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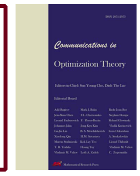

MULTI-TERM FRACTIONAL q-DIFFERENCE EQUATIONS WITH q-INTEGRAL BOUNDARY CONDITIONS VIA TOPOLOGICAL DEGREE THEORY

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Abstract. In this paper, we discuss the existence of solutions for a new class of multi-term nonlinear fractional q-difference equations involving q-derivative of the Caputo sense. The main results are obtained by employing the topological degree for condensing maps via a priori estimate method and the Banach contraction principle fixed point theorem. Finally, two illustrations examples are presented to show the validity and efficacity of the obtained results.

Keywords. Multi-term fractional differential equations; Fractional q-derivative; Fractional q-integral; Topological degree theory; Condensing maps.

1. Introduction

Recently, the fractional calculus received much attention due to its vast applications in the mathematical modeling of the genuine world phenomena occurring in scientific and engineering disciplines. Fractional-order operators give rise to more realistic models than the ones relying on classical calculus due to their nonlocal nature, which takes into account the past effects of the phenomena under consideration. The applications of fractional calculus appear in numerous fields, such, as continuum mechanics [1], bioengineering [2], financial and economy dynamics [3], physics [4], ecology [5], disease models [6].

Since the pioneering works on fractional q-difference equations presented in [7, 8] in nineteenth century, many authors have considered them [9, 10]. In particular, we refer to [9, 11–17] for recent results on initial and boundary value problems of q-difference and fractional q-difference equations.

The topological methods were proved to be a powerful tool in the study of various problems which appear in nonlinear analysis. In particular, the a priori estimate method (or the method of a priori bounds) has been often used in order to prove the existence of solutions for some boundary value problems for nonlinear differential equations or nonlinear partial differential

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equations. Recently, Isaia [18] proved a new fixed theorem that was obtained via the coincidence degree theory for condensing maps. To see more applications of the coincidence degree theory for condensing maps in the study for the existence of solutions of certain integral equations, the reader can be referred to [18–25] and the references therein.

In this paper, we introduce and study a nonlinear nonlocal-terminal value problem consisting of triple Caputo fractional q-derivatives and Riemann-Liouville q-integral boundary conditions. Precisely, we investigate the following problem:

$$\begin{cases} {}^{C}\mathscr{D}_q^{\alpha}\left[{}^{C}\mathscr{D}_q^{\beta}\left({}^{C}\mathscr{D}_q^{\gamma}u(t)-h(t,u(t))\right)-g(t,u(t))\right]=f(t,u(t)),\ t\in \mathrm{J}:=[0,T], \\ \\ \mathscr{D}_q^{\gamma}u(0)=u(0)=0, \quad au(\eta)+bu(T)=\sum_{i=1}^{m}\lambda_i\mathscr{I}_q^{\sigma_i}u(\xi_i),\, 0<\eta,\xi_i<T,\ \sigma_i>0. \end{cases} \tag{1.1}$$

where $\mathscr{D}_q^{z}$ is the fractional $q-$derivative of the Riemann-Liouville type $z\in\{\alpha,\beta,\gamma\}$ with $0<z\leq 1$ and $f,g,h: J\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

The rest of the paper is arranged as follows. In Section 2, we recall some basic concepts of fractional calculus and prove a lemma relating the problem in Equation (1.1) with an integral equation. In Section 3, based on the coincidence degree theory for condensing maps, we establish a theorem on the existence of solutions for problem (1.1) by using the Banach contraction principle. We also give a uniqueness results for problem (1.1). Finally, Section 4 provides illustrative examples for the main results.

2. Preliminaries

We start this section by introducing some necessary definitions and basic results.

Consider the space of real and continuous functions $\mathscr{C}=C([0,T],\mathbb{R})$ with topological norm

$$\|u\|_{\infty}=\sup\{|u(t)|,t\in \mathrm{J}\},\quad \forall u\in\mathscr{C}.$$

$\mathfrak{M}_{\mathscr{C}}$ represents the class of all bounded mappings in $\mathscr{C}$. Let us recall some definitions and properties about q-derivative and q-integral. These details can be found in the recent literature [6, 26–29]. For $a\in\mathbb{R}$, we set

$$[a]_q=\frac{1-q^a}{1-q}.$$

The q-analogue of the power $((a-b)^n)$ is

$$(a-b)^{(0)}=1,\qquad (a-b)^{(n)}=\prod_{k=0}^{n-1}\left(a-bq^k\right),\quad a,b\in\mathbb{R},n\in\mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)}=a^{\alpha}\prod_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right),\quad a,b,\alpha\in\mathbb{R}.$$

Definition 2.1. [27] The q-gamma function is defined by

$$\Gamma_q(\alpha)=\frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}},\quad \alpha\in\mathbb{R}-\{0,-1,-2,\dots\}.$$

Notice that the q-gamma function satisfies $(\Gamma_q(1+\alpha)=[\alpha]_q\Gamma_q(\alpha))$.

