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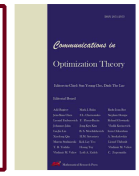

EXISTENCE OF SOLUTIONS FOR NEUTRAL CAPUTO-TYPE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. This paper investigates the existence of solutions for the neutral of nonlinear integro-differential equation of fractional order with nonlocal boundary conditions. The existence and uniqueness results are established by Banach contraction principle. Based on the O'Regan fixed point theorem, we prove an existence result. Finally, illustrative examples are provided to support of our main results presented in this paper.

Keywords. Boundary value problem; Caputo derivative; Existence; Fixed point theorem; Neutral equations.

1. Introduction

In recent years, fractional-order differential equations have emerged as an important research since this theory has many real applications in mathematics, physics, bio-engineering, and other applied sciences. For more details, we refer the reader to [1, 2, 3, 4, 5, 6]. Recently, many authors obtained the existence results for a boundary value problems of fractional-order; see, e.g., [7, 8, 2, 9, 10, 11, 12] and the references therein. Using the techniques of nonlinear analysis, many authors investigated the existence of solutions for the boundary value problems of fractional differential equations with local boundary value problems; see, e.g., [13, 14, 15, 16] and the references. On the other hand, differential equations of fractional order with nonlocal boundary conditions arise in different areas of applied mathematics and physics [17, 18, 19]. For recent results on nonlocal integro-differential boundary value problems, the reader is referred to [20, 21, 22] and the references therein.

In this paper, we study the existence of solutions for the following boundary value problem of Caputo type neutral fractional integro-differential equations with nonlocal boundary conditions

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given by

$$\begin{cases} D^{\alpha}\left[u(t)-f(t,u(t))\right]=\sum\limits_{i=1}^{n}P_iI^{\sigma_i}g_i(t,u(t)),\ t\in J=[0,T],\\ u(0)-u_0=h(u(t)),\ I^{\beta}u(T)=\sum\limits_{j=1}^{m}Q_jI^{\delta_j}u(\xi_j),\ \in 0<\xi_j<T,\end{cases} \tag{1.1}$$

where D^{α} denotes the Caputo fractional derivative with $1<\alpha\leq 2$, I^{v} is the Riemann-Liouville integral of order $v>0, v\in\left\{\sigma_i,\beta,\delta_j\right\}$, $g_i: J\times\mathbb{R}\to\mathbb{R}$ is continuously differentiable, $P_i, Q_j\in\mathbb{R}$, $i=1,2,...,n$, $j=1,2,...,m$, and $f: J\times\mathbb{R}\to\mathbb{R}$, $h: C(J,\mathbb{R})\to\mathbb{R}$ are continuous functions.

The paper is organized as follows. In Section 2, we recall some preliminaries and lemmas that we need in the sequel. In Section 3, we present our main results for existence and uniqueness of solutions for problem (1.1). Some examples to illustrate our results are presented in Section 4.

2. Preliminaries

In this section, we present some useful definitions and lemmas [3, 4, 5].

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha\geq 0$, for a continuous function f on $[a,b]$, is defined by

$$I^{\vartheta}f(t)=\frac{1}{\Gamma(\vartheta)}\int_a^t(t-\tau)^{\vartheta-1}f(\tau)\,d\tau,\ \vartheta>0,\ a\leq t\leq b.$$

$$I^{0}f(t)=f(t),$$

where $\Gamma(\vartheta):=\int_0^{\infty}e^{-u}u^{\vartheta-1}du$.

Definition 2.2. The fractional derivative of $f\in C^{n}([a,b])$ in the Caputo's sense is defined by

$$D^{\vartheta}f(t)=\frac{1}{\Gamma(n-\vartheta)}\int_a^t(t-\tau)^{n-\vartheta-1}f^{(n)}(\tau)\,d\tau,\ n-1<\vartheta,\ n\in N^{*},\ a\leq t\leq b.$$

The following lemmas give some properties of Riemann-Liouville fractional integrals and Caputo fractional derivative [3, 4]:

Lemma 2.3. *Let $r,s>0$, and $f\in L_1([a,b])$. Then $I^{r}I^{s}f(t)=I^{r+s}f(t)$ and $D^{s}I^{s}f(t)=f(t)$, $t\in[a,b]$.*

We also give the following lemmas [3].

Lemma 2.4. *For $\vartheta>0$, the general solution of the fractional differential equation $D^{\vartheta}x(t)=0$ is given by$x(t)=c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1}$, where $c_i\in\mathbb{R}, i=0,1,2,..,n-1$, $n=[\vartheta]+1$.*

Lemma 2.5. *Let $\vartheta>0$. Then$I^{\vartheta}D^{\vartheta}x(t)=x(t)+c_0+c_1t+c_2t^2+...+c_{n-1}t^{n-1}$, for some $c_i\in\mathbb{R}, i=0,1,2,...,n-1$, $n=[\vartheta]+1$.*

We need also the following auxiliary result.

