^{k+1})\rangle. \end{aligned}$$

Then, using $\mathbf{Ass}_\Psi(\Psi_2)$, and the strong monotonicity of Ψ in $\mathbf{Ass}_\Psi(\Psi_1)$ that

$$\Psi(x, y) + \Psi(y, x) \leq -\nu\|x - y\|^2, \quad \forall x, y \in C,$$

we get

$$\begin{aligned} &\Psi(t^k, y_*^k) - \Psi(t^k, x^{k+1}) + \Psi(x^*, x^{k+1}) - \Psi(x^*, y_*^k) \\ &\leq -\nu\|t^k - x^*\|^2 + \sum_{i=1}^m \langle \bar{\Psi}_i(t^k, x^*), \hat{\psi}_i(t^k - x^{k+1}) - \hat{\psi}_i(x^* - y_*^k)\rangle \\ &\leq -\nu\|t^k - x^*\|^2 + \sum_{i=1}^m \|\bar{\Psi}_i(t^k, x^*)\| \|\hat{\psi}_i(t^k - x^{k+1}) - \hat{\psi}_i(x^* - y_*^k)\| \\ &\leq -\nu\|t^k - x^*\|^2 + \sum_{i=1}^m \bar{L}_i\hat{L}_i\|t^k - x^*\| \|t^k - x^{k+1} - x^* + y_*^k\| \\ &= -\nu\|t^k - x^*\|^2 + S\|t^k - x^*\| \|t^k - x^{k+1} - x^* + y_*^k\|. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we get

$$\begin{aligned} 0 \ &\leq (1 - 2s_k\nu)\|t^k - x^*\|^2 + 2s_kS\|t^k - x^*\| \|t^k - x^{k+1} - x^* + y_*^k\| - \|x^{k+1} - t^k - y_*^k + x^*\|^2 \\ &-\|x^{k+1} - y_*^k\|^2 \\ &= (1 - 2s_k\nu)\|t^k - x^*\|^2 - (\|x^{k+1} - t^k - y_*^k + x^*\| - s_kS\|t^k - x^*\|)^2 + s_k^2S^2\|t^k - x^*\|^2 \\ &-\|x^{k+1} - y_*^k\|^2 \\ &\leq (1 - 2s_k\nu + s_k^2S^2)\|t^k - x^*\|^2 - \|x^{k+1} - y_*^k\|^2. \end{aligned}$$

From

$$0 < \lambda < \min\{\nu, S\} \text{ and } 0 < s_k < \min\{\frac{1}{\lambda}, \frac{2\nu - 2\lambda}{S^2 - \lambda^2}\},$$

it follows that $0 \leq \eta_k = \sqrt{1 - 2s_k\nu + s_k^2S^2} < 1 - \lambda s_k$. This ensures the desired result.

Step 4. Show that the sequence $\{x^k\}$ is bounded.
In fact, setting

$$\begin{aligned}&X := C,\ Y := [0,1],\ \mathscr{G}(s) := C\ \forall s \in Y,\\ &s := s_k,\ W(x,s) := -s\Psi(x^*,x) - \tfrac{1}{2}\|x-x^*\|^2\ \forall (x,s) \in X \times Y,\end{aligned}$$

we have that

$$\begin{aligned} M(s_k) &= \operatorname{argmax}\{W(x,s_k) : x \in C\}\\ &= \operatorname{argmin}\{s_k\Psi(x^*,x) + \tfrac{1}{2}\|x-x^*\|^2 : x \in C\}\\ &= \{y_*^k\}. \end{aligned}$$

Note that M is continuous and $\lim_{k\to\infty} y_*^k = x^*$. Since Ψ is continuous on C, we get

$$\lim_{k\to\infty} \Psi(x^*,y_*^k) = \Psi(x^*,x^*) = 0.$$

In terms of $\mathrm{Ass}_\Psi(\Psi_3)$, there exists a constant $\bar{M}(x^*) > 0$ such that

$$|\Psi(x^*,y_*^k)| \le \bar{M}(x^*)\|y_*^k - x^*\|, \quad \forall k \ge 1.$$

Putting $x = x^*$ in (3.9), and using $\Psi(x^*,x^*) = 0$, we get

$$-s_k\Psi(x^*,y_*^k) + \langle y_*^k - x^*, x^* - y_*^k\rangle \ge 0,$$

which hence yields

$$\|y_*^k - x^*\|^2 \le s_k[-\Psi(x^*,y_*^k)] \le s_k\bar{M}(x^*)\|y_*^k - x^*\|, \quad \forall k \ge 1.$$