Definition 2.2. [27] For any $\alpha, \beta > 0$, the q-beta function is defined by

$$B_q(\alpha,\beta) = \int_0^1 f^{(\alpha-1)}(1-qf)^{(\beta-1)} d_q f, \quad q \in (0,1),$$

where the expression of q-beta function in terms of the q-gamma function is

$$B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$$

Definition 2.3. [27] The q-derivative of order $n \in \mathbb{N}$ of a function f is defined by $D_q^0 f(t) = f(t)$,

$$D_q f(t) := D_q^1 f(t) = \frac{f(t)-f(qt)}{(1-q)t}, \quad t \neq 0, \qquad D_q f(0) = \lim_{t\to 0} D_q f(t),$$

and

$$D_q^n f(t) = D_q D_q^{n-1} f(t), \quad t \in I, n \in \{1,2,\ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.4. [27] The q-integral of a function $f : I_t \to \mathbb{R}$ is defined by

$$I_q f(t) = \int_0^t f(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f\big(tq^n\big),$$

provided that the series converges.

We note that $D_q I_q f(t) = f(t)$. If f is continuous at 0, then

$$I_q D_q f(t) = f(t) - f(0).$$

Definition 2.5. [4] The fractional q-integral of the Riemann–Liouville type of order $\alpha \geq 0$ of a function $f : I \to \mathbb{R}$ is defined by $I_q^0 f(t) = f(t)$, and

$$I_q^\alpha f(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s)\, d_q s, \quad t \in I.$$

Lemma 2.6. *[28] Let $\alpha, \beta \geq 0$ and let f be a function defined on $[0,1]$. Then*

$$I_q^\alpha I_q^\beta f(t) = I_q^{\alpha+\beta} f(t) = I_q^\beta I_q^\alpha f(t),$$

and

$$I_q^\alpha t^\beta = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\alpha+\beta+1)} t^{\alpha+\beta}, \;\; \beta \in (-1,\infty), \; \alpha \geq 0, \; t > 0$$

In particular, if $f \equiv 1$, then

$$I_q^\alpha 1(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}, \quad \textit{for all } t > 0.$$

Definition 2.7. [29] The fractional q-derivative of the Riemann–Liouville type of order $\alpha \geq 0$ of a function $f : I \to \mathbb{R}$ is defined by $(D_q^0 f)(t) = f(t)$, and

$$\begin{aligned} D_q^\alpha f(t) &= D_q^{[\alpha]} I_q^{[\alpha]-\alpha} f(t) \\ &= \frac{1}{\Gamma_q(n-\alpha)} \int_0^t \frac{f(s)}{(t-qs)^{\alpha-n+1}} d_q s. \end{aligned}$$

where $[\alpha]$ is the integer part of α.

Lemma 2.8. *[26] Let $\alpha > 0$ and $n \in \mathbb{N}$. Then, the following equality holds*

$$I_q^{\alpha} D_q^n f)(t) = D_q^n I_q^{\alpha} f(t) - \sum_{k=0}^{\alpha-1} \frac{t^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (D_q^k f)(0).$$

Definition 2.9. [29] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $f : I \to \mathbb{R}$ is defined by ${}^C D_q^0 u(t) = u(t)$ and

$${}^C D_q^{\alpha} u(t) = I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u(t), \quad t \in I.$$

Lemma 2.10. *[29] Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:*

$$I_q^{\alpha C} D_q^{\alpha} u(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} D_q^k u(0).$$

In particular, if $\alpha \in (0,1)$, then

$$I_q^{\alpha C} D_q^{\alpha} u(t) = u(t) - u(0).$$

We state here the results given below from [30, 31].

Definition 2.11. The mapping $\kappa : \mathfrak{M}_{\mathscr{C}} \to [0,\infty)$ for Kuratowski measure of non-compactness is defined as:

$$\kappa(B) = \inf\Big\{\varepsilon > 0 : B \text{ can be covered by finitely many sets with diameter} \leq \varepsilon\Big\}.$$

Proposition 2.12. *The Kuratowski measure of noncompactness satisfies some properties.*

- $A \subset B \Rightarrow \kappa(A) \leq \kappa(B)$,
- $\kappa(A) = 0$ *if and only if A is relatively compact,*
- $\kappa(A) = \kappa(\overline{A}) = \kappa(conv(A))$, *where $\overline{A}$ and* $\text{conv}(A)$ *represent the closure and the convex hull of A respectively,*
- $\kappa(A+B) \leq \kappa(A) + \kappa(B)$,
- $\kappa(\lambda A) = |\lambda| \kappa(A), \lambda \in \mathbb{R}$.

Definition 2.13. Let $\mathscr{T} : A \longrightarrow \mathscr{C}$ be a continuous bounded map and $A \subset \mathscr{C}$. The operator $\mathscr{T}$ is said to be κ-Lipschitz if we can find a constant $\ell \geq 0$ such that

$$\kappa(\mathscr{A}(B)) \leq \ell \kappa(B), \text{ for every } B \subset A.$$

Moreover, $\mathscr{A}$ is called a strict κ-contraction provided $\ell < 1$.