Lemma 2.6. *Let* $\sum_{j=1}^{m} \frac{Q_j \xi_j^{\delta_j+1}}{\Gamma(\delta_j+2)} \neq \frac{T^{\beta+1}}{\Gamma(\beta+2)}$. *For any* $\varphi \in C([0,T],\mathbb{R})$, *the general solution of the boundary value problem*

$$\begin{cases} D^{\alpha}(u(t)-f(t,u(t))) = \varphi(t),\ t\in J=[0,T], \\ u(0)=u_0+h(u(t)), I^{\beta}u(T)=\sum_{j=1}^{m} Q_j I^{\delta_j} u(\xi_j),\ 0<\xi_i<T, \end{cases} \tag{2.1}$$

is given by

$$\begin{aligned} & u(t) \\ = & f(t,u(t)) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s)\,ds + \frac{t}{\Delta}\left\{\sum_{j=1}^{m} Q_j \left(\int_0^{\xi_j} \frac{(\xi_j-s)^{\delta_j-1}}{\Gamma(\delta_j)} f(s,u(s))\,ds\right.\right. \\ & \left. + \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha+\delta_j-1}}{\Gamma(\alpha+\delta_j)}\varphi(s)\,ds\right) - \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\,ds \\ & \left. - \int_0^T \frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\varphi(s)\,ds\right\} - f(0,u(0))\left[1+\frac{t}{\Delta}\left(\sum_{j=1}^{m} Q_j \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} - \frac{T^{\beta}}{\Gamma(\beta+1)}\right)\right] \\ & + \left[1+\frac{t}{\Delta}\left(\sum_{j=1}^{m} Q_j \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} - \frac{T^{\beta}}{\Gamma(\beta+1)}\right)\right](u_0+h(u(t))). \end{aligned} \tag{2.2}$$

where $\Delta = \frac{T^{\beta+1}}{\Gamma(\beta+2)} - \sum_{j=1}^{m} Q_j \frac{\xi_j^{\delta_j+1}}{\Gamma(\delta_j+2)}$.

Proof. From Lemma 2.4 and Lemma 2.5, the general solution of (2.1) can written as

$$u(t)-f(t,u(t)) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi(s)\,ds - c_0 - c_1 t \tag{2.3}$$

where c_0 and c_1 are arbitrary constants. Applying the condition $u(0)=u_0+h(u(t))$, we obtain $c_0=-u_0-h(u(t))+f(0,u(0))$. Taking Riemann-Liouville fractional integral of order β,δ_j and the condition $I^{\beta}u(T)=\sum_{j=1}^{m} Q_j I^{\delta_j} u(\xi_j)$, we have

$$\begin{aligned} c_1 = & \frac{1}{\Delta}\left\{\sum_{j=1}^{m} Q_j \left[I^{\delta_j} f(\xi_j,u(\xi_j)) + I^{\alpha+\delta_j}\varphi(\xi_j)\right] - I^{\beta} f(T,u(T)) - I^{\alpha+\beta}\varphi(T)\right. \\ & \left. + [u_0 - f(0,u(0)) + h(u(t))]\left(\sum_{j=1}^{m} Q_j \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} - \frac{T^{\beta}}{\Gamma(\beta+1)}\right)\right\}. \end{aligned}$$

Substituting the values of c_0 and c_1 into (2.3) yields the solution. □

To prove the second main result, we introduce the following O'Regan Lemma [23].

Lemma 2.7. *Let D be an open set in a closed, convex set U of a Banach space E. Assume that $0\in D$, and $\Lambda(\bar{D})$ is bounded, where $\Lambda:\bar{D}\to U$ is given by $\Lambda=\Lambda_1+\Lambda_2$, in which $\Lambda_1:\bar{D}\to E$*

is continuous and completely continuous, and $\Lambda_2 : \bar{D} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\omega : (0,\infty) \to (0,\infty)$ satisfying $\omega(v) < v$ for $v > 0$ such that $\|\Lambda_2(u) - \Lambda_2(v)\| \leq \omega \|u - v\|$ for all $u, v \in \bar{D}$). Then, either

(I) : Λ has a fixed point $u \in \bar{D}$; or

(II) : there exist a point $u \in \partial D$ and $0 < \mu < 1$ with $x = \mu\Lambda(u)$,where $\bar{D}$.

(respectively ∂D) represents the closure, (respectively the boundary) of D.

3. Main Results

We denote by $E = C([0,T], \mathbb{R})$ the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|u\| = \sup_{t\in[0,T]} |u(t)|$.

In view of Lemma 2.6, we define an operator $\Lambda : E \to E$ by

$$\begin{aligned}
&\Lambda u(t)\\
&= ft,u(t)) + \sum_{i=1}^{n} P_i \int_0^t \frac{(t-s)^{\alpha+\sigma_i-1}}{\Gamma(\alpha+\sigma_i)} g_i(s,u(s))ds + \frac{t}{\Delta}\Big\{ \sum_{j=1}^{m} Q_j \big[\int_0^{\xi_j} \frac{(\xi_j-s)^{\delta_j-1}}{\Gamma(\delta_j)} f(s,u(s))ds \\
&+ \sum_{i=1}^{n} P_i \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha+\delta_j+\sigma_i-2}}{\Gamma(\alpha+\delta_j+\sigma_i-1)} g_i(s,u(s))ds\big] - \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta} f(s,u(s))ds \\
&- \sum_{i=1}^{n} P_i \int_0^T \frac{(T-s)^{\alpha+\beta+\sigma_i-2}}{\Gamma(\alpha+\beta+\sigma_i-1)} g_i(s,u(s))ds\Big\} - f(0,u(0))\big[1 + \frac{t}{\Delta}\big(\sum_{j=1}^{m} Q_j \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1} - \frac{T^\beta}{\Gamma(\beta+1)}\big)\big] \\
&+ \big[1 + \frac{t}{\Delta}\big(\sum_{j=1}^{m} Q_j \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} - \frac{T^\beta}{\Gamma(\beta+1)}\big)\big](u_0 + h(u(t))).
\end{aligned} \tag{3.1}$$