This immediately implies that

$$\|y_*^k - x^*\| \le s_k\bar{M}(x^*), \quad \forall k \ge 1.$$

Note that

$$\begin{aligned} \|u^k - x^*\|^2 &= \varepsilon_k\|x^k - x^*\|^2 + (1-\varepsilon_k)\|T_kx^k - x^*\|^2 - \varepsilon_k(1-\varepsilon_k)\|x^k - T_kx^k\|^2\\ &\le \|x^k - x^*\|^2 - \varepsilon_k(1-\varepsilon_k)\|x^k - T_kx^k\|^2. \end{aligned} \tag{3.12}$$

Note that the mapping $G : \mathscr{H} \to C$ is defined as $G := J_{\lambda_1}^{B_1}(I-\lambda_1A_1)J_{\lambda_2}^{B_2}(I-\lambda_2A_2)$, where $\lambda_1 \in (0,2\alpha)$ and $\lambda_2 \in (0,2\beta)$. From Lemma 2.4, we know that G is nonexpansive. Since $v^k = J_{\lambda_2}^{B_2}(I-\lambda_2A_2)u^k$, and $p^k = J_{\lambda_1}^{B_1}(I-\lambda_1A_1)^k$, we have $p^k = Gu^k$. Thus,

$$\|p^k - x^*\| = \|Gu^k - x^*\| \le \|u^k - x^*\|,$$

which together with (3.12) yields

$$\begin{aligned} \|p^k - x^*\|^2 &\le \|u^k - x^*\|^2\\ &\le \|x^k - x^*\|^2 - \varepsilon_k(1-\varepsilon_k)\|x^k - T_kx^k\|^2. \end{aligned} \tag{3.13}$$

Utilizing the result in Step 1, we find from (3.13) that

$$\begin{aligned} \|z^k - x^*\|^2 &\le \|p^k - x^*\|^2\\ &\le \|u^k - x^*\|^2\\ &\le \|x^k - x^*\|^2 - \varepsilon_k(1-\varepsilon_k)\|x^k - T_kx^k\|^2, \quad \forall k \ge 1. \end{aligned} \tag{3.14}$$

From Lemma 2.1, and $(\gamma_k+\delta_k)\xi \le \gamma_k$, we deduce from (3.14) that

$$\begin{aligned}\|t^k-x^*\| &\le \beta_k\|x^k-x^*\|+(1-\beta_k)\|\tfrac{1}{1-\beta_k}[\gamma_k(z^k-x^*)+\delta_k(Tz^k-x^*)]\| \\ &\le \beta_k\|x^k-x^*\|+(1-\beta_k)\|z^k-x^*\| \\ &\le \|x^k-x^*\|,\end{aligned}$$

and hence

$$\begin{aligned}\|x^{k+1}-x^*\| &\le \|x^{k+1}-y_*^k\|+\|y_*^k-x^*\| \\ &\le (1-\lambda s_k)\|t^k-x^*\|+\|y_*^k-x^*\| \\ &\le (1-\lambda s_k)\|x^k-x^*\|+\lambda s_k\tfrac{\bar{M}(x^*)}{\lambda} \\ &\le \max\{\|x^k-x^*\|,\tfrac{\bar{M}(x^*)}{\lambda}\}.\end{aligned} \tag{3.15}$$

By induction, we get

$$\|x^k-x^*\| \le \max\{\|x^1-x^*\|,\frac{\bar{M}(x^*)}{\lambda}\},\ \forall k\ge 1.$$

Thus, $\{x^k\}$ is bounded, and so are the sequences $\{t^k\},\{p^k\},\{y^k\},\{z^k\},\{u^k\},\{v^k\}$.

Step 5. Show that if $x^{k_i}\rightharpoonup \hat{x}$, $p^{k_i}-x^{k_i}\to 0$, and $p^{k_i}-y^{k_i}\to 0$ for $\{k_i\}\subset\{k\}$, then $\hat{x}\in \mathrm{Sol}(C,\Phi)$.

From $p^{k_i}-x^{k_i}\to 0$ and $p^{k_i}-y^{k_i}\to 0$, we get

$$\|x^{k_i}-y^{k_i}\| \le \|x^{k_i}-p^{k_i}\|+\|p^{k_i}-y^{k_i}\|\to 0 \quad (i\to\infty). \tag{3.16}$$

So, it follows from $x^{k_i}\rightharpoonup\hat{x}$ that $p^{k_i}\rightharpoonup\hat{x}$, and $y^{k_i}\rightharpoonup\hat{x}$. Since $\{y^k\}\subset C$, $y^{k_i}\rightharpoonup\hat{x}$, and C is weakly closed, we know that $\hat{x}\in C$. By (3.3), we have

$$\alpha_{k_i}\Phi(p^{k_i},x)\ge \alpha_{k_i}\Phi(p^{k_i},y^{k_i})+\langle y^{k_i}-p^{k_i},y^{k_i}-x\rangle,\quad \forall x\in C.$$

Taking the limit as $i\to\infty$, and using the assumptions that $\lim_{k\to\infty}\alpha_k=\tilde{\alpha}>0$, $\Phi(\hat{x},\hat{x})=0$, $\{y^{k_i}\}$ is bounded, and Φ is weakly continuous, we obtain that $\tilde{\alpha}\Phi(\hat{x},x)\ge 0\ \forall x\in C$. This implies that $\hat{x}\in \mathrm{sol}(C,\Phi)$.