Definition 2.14. The function $\mathscr{A}$ is said to be κ-condensing if

$$\kappa(\mathscr{A}(B)) < \kappa(B),$$

for every bounded and nonprecompact subset B of A. That is,

$$\kappa(\mathscr{A}(B)) \geq \kappa(B), \text{ implies } \kappa(B) = 0.$$

Further, we have that $\mathscr{A} : A \longrightarrow \mathscr{C}$ is Lipschitz if there exists $\ell > 0$ such that

$$\|\mathscr{A}(u) - \mathscr{A}(v)\| \leq \ell \|u - v\|, \text{ for all } u, v \in A,$$

if $\ell < 1$, $\mathscr{A}$ is said to be strict contraction.

From [18], we have the following propositions.

Proposition 2.15. *If $\mathscr{A},\mathscr{B}: A\longrightarrow\mathscr{C}$ are κ-Lipschitz with constants ℓ_1 and ℓ_2, respectively, then $\mathscr{A}+\mathscr{B}: A\longrightarrow\mathscr{C}$ are κ-Lipschitz with constants $\ell_1+\ell_2$.*

Proposition 2.16. *If $\mathscr{A}: A\longrightarrow\mathscr{C}$ is compact, then $\mathscr{A}$ is κ-Lipschitz with constant $\ell=0$.*

Proposition 2.17. *If $\mathscr{A}: A\longrightarrow\mathscr{C}$ is Lipschitz with constant ℓ, then $\mathscr{A}$ is κ-Lipschitz with the same constant ℓ.*

Isaia [18] present the following results based on topological degree theory.

Theorem 2.18. *Let $\mathscr{F}: A\longrightarrow\mathscr{C}$ be κ-condensing and*

$$\Theta=\{u\in\mathscr{C}:\ \text{there exist } \varsigma\in[0,1] \text{ such that } x=\varsigma\mathscr{F}u\}.$$

If Θ is a bounded set in $\mathscr{C}$, then there exists $r>0$ such that $\Theta\subset \mathrm{B}_r(0)$, and the degree

$$\deg(I-\varsigma\mathscr{F},\mathrm{B}_r(0),0)=1, \text{ for all } \ \varsigma\in[0,1].$$

Consequently, $\mathscr{F}$ has at least one fixed point and the set of the fixed points of $\mathscr{F}$ lies in $\mathrm{B}_r(0)$.

3. Main results

To establish our main results for problem (1.1), we need the following lemma.

Lemma 3.1. *For a given $h\in C(\mathrm{J},\mathbb{R})$, the unique solution of the linear fractional boundary value problem*

$$\begin{cases} {}^{C}\mathscr{D}_q^{\alpha}\left[{}^{C}\mathscr{D}_q^{\beta}\left({}^{C}\mathscr{D}_q^{\gamma}u(t)-h(t,u(t))\right)-g(t,u(t))\right]=f(t,u(t)),\ t\in\mathrm{J}:=[0,T],\\ \mathscr{D}_q^{\gamma}u(0)=u(0)=0, au(\eta)+bu(T)=\sum_{i=1}^{m}\lambda_i\mathscr{I}_q^{\sigma_i}u(\xi_i), 0<\eta,\xi_i<T,\ \sigma_i>0,\end{cases} \tag{3.1}$$

is given by

$$\begin{aligned} u(t)&=\mathscr{I}_q^{\gamma}h_u(t)+\mathscr{I}_q^{\beta+\gamma}g_u(t)+\mathscr{I}_q^{\alpha+\beta+\gamma}f_u(t)\\ &\frac{t^{\beta+\gamma}}{\Lambda}\left\{\sum_{i=0}^{m}\lambda_i\mathscr{I}_q^{\gamma+\sigma_i}h_u(\xi_i)+\sum_{i=0}^{m}\lambda_i\mathscr{I}_q^{\beta+\gamma+\sigma_i}g_u(\xi_i)+\sum_{i=0}^{m}\lambda_i\mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i}f_u(\xi_i)\right.\\ &-a\mathscr{I}_q^{\gamma}h_u(\eta)-a\mathscr{I}_q^{\beta+\gamma}g_u(\eta)-a\mathscr{I}_q^{\alpha+\beta+\gamma}f_u(\eta)\\ &\left.-b\mathscr{I}_q^{\gamma}h_u(T)-b\mathscr{I}_q^{\beta+\gamma}g_u(T)-b\mathscr{I}_q^{\alpha+\beta+\gamma}f_u(T)\right\},\end{aligned} \tag{3.2}$$

where

$$\Omega=\left(\frac{a\eta^{\beta+\gamma}}{\Gamma_q(\beta+\gamma+1)}+\frac{bT^{\beta+\gamma}}{\Gamma_q(\beta+\gamma+1)}+\sum_{i=0}^{m}\lambda_i\frac{\xi^{\beta+\gamma+\sigma_i}}{\Gamma_q(\beta+\gamma+\sigma_i+1)}\right) \tag{3.3}$$

and

$$\Lambda=\Omega\Gamma_q(\beta+\gamma+1)=\left(a\eta^{\beta+\gamma}+bT^{\beta+\gamma}+\sum_{i=0}^{m}\lambda_i\frac{\Gamma_q(\beta+\gamma+1)\xi^{\beta+\gamma+\sigma_i}}{\Gamma_q(\beta+\gamma+\sigma_i+1)}\right). \tag{3.4}$$