Observe that problem (1.1) has solutions if and only if $\Lambda u = u$. Define two operators from $E \to E$, respectively, by

$$\begin{aligned}
\Lambda_1 u(t) &= f(t,u(t)) + \sum_{i=1}^{n} P_i \int_0^t \frac{(t-s)^{\alpha+\sigma_i-1}}{\Gamma(\alpha+\sigma_i)} g_i(s,u(s))ds \\
&+ \frac{t}{\Delta}\left\{ \sum_{j=1}^{m} Q_j \left[\int_0^{\xi_j} \frac{(\xi_j-s)^{\delta_j-1}}{\Gamma(\delta_j)} f(s,u(s))ds \right.\right. \\
&\left. + \sum_{i=1}^{n} P_i \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha+\delta_j+\sigma_i-2}}{\Gamma(\alpha+\delta_j+\sigma_i-1)} g_i(s,u(s))ds \right] - \int_0^T \frac{(T-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds \\
&\left. - \sum_{i=1}^{n} P_i \int_0^T \frac{(T-s)^{\alpha+\beta+\sigma_i-2}}{\Gamma(\alpha+\beta+\sigma_i-1)} g_i(s,u(s))ds \right\} \\
&- f(0,u(0))\big[1 + \frac{t}{\Delta}\big(\sum_{j=1}^{m} Q_j \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} - \frac{T^\beta}{\Gamma(\beta+1)}\big)\big],
\end{aligned} \tag{3.2}$$

and

$$\Lambda_2 u(t) = \left[1+\frac{t}{\Delta}\left(\sum_{j=1}^{m} Q_j \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} - \frac{T^{\beta}}{\Gamma(\beta+1)}\right)\right](u_0 + h(u(t))). \tag{3.3}$$

Now, we list the following hypotheses:

(H_1) There exist nonnegative constants K_i, $i=1,...,n+1$ such that, for all $t\in[0,T]$ and $(u,v)\in\mathbb{R}^2$,

$$|f(t,u(t)) - f(t,v(t))| \leq K_1 |u-v|,$$

and

$$|g_i(t,u(t)) - g_i(t,v(t))| \leq K_{i+1}\times|u-v|.$$

(H_2) There exist a positive constant $\overline{K}<1$ and a continuous function $h: C([0,T],\mathbb{R})\to\mathbb{R}$, such that

$$|h(u(t)) - h(v(t))| \leq \overline{K}|u-v|$$

for all $u,v\in C([0,T],\mathbb{R})$,

(H_3) $h(0)=0$.

(H_4) There exist a functions $\gamma_1,\gamma_{i+1}\in C([0,T],\mathbb{R})$, and nondecreasing functions ψ_1,ψ_{i+1} : $\mathbb{R}^+\to\mathbb{R}^+$, $i=1,...,n$ such that

$$|f(t,u)| \leq \|\gamma_1\|\,\psi_1(\|u\|),$$

and

$$|g_i(t,u)| \leq \|\gamma_{i+1}\|\,\psi_{i+1}(\|u\|)$$

for all $(t,u)\in[0,T]\times\mathbb{R}$.

(H_5)

$$\sup_{\delta\in(0.+\infty)} \frac{\delta}{\sum_{i=1}^{n+1} L_i\psi_i(\delta) + \overline{\rho}(|u_0| + |f(0,u)|)} > \frac{1}{1-\overline{\rho}\overline{K}},$$

where

$$L_1 = \left(\frac{T}{|\Delta|}\sum_{j=1}^{m}|Q_j|\frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} + \frac{T^{\beta}}{\Gamma(\beta+1)} + 1\right)\|\gamma_1\|,$$

and

$$L_{i+1} = |P_i|\left(\frac{T}{|\Delta|}\sum_{j=1}^{m}|Q_j|\frac{\xi_j^{\alpha+\delta_j+\sigma_i-1}}{\Gamma(\alpha+\delta_j+\sigma_i)} + \frac{T^{\alpha+\sigma_i-1}}{\Gamma(\alpha+\sigma_i)} + \frac{T^{\alpha+\beta+\sigma_i-1}}{\Gamma(\alpha+\beta+\sigma_i)}\right)\|\gamma_{i+1}\|,$$

for $i=1,2,...n$.

Our first result is based on the Banach contraction principle.

Theorem 3.1. *Assume that f, g_i :$[0,T]\times\mathbb{R}\to\mathbb{R}$, $i=1,...,n$ are continuous fonctions. Suppose that (H_1) and (H_2) hold. If*

$$\sum_{i=1}^{n+1} K_i\rho_i < 1-\overline{K}\overline{\rho}, \tag{3.4}$$

where

$$\begin{aligned}
\rho_1 &: = \frac{T}{|\Delta|}\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}+\frac{T^{\beta}}{\Gamma(\beta+1)}+1,\\
\rho_{i+1} &: = |P_i|\left(\frac{T}{|\Delta|}\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\alpha+\delta_j+\sigma_i-1}}{\Gamma\left(\alpha+\delta_j+\sigma_i\right)}+\frac{T^{\alpha+\sigma_i-1}}{\Gamma(\alpha+\sigma_i)}+\frac{T^{\alpha+\beta+\sigma_i-1}}{\Gamma(\alpha+\beta+\sigma_i)}\right), i=1,2,...n,\\
\overline{\rho} &: = 1+\tfrac{T}{|\Delta|}\left|\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|,
\end{aligned}$$

then fractional problem (1.1) has a unique solution on $[0,T]$.