Step 6. Show that $x^k\to x^*$, a unique solution of the MBEP with the GSVI and CFPP constraint.

In fact, setting $\Gamma_k=\|x^k-x^*\|^2$ and using Lemma 2.1, we obtain

$$\begin{aligned}&\|t^k-x^*\|^2 \\ &=\beta_k\|x^k-x^*\|^2+(1-\beta_k)\|\frac{\gamma_k}{1-\beta_k}(z^k-x^*)+\frac{\delta_k}{1-\beta_k}(Tz^k-x^*)\|^2 \\ &\quad -\beta_k(1-\beta_k)\|\frac{\gamma_k}{1-\beta_k}(x^k-z^k)+\frac{\delta_k}{1-\beta_k}(x^k-Tz^k)\|^2 \\ &\le \beta_k\|x^k-x^*\|^2+(1-\beta_k)\|z^k-x^*\|^2 \\ &\quad -\beta_k(1-\beta_k)\|\frac{\gamma_k}{1-\beta_k}(x^k-z^k)+\frac{\delta_k}{1-\beta_k}(x^k-Tz^k)\|^2.\end{aligned}$$

By the results in Steps 1 and 2, we deduce from (3.14) that

$$\begin{aligned}
&\|x^{k+1}-x^*\|^2\\
&\leq \|t^k-x^*\|^2-\|x^{k+1}-t^k\|^2+2s_k[\Psi(t^k,x^*)-\Psi(t^k,x^{k+1})]\\
&\leq \beta_k\|x^k-x^*\|^2+(1-\beta_k)\|z^k-x^*\|^2-\beta_k(1-\beta_k)\|\tfrac{\gamma_k}{1-\beta_k}(x^k-z^k)+\tfrac{\delta_k}{1-\beta_k}(x^k-Tz^k)\|^2\\
&\quad -\|x^{k+1}-t^k\|^2+2s_k[\Psi(t^k,x^*)-\Psi(t^k,x^{k+1})]\\
&\leq \beta_k\|x^k-x^*\|^2+(1-\beta_k)\{\|p^k-x^*\|^2-(1-2\alpha_kc_1)\|y^k-p^k\|^2-(1-2\alpha_kc_2)\|z^k-y^k\|^2\}\\
&\quad -\beta_k(1-\beta_k)\|\tfrac{\gamma_k}{1-\beta_k}(x^k-z^k)+\tfrac{\delta_k}{1-\beta_k}(x^k-Tz^k)\|^2-\|x^{k+1}-t^k\|^2\\
&\quad +2s_k[\Psi(t^k,x^*)-\Psi(t^k,x^{k+1})]\\
&\leq \|x^k-x^*\|^2-(1-\beta_k)\{\varepsilon_k(1-\varepsilon_k)\|x^k-T_kx^k\|^2+(1-2\alpha_kc_1)\|y^k-p^k\|^2\\
&\quad +(1-2\alpha_kc_2)\|z^k-y^k\|^2\}-\beta_k(1-\beta_k)\|\tfrac{\gamma_k}{1-\beta_k}(x^k-z^k)+\tfrac{\delta_k}{1-\beta_k}(x^k-Tz^k)\|^2\\
&\quad -\|x^{k+1}-t^k\|^2+2s_k[\Psi(t^k,x^*)-\Psi(t^k,x^{k+1})]\\
&\leq \|x^k-x^*\|^2-(1-\beta_k)\{\varepsilon_k(1-\varepsilon_k)\|x^k-T_kx^k\|^2+(1-2\alpha_kc_1)\|y^k-p^k\|^2\\
&\quad +(1-2\alpha_kc_2)\|z^k-y^k\|^2\}-\beta_k(1-\beta_k)\|\tfrac{\gamma_k}{1-\beta_k}(x^k-z^k)+\tfrac{\delta_k}{1-\beta_k}(x^k-Tz^k)\|^2\\
&\quad -\|x^{k+1}-t^k\|^2+s_kK,
\end{aligned} \tag{3.17}$$

where $\sup_{k\geq 1}\{2|\Psi(t^k,x^*)-\Psi(t^k,x^{k+1})|\}\leq K$ for some $K>0$.