Proof. Applying the operator $\mathscr{I}_q^{\alpha},\mathscr{I}_q^{\beta}$ and $\mathscr{I}_q^{\gamma}$ on both sides of the fractional differential equation in Equation (3.1), we see that (3.1) is reduced to an equivalent integral equation

$$u(t)=\mathscr{I}_q^{\gamma}h_u(t)+\mathscr{I}_q^{\beta+\gamma}g_u(t)+\mathscr{I}_q^{\alpha+\beta+\gamma}f_u(t)+k_1\frac{t^{\beta+\gamma}}{\Gamma_q(\beta+\gamma+1)}+k_2\frac{t^{\gamma}}{\Gamma_q(\gamma+1)}+k_3, \tag{3.5}$$

where $k_1, k_2, k_3 \in \mathbb{R}$. Using the first boundary conditions of the problem (3.1) in (3.5), we obtain $k_2 = k_3 = 0$. Therefore,

$$u(t) = \mathscr{I}_q^{\gamma} h_u(t) + \mathscr{I}_q^{\beta+\gamma} g_u(t) + \mathscr{I}_q^{\alpha+\beta+\gamma} f_u(t) + k_1 \frac{t^{\beta+\gamma}}{\Gamma_q(\beta+\gamma+1)}, \quad k_1 \in \mathbb{R}. \tag{3.6}$$

By the second condition of (3.1), we get

$$\begin{aligned} au(\eta) &= a\mathscr{I}_q^{\gamma} h_u(\eta) + a\mathscr{I}_q^{\beta+\gamma} g_u(\eta) + a\mathscr{I}_q^{\alpha+\beta+\gamma} f_u(\eta) + k_1 \frac{a\eta^{\beta+\gamma}}{\Gamma_q(\beta+\gamma+1)}, \\ bu(T) &= b\mathscr{I}_q^{\gamma} h_u(T) + b\mathscr{I}_q^{\beta+\gamma} g_u(T) + b\mathscr{I}_q^{\alpha+\beta+\gamma} f_u(T) + k_1 \frac{bT^{\beta+\gamma}}{\Gamma_q(\beta+\gamma+1)}, \\ \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\sigma_i} u(\xi_i) &= \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\gamma+\sigma_i} h_u(\xi_i) + \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\beta+\gamma+\sigma_i} g_u(\xi_i), \\ &\quad + \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i} f_u(\xi_i) + k_1 \sum_{i=0}^{m} \lambda_i \frac{\xi^{\beta+\gamma+\sigma_i}}{\Gamma_q(\beta+\gamma+\sigma_i+1)}. \end{aligned}$$

After collecting the similar terms in one part, we have the following equations:

$$\begin{aligned} &k_1 \left(\frac{a\eta^{\beta+\gamma}}{\Gamma_q(\beta+\gamma+1)} + \frac{bT^{\beta+\gamma}}{\Gamma_q(\beta+\gamma+1)} - \sum_{i=0}^{m} \lambda_i \frac{\xi^{\beta+\gamma+\sigma_i}}{\Gamma_q(\beta+\gamma+\sigma_i+1)} \right) \\ &= \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\gamma+\sigma_i} h_u(\xi_i) + \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\beta+\gamma+\sigma_i} g_u(\xi_i) + \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i} f_u(\xi_i) \\ &- a\mathscr{I}_q^{\gamma} h_u(\eta) - a\mathscr{I}_q^{\beta+\gamma} g_u(\eta) - a\mathscr{I}_q^{\alpha+\beta+\gamma} f_u(\eta) \\ &- b\mathscr{I}_q^{\gamma} h_u(T) - b\mathscr{I}_q^{\beta+\gamma} g_u(T) - b\mathscr{I}_q^{\alpha+\beta+\gamma} f_u(T). \end{aligned} \tag{3.7}$$

Therefore,

$$\begin{aligned} k_1 &= \frac{1}{\Omega} \left\{ \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\gamma+\sigma_i} h_u(\xi_i) + \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\beta+\gamma+\sigma_i} g_u(\xi_i) + \sum_{i=0}^{m} \lambda_i \mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i} f_u(\xi_i) \right. \\ &- a\mathscr{I}_q^{\gamma} h_u(\eta) - a\mathscr{I}_q^{\beta+\gamma} g_u(\eta) - a\mathscr{I}_q^{\alpha+\beta+\gamma} f_u(\eta) \\ &- \left. b\mathscr{I}_q^{\gamma} h_u(T) - b\mathscr{I}_q^{\beta+\gamma} g_u(T) - b\mathscr{I}_q^{\alpha+\beta+\gamma} f_u(T) \right\}, \end{aligned}$$

where Ω is given by (3.3). Substituting the value of k_1, k_2, k_3 into (3.5), we get (3.2). The proof is completed. □

We now introduce the following hypotheses for use later.

