Commun. Optim. Theory 2021 (2021), Article ID 3 https://doi.org/10.23952/cot.2021.3

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

RANDOM COUPLED CAPUTO-TYPE MODIFICATION OF ERDÉLYI-KOBER FRACTIONAL DIFFERENTIAL SYSTEMS IN GENERALIZED BANACH SPACES WITH RETARDED AND ADVANCED ARGUMENTS

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Abstract. In this paper, we deal with existence and uniqueness of random solutions of a coupled system of nonlinear implicit fractional differential equations of the Caputo-type modification of the Erdélyi-Kober equation, involving both retarded and advanced arguments. The main tool used to carry out our results is a random fixed point theorem. An example is included to show the applicability of the outcomes.

Keywords. Random coupled implicit system; Caputo-type modification of the Erdélyi-Kober fractional differential system ; existence; retarded argument; advanced argument; fixed point.

1. Introduction

Differential equations of fractional order are valuable in modeling phenomena in various fields of science and engineering. They can be found in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc. For examples and details, we refer the reader to the monographs [1, 2, 3, 4]. On the other hand, coupled systems of fractional differential equations arise in various problems of applied nature. In recent years, some authors investigated the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations; see [5, 6] and the references therein.

Random differential equations arise in many applications and have been investigated by many mathematicians, We refer the reader to the monographs [7, 8, 9], the papers [10, 11, 12, 13, 14], and references therein.

In [15], the authors provided some properties of the Caputo-type modification of the Erdélyi-Kober fractional derivative. More details on the Erdélyi-Kober fractional integral and fractional

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Received September 27, 2020; Accepted February 11, 2021.

derivative are given in [16, 17, 18, 19, 20]. Implicit differential equations were also considered by many authors [21, 22, 23]. In [24, 25, 26, 27, 28], the authors studied the existence and uniqueness of solutions for boundary value problems of Hadamard-type fractional functional differential equations and inclusions involving both retarded and advanced arguments. In [29], Abbas *et al.* studied a class of coupled Hilfer and Hadamard random fractional differential systems with finite delay in generalized Banach spaces given by

$$\begin{cases} (u(t,w),v(t,w)) = (\phi_1(t,w),\phi_2(t,w));\ t\in[-h,0], \\ (D_0^{\alpha_1,\beta_1}u)(t,w) = f_1(t,u_t(w),v_t(w),w);\ t\in I, \\ (D_0^{\alpha_2,\beta_2}v)(t,w) = f_2(t,u_t(w),v_t(w),w);\ t\in I, \\ ((I_0^{1-\gamma_1}u)(0,w),(I_0^{1-\gamma_2}v)(0,w)) = (\Phi_1(w),\Phi_2(w)), \end{cases} ;\ w\in\Omega,$$

where $(\Omega,\mathscr{A})$ is a measurable space, $I:=[0,T],\ T>0, \alpha_i\in(0,1),\ \beta_i\in[0,1],\ \gamma_i=\alpha_i+\beta_i-\alpha_i\beta_i, \Phi_i:\Omega\to\mathbb{R}^m,\ f_i:I\times C_h\times C_h\times\Omega\to\mathbb{R}^m,\ i=1,2$, are given functions, $C_h:=C[-h,0],\ h>0,\ \phi_i(\cdot,w)\in C_h$ such that $(I_0^{1-\gamma_i}\phi_i)(0,w)=\Phi_i(w),\ i=1,2.$

In [30], Abbas *et al.* studied coupled Hilfer fractional differential systems with random effects given by

$$\begin{cases} {}^HD_1^{\alpha_1,\beta_1}u(t,w) = f_1(t,u(t,w),v(t,w),w) \\ {}^HD_1^{\alpha,\beta}v(t,w) = f_2(t,u(t,w),v(t,w),w) \end{cases} \quad t\in I:=[1,T],\ w\in\Omega,$$

with the initial conditions

$$\begin{cases} ({}^HI^{1-\gamma}u_1)(t,w)\mid_{t=1}=\psi_1(w) \\ ({}^HI^{1-\gamma}u_2)(t,w)\mid_{t=1}=\psi_2(w) \end{cases} \quad w\in\Omega,$$

where $T>1, t\in I=[1,T], \alpha_i\in(0,1), \beta_i\in[0,1]$, $(\Omega,\mathscr{A})$ is a measurable space, $\psi_i(w):I\to\Omega$, $f_i:I\times\mathbb{R}^2\times\Omega\to\mathbb{R},\ i=1,2$, are given continuous functions, and ${}^HD_1^{\alpha_i,\beta_i}$ is the Hilfer-Hadamard fractional derivative of order α_i and type β_i.