Finally, we show the convergence of $\{\Gamma_k\}$ to zero by the following two cases:

Case 1. Suppose that there exists an integer $k_0\geq 1$ such that $\{\Gamma_k\}$ is non-increasing. Then the limit $\lim_{k\to\infty}\Gamma_k=\hbar<+\infty$, and $\Gamma_k-\Gamma_{k+1}\to 0 \quad (k\to\infty)$. From (3.17), we get

$$\begin{aligned}
&(1-\beta_k)\{\varepsilon_k(1-\varepsilon_k)\|x^k-T_kx^k\|^2+(1-2\alpha_kc_1)\|y^k-p^k\|^2+(1-2\alpha_kc_2)\|z^k-y^k\|^2\}\\
&\quad +\beta_k(1-\beta_k)\|\tfrac{\gamma_k}{1-\beta_k}(x^k-z^k)+\tfrac{\delta_k}{1-\beta_k}(x^k-Tz^k)\|^2+\|x^{k+1}-t^k\|^2\leq \Gamma_k-\Gamma_{k+1}+s_kK,
\end{aligned} \tag{3.18}$$

Since $s_k\to 0$, $\Gamma_k-\Gamma_{k+1}\to 0$, $0<\liminf_{k\to\infty}\beta_k\leq\limsup_{k\to\infty}\beta_k<1$, and $0<\liminf_{k\to\infty}\varepsilon_k\leq\limsup_{k\to\infty}\varepsilon_k<1$, we obtain from $\{\alpha_k\}\subset(a,b)\subset(0,\min\{\frac{1}{2c_1},\frac{1}{2c_2}\})$ that

$$\lim_{k\to\infty}\|\frac{\gamma_k}{1-\beta_k}(x^k-z^k)+\frac{\delta_k}{1-\beta_k}(x^k-Tz^k)\|=\lim_{k\to\infty}\|x^k-T_kx^k\|=0, \tag{3.19}$$

and

$$\lim_{k\to\infty}\|y^k-p^k\|=\lim_{k\to\infty}\|z^k-y^k\|=\lim_{k\to\infty}\|x^{k+1}-t^k\|=0. \tag{3.20}$$

We now show that $\|u^k-p^k\|\to 0$ as $k\to\infty$. Indeed, we set $y^*=J_{\lambda_2}^{B_2}(x^*-\lambda_2A_2x^*)$. Note that $v^k=J_{\lambda_2}^{B_2}(u^k-\lambda_2A_2u^k)$, and $p^k=J_{\lambda_1}^{B_1}(v^k-\lambda_1A_1v^k)$. Then $p^k=Gu^k$. It follows that

$$\|v^k-y^*\|^2\leq\|u^k-x^*\|^2-\lambda_2(2\beta-\lambda_2)\|A_2u^k-A_2x^*\|^2, \tag{3.21}$$

and

$$\|p^k-x^*\|^2\leq\|v^k-y^*\|^2-\lambda_1(2\alpha-\lambda_1)\|A_1v^k-A_1y^*\|^2. \tag{3.22}$$

Substituting (3.21) into (3.22), we get

$$\|p^k-x^*\|^2\leq\|x^k-x^*\|^2-\lambda_2(2\beta-\lambda_2)\|A_2u^k-A_2x^*\|^2-\lambda_1(2\alpha-\lambda_1)\|A_1v^k-A_1y^*\|^2. \tag{3.23}$$

Also, substituting (3.23) into (3.17), we get

$$\begin{aligned}&\|x^{k+1}-x^*\|^2 \le \beta_k\|x^k-x^*\|^2+(1-\beta_k)\|p^k-x^*\|^2+s_kK\\ &\le \beta_k\|x^k-x^*\|^2+(1-\beta_k)[\|x^k-x^*\|^2-\lambda_2(2\beta-\lambda_2)\|A_2u^k-A_2x^*\|^2\\ &-\lambda_1(2\alpha-\lambda_1)\|A_1v^k-A_1y^*\|^2]+s_kK\\ &=\|x^k-x^*\|^2-(1-\beta_k)[\lambda_2(2\beta-\lambda_2)\|A_2u^k-A_2x^*\|^2+\lambda_1(2\alpha-\lambda_1)\|A_1v^k-A_1y^*\|^2]+s_kK,\end{aligned}$$

which immediately yields

$$(1-\beta_k)[\lambda_2(2\beta-\lambda_2)\|A_2u^k-A_2x^*\|^2+\lambda_1(2\alpha-\lambda_1)\|A_1v^k-A_1y^*\|^2]\le\Gamma_k-\Gamma_{k+1}+s_kK.$$