Proof. Assume that $M_1=\sup_{t\in[0,T]}|f(t,0)|$ and for $i=1,...,n$, $M_{i+1}=\sup_{t\in[0,T]}|g_i(t,0)|$, and

$$\frac{\sum_{i=1}^{n+1}M_i\rho_i+\overline{\rho}\left(|u_0|+|f(0,u(0))|\right)}{1-\left(\sum_{i=1}^{n+1}K_i\rho_i+\overline{K}\overline{\rho}\right)}\leq r.$$

We show that $\Lambda B_r\subset B_r$, where $B_r=\{u\in E:\|u\|\leq r\}$. For $u\in B_r$ and for each $t\in[0,T]$, from the definition of Λ and hypothesis (H_1) and (H_2), we obtain

$$\begin{aligned}
&||\Lambda u||\\
\leq\ & \sup_{t\in[0,T]}\left\{\frac{t}{|\Delta|}\left(\sum_{j=1}^{m}\left|Q_j\right|\left(\int_0^{\xi_j}\frac{\left(\xi_j-s\right)^{\delta_j-1}}{\Gamma\left(\delta_j\right)}|f(s,u(s))-f(s,0)+f(s,0)|ds\right)\right.\right.\\
&+\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}|f(s,u(s))-f(s,0)+f(s,0)|ds+|f(t,u(s))-f(t,0)+f(t,0)|\\
&+|f(0,u(0))|\left(1+\frac{t}{|\Delta|}\left|\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right)\\
&+\frac{t}{|\Delta|}\sum_{j=1}^{m}\left|Q_j\right|\sum_{i=1}^{n}|P_i|\int_0^{\xi_j}\frac{\left(\xi_j-s\right)^{\alpha+\delta_j+\sigma_i-2}}{\Gamma\left(\alpha+\delta_j+\sigma_i-1\right)}|g_i(s,u(s))-g_i(s,0)+g_i(s,0)|ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\sum_{i=1}^{n}|P_i|\int_0^s\frac{(s-\tau)^{\sigma_i-1}}{\Gamma(\sigma_i)}|g_i(\tau,u(\tau))-g_i(\tau,0)+g_i(\tau,0)|d\tau\right)ds\\
&+\int_0^T\frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left(\sum_{i=1}^{n}|P_i|\int_0^s\frac{(s-\tau)^{\sigma_i-1}}{\Gamma(\sigma_i)}|g_i(\tau,u(\tau))-g_i(\tau,0)+g_i(\tau,0)|d\tau\right)ds\\
&\left.+\left(1+\frac{T}{|\Delta|}\left|\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right)\left(|u_0|+|h(u(t))|\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
\leq\;& \left(\frac{T}{|\Delta|}\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}+\frac{T^{\beta}}{\Gamma(\beta+1)}+1\right)(K_1r+M_1)\\
&+|f(0,u(0))|\left(1+\frac{T}{|\Delta|}\left|\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right)\\
&+\sum_{i=1}^{n}(K_{i+1}r+M_{i+1})|P_i|\left(\frac{T}{|\Delta|}\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\alpha+\delta_j+\sigma_i-1}}{\Gamma\left(\alpha+\delta_j+\sigma_i\right)}+\frac{T^{\alpha+\sigma_i-1}}{\Gamma(\alpha+\sigma_i)}+\frac{T^{\alpha+\beta+\sigma_i-1}}{\Gamma(\alpha+\beta+\sigma_i)}\right)\\
&+\left(1+\frac{T}{|\Delta|}\left|\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right)\left(|u_0|+\overline{K}r\right)\\
\leq\;& r\sum_{i=1}^{n+1}K_i\rho_i+\sum_{i=1}^{n+1}M_i\rho_i+r\overline{K}\overline{\rho}+\overline{\rho}\left(|u_0|+|f(0,u(0))|\right).
\end{aligned}
$$

Consequently $\|\Lambda u\|\leq r$. This shows that $\Lambda B_r\subset B_r$. Now, for $u,v\in B_r$ and for all $t\in[0,T]$, we have

$$
\begin{aligned}
&||\Lambda u-\Lambda v||\\
\leq\;& \sup_{t\in[0,T]}\left\{\frac{t}{|\Delta|}\left(\sum_{j=1}^{m}\left|Q_j\right|\left(\int_0^{\xi_j}\frac{\left(\xi_j-s\right)^{\delta_j-1}}{\Gamma\left(\delta_j\right)}|f(s,u(s))-f(s,v(s))|ds\right)\right.\right.\\
&+\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}|f(s,u(s))-f(s,v(s))|ds+|f(t,u(t))-f(t,v(t))|\\
&+\frac{t}{|\Delta|}\sum_{j=1}^{m}\left|Q_j\right|\sum_{i=1}^{n}|P_i|\int_0^{\xi_j}\frac{\left(\xi_j-s\right)^{\alpha+\delta_j+\sigma_i-2}}{\Gamma\left(\alpha+\delta_j+\sigma_i-1\right)}|g_i(s,u(s))-g_i(s,v(s))|ds\\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\sum_{i=1}^{n}|P_i|\int_0^s\frac{(s-\tau)^{\sigma_i-1}}{\Gamma(\sigma_i)}|g_i(\tau,u(\tau))-g_i(\tau,v(\tau))|d\tau\right)ds\\
&+\int_0^T\frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left(\sum_{i=1}^{n}|P_i|\int_0^s\frac{(s-\tau)^{\sigma_i-1}}{\Gamma(\sigma_i)}|g_i(\tau,u(\tau))-g_i(\tau,v(\tau))|d\tau\right)ds\\
&\left.+\left(1+\frac{T}{|\Delta|}\left|\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right)(|h(u(t))-h(v(t))|)\right\}\\
\leq\;& \left(\frac{T}{|\Delta|}\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}+\frac{T^{\beta}}{\Gamma(\beta+1)}+1\right)K_1\|u-v\|\\
&+|f(0,u(0))-f(0,v(0))|\left(1+\frac{T}{|\Delta|}\left|\sum_{j=1}^{m}\left|Q_j\right|\frac{\xi_j^{\delta_j}}{\Gamma\left(\delta_j+1\right)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right)
\end{aligned}
$$