In this paper, we study the existence and uniqueness of random solutions for the following coupled nonlinear implicit system of the Caputo-type modification of the Erdélyi-Kober fractional differential equations involving both retarded and advanced arguments:

$$\begin{cases} ({}^\rho_c D_{a^+}^{\alpha_1}u)(t,w) = f_1(t,u^t(w),v^t(w),({}^\rho_c D_{a^+}^{\alpha_1}u)(t,w),w) \\ ({}^\rho_c D_{a^+}^{\alpha_2}v)(t,w) = f_2(t,u^t(w),v^t(w),({}^\rho_c D_{a^+}^{\alpha_2}v)(t,w),w) \end{cases} \quad t\in I:=[a,T], w\in\Omega, \tag{1.1}$$

$$\begin{cases} (u(t,w),v(t,w)) = (\phi_1(t,w),\phi_2(t,w)),\ t\in[a-r,a],\ r>0, \\ (u(t,w),v(t,w)) = (\psi_1(t,w),\psi_2(t,w)),\ t\in[T,T+\beta],\ \beta>0, \end{cases} \quad w\in\Omega \tag{1.2}$$

where $\alpha_i\in(1,2]$, ${}^\rho_c D_{a^+}^{\alpha_i}, i=1,2$, is the Caputo-type modification of the Erdélyi-Kober fractional derivative and $f_i: I\times C([-r,\beta],\mathbb{R}^n)\times C([-r,\beta],\mathbb{R}^n)\times\mathbb{R}^n\times\Omega\to\mathbb{R}^n$ is a given function, $\phi_i\in C([a-r,a],\mathbb{R}^n)$ with $\phi_i(a,w)=0$ and $\psi_i\in C([T,T+\beta],\mathbb{R}^n)$ with $\psi_i(T,w)=0$, $i=1,2$.

We denote by u_t the element of $C([-r,\beta])$ defined by

$$u_t(s)=u(t+s): s\in[-r,\beta].$$

2. Preliminaries

In this section, we present notations and definitions we will use throughout this paper. By $C([-r,\beta],\mathbb{R}^n)$ we denote the Banach space of all continuous functions from $[-r,\beta]$ into $\mathbb{R}^n$ equipped with the norm

$$\|u\|_{[-r,\beta]} = \sup\{\|u(t)\| : -r \le t \le \beta\}$$

and $C([a,T],\mathbb{R}^n)$ is the Banach space endowed with the norm.

$$\|u\|_{[a,T]} = \sup\{\|u(t)\| : a \le t \le T\}.$$

Also, let $E_1 = C([a-r,a],\mathbb{R}^n)$, $E_2 = C([T,T+\beta],\mathbb{R}^n)$,
and

$$AC^1(I) := \{g : I \longrightarrow \mathbb{R}^n : g' \in AC(I)\},$$

where

$$g'(t) = t\frac{d}{dt}g(t), \ \ t \in I.$$

$AC(I,\mathbb{R}^n)$ is the space of absolutely continuous functions on I. Let $\mathscr{C} = \{u : [a-r,T+\beta] \longmapsto \mathbb{R}^n : u\,|_{[a-r,a]} \in C([a-r,a]), u\,|_{[a,T]} \in AC^1([a,T])$

$$\text{and } u\,|_{[T,T+\beta]} \in C([T,T+\beta])\},$$

where the listed spaces are endowed, respectively, with the norms

$$\|u\|_{[a-r,a]} = \sup\{\|u(t)\| : a-r \le t \le a\},$$

$$\|u\|_{[T,T+\beta]} = \sup\{\|u(t)\| : T \le t \le T+\beta\},$$

$$\|u\|_{\mathscr{C}} = \sup\{\|u(t)\| : a-r \le t \le T+\beta\}.$$

Define the weighted product space $\overline{\mathscr{C}} := \mathscr{C} \times \mathscr{C}$ with the norm

$$\|(u,v)\|_{\overline{\mathscr{C}}} := \|u\|_{\mathscr{C}} + \|v\|_{\mathscr{C}}.$$

Consider the space $X^p_c(a,b)$, $(c \in \mathbb{R}, 1 \le p \le \infty)$ of those complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X^p_c} < \infty$, where the norm is defined by

$$\|f\|_{X^p_c} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}}, \ \ (1 \le p < \infty, c \in \mathbb{R}).$$

In particular, when $c = \frac{1}{p}$, the space $X^p_c(a,b)$ coincides with $L^p(a,b)$, i.e.,

$$X^p_{\frac{1}{p}}(a,b) = L^p(a,b).$$

And, for $p = \infty$, we have

$$L^\infty(I) = \left\{ f : I \longrightarrow \mathbb{R} \,\middle|\, \begin{array}{c} f \text{ is measurable and there is a constant } C \\ \text{such that } |f(x)| \le C \text{ a.e. on } \ I \end{array} \right\},$$

with the norm

$$\|f\|_{L^\infty} = \inf\{C; \ \ |f(x)| \le C \text{ a.e. on } \ I\}.$$

Definition 2.1. ([15]): **(Erdélyi-Kober fractional integral)** For $\alpha \in \mathbb{R}$, $c \in \mathbb{R}$ and $g \in X_c^p(a,b)$, the Erdélyi-Kober fractional integral of order α is defined by

$$({}^\rho I_{a^+}^\alpha g)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} g(s)ds, \qquad t > a, \rho > 0, \tag{2.1}$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0$.

Definition 2.2. ([15]) The generalized fractional derivative, corresponding to the generalized fractional integral (2.1), is defined, for $0 \le a < t$, by

$$\begin{aligned} ({}^\rho D_{a^+}^\alpha g)(t) &:= \frac{\rho^{1-n+\alpha}}{\Gamma(n-\alpha)} \left(s^{1-\rho} \frac{d}{ds} \right)^n \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-n+\alpha}} g(s)ds \\ &= \delta_\rho^n ({}^\rho I_{a^+}^{n-\alpha} g)(t), \end{aligned} \tag{2.2}$$

where $\delta_\rho^n = \left(s^{1-\rho} \frac{d}{ds} \right)^n$.