Since $\lambda_1\in(0,2\alpha)$, $\lambda_2\in(0,2\beta)$, $s_k\to0$, $\Gamma_k-\Gamma_{k+1}\to0$, and $\liminf_{n\to\infty}(1-\beta_k)>0$, we get

$$\lim_{k\to\infty}\|A_2u^k-A_2x^*\|=0 \quad\text{and}\quad \lim_{k\to\infty}\|A_1v^k-A_1y^*\|=0. \tag{3.24}$$

On the other hand, from Proposition 2.2 (ii), we get

$$\begin{aligned}&\|p^k-x^*\|^2\le\langle v^k-y^*,p^k-x^*\rangle+\lambda_1\langle A_1y^*-A_1v^k,p^k-x^*\rangle\\ &\le\tfrac12[\|v^k-y^*\|^2+\|p^k-x^*\|^2-\|v^k-p^k+x^*-y^*\|^2]+\lambda_1\|A_1y^*-A_1v^k\|\|p^k-x^*\|.\end{aligned}$$

This ensures that

$$\|p^k-x^*\|^2\le\|v^k-y^*\|^2-\|v^k-p^k+x^*-y^*\|^2+2\lambda_1\|A_1y^*-A_1v^k\|\|p^k-x^*\|. \tag{3.25}$$

Similarly, we get

$$\|v^k-y^*\|^2\le\|u^k-x^*\|^2-\|u^k-v^k+y^*-x^*\|^2+2\lambda_2\|A_2x^*-A_2u^k\|\|v^k-y^*\|. \tag{3.26}$$

Combining (3.25) and (3.26), we have

$$\begin{aligned}\|p^k-x^*\|^2 &\le\|x^k-x^*\|^2-\|u^k-v^k+y^*-x^*\|^2-\|v^k-p^k+x^*-y^*\|^2\\ &+2\lambda_1\|A_1y^*-A_1v^k\|\|p^k-x^*\|+2\lambda_2\|A_2x^*-A_2u^k\|\|v^k-y^*\|.\end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.17), we get

$$\begin{aligned}&\|x^{k+1}-x^*\|^2\le\beta_k\|x^k-x^*\|^2+(1-\beta_k)\|p^k-x^*\|^2+s_kK\\ &\le\beta_k\|x^k-x^*\|^2+(1-\beta_k)[\|x^k-x^*\|^2-\|u^k-v^k+y^*-x^*\|^2-\|v^k-p^k+x^*-y^*\|^2\\ &+2\lambda_1\|A_1y^*-A_1v^k\|\|p^k-x^*\|+2\lambda_2\|A_2x^*-A_2u^k\|\|v^k-y^*\|]+s_kK\\ &\le\|x^k-x^*\|^2-(1-\beta_k)[\|u^k-v^k+y^*-x^*\|^2+\|v^k-p^k+x^*-y^*\|^2]\\ &+2\lambda_1\|A_1y^*-A_1v^k\|\|p^k-x^*\|+2\lambda_2\|A_2x^*-A_2u^k\|\|v^k-y^*\|+s_kK.\end{aligned}$$

This immediately leads to

$$\begin{aligned}&(1-\beta_k)[\|u^k-v^k+y^*-x^*\|^2+\|v^k-p^k+x^*-y^*\|^2]\\ &\le\Gamma_k-\Gamma_{k+1}+2\lambda_1\|A_1y^*-A_1v^k\|\|p^k-x^*\|+2\lambda_2\|A_2x^*-A_2u^k\|\|v^k-y^*\|+s_kK.\end{aligned}$$

Since $s_k\to0$, $\Gamma_k-\Gamma_{k+1}\to0$ and $\liminf_{k\to\infty}(1-\beta_k)>0$, we deduce from (3.24) that

$$\lim_{k\to\infty}\|u^k-v^k+y^*-x^*\|=0 \quad\text{and}\quad \lim_{k\to\infty}\|v^k-p^k+x^*-y^*\|=0.$$

Thus,

$$\|u^k-Gu^k\|=\|u^k-p^k\|\le\|u^k-v^k+y^*-x^*\|+\|v^k-p^k+x^*-y^*\|\to0\quad(k\to\infty). \tag{3.28}$$

Noticing $u^k=\varepsilon_kx^k+(1-\varepsilon_k)T_kx^k$, we obtain from (3.19) that

$$\lim_{k\to\infty}\|u^k-x^k\|=\lim_{k\to\infty}(1-\varepsilon_k)\|T_kx^k-x^k\|=0, \tag{3.29}$$

which together with (3.28) yields

$$\|x^k - p^k\| \le \|x^k - u^k\| + \|u^k - p^k\| \to 0 \quad (k \to \infty). \tag{3.30}$$

Combining (3.28) and (3.29), we get

$$\begin{aligned} \|x^k - Gx^k\| &\le \|x^k - u^k\| + \|u^k - Gu^k\| + \|Gu^k - Gx^k\| \\ &\le 2\|x^k - u^k\| + \|u^k - Gu^k\| \to 0 \quad (k \to \infty). \end{aligned} \tag{3.31}$$