$$+\sum_{i=1}^{n}K_{i+1}\left|P_{i}\right|\left(\frac{T}{|\Delta|}\sum_{j=1}^{m}\left|Q_{j}\right|\frac{\xi_{j}^{\alpha+\delta_{j}+\sigma_{i}-1}}{\Gamma\left(\alpha+\delta_{j}+\sigma_{i}\right)}+\frac{T^{\alpha+\sigma_{i}-1}}{\Gamma\left(\alpha+\sigma_{i}\right)}+\frac{T^{\alpha+\beta+\sigma_{i}-1}}{\Gamma\left(\alpha+\beta+\sigma_{i}\right)}\right)\|u-v\|$$
$$+\left(1+\frac{T}{|\Delta|}\left|\sum_{j=1}^{m}\left|Q_{j}\right|\frac{\xi_{j}^{\delta_{j}}}{\Gamma\left(\delta_{j}+1\right)}-\frac{T^{\beta}}{\Gamma\left(\beta+1\right)}\right|\right)\overline{K}\|u-v\|$$
$$\leq\left(\sum_{i=1}^{n+1}K_{i}\rho_{i}+\overline{K\rho}\right)\|u-v\|.$$

Hence Λ is a contraction mapping. As a consequence of Banach fixed point theorem, we deduce that Λ has a fixed point which is a solution of problem (1.1). □

Now, we use Lemma 2.7 to prove the following result.

Theorem 3.2. *Let f, g_i :$[0,T]\times\mathbb{R}\rightarrow\mathbb{R}$, $i=1,...,n$ be a continuous fonctions. Suppose that $(H_1),(H_2),(H_3)$ and (H_4) are satisfied. Then boundary value problem (1.1) has at least one solution on $[0,T]$.*

Proof. Consider the operator $\Lambda:E\rightarrow E$ defined by$\Lambda\left(u\right)\left(t\right):=\Lambda_{1}\left(u\right)\left(t\right)+\Lambda_{2}\left(u\right)\left(t\right),\ t\in J$, where the operators Λ_1 and Λ_2 are defined respectively in (3.2) and (3.3). From (H_5), we see that there exist a number $\delta_0>0$ such that

$$\frac{\delta_{0}}{\sum_{i=1}^{n+1}L_{i}\psi_{i}(\delta)+\overline{\rho}(|u_{0}|+|f(0,u)|)}>\frac{1}{1-\overline{\rho}\overline{K}}. \tag{3.5}$$

We next prove that Λ_1 and Λ_2 satisfy all the conditions in Lemma 2.7.

Step 1. Show that the operator $\Lambda_{1}:\overline{D}_{\delta_{0}}\rightarrow E$ is continuous and completely continuous. Let us consider the set $\overline{D}_{\delta_{0}}:=\{u\in C([0,T],\mathbb{R}):\|u\|\leq\delta_{0}\}$, and show that $\Lambda_{1}\left(\overline{D}_{\delta_{0}}\right)$ is bounded. For each $u\in\overline{D}_{\delta_{0}}$ and $t\in J$, we have

$$||\Lambda_{1}u||$$
$$\leq\sup_{t\in[0,T]}\left\{\frac{t}{|\Delta|}\left(\sum_{j=1}^{m}\left|Q_{j}\right|\left(\int_{0}^{\xi_{j}}\frac{\left(\xi_{j}-s\right)^{\delta_{j}-1}}{\Gamma\left(\delta_{j}\right)}\left|f\left(s,u\left(s\right)\right)\right|ds\right)+\int_{0}^{T}\frac{\left(T-s\right)^{\beta-1}}{\Gamma\left(\beta\right)}\left|f\left(s,u\left(s\right)\right)\right|ds\right.\right.$$
$$+\left|f\left(t,u\left(t\right)\right)\right|+\frac{t}{|\Delta|}\sum_{j=1}^{m}\left|Q_{j}\right|\sum_{i=1}^{n}\left|P_{i}\right|\int_{0}^{\xi_{j}}\frac{\left(\xi_{j}-s\right)^{\alpha+\delta_{j}+\sigma_{i}-2}}{\Gamma\left(\alpha+\delta_{j}+\sigma_{i}-1\right)}\left|g_{i}\left(s,u\left(s\right)\right)\right|ds$$
$$+\int_{0}^{t}\frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left(\sum_{i=1}^{n}\left|P_{i}\right|\int_{0}^{s}\frac{\left(s-\tau\right)^{\sigma_{i}-1}}{\Gamma\left(\sigma_{i}\right)}\left|g_{i}\left(\tau,u\left(\tau\right)\right)\right|d\tau\right)ds$$
$$+\int_{0}^{T}\frac{\left(T-s\right)^{\alpha+\beta-1}}{\Gamma\left(\alpha+\beta\right)}\left(\sum_{i=1}^{n}\left|P_{i}\right|\int_{0}^{s}\frac{\left(s-\tau\right)^{\sigma_{i}-1}}{\Gamma\left(\sigma_{i}\right)}\left|g_{i}\left(\tau,u\left(\tau\right)\right)\right|d\tau\right)ds$$
$$\leq\left(\frac{T}{|\Delta|}\sum_{j=1}^{m}\left|Q_{j}\right|\frac{\xi_{j}^{\delta_{j}}}{\Gamma\left(\delta_{j}+1\right)}+\frac{T^{\beta}}{\Gamma\left(\beta+1\right)}+1\right)\|\gamma_{1}\|\,\psi_{1}\left(\|u\|\right)+\sum_{i=1}^{n}\|\gamma_{i+1}\|\,\psi_{i+1}\left(\|u\|\right)$$