Definition 2.3. ([15]) The Caputo-type generalized fractional derivative ${}_c^\rho D_{a^+}^\alpha$ is defined, via the above generalized fractional derivative (2.2), by

$${}_c^\rho D_{a^+}^\alpha g(t) = \left({}^\rho D_{a^+}^\alpha \left[g(s) - \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!} (s-a)^k \right] \right)(t). \tag{2.3}$$

Lemma 2.4. *([15]) Let $\alpha, \rho \in \mathbb{R}^+$. Then*

$$({}^\rho I_{a^+}^\alpha {}_c^\rho D_{a^+}^\alpha g)(t) = g(t) - \sum_{k=0}^{n-1} c_k \left(\frac{t^\rho - a^\rho}{\rho} \right)^k, \tag{2.4}$$

for some $c_k \in \mathbb{R}$, $n = [\alpha] + 1$.

Let $B_{\mathbb{R}^m}$ be the σ-algebra of Borel subsets of $\mathbb{R}^m$. A mapping $v : \Omega \to \mathbb{R}^m$ is said to be measurable if, for any $D \in B_{\mathbb{R}^m}$,

$$v^{-1}(D) = \{w \in \Omega : v(w) \in D\} \subset \mathscr{A}.$$

To define integrals of sample paths of a random process, it is necessary to define a jointly measurable map.

Definition 2.5. A mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if, for any $D \in B_{\mathbb{R}^m}$,

$$T^{-1}(D) = \{(w,v) \in \Omega \times E : T(w,v) \in D\} \subset \mathscr{A} \times B_{\mathbb{R}^m},$$

where $\mathscr{A} \times B_{\mathbb{R}^m}$ is the direct product of the σ-algebras $\mathscr{A}$ and $B_{\mathbb{R}^m}$, those defined on Ω and $\mathbb{R}^m$, respectively.

Definition 2.6. A function $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in \mathbb{R}^m$ and $T(w, \cdot)$ is continuous for all $w \in \Omega$.

A mapping $T : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$ is called a random operator if $T(w,u)$ is measurable in w for all $u \in \mathbb{R}^m$, and it is expressed as $T(w)u = T(w,u)$. In this case, we also say that $T(w)$ is a random operator on $\mathbb{R}^m$. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w,u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [31].

Definition 2.7. [32] Let $\mathscr{P}(Y)$ be the family of all nonempty subsets of Y and let C be a mapping from Ω into $\mathscr{P}(Y)$. A mapping $T : \{(w,y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called a random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and, for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w,y) \in D\}$ is measurable. T is called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called a random (stochastic) fixed point of T if for $P-$almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$, and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Definition 2.8. A function $f : I \times C([-r,\beta],\mathbb{R}^n) \times C([-r,\beta],\mathbb{R}^n) \times \mathbb{R}^n \times \Omega \to \mathbb{R}^m$ is called random Carathéodory if the following conditions are satisfied:

(i) the map $(t,w) \to f(t,u,v,x,w)$ is jointly measurable for all $(u,v,x) \in C([-r,\beta],\mathbb{R}^n) \times C([-r,\beta],\mathbb{R}^n) \times \mathbb{R}^n$ and

(ii) the map $(u,v,x) \to f(t,u,v,x,w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let $x,y \in \mathbb{R}^m$ with $x = (x_1,x_2,\ldots,x_m),\ y = (y_1,y_2,\ldots,y_m)$. By $x \leq y$, we mean $x_i \leq y_i,\ i = 1,\ldots,m$. Also $|x| = (|x_1|,|x_2|,\ldots,|x_m|)$, $\max(x,y) = (\max(x_1,y_1),\ldots,\max(x_m,y_m))$, and $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x_i \in \mathbb{R}_+,\ i = 1,\ldots,m\}$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c,\ i = 1,\ldots,m$.

Definition 2.9. Let X be a nonempty set. By a vector-valued metric on X, we mean a map $d : X \times X \to \mathbb{R}^m$ with the following properties:

(i) $d(x,y) \geq 0$ for all $x,y \in X$, and if $d(x,y) = 0$, then $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

We call the pair (X,d) a generalized metric space with $d(x,y) := \begin{pmatrix} d_1(x,y) \\ d_2(x,y) \\ \cdot \\ \cdot \\ \cdot \\ d_m(x,y) \end{pmatrix}$.

Notice that d is a generalized metric space on X if and only if $d_i,\ i = 1,\ldots,m$, are metrics on X.

Definition 2.10. [33] A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc, i.e. $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $det(M - \lambda I) = 0$, where I denotes the unit matrix of $M_{m\times m}(\mathbb{R})$.

Example 2.11. The matrix $A \in M_{2\times 2}(\mathbb{R})$ defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

converges to zero in the following cases:

(1) $b = c = 0,\ a,d > 0$ and $\max\{a,d\} < 1$.

(2) $c = 0,\ a,d > 0,\ a+d < 1$ and $-1 < b < 0$.