(1) For each $t \in$ J, and for each, $u, v \in \mathscr{C}$, there exist constant $L_1, L_2, L_3 > 0$ such that

$$\begin{aligned} \|f(t,u) - f(t,v)\| &\leq L_1 \|u - v\|, \\ \|h(t,u) - h(t,v)\| &\leq L_2 \|u - v\|, \\ \|g(t,u) - g(t,v)\| &\leq L_3 \|u - v\|. \end{aligned} \tag{3.8}$$

(2) The functions f, g and h satisfy the following growth conditions for constants $M_i, N_i > 0$, $i = 1, 2, 3,$ where $M = \max\{M_1, M_2, M_3\}$ and $N = \max\{N_1, N_2, N_3\}$, $i = 1, 2, 3,$ $p \in$

$(0,1)$,

$$\begin{aligned}\|f(t,u)\| &\leq M_1\|u\|^p+N_1,\\ \|h(t,u)\| &\leq M_2\|u\|^p+N_2,\\ \|g(t,u)\| &\leq M_3\|u\|^p+N_3.\end{aligned} \tag{3.9}$$

for each $t\in \mathrm{J}$ and each $u\in\mathscr{C}$.

In the following, we set an abbreviated notation for the fractional q-integral of the Caputo type of order $\alpha>0$, for a function with two variables as

$$\mathscr{I}_q^{\alpha} f_u(t)=\frac{1}{\Gamma(\alpha)}\int_1^t (t-qs)^{\alpha-1} f(s,u(s))ds.$$

In view of Lemma 3.1, we consider two operators $\mathscr{A},\mathscr{B}:\mathscr{C}\longrightarrow\mathscr{C}$ as follows:

$$\mathscr{A}u(t)=\mathscr{I}_q^{\gamma}h_u(t)+\mathscr{I}_q^{\beta+\gamma}g_u(t)+\mathscr{I}_q^{\alpha+\beta+\gamma}f_u(t),\ t\in\mathrm{J}, \tag{3.10}$$

and

$$\begin{aligned}\mathscr{B}u(t)=\frac{t^{\beta+\gamma}}{\Lambda}\Big\{&\sum_{i=0}^m\lambda_i\mathscr{I}_q^{\gamma+\sigma_i}h_u(\xi_i)+\sum_{i=0}^m\lambda_i\mathscr{I}_q^{\beta+\gamma+\sigma_i}g_u(\xi_i)+\sum_{i=0}^m\lambda_i\mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i}f_u(\xi_i)\\ &-a\mathscr{I}_q^{\gamma}h_u(\eta)-a\mathscr{I}_q^{\beta+\gamma}g_u(\eta)-a\mathscr{I}_q^{\alpha+\beta+\gamma}f_u(\eta)\\ &-b\mathscr{I}_q^{\gamma}h_u(T)-b\mathscr{I}_q^{\beta+\gamma}g_u(T)-b\mathscr{I}_q^{\alpha+\beta+\gamma}f_u(T)\Big\},\ t\in\mathrm{J}.\end{aligned} \tag{3.11}$$

Then the integral equation (3.2) can be written as an operator equation

$$\mathscr{F}u(t)=\mathscr{A}u(t)+\mathscr{B}u(t),\ \ t\in\mathrm{J}, \tag{3.12}$$

The continuity of f shows that $\mathscr{F}$ is well define and fixed points of the operator equation are solutions of the integral equations (3.2) in Lemma 3.1.

Lemma 3.2. *$\mathscr{A}$ is Lipschitz with constant $\mathscr{Q}=\sum_{i=1}^3\varpi_iL_i$. Moreover, $\mathscr{A}$ satisfies the following condition*

$$\|\mathscr{A}u\|\leq\sum_{i=1}^3\varpi_i(M\|u\|^p+N),\quad \textit{for every } u\in\mathscr{C}.$$

where

$$\varpi_1=\frac{T^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)},\quad \varpi_2=\frac{T^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)},\quad \varpi_3=\frac{T^{\gamma}}{\Gamma(\gamma+1)}. \tag{3.13}$$