Also, from (3.19), we have

$$\|x^k - t^k\| = (1-\beta_k)\|\frac{\gamma_k}{1-\beta_k}(x^k - z^k) + \frac{\delta_k}{1-\beta_k}(x^k - Tz^k)\| \to 0, \quad (k \to \infty), \tag{3.32}$$

which together with (3.20) implies that

$$\|x^k - x^{k+1}\| \le \|x^k - t^k\| + \|t^k - x^{k+1}\| \to 0 \quad (k \to \infty). \tag{3.33}$$

From (3.20), (3.30), (3.32), and $0 < \liminf_{k\to\infty} \delta_k$, we conclude that

$$\|z^k - x^k\| \le \|z^k - y^k\| + \|y^k - p^k\| + \|p^k - x^k\| \to 0 \quad (k \to \infty), \tag{3.34}$$

$$\delta_k\|Tz^k - z^k\| = \|t^k - z^k - \beta_k(x^k - z^k)\| \le \|t^k - x^k\| + \|x^k - z^k\| \to 0 \quad (k \to \infty), \tag{3.35}$$

and

$$\|y^k - x^k\| \le \|y^k - z^k\| + \|z^k - x^k\| \to 0 \quad (k \to \infty). \tag{3.36}$$

Next, we show that $\lim_{k\to\infty}\|x^k - x^*\| = 0$. In fact, since the sequences $\{t^k\}$ and $\{x^k\}$ are bounded, we know that there exists a subsequence $\{t^{k_i}\}$ of $\{t^k\}$ converging weakly to $\hat{x} \in C$ and satisfying the equality

$$\liminf_{k\to\infty}[\Psi(x^*, t^k) + \Psi(t^k, x^{k+1})] = \lim_{i\to\infty}[\Psi(x^*, t^{k_i}) + \Psi(t^{k_i}, x^{k_i+1})]. \tag{3.37}$$

From (3.20), (3.32) and (3.34), it follows that $x^{k_i} \rightharpoonup \hat{x}$, $x^{k_i+1} \rightharpoonup \hat{x}$, and $z^{k_i} \rightharpoonup \hat{x}$. Then, by the result in Step 5, we deduce that $\hat{x} \in \mathrm{Sol}(C, \Phi)$. We now show that $\lim_{k\to\infty}\|x^k - T_j x^k\| = 0$ for $j = 1, \ldots, N$. Note that, for $j = 1, \ldots, N$,

$$\begin{aligned} \|x^k - T_{k+j}x^k\| &\le \|x^k - x^{k+j}\| + \|x^{k+j} - T_{k+j}x^{k+j}\| + \|T_{k+j}x^{k+j} - T_{k+j}x^k\| \\ &\le 2\|x^k - x^{k+j}\| + \|x^{k+j} - T_{k+j}x^{k+j}\|. \end{aligned}$$

From (3.19) and (3.33), we get $\lim_{k\to\infty}\|x^k - T_{k+j}x^k\| = 0$ for $j = 1, \ldots, N$. This immediately implies that $\lim_{k\to\infty}\|x^k - T_j x^k\| = 0$ for $j = 1, \ldots, N$. It is clear that $x^{k_i} - T_j x^{k_i} \to 0$ for $j = 1, \ldots, N$. Note that Lemma 2.6 guarantees the demiclosedness of $I - T_j$ at zero for $j = 1, \ldots, N$. So, we know that $\hat{x} \in \mathrm{Fix}(T_j)$. Since j is an arbitrary element in the finite set $\{1, \ldots, N\}$, we get $\hat{x} \in \cap_{j=1}^N \mathrm{Fix}(T_j)$. Also, note that Lemma 2.6 guarantees the demiclosedness of both $I - T$ and $I - G$ at zero. Since $\lim_{k\to\infty}\|z_k - Tz_k\| = 0$ (due to (3.35)), we deduce from $z^{k_i} \rightharpoonup \hat{x}$ that $\hat{x} \in \mathrm{Fix}(T)$, which hence yields $\hat{x} \in \bigcap_{j=0}^N \mathrm{Fix}(T_j)$. Meantime, from $x^{k_i} \rightharpoonup \hat{x}$ and $x^k - Gx^k \to 0$ (due to (3.31)), it follows that $\hat{x} \in \mathrm{Fix}(G)$. Consequently, $\hat{x} \in \bigcap_{j=0}^N \mathrm{Fix}(T_j) \cap \mathrm{Fix}(G) \cap \mathrm{Sol}(C, \Phi) = \Omega$. In terms of (3.37), we obtain

$$\liminf_{k\to\infty}[\Psi(x^*, t^k) + \Psi(t^k, x^{k+1})] = \Psi(x^*, \hat{x}) \ge 0. \tag{3.38}$$