(3) $a+b = c+d = 0,\ a > 1,\ c > 0$ and $|a-c| < 1$.

In the sequel, we will make use of the following random fixed point theorems.

Theorem 2.12. [34, 35, 36] *Let $(\Omega,\mathscr{F})$ be a measurable space. Let X be a real separable generalized Banach space and let $F:\Omega\times X\to X$ be a continuous random operator. Let $M(w)\in\mathscr{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matric such that, for every $w\in\Omega$, the matrix $M(w)$ converges to 0 and*

$$d(F(w,x_1),F(w,x_2))\le M(w)d(x_1,x_2);\ for\ each\ x_1,x_2\in X\ and\ w\in\Omega.$$

Then there exists a random variable $x:\Omega\to X$, which is the unique random fixed point of F.

Theorem 2.13. *([31]). Let X be a nonempty, closed convex bounded subset of the separable Banach space E and let $N:\Omega\times X\longmapsto X$ be a compact and continuous random operator. Then the random equation $N(w,u(w))=u(w)$ has a random solution.*

3. Existence of Solutions

Lemma 3.1. *Let $1<\alpha\le 2$, $\phi\in C([a-r,a],\mathbb{R})$ with $\phi(a)=0$, $\psi\in C([T,T+\beta],\mathbb{R})$ with $\psi(T)=0$ and $h:I\to\mathbb{R}$ be a continuous function. Then the linear problem*

$$ {}_c^\rho D^\alpha_{a^+}u(t)=h(t),\ for\ a.e.\ t\in I:=[a,T],\ 1<\alpha\le 2, \tag{3.1}$$

$$u(t)=\phi(t),\ t\in[a-r,a],\ r>0, \tag{3.2}$$

$$u(t)=\psi(t),\ t\in[T,T+\beta],\ \beta>0, \tag{3.3}$$

has a unique solution, which is given by

$$u(t)=\begin{cases}\phi(t),\ if\ \ t\in[a-r,a],\\ \\ -\int_a^T G(t,s)h(s)ds,\ if\ \ t\in I,\\ \\ \psi(t),\ if\ \ t\in[T,T+\beta],\end{cases} \tag{3.4}$$

where

$$G(t,s)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\begin{cases}\frac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho-a^\rho)}-s^{\rho-1}(t^\rho-s^\rho)^{\alpha-1}, & a\le s\le t\le T,\\ \\ \frac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}}{(T^\rho-a^\rho)}, & a\le t\le s\le T,\end{cases} \tag{3.5}$$

where $G(t,s)$ is called the Green's function for the boundary value problem (3.1)-(3.3).

Proof. From (2.4), we have

$$u(t)=c_0+c_1\left(\frac{t^\rho-a^\rho}{\rho}\right)+{}^\rho I^\alpha_{a^+}h(s),\quad c_0,c_1\in\mathbb{R}. \tag{3.6}$$

Therefore,

$$u(a)=c_0=0,$$

$$u(T)=c_1\left(\frac{T^\rho-a^\rho}{\rho}\right)+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^T(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds,$$

and

$$c_1=-\frac{\rho^{2-\alpha}}{(T^\rho-a^\rho)\Gamma(\alpha)}\int_a^T(T^\rho-s^\rho)^{\alpha-1}s^{\rho-1}h(s)ds.$$

Substituting the values of c_0 and c_1 into equation (3.6), we get equation (3.4), that is,

$$u(t)=\begin{cases}\phi(t), \text{ if } \ t\in[a-r,a],\\ -\int_a^T G(t,s)h(s)ds, \text{ if } \ t\in I,\\ \psi(t), \text{ if } \ t\in[T,T+\beta],\end{cases}$$

where G is defined by equation (3.5), and the proof is complete. □

Lemma 3.2. *Let $f_i : I\times C([-r,\beta],\mathbb{R}^n)\times C([-r,\beta],\mathbb{R}^n)\times\mathbb{R}^n\times\Omega\to\mathbb{R}^n,\ i=1,2$ be continuous functions. A functions $(u,v)\in\mathscr{C}^2$ is a random solution of system (1.1)-(1.2) if and only if (u,v) satisfies the following random coupled system integral equations,*

$$u(t,w)=\begin{cases}\phi_1(t,w), \ \textit{if} \ \ t\in[a-r,a],\\ -\int_a^T G_{\alpha_1}(t,s)h_1(s,w)ds, \ \textit{if} \ \ t\in I, \quad w\in\Omega,\\ \psi_1(t,w), \ \textit{if} \ \ t\in[T,T+\beta],\end{cases}$$

$$v(t)=\begin{cases}\phi_2(t,w), \ \textit{if} \ \ t\in[a-r,a],\\ -\int_a^T G_{\alpha_2}(t,s)h_2(s,w)ds, \ \textit{if} \ \ t\in I, \quad w\in\Omega,\\ \psi_2(t,w), \ \textit{if} \ \ t\in[T,T+\beta],\end{cases}$$

where $h_i(\cdot,w)\in C(I), w\in\Omega$, satisfies the system of functional equations,

$$\begin{cases}h_1(t,w)=f_1(t,u^t(w),v^t(w),h_1(t,w)),\\ h_2(t,w)=f_2(t,u^t(w),v^t(w),h_2(t,w)),\end{cases} \quad w\in\Omega,$$

and the Green's function $G_{\alpha_i},\ i=1,2$ is given by

$$G_{\alpha_i}(t,s)=\frac{\rho^{1-\alpha_i}}{\Gamma(\alpha_i)}\begin{cases}\frac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha_i-1}s^{\rho-1}}{(T^\rho-a^\rho)}-s^{\rho-1}(t^\rho-s^\rho)^{\alpha_i-1}, & a\le s\le t\le T,\\ \frac{(t^\rho-a^\rho)(T^\rho-s^\rho)^{\alpha_i-1}s^{\rho-1}}{(T^\rho-a^\rho)}, & a\le t\le s\le T.\end{cases} \tag{3.7}$$