$$\times |P_i| \left(\frac{T}{|\Delta|} \sum_{j=1}^{m} |Q_j| \frac{\xi_j^{\alpha+\delta_j+\sigma_i-1}}{\Gamma(\alpha+\delta_j+\sigma_i)} + \frac{T^{\alpha+\sigma_i-1}}{\Gamma(\alpha+\sigma_i)} + \frac{T^{\alpha+\beta+\sigma_i-1}}{\Gamma(\alpha+\beta+\sigma_i)} \right)$$
$$\leq \sum_{i=1}^{n+1} \rho_i \|\gamma_i\| \psi_i(\|u\|).$$

Thus the operator $\Lambda_1\left(\overline{D}_{\delta_0}\right)$ is uniformly bounded. For any $0 \leq t_1 < t_2 \leq T$ we have

$$\begin{aligned}
& |\Lambda_1 u(t_2) - \Lambda_1 u(t_1)| \\
\leq\ & |f(t_2, u(t_2))| - |f(t_1, u(t_1))| \\
& + \sum_{i=1}^{n} \frac{|P_i|}{\Gamma(\alpha+\sigma_i)} \left[\int_0^{t_1} \left[(t_2-s)^{\alpha+\sigma_i-1} - (t_1-s)^{\alpha+\sigma_i-1} \right] |g_i(s,u(s))| \, ds \right. \\
& + \int_{t_1}^{t_2} (t-s)^{\alpha+\sigma_i-1} \left[(t_2-s)^{\alpha+\sigma_i-1} \right] |g_i(s,u(s))| \, ds \\
& + \left| \frac{t_2-t_1}{\Delta} \right| \sum_{j=1}^{m} |Q_j| \left[\int_0^{\xi_j} \frac{(\xi_j-s)^{\delta_j-1}}{\Gamma(\delta_j)} |f(s,u(s))| \, ds \right. \\
& \left. + \sum_{i=1}^{n} P_i \int_0^{\xi_j} \frac{(\xi_j-s)^{\alpha+\delta_j+\sigma_i-2}}{\Gamma(\alpha+\delta_j+\sigma_i-1)} |g_i(s,u(s))| \, ds \right] \\
\leq\ & |f(t_2, u(t_2)) - f(t_1, u(t_1))| \\
& + \sum_{i=1}^{n} \frac{|P_i|}{\Gamma(\alpha+\sigma_i+1)} \|\gamma_{i+1}\| \psi_{i+1}(\delta_0) \left| t_2^{\alpha+\sigma_i} - t_1^{\alpha+\sigma_i} \right| \\
& + \left| \frac{t_2-t_1}{\Delta} \right| \sum_{j=1}^{m} |Q_j| \left[\|\gamma_1\| \psi_1(\delta_0) \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} + \sum_{i=1}^{n} |P_i| \|\gamma_{i+1}\| \psi_{i+1}(\delta_0) \frac{\xi_j^{\alpha+\delta_j+\sigma_i-1}}{\Gamma(\alpha+\delta_j+\sigma_i)} \right],
\end{aligned}$$

which is independent of u and tends to zero as $t_2 \to t_1$. Thus, Λ_1 is equicontinuous. Hence, by Arzela–Ascoli's theorem, $\Lambda_1\left(\overline{D}_{\delta_0}\right)$ is a relatively compact set. Now, let sequence $u_n \subset \overline{D}_{\delta_0}$ with $u_n \to u$. Then $u_n(t) \to u(t)$ uniformly valid on J. From the uniform continuitly of $f(t,u)$ on the compact set $J \times \overline{D}_{\delta_0}$, we have that

$$\|f(t, u_n(t)) - f(t, u(t))\| \to 0$$

is uniformly valid on J. Hence

$$\|\Lambda_1(u_n)(t) - \Lambda_1(u)(t)\| \to 0$$

as $n \to \infty$, which proves the continuity of $\Lambda_1\left(\overline{D}_{\delta_0}\right)$.

Step 2. The operator $\Lambda_2 : \overline{D}_{\delta_0} \to X$ is contractive, which is consequence of (H_2).

Step 3. The set $\Lambda_2\left(\overline{D}_{\delta_0}\right)$ is bounded. By (H_2) and (H_3), we have

$$\|\Lambda_2(u)\| \leq \overline{\rho}\left(|u_0| + \overline{K}\delta_0\right), \ u \in \overline{D}_{\delta_0},$$

which together with the fact that $\Lambda_1\left(\overline{D}_{\delta_0}\right)$ is bounded yields that $\Lambda\left(\overline{D}_{\delta_0}\right)$ is bounded.

Step 4. Finally, we shown that the case (II) in Lemma 2.7 does not hold.