Since Ψ is ν-strongly monotone, we have

$$\limsup_{k\to\infty}[\Psi(x^*, t^k) + \Psi(t^k, x^*)] \le \limsup_{k\to\infty}(-\nu\|t^k - x^*\|^2) = -\nu\hbar. \tag{3.39}$$

Combining (3.38) and (3.39), we get

$$\begin{aligned}&\limsup_{k\to\infty}[\Psi(t^k,x^*)-\Psi(t^k,x^{k+1})]\\&=\limsup_{k\to\infty}[\Psi(t^k,x^*)+\Psi(x^*,t^k)-\Psi(x^*,t^k)-\Psi(t^k,x^{k+1})]\\&\le\limsup_{k\to\infty}[\Psi(t^k,x^*)+\Psi(x^*,t^k)]+\limsup_{k\to\infty}[-\Psi(x^*,t^k)-\Psi(t^k,x^{k+1})]\\&=\limsup_{k\to\infty}[\Psi(t^k,x^*)+\Psi(x^*,t^k)]-\liminf_{k\to\infty}[\Psi(x^*,t^k)+\Psi(t^k,x^{k+1})]\\&\le-\nu\hbar.\end{aligned}$$

We now claim that $\hbar=0$. On the contrary, we assume $\hbar>0$. Without loss of generality, we may assume that $\exists k_0\ge1$ such that

$$\Psi(t^k,x^*)-\Psi(t^k,x^{k+1})\le-\frac{\nu\hbar}{2}\quad\forall k\ge k_0,$$

which together with (3.17) implies that, for all $k\ge k_0$,

$$\begin{aligned}&\|x^{k+1}-x^*\|^2\le\|x^k-x^*\|^2-(1-\beta_k)\{\varepsilon_k(1-\varepsilon_k)\|x^k-T_kx^k\|^2+(1-2\alpha_kc_1)\|y^k-p^k\|^2\\&+(1-2\alpha_kc_2)\|z^k-y^k\|^2\}-\beta_k(1-\beta_k)\|\tfrac{\gamma_k}{1-\beta_k}(x^k-z^k)+\tfrac{\delta_k}{1-\beta_k}(x^k-Tz^k)\|^2\\&-\|x^{k+1}-t^k\|^2+2s_k[\Psi(t^k,x^*)-\Psi(t^k,x^{k+1})]\\&\le\|x^k-x^*\|^2+2s_k[\Psi(t^k,x^*)-\Psi(t^k,x^{k+1})]\\&\le\|x^k-x^*\|^2-s_k\nu\hbar.\end{aligned}$$

So it follows that, for all $k\ge k_0$, $\Gamma_k-\Gamma_{k_0}\le-\nu\hbar\sum_{j=k_0}^{k}s_j$. Since $\sum_{j=1}^{\infty}s_j=+\infty$, and $\lim_{k\to\infty}\Gamma_k=\hbar$, we obtain

$$\begin{aligned}-\infty\quad&<\hbar-\Gamma_{k_0}=\lim_{k\to\infty}(\Gamma_k-\Gamma_{k_0})\\&\le\lim_{k\to\infty}[-\nu\hbar\sum_{j=k_0}^{k}s_j]=-\infty.\end{aligned}$$

This reaches a contradiction. Therefore, $\lim_{k\to\infty}\Gamma_k=0$. Hence $\{x^k\}$ converges strongly to the unique solution x^* of the problem $\mathrm{EP}(\Omega,\Psi)$.

Case 2. Suppose that $\exists\{\Gamma_{k_j}\}\subset\{\Gamma_k\}$ such that $\Gamma_{k_j}<\Gamma_{k_j+1}$, $\forall j\in\mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. Define the mapping $\tau:\mathbf{N}\to\mathbf{N}$ by

$$\tau(k):=\max\{j\le k:\Gamma_j<\Gamma_{j+1}\}.$$

By Lemma 2.7, we get $\Gamma_{\tau(k)}\le\Gamma_{\tau(k)+1}$, and $\Gamma_k\le\Gamma_{\tau(k)+1}$. Utilizing the same inferences as in (3.20) and (3.33), we can obtain that

$$\lim_{k\to\infty}\|x^{\tau(k)+1}-t^{\tau(k)}\|=\lim_{k\to\infty}\|p^{\tau(k)}-y^{\tau(k)}\|=\lim_{k\to\infty}\|y^{\tau(k)}-z^{\tau(k)}\|=0,\tag{3.40}$$

and

$$\lim_{k\to\infty}\|x^{\tau(k)+1}-x^{\tau(k)}\|=0.\tag{3.41}$$

Since $\{t^k\}$ is bounded, there exists a subsequence of $\{t^{\tau(k)}\}$ converging weakly to $\hat{x}$. Without loss of generality, we may assume that $t^{\tau(k)}\rightharpoonup\hat{x}$. Then, utilizing the same inferences as in Case 1, we can obtain that $\hat{x}\in\Omega=\bigcap_{i=0}^{N}\mathrm{Fix}(T_i)\cap\mathrm{Fix}(G)\cap\mathrm{Sol}(C,\Phi)$. From $t^{\tau(k)}\rightharpoonup\hat{x}$ and