The following hypotheses will be used in the sequel:

(H_1): The functions $f_i, i=1,2$, are random Carathéodory.

(H_2): There exist continuous functions $p_i,q_i,r_i : I\longrightarrow L^\infty(\Omega,\mathbb{R}_+)$, with $\|r_i(\cdot,w)\|_{[a,T]}<1$ such that

$$\begin{aligned}\|f_i(t,u,v,x,w)-f_i(t,\bar u,\bar v,\bar x,w)\| &\le p_i(t,w)\|u-\bar u\|_{[-r,\beta]}+q_i(t,w)\|v-\bar v\|_{[-r,\beta]}\\ &+ r_i(t,w)\|x-\bar x\|\end{aligned}$$

for any $u,\bar u,v,\bar v\in C([-r,\beta])$ and $x,\bar x\in\mathbb{R}^n, i=1,2$.

(H_3): There exist measurable functions $a_i,b_i,c_i,d_i : I\longrightarrow L^\infty(\Omega,\mathbb{R}_+)$; $i=1,2$, with $d_i^*(\cdot,w)<1$, such that

$$\|f_i(t,u,v,x,w)\|\le a_i(t,w)+b_i(t,w)\|u\|_{[-r,\beta]}+c_i(t,w)\|v\|_{[-r,\beta]}+d_i(t,w)\|x\|,$$

for a.e. $t\in I,\ w\in\Omega$, and each $u,v\in C_{[-r,\beta]}, x\in\mathbb{R}^n$.

Set

$$a_i^*(\cdot,w)=ess\sup_{t\in I}a_i(t,w), \quad b_i^*(\cdot,w)=ess\sup_{t\in I}b_i(t,w),$$
$$c_i^*(\cdot,w)=ess\sup_{t\in I}c_i(t,w), \quad d_i^*(\cdot,w)=ess\sup_{t\in I}d_i(t,w), \quad i=1,2$$

$$\widetilde{G_{\alpha_i}} = \sup\left\{\int_a^T |G_{\alpha_i}(t,s)|ds, t\in I\right\}.$$

Now, we state and prove our existence and uniqueness of random solutions result for of the problem (1.1)-(1.2)

Theorem 3.3. *Assume (H_1) and (H_2) hold. If, for every $w\in\Omega$, the matrix*

$$M(w) := \begin{pmatrix} \frac{\widetilde{G_{\alpha_1}}\|p_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}} & \frac{\widetilde{G_{\alpha_1}}\|q_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}} \\ \\ \frac{\widetilde{G_{\alpha_2}}\|p_2(\cdot,w)\|_{[a,T]}}{1-\|r_2(\cdot,w)\|_{[a,T]}} & \frac{\widetilde{G_{\alpha_2}}\|q_2(\cdot,w)\|_{[a,T]}}{1-\|r_2(\cdot,w)\|_{[a,T]}} \end{pmatrix}$$

converges to 0*, then problem (1.1)-(1.2) has a unique solution.*

Proof. Let the operator $N:\mathscr{C}^2\times\Omega\longmapsto\mathscr{C}^2$ be defined by

$$N(u,v)(t,w) = (N_1(u,v)(t,w), N_2(u,v)(t,w))$$

$$= \begin{cases} (\phi_1(t,w),\phi_2(t,w)), \text{ if } & t\in[a-r,a],\\ -\left(\int_a^T G_1(t,s)h_1(s,w)ds, \int_a^T G_2(t,s)h_2(s,w)ds\right), \text{ if } & t\in I,\\ (\psi_1(t,w),\psi_2(t,w)), \text{ if } & t\in[T,T+\beta]. \end{cases} \tag{3.8}$$

By Lemma 3.2, it is clear that the fixed points of N are solutions (1.1)-(1.2).

Let $(u_2,v_2),(u_1,v_1)\in\mathscr{C}^2$ *and* $w\in\Omega$. If $t\in[a-r,a]$ or $t\in[T,T+\beta]$, then

$$\|N(u_2,v_2)(t,w)-N(u_1,v_1)(t,w)\| = 0.$$

For $t\in I$, we have

$$\|N_1(u_2,v_2)(t,w)-N_1(u_1,v_1)(t,w)\| \le \int_a^T |G(t,s)|\|h_1(t,w)-\bar{h_1}(t,w)\|ds, \tag{3.9}$$