On the contrary, we suppose that (II) holds. Then, we have that there exist $\mu\in(0,1)$ and $u\in\partial D_{\delta_0}$ such that $u=\mu\phi(u)$. So, $\|u\|=\delta_0$. Using the hypothesis (H_3) and $(H_4)-(H_5)$, we have

$$\begin{aligned}
|u(t)| \leq\ & \mu\Bigg\{|f(t,u)|+\sum_{i=1}^{n}|P_i|\int_0^t\frac{(t-s)^{\alpha+\sigma_i-1}}{\Gamma(\alpha+\sigma_i)}|g_i(s,u(s))|\,ds \\
&+\frac{t}{|\Delta|}\Bigg[\sum_{j=1}^{m}|Q_j|\Bigg[\int_0^{\xi_j}\frac{(\xi_j-s)^{\delta_j-1}}{\Gamma(\delta_j)}|f(s,u(s))|\,ds \\
&+\sum_{i=1}^{n}|P_i|\int_0^{\xi_j}\frac{(\xi_j-s)^{\alpha+\delta_j+\sigma_i-2}}{\Gamma(\alpha+\delta_j+\sigma_i-1)}|g_i(s,u(s))|\,ds\Bigg] \\
&+\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}|f(s,u(s))|\,ds \\
&+\sum_{i=1}^{n}|P_i|\int_0^T\frac{(T-s)^{\alpha+\beta+\sigma_i-2}}{\Gamma(\alpha+\beta+\sigma_i-1)}|g_i(s,u(s))|\,ds\Bigg] \\
&+|f(0,u(0))|\left[1+\frac{t}{|\Delta|}\left(\left|\sum_{j=1}^{m}|Q_j|\frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right)\right] \qquad (3.6)\\
&+\left[1+\frac{t}{|\Delta|}\left(\left|\sum_{j=1}^{m}|Q_j|\frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right)\right](u_0+h(u(t)))\Bigg\}. \\
\leq\ & \Bigg[\frac{t}{|\Delta|}\left(\sum_{j=1}^{m}|Q_j|\left(\int_0^{\xi_j}\frac{(\xi_j-s)^{\delta_j-1}}{\Gamma(\delta_j)}\gamma_1(s)\,ds\right)\right. \\
&\left.+\int_0^T\frac{(T-s)^{\beta-1}}{\Gamma(\beta)}\gamma_1(s)\,ds+\gamma_1(s)\right]\psi_1(\|u\|) \\
&+\sum_{i=1}^{n}|P_i|\Bigg[\frac{t}{|\Delta|}\sum_{j=1}^{m}|Q_j|\int_0^{\xi_j}\frac{(\xi_j-s)^{\alpha+\delta_j+\sigma_i-2}}{\Gamma(\alpha+\delta_j+\sigma_i-1)}\gamma_{i+1}(s)\,ds \\
&+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\int_0^s\frac{(s-\tau)^{\sigma_i-1}}{\Gamma(\sigma_i)}\gamma_{i+1}(s)\,d\tau\right)ds \\
&+\int_0^T\frac{(T-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\left(\int_0^s\frac{(s-\tau)^{\sigma_i-1}}{\Gamma(\sigma_i)}\gamma_{i+1}(s)\,d\tau\right)ds\Bigg]\psi_{i+1}(\|u\|) \\
&+|f(0,u(0))|\left(1+\frac{T}{|\Delta|}\left|\sum_{j=1}^{m}|Q_j|\frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right) \\
&\left(1+\frac{T}{|\Delta|}\left|\sum_{j=1}^{m}|Q_j|\frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)}-\frac{T^{\beta}}{\Gamma(\beta+1)}\right|\right)(|u_0|+\overline{K}\delta_0).
\end{aligned}$$

Taking the supremum over $t \in [0,T]$, and using the definition of $\bar{D}_{\delta_0}$, we obtain

$$\begin{aligned}
\delta_0 \leq & \left(\frac{T}{|\Delta|} \sum_{j=1}^{m} |Q_j| \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} + \frac{T^\beta}{\Gamma(\beta+1)} + 1 \right) \|\gamma_1\| \psi_1(\delta_0) \\
& + |f(0,u(0))| \left(1 + \tfrac{T}{|\Delta|} \left| \sum_{j=1}^{m} |Q_j| \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} - \frac{T^\beta}{\Gamma(\beta+1)} \right| \right) \\
& + \sum_{i=1}^{n} \|\gamma_{i+1}\| \psi_{i+1}(\delta_0) |P_i| \\
& \times \left(\frac{T}{|\Delta|} \sum_{j=1}^{m} |Q_j| \frac{\xi_j^{\alpha+\delta_j+\sigma_i-1}}{\Gamma(\alpha+\delta_j+\sigma_i)} + \frac{T^{\alpha+\sigma_i-1}}{\Gamma(\alpha+\sigma_i)} + \frac{T^{\alpha+\beta+\sigma_i-1}}{\Gamma(\alpha+\beta+\sigma_i)} \right) \\
& + \left(1 + \frac{T}{|\Delta|} \left| \sum_{j=1}^{m} |Q_j| \frac{\xi_j^{\delta_j}}{\Gamma(\delta_j+1)} - \frac{T^\beta}{\Gamma(\beta+1)} \right| \right) \left(|u_0| + \overline{K}\delta_0 \right),
\end{aligned}$$

which implies

$$\delta_0 \leq \sum_{i=1}^{n+1} L_i \psi(\delta_0) + \delta_0 \overline{K}\overline{\rho} + \overline{\rho}\left(|u_0| + |f(0,u(0))|\right).$$

However,

$$\frac{\delta_0}{\sum_{i=1}^{n+1} L_i \psi_i(\delta) + \overline{\rho}(|u_0| + |f(0,u(0))|)} \leq \frac{1}{1-\overline{\rho}\overline{K}},$$

which contradicts (H_5). Consequently, we have proved that Λ_1 and Λ_2 satisfy all the conditions in Lemma 2.7. Hence, Λ has at least one fixed point $u \in \bar{D}$, which is the solution of fractional boundary value problem (1.1). In view of Lemma 2.7, we can transform problem (1.1) into an equivalent fixed point problem $\Lambda u = u$. Observe that the existence of a fixed point for the operator Λ implies the existence of a solution to fractional problem (1.1). □

4. Examples

To illustrate our main results, we give the following examples.