Proof. To show that $\mathscr{A}$ is Lipschitz with constant $\mathscr{Q}$. Letting $u,v\in\mathscr{C}$, we have

$$\begin{aligned}&|\mathscr{A}u(t)-\mathscr{A}v(t)|\\ &=\left|\mathscr{I}_q^{\gamma}f_u(t)-\mathscr{I}_q^{\gamma}f_v(t)\right|+\left|\mathscr{I}_q^{\beta+\gamma}f_u(t)-\mathscr{I}_q^{\beta+\gamma}f_v(t)\right|+\left|\mathscr{I}_q^{\alpha+\beta+\gamma}f_u(t)-\mathscr{I}_q^{\alpha+\beta+\gamma}f_v(t)\right|\\ &\leq\mathscr{I}_q^{\gamma}|f_u(t)-f_v(t)|+\mathscr{I}_q^{\beta+\gamma}|f_u(t)-f_v(t)|+\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u(t)-f_v(t)|\\ &\leq\mathscr{I}_q^{\gamma}(1)(T)L_3\|u-v\|+\mathscr{I}_q^{\beta+\gamma}(1)(T)L_2\|u-v\|+\mathscr{I}_q^{\alpha+\beta+\gamma}(1)(T)L_1\|u-v\|\\ &=\left(\frac{L_3T^{\gamma}}{\Gamma(\gamma+1)}+\frac{L_2T^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{L_1T^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}\right)\|u-v\|,\end{aligned}$$

for all $t \in$ J. Taking supremum over t, we obtain

$$\|\mathscr{A}u-\mathscr{A}v\| \leq \left(\frac{L_3 T^{\gamma}}{\Gamma(\gamma+1)}+\frac{L_2 T^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}+\frac{L_1 T^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}\right)\|u-v\|.$$

Hence, $\mathscr{A}:\mathscr{C}\longrightarrow\mathscr{C}$ is a Lipschitzian on $\mathscr{C}$ with Lipschitz constant $\mathscr{Q}$. By Proposition 2.17, we have that $\mathscr{A}$ is κ–Lipschitz with constant $\mathscr{Q}$. Moreover, we have

$$\begin{aligned}
|\mathscr{A}u(t)| &\leq \mathscr{I}_q^{\gamma}|f_u|(t)+\mathscr{I}_q^{\beta+\gamma}|f_u|(t)+\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u|(t)\\
&=\frac{T^{\gamma}}{\Gamma(\gamma+1)}(M_3\|u\|^p+N_3)+\frac{T^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)}(M_2\|u\|^p+N_2)\\
&+\frac{T^{\alpha+\beta+\gamma}}{\Gamma(\alpha+\beta+\gamma+1)}(M_1\|u\|^p+N_1),
\end{aligned}$$

Hence $\|\mathscr{A}u\| \leq \sum_{i=1}^{3}\varpi_i(M\|u\|^p+N)$. □

Lemma 3.3. *$\mathscr{B}$ is continuous and satisfies the growth condition given as below,*

$$\|\mathscr{B}u\| \leq \frac{T^{\beta+\gamma}}{|\Lambda|}\sum_{i=1}^{3}(M\|u\|^p+N)\omega_i,\ \textit{for every } u\in\mathscr{C},$$

where ω_i is defined by

$$\begin{aligned}
\omega_1 &= \frac{T^{\alpha+\gamma+\beta}}{\Gamma_q(\alpha+\gamma+\beta+1)}+\frac{T^{\alpha+\gamma+\beta}}{|\Lambda|}\left\{\sum_{i=1}^{m}\frac{\xi_i^{\alpha+\gamma+\beta+\sigma_i}}{\Gamma_q(\alpha+\gamma+\beta+\sigma_i+1)}\right.\\
&\left.+\frac{|a|\eta^{\alpha+\gamma+\beta}}{\Gamma_q(\alpha+\gamma+\beta+1)}+\frac{|b|T^{\alpha+\gamma+\beta}}{\Gamma_q(\alpha+\gamma+\beta+1)}\right\},\\
\omega_2 &= \frac{T^{\gamma+\beta}}{\Gamma_q(\gamma+\beta+1)}\\
&+\frac{T^{\gamma+\beta}}{|\Lambda|}\left\{\sum_{i=1}^{m}\frac{\xi_i^{\gamma+\beta+\sigma_i}}{\Gamma_q(\gamma+\beta+\sigma_i+1)}+\frac{|a|\eta^{\gamma+\beta}}{\Gamma_q(\gamma+\beta+1)}+\frac{|b|T^{\gamma+\beta}}{\Gamma_q(\gamma+\beta+1)}\right\},\\
\omega_3 &= \frac{T^{\gamma}}{\Gamma_q(\gamma+1)}+\frac{T^{\gamma}}{|\Lambda|}\left\{\sum_{i=1}^{m}\frac{\xi_i^{\gamma+\sigma_i}}{\Gamma_q(\gamma+\sigma_i+1)}+\frac{|a|\eta^{\gamma}}{\Gamma_q(\gamma+1)}+\frac{|b|T^{\gamma}}{\Gamma_q(\gamma+1)}\right\}
\end{aligned} \tag{3.14}$$