where $h_i(\cdot,w),\bar{h_i}(\cdot,w)\in C(I)$ for $w\in\Omega$ are given by

$$h_i(t,w) = f_i(t,u^t(w),v^t(w),h_i(t,w)), i=1,2,$$

and

$$\bar{h_i}(t,w) = f_i(t,\bar{u^t}(w),\bar{v^t}(w),\bar{h_i}(t,w)), i=1,2.$$

From (H_2), we have

$$\begin{aligned} \|h_1(t,w)-\bar{h_1}(t,w)\| &\le p_i(t,w)\|u-\bar{u}\|_{[-r,\beta]} + q_i(t,w)\|v-\bar{v}\|_{[-r,\beta]}\\ &+ r_i(t,w)\|h_1(t,w)-\bar{h_1}(t,w)\|\\ &\le \|p_i(\cdot,w)\|_{[a,T]}\|u-\bar{u}\|_{[-r,\beta]} + \|q_i(\cdot,w)\|_{[a,T]}\|v-\bar{v}\|_{[-r,\beta]}\\ &+ \|r_i(\cdot,w)\|_{[a,T]}\|h_1(t,w)-\bar{h_1}(t,w)\|. \end{aligned}$$

Then

$$\begin{aligned} \|h_1(t,w)-\bar{h_1}(t,w)\| &\le \frac{\|p_i(\cdot,w)\|_{[a,T]}}{1-\|r_i(\cdot,w)\|_{[a,T]}}\|u-\bar{u}\|_{[-r,\beta]}\\ &+ \frac{\|q_i(\cdot,w)\|_{[a,T]}}{1-\|r_i(\cdot,w)\|_{[a,T]}}\|v-\bar{v}\|_{[-r,\beta]}, \end{aligned}$$

from which we conclude

$$\begin{aligned}\|N_1(u,v)(t,w)-N_1(\bar{u},\bar{v})(t,w)\| &\leq \frac{\|p_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\int_a^T |G_1(t,s)|\|u-\bar{u}\|_{[-r,\beta]}ds \\ &+ \frac{\|q_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\int_a^T |G_1(t,s)|\|v-\bar{v}\|_{[-r,\beta]}ds \\ &\leq \frac{\widetilde{G_1}\|p_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\|u-\bar{u}\|_{[-r,\beta]} \\ &+ \frac{\widetilde{G_1}\|q_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\|v-\bar{v}\|_{[-r,\beta]}.\end{aligned}$$

Therefore, for each $t\in I$, and $w\in\Omega$,

$$\begin{aligned}\|N_1(u,v)(\cdot,w)-N_1(\bar{u},\bar{v})(\cdot,w)\|_{\mathscr{C}} &\leq \frac{\widetilde{G_1}\|p_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\|u-\bar{u}\|_{\mathscr{C}} \\ &+ \frac{\widetilde{G_1}\|q_1(\cdot,w)\|_{[a,T]}}{1-\|r_1(\cdot,w)\|_{[a,T]}}\|v-\bar{v}\|_{\mathscr{C}}.\end{aligned}$$

Also, for any $w\in\Omega$ and each $(u,v),(\bar{u},\bar{v})\in\mathscr{C}^2$ and $t\in I$, we get

$$\begin{aligned}\|N_2(u,v)(\cdot,w)-N_2(\bar{u},\bar{v})(\cdot,w)\|_{\mathscr{C}} &\leq \frac{\widetilde{G_2}\|p_2(\cdot,w)\|_{[a,T]}}{1-\|r_2(\cdot,w)\|_{[a,T]}}\|u-\bar{u}\|_{\mathscr{C}} \\ &+ \frac{\widetilde{G_2}\|q_2(\cdot,w)\|_{[a,T]}}{1-\|r_2(\cdot,w)\|_{[a,T]}}\|v-\bar{v}\|_{\mathscr{C}}.\end{aligned}$$

Thus,

$$d(N(u,v)(\cdot,w),N(\bar{u},\bar{v})(\cdot,w))\leq M(w)d((u(\cdot,w),v(\cdot,w)),(\bar{u}(\cdot,w),\bar{v}(\cdot,w)),$$

where

$$d((u(\cdot,w),v(\cdot,w)),(\bar{u}(\cdot,w),\bar{v}(\cdot,w)))=\begin{pmatrix}\|u(\cdot,w)-v(\cdot,w)\|_{\mathscr{C}}\\ \\ \|\bar{u}(\cdot,w)-\bar{v}(\cdot,w)\|_{\mathscr{C}}\end{pmatrix}.$$

Since, for every $w\in\Omega$, the matrix $M(w)$ converges to zero, then, by Theorem 2.12, N has a unique random fixed point, which is a solution to problem (1.1)-(1.2). □

Theorem 3.4. *Suppose that (H_1) and (H_3) hold. Then problem (1.1)-(1.2) has at least one solution.*

Proof. The proof is split into steps.

Step 1: $N(\cdot,\cdot,w)$ is continuous.