Example 4.1. Let us consider the following problem:

$$\begin{cases} D^{\frac{4}{3}}\left(u(t) - f(t,u(t))\right) = \sum_{i=1}^{2} P_i I^{\sigma_i} g_i(t,u(t)), \; t \in [0,1], \\ u(0) = \frac{1}{11} + \frac{\sqrt{3}}{8} u(t), \; I^\beta u(1) = Q I^\delta u(\xi), 0 < \xi < 1. \end{cases} \tag{4.1}$$

For this example, we have $\alpha = \frac{4}{3}, \sigma_1 = \frac{5}{2}, \sigma_2 = \frac{7}{3}, \beta = \frac{3}{2}, \delta = \frac{7}{4}, P_1 = \frac{1}{12}, P_2 = \frac{3}{17}, Q = \frac{2}{9}, \xi = \frac{1}{4}$,

$$f(t,u(t)) = \frac{(e^{-t}+1)\sin(u-1)}{20(t+1)},$$

$$g_1(t,u(t)) = \frac{e^{-\pi t}|u(t)|}{\left(19\sqrt{\pi}+2\right)(1+|u(t)|)},$$

$$g_2(t,u(t)) = \frac{\left(1-\frac{t}{3}\right)u(t)}{25\pi + t^2 + \cos^2 u(t)} + \frac{1+e^{t+1}}{5\pi+1},$$

and

$$h(u(t)) = \frac{\sqrt{3}}{8}u(t).$$

For $u,v \in \mathbb{R}, t \in [0,1]$, we have

$$|f(t,u(t)) - f(t,v(t))| \leq \tfrac{1}{10}|u-v|,$$

$$|g_1(t,u(t)) - g_1(t,v(t))| \leq \tfrac{1}{19\sqrt{\pi}+2}|u-v|,$$

$$|g_2(t,u(t)) - g_2(t,v(t))| \leq \frac{1}{25\pi}|u-v|,$$

$\Delta = 0.29979$, $\rho_1 = 1.7930$, $\rho_2 = 1.9133 \times 10^{-2}$, $\rho_3 = 4.9693 \times 10^{-2}$, and $\overline{\rho} = 3.4685$. It is clear that

$$\sum_{i=1}^{n+1} K_i\rho_i + \overline{K\rho} = 0.93142 < 1.$$

Hence, all the hypotheses of Theorem 3.1 are satisfied. Thus, problem (4.1) has a unique solution on $[0,1]$.

Example 4.2. Consider the following problem:

$$\begin{cases} D^{\alpha}(u(t) - f(t,u(t))) = \sum\limits_{i=1}^{2} P_i I^{\sigma_i} g_i(t,u(t)),\ t \in [0,1], \\ x(0) = \frac{1}{11\pi^2} + \frac{1}{17e^{2\pi}}u(t),\ I^{\beta}u(1) = \sum\limits_{j=1}^{2} Q_j I^{\delta_j} u(\xi_j),\ 0 < \xi_j < 1. \end{cases} \tag{4.2}$$

Now, we need to verify that all the conditions are satisfied. We have $\alpha = \frac{7}{6}$, $\sigma_1 = \frac{3}{4}$, $\sigma_2 = \frac{5}{7}$,

$\beta = \frac{4}{3}$, $\delta_1 = \frac{6}{7}$, $\delta_2 = \frac{2}{5}, P_1 = \frac{2}{3\pi}, P_2 = \frac{1}{15}, Q_1 = \frac{1}{21}, Q_2 = \frac{1}{22}, \xi_1 = \frac{3}{4}, \xi_2 = \frac{2}{7}$,

$$f(t,u(t)) = \frac{t\sin(u-1)}{(27e^{3t}+1)(t^2+1)},$$

$$g_1(t,u(t)) = \frac{te^{-3t}|u(t)+2|}{(17e^{t}+2)(1+|u(t)|)},$$

$$g_2(t,u(t)) = \frac{(1+t)u(t)}{13e^{\pi+t}+1+\sin^2 u(t)},$$

and

$$h(u(t)) = \frac{1}{17e^{2\pi}}u(t).$$

For all $u \in \mathbb{R}, t \in [0,1]$, we have

$$|f(t,u(t))| \leq \tfrac{1}{28}(||u||+1),$$

$$|g_1(t,u(t))| \leq \tfrac{1}{19}(||u||+2),$$

and

$$|g_2(t,u(t))| \leq \tfrac{2}{13e^{\pi}}\|u\|,$$

where $\psi_1(||u||) = ||u|| + 1$, $\psi_2(||u||) = ||u|| + 2$, $\psi_3(||u||) = ||u||$ and $\Delta = 0.33776, L_1 = 7.3143 \times 10^{-2}$, $L_2 = 1.6745 \times 10^{-2}$, $L_3 = 6.7833 \times 10^{-4}, \overline{\rho} = 3.2783$, Clearly,

$$\delta > \frac{L_1 + 2L_2 + \overline{\rho}\left(|u_0| + |f(0, u(0))|\right)}{1 - \left(\overline{\rho}\overline{K} + L_1 + L_2 + L_3\right)} = 0.15091.$$

Hence, the conditions in Theorem 3.2 are satisfied, So, boundary value problem (4.2) has at least one solution on $[0,1]$.

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