Proof. Choose a bounded subset $D_r=\{u\in\mathscr{C}:\|u\|\leq r\}\subset\mathscr{C}$ and consider a sequence $\{u_n\}\in D_r$ such that u_n as $n\to\infty$ in D_r. We need to show that $\|\mathscr{S}u_n-\mathscr{S}u|\to 0, n\to\infty$. From the continuity of $\mathscr{K}\in\{f,g,h\}$, it follows that $\mathscr{K}(s,u_n)\to\mathscr{K}(t,u)$, as $n\to\infty$. In view of (H_2), we obtain the following relations:

$$\begin{aligned}
&(\varepsilon-s)^{\gamma-1}\|\mathscr{K}(s,u_n(s))-\mathscr{K}(s,u(s))\| \leq (M_i\|u\|^p+N_i)(\varepsilon-s)^{\gamma-1},\\
&(\varepsilon-s)^{\gamma+\beta_i-1}\mapsto (M_i\|u\|^p+N_i)(\varepsilon-s)^{\gamma+\beta_i-1},\\
&(\varepsilon-s)^{\gamma+\beta+\alpha-1}\mapsto (M_i\|u\|^p+N_i)(\varepsilon-s)^{\gamma+\beta+\alpha-1},
\end{aligned}$$

and

$$(\xi_i - s)^{\tau-1} \mapsto (M_i\|u\|^p + N_i)\,(\xi_i - s)^{\tau_i - 1}.$$

where $\varepsilon \in \{t, \eta, T\}$ and $\tau \in \{\gamma+\sigma_i, \gamma+\beta+\sigma_i, \gamma+\beta+\alpha+\sigma_i\}$, which implies that each term on the left is integrable. By Lebesgue Dominated convergent theorem, we obtain

$$\begin{aligned}
&\mathscr{I}_q^{\gamma}|\mathscr{K}_{u_n} - \mathscr{K}_u|(\eta_i) \to 0 \ \text{ as } \ n \to +\infty,\\
&\mathscr{I}_q^{\gamma+\beta}|\mathscr{K}_{u_n} - \mathscr{K}_u|(\sigma_i) \to 0 \ \text{ as } \ n \to +\infty,\\
&\mathscr{I}_q^{\alpha+\beta+\gamma}|\mathscr{K}_{u_n} - \mathscr{K}_u|(1) \to 0 \ \text{ as } \ n \to +\infty,
\end{aligned}$$

and

$$\mathscr{I}_q^{\tau}|\mathscr{K}_{u_n} - \mathscr{K}_u|(\xi_i) \to 0 \ \text{ as } \ n \to +\infty,$$

It follows that $\|\mathscr{B}u_n - \mathscr{B}u\| \to 0$ as $n \to +\infty$, which implies the continuity of the operator $\mathscr{B}$. Using the assumption (H2) we have

$$\begin{aligned}
&|\mathscr{B}u(t)|\\
&\le \frac{t^{\beta+\gamma}}{|\Lambda|}\Bigg\{\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\gamma+\sigma_i}|h_u|(\xi_i)\\
&+\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\beta+\gamma+\sigma_i}|g_u|(\xi_i) + \sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i}|f_u|(\xi_i)\\
&+|a|\mathscr{I}_q^{\gamma}|h_u|(\eta) + |a|\mathscr{I}_q^{\beta+\gamma}|g_u|(\eta) + |a|\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u|(\eta)\\
&+|b|\mathscr{I}_q^{\gamma}|h_u|(T) + |b|\mathscr{I}_q^{\beta+\gamma}|g_u|(T) + |b|\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u|(T)\Bigg\},\\
&\le \frac{t^{\beta+\gamma}}{|\Lambda|}\Bigg\{\left(\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\gamma+\sigma_i}|h_u|(\xi_i) + |a|\mathscr{I}_q^{\gamma}|h_u|(\eta) + |b|\mathscr{I}_q^{\gamma}|h_u|(T)\right)(M_3\|u\|^p + N_3)\\
&+\left(\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\beta+\gamma+\sigma_i}|g_u|(\xi_i) + |a|\mathscr{I}_q^{\beta+\gamma}|g_u|(\eta) + |b|\mathscr{I}_q^{\beta+\gamma}|g_u|(T)\right)(M_2\|u\|^p + N_2)\\
&+\left(\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i}|f_u|(\xi_i) + |a|\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u|(\eta) + |b|\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u|(T)\right)(M_1\|u\|^p + N_1)\Bigg\},
\end{aligned}$$

Therefore,

$$\|\mathscr{B}u\| \le \frac{T^{\beta+\gamma}}{|\Lambda|}\sum_{i=1}^{3}(M\|u\|^p + N)\omega_i, \tag{3.15}$$

where ω_i is given by (3.14). This completes the proof of Lemma 3.3. □

Lemma 3.4. *The operator $\mathscr{B} : \mathscr{C} \longrightarrow \mathscr{C}$ is compact. Consequently, $\mathscr{B}$ is κ-Lipschitz with zero constant.*