(3.46), we get $x^{\tau(k)+1} \rightharpoonup \hat{x}$. Using the condition $\{\alpha_k\} \subset (a,b) \subset (0, \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$, we have $1-2\alpha_{\tau(k)}c_1 > 0$ and $1-2\alpha_{\tau(k)}c_2 > 0$. So it follows from (3.17) that

$$\begin{aligned}
&2s_{\tau(k)}[\Psi(t^{\tau(k)}, x^{\tau(k)+1}) - \Psi(t^{\tau(k)}, x^*)] \\
&\leq \Gamma_{\tau(k)} - \Gamma_{\tau(k)+1} - (1-\beta_{\tau(k)})\{\varepsilon_{\tau(k)}(1-\varepsilon_{\tau(k)})\|x^{\tau(k)} - T_{\tau(k)}x^{\tau(k)}\|^2 \\
&\ +(1-2\alpha_{\tau(k)}c_1)\|y^{\tau(k)} - p^{\tau(k)}\|^2 + (1-2\alpha_{\tau(k)}c_2)\|z^{\tau(k)} - y^{\tau(k)}\|^2\} \\
&\ -\beta_{\tau(k)}(1-\beta_{\tau(k)})\|\tfrac{\gamma_{\tau(k)}}{1-\beta_{\tau(k)}}(x^{\tau(k)} - z^{\tau(k)}) + \tfrac{\delta_{\tau(k)}}{1-\beta_{\tau(k)}}(x^{\tau(k)} - Tz^{\tau(k)})\|^2 - \|x^{\tau(k)+1} - t^{\tau(k)}\|^2 \\
&\leq 0,
\end{aligned}$$

which hence leads to

$$\Psi(t^{\tau(k)}, x^{\tau(k)+1}) - \Psi(t^{\tau(k)}, x^*) \leq 0. \tag{3.42}$$

Since Ψ is ν-strongly monotone on C, we have

$$\nu\|t^{\tau(k)} - x^*\|^2 \leq -\Psi(t^{\tau(k)}, x^*) - \Psi(x^*, t^{\tau(k)}). \tag{3.43}$$

Combining (3.42) and (3.43), we deduce from $\mathrm{Ass}_{\Psi}(\Psi_1)$, and $\hat{x} \in \Omega$ that

$$\begin{aligned}
\nu \limsup_{k\to\infty} \|t^{\tau(k)} - x^*\|^2 &\leq \limsup_{k\to\infty}[-\Psi(t^{\tau(k)}, x^{\tau(k)+1}) - \Psi(x^*, t^{\tau(k)})] \\
&= -\Psi(\hat{x}, \hat{x}) - \Psi(x^*, \hat{x}) \\
&\leq 0.
\end{aligned}$$

Hence, $\limsup_{k\to\infty}\|x^{\tau(k)} - x^*\|^2 \leq 0$. Thus, $\lim_{k\to\infty}\|x^{\tau(k)} - x^*\|^2 = 0$. From (3.41), we have

$$\begin{aligned}
&\|x^{\tau(k)+1} - x^*\|^2 - \|x^{\tau(k)} - x^*\|^2 \\
&= 2\langle x^{\tau(k)+1} - x^{\tau(k)}, x^{\tau(k)} - x^*\rangle + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2 \\
&\leq 2\|x^{\tau(k)+1} - x^{\tau(k)}\|\|x^{\tau(k)} - x^*\| + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2 \to 0 \quad (k\to\infty).
\end{aligned}$$

Thanks to $\Gamma_k \leq \Gamma_{\tau(k)+1}$, we have

$$\begin{aligned}
&\|x^k - x^*\|^2 \leq \|x^{\tau(k)+1} - x^*\|^2 \\
&\leq \|x^{\tau(k)} - x^*\|^2 + 2\|x^{\tau(k)+1} - x^{\tau(k)}\|\|x^{\tau(k)} - x^*\| + \|x^{\tau(k)+1} - x^{\tau(k)}\|^2.
\end{aligned}$$

So it follows from (3.41) that $x^k \to x^*$ as $k \to \infty$. This completes the proof. □

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