Let $\{(u_n,v_n)\}$ be a sequence such that $(u_n,v_n)\to(u,v)$ in $\mathscr{C}\times\mathscr{C}$, for any $w\in\Omega$. If $t\in[a-r,a]$ or $t\in[T,T+\beta]$, then

$$\|(N(u_n,v_n))(t,w)-(N(u,v))(t,w)\|=0.$$

For $t \in I$, we have

$$\begin{aligned}\|(N_i(u_n,v_n)(t,w)) - (N_i(u,v))(t,w)\| &\leq \int_a^T |G_{\alpha_i}(t,s)| \|f_i(t,u_{t,n}(w),v_{t,n}(w),h_{i,n}(t,w)) \\ &\quad - f_i(t,u_t(w),v_t(w),h_i(t,w))\|ds \\ &\leq \widetilde{G_{\alpha_i}} \|f_i(\cdot,u_n(\cdot,w),v_n(\cdot,w),h_{i,n}(\cdot,w)) \\ &\quad - f_i(\cdot,u_t(w),v(\cdot,w),h_i(\cdot,w))\|_{\mathscr{C}}, i=1,2,\end{aligned}$$

where

$$h_i(t,w) = f_i(t,u^t(w),v^t(w),h_i(t,w)).$$

Since f_i is Carathéodory, we have

$$\|(N_i(u_n,v_n))(\cdot,w) - (N_i(u,v))(\cdot,w)\|_{\mathscr{C}} \longrightarrow 0 \; as \; n \longrightarrow \infty, i=1,2,$$

and hence

$$\|N(u_n,v_n)(\cdot,w) - N(u,v)(\cdot,w)\|_{\overline{\mathscr{C}}} \longrightarrow 0 \; as \; n \longrightarrow \infty.$$

Consequently, N is continuous. Let the constant $R(w)$ be such that

$$\begin{aligned}R(w) \geq \max\{&L_1(w)+L_2(w), \|\phi_1(\cdot,w)\|_{[a-r,a]} + \|\phi_2(\cdot,w)\|_{[a-r,a]}, \|\psi_1(\cdot,w)\|_{[T,T+\beta]} \\ &+ \|\psi_2(\cdot,w)\|_{[T,T+\beta]}\},\end{aligned}$$

and define

$$D_{R(w)} = \{(u,v) \in \mathscr{C} \times \mathscr{C} : \|u\|_{\mathscr{C}} \leq R(w) \; and \; \|v\|_{\mathscr{C}} \leq R(w)\}.$$

It is clear that D_R is a bounded, closed and convex subset of $\overline{\mathscr{C}}$.

Step 2: $N(D_R(w)) \subset D_R(w)$.

Let $(u,v) \in D_R(w)$. We show that $N(u,v) = (N_1(u,v), N_2(u,v)) \in D_R(w)$.
For any $w \in \Omega$, if $t \in [a-r,a]$, then

$$\|N(u,v)(t,w)\| \leq \|\phi_1(\cdot,w)\|_{[a-r,a]} + \|\phi_2(\cdot,w)\|_{[a-r,a]} \leq R(w),$$

and if $t \in [T,T+\beta]$, then

$$\|N(u,v)(t,w)\| \leq \|\psi_1(\cdot,w)\|_{[T,T+\beta]} + \|\psi_2(\cdot,w)\|_{[T,T+\beta]} \leq R(w).$$

For any $w \in \Omega$ and each $t \in I$, we have

$$\|(N_i(u,v))(t,w)\| \leq \int_a^T |G_{\alpha_i}(t,s)| \|h_{\alpha_i}(s,w)\|ds, i=1,2.$$

By (H_3), we have, for any $w \in \Omega$ and each $t \in I$,

$$\begin{aligned}\|h_i(t,w)\| &\leq a_i(t,w) + b_i(t,w)\|u\|_{[-r,\beta]} + c_i(t,w)\|v\|_{[-r,\beta]} + d_i(t,w)\|h_i(t,w)\| \\ &\leq a_i^*(\cdot,w) + b_i^*(\cdot,w)\|u\|_{[-r,\beta]} + c_i^*(\cdot,w)\|v\|_{[-r,\beta]} + d_i^*(\cdot,w)\|h_i(t,w)\|,\end{aligned}$$

where

$$h_i(t,w) = f_i(t,u^t(w),v^t(w),h_i(t,w)), i=1,2.$$

Then

$$\|h_i(t,w)\| \leq \frac{a_i^*(\cdot,w) + (b_i^*(\cdot,w) + c_i^*(\cdot,w))R}{1 - d_i^*(\cdot,w)} = A(w). \tag{3.10}$$

By (3.10), for any $w \in \Omega$ and $t \in I$, we have

$$\begin{aligned}\|(N_i(u,v))(t,w)\| &\leq A(w)\int_a^T |G_{\alpha_i}(t,s)|ds \\ &\leq A(w)\widetilde{G_{\alpha_i}} \\ &= L_i(w),\end{aligned}$$

from which it follows that, for each $t \in [a-r, T+\beta]$,

$$\|N_i(u,v)(t,w)\| \leq L_i(w),$$

which implies that $\|N_i(u,v)(\cdot,w)\|_{\mathscr{C}} \leq L_i(w)$. Hence,

$$\begin{aligned}\|N(u,v)(\cdot,w)\|_{\overline{\mathscr{C}}} &\leq L_1(w)+L_2(w) \\ &\leq R(w).\end{aligned}$$

Consequently,

$$N(D_R(w)) \subset D_R(w).$$

Step 3: $N(D_R(w))$ is bounded and equicontinuous.