Proof. In order to show that $\mathscr{B}$ is compact. Let us take a bounded set $\Omega \subset B_r$, i=1,2. We need to show that $\mathscr{B}(\Omega)$ is relatively compact in $\mathscr{C}$. For arbitrary $u_i \in \Omega \subset B_r$, with the help of the estimates (3.15), we can obtain

$$\|\mathscr{B}u\| \leq \frac{T^{\beta+\gamma}}{|\Lambda|}\sum_{i=1}^{3}(Mr^p+N)\,\omega_i,$$

where ω is given by (3.3). This shows that $\mathscr{B}(\Omega)$ is uniformly bounded. Now, for equicontinuity of $\mathscr{B}$, we take $t_1, t_2 \in \mathrm{J}$ with $t_1 < t_2$, and let $u \in \Omega$. Thus,

$$\begin{aligned}
&|\mathscr{B}u(t_2)-\mathscr{B}u(t_1)|\\
&\leq \frac{\left(t_2^{\beta+\gamma}-t_1^{\beta+\gamma}\right)}{|\Lambda|}\left\{\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\gamma+\sigma_i}|h_u|(\xi_i)+\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\beta+\gamma+\sigma_i}|g_u|(\xi_i)\right.\\
&+\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i}|f_u|(\xi_i)+|a|\mathscr{I}_q^{\gamma}|h_u|(\eta)+|a|\mathscr{I}_q^{\beta+\gamma}|g_u|(\eta)+|a|\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u|(\eta)\\
&\left.+|b|\mathscr{I}_q^{\gamma}|h_u|(T)+|b|\mathscr{I}_q^{\beta+\gamma}|g_u|(T)+|b|\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u|(T)\right\},\\
&\leq \frac{\left(t_2^{\beta+\gamma}-t_1^{\beta+\gamma}\right)}{|\Lambda|}\\
&\times\left\{\left(\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\gamma+\sigma_i}|h_u|(\xi_i)+|a|\mathscr{I}_q^{\gamma}|h_u|(\eta)+|b|\mathscr{I}_q^{\gamma}|h_u|(T)\right)(M_3\|u\|^p+N_3)\right.\\
&+\left(\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\beta+\gamma+\sigma_i}|g_u|(\xi_i)+|a|\mathscr{I}_q^{\beta+\gamma}|g_u|(\eta)+|b|\mathscr{I}_q^{\beta+\gamma}|g_u|(T)\right)(M_2\|u\|^p+N_2)\\
&\left.+\left(\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i}|f_u|(\xi_i)+|a|\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u|(\eta)+|b|\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u|(T)\right)(M_1\|u\|^p+N_1)\right\},\\
&\leq \frac{\left(t_2^{\beta+\gamma}-t_1^{\beta+\gamma}\right)}{|\Lambda|}\sum_{i=1}^{3}\omega_i\,(M_i\|u\|^p+N_i)\,,
\end{aligned}$$

which implies that

$$|\mathscr{B}u(t_2)-\mathscr{B}u(t_1)| \leq \frac{\left(t_2^{\beta+\gamma}-t_1^{\beta+\gamma}\right)}{|\Lambda|}\sum_{i=1}^{3}\omega_i\,(M\|u\|^p+N)\,,$$

where ω_i is given by (3.14). We deduce that $\|\mathscr{B}u(t_2)-\mathscr{S}u(t_1)\| \to 0$ as $t_2 \to t_1$. Consequently, $\mathscr{B}$ is equicontinuous. Thus, by Ascoli–Arzelà theorem, the operator $\mathscr{B}$ is compact. Using Proposition 2.16, we have that $\mathscr{B}$ is κ–Lipschitz with zero constant. □

Theorem 3.5. *Assume that (H1)–(H2) are satisfied. Then Problem* (1.1) *has at least one solution $u \in C(\mathrm{J},\mathbb{R})$ provided that $\mathscr{Q} < 1$ and the set of the solutions is bounded in $C(\mathrm{J},\mathbb{R})$.*

Proof. Let $\mathscr{A}, \mathscr{B}, \mathscr{F}$ be the operators defined by (3.10), (3.11) and (3.12), respectively. The operators are continuous and bounded. Moreover, using Lemma 3.2, we have that $\mathscr{A}$ is κ–Lipschitz with constant $\mathscr{Q}$. Using Lemma 3.4, we see that $\mathscr{S}$ is κ–Lipschitz with constant 0.

Thus, $\mathscr{F}$ is κ–Lipschitz with constant ℓ_f. Hence $\mathscr{F}$ is strict κ–contraction with constant $\mathscr{Q}$. Since $\mathscr{Q} < 1$, we have that $\mathscr{F}$ is κ-condensing.

Now, we consider the following set

$$\Theta = \{u \in \mathscr{C} : \text{ there exist } \varsigma \in [0,1] \text{ such that } x = \varsigma \mathscr{F} u\}.$$

We will show that the set Θ is bounded. For $u \in \Theta$, we have $u = \varsigma \mathscr{F} u = \varsigma(\mathscr{A}(u) + \mathscr{B}(u))$, which implies that

$$\begin{aligned} \|u\| &\leq \varsigma(\|\mathscr{A} u\| + \|\mathscr{B} u\|) \\ &\leq \sum_{i=1}^{3} \left[\frac{T^{\beta+\gamma}}{|\Lambda|} \omega_i + \varpi_i \right] (M\|u\|^p + N), \end{aligned}$$

where ω_i and ϖ_i are given by (3.14) and (3.13), respectively, for $i = 1,2,3$. From the above inequalities, we conclude that Θ is bounded in $C(\mathrm{J},\mathbb{