By Step 2, we have that $N(D_R(w))$ is bounded. For $t_1, t_2 \in I = [a,T]$, $t_1 < t_2$, $(u,v) \in D_R(w)$, $w \in \Omega$, we have

$$\begin{aligned}\|(N_i(u,v))(t_2,w)-(N_i(u,v))(t_1,w)\| &\leq \int_a^T |G_{\alpha_i}(t_2,s)-G_{\alpha_i}(t_1,s)|\|h_i(s,w)\|ds \\ &\leq A(w)\int_a^T |G_{\alpha_i}(t_2,s)-G_{\alpha_i}(t_1,s)|ds.\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right hand side of the above inequality tends to zero. Therefore, the operator $N(u,v)(\cdot,w)$ is equicontinuous. As consequence of Step 1 to Step 3, together withe the Arzela-Ascoli theorem, we can conclude that N is continuous and completely continuous. Theorem 3.4 implies that the operator equation $N(u,v)(\cdot,w) = (u,v)$ has a random solution. This shows that (1.1)-(1.2) has a random solution. □

4. An Example

We equip the space $\mathbb{R}_-^* := (-\infty, 0)$ with the usual σ-algebra consisting of Lebesgue measurable subsets of $\mathbb{R}_-^*$. Consider the boundary value problem of implicit Caputo-type modification of the Erdélyi-Kober fractional differential equation:

$$\begin{cases}(u(t,w),v(t,w)) = (e^{t-1}-1, 2t-2), \ \ t\in[0,1], \\ {}^1_cD^{\frac{3}{2}}_{1^+}u(t,w) = \dfrac{\frac{1}{2}t}{(w^2+10)\left(1+\|u^t(\cdot,w)\|+\|v^t(\cdot,w)\|+|{}^1_cD^{\frac{3}{2}}_{1^+}u(t,w)|\right)}, \ t\in I=[1,2], \\ {}^1_cD^{\frac{3}{2}}_{1^+}v(t,w) = \dfrac{(t-1)\cos(t)}{(w^2+10)\left(1+\|u^t(\cdot,w)\|+\|v^t(\cdot,w)\|+|{}^1_cD^{\frac{3}{2}}_{1^+}v(t,w)|\right)}, \ t\in I=[1,2] \\ (u(t,w),v(t,w)) = (\ln(t-1), t-2), \ \ \ t\in[2,3].\end{cases} \tag{4.1}$$

Set

$$f_1(t,u,v,\bar{u},w)=\frac{\frac{1}{2}t}{(w^2+10)\,(1+|u^t|+|v^t|+|\bar{u}|)},\ t\in[1,2],u,v\in C([-1,1]),\bar{u}\in\mathbb{R}^n,$$

$$f_2(t,u,v,\bar{v},w)=\frac{(t-1)\cos(t)}{(w^2+10)\,(1+|u^t|+|v^t|+|\bar{v}|)},\ \ t\in[1,2],u,v\in C([-1,1]),\bar{v}\in\mathbb{R}^n,$$

and $\alpha_1=\alpha_2=\frac{3}{2},\rho=1,r=1,\beta=1$. Indeed for each $u,v,\bar{u},\bar{v}\in C([-1,1]),x,\bar{x}\in\mathbb{R}^n$ and $t\in[1,2]$, we have

$$\begin{aligned}\|f_1(t,u,v,x,w)-f_1(t,\bar{u},\bar{v},\bar{x},w)\| &\leq \frac{1}{(w^2+10)}(\|u-\bar{u}\|_{[-r,\beta]}+\|v-\bar{v}\|_{[-r,\beta]}\\ &+ \|x-\bar{x}\|),\end{aligned}$$

and

$$\begin{aligned}\|f_2(t,u,v,x,w)-f_2(t,\bar{u},\bar{v},\bar{x},w)\| &\leq \frac{1}{(w^2+10)}(\|u-\bar{u}\|_{[-r,\beta]}+\|v-\bar{v}\|_{[-r,\beta]}\\ &+ \|x-\bar{x}\|).\end{aligned}$$

Therefore, (H_2) is verified with

$$\|p_i(\cdot,w)\|_{[-r,\beta]}=\|q_i(\cdot,w)\|_{[-r,\beta]}=\|r_i(\cdot,w)\|_{[-r,\beta]}=\frac{1}{w^2+10}.$$

For each $t\in I,\ i=1,2$, we have

$$\begin{aligned}\int_a^T|G_{\alpha_i}(t,s)|ds &\leq \frac{1}{\Gamma(\alpha_i)}\left(\frac{t^\rho-a^\rho}{T^\rho-a^\rho}\right)\int_a^T\left|\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha_i-1}s^{\rho-1}\right|ds\\ &+ \frac{1}{\Gamma(\alpha_i)}\int_a^t\left|\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha_i-1}s^{\rho-1}\right|ds\\ &\leq \frac{2}{\Gamma(\alpha_i+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha_i}.\end{aligned}$$

Therefore

$$\widetilde{G_{\alpha_i}}\leq\frac{2}{\Gamma(\alpha_i+1)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha_i}.$$

Furthermore, for every $w\in\Omega$, the matrix

$$\frac{1}{(w^2+9)\frac{3}{4}\sqrt{\pi}}\begin{pmatrix}1&1\\1&1\end{pmatrix},$$

converges to 0. Hence, Theorem 3.3 implies that system (4.1) has a unique random solution defined on $[1,2]$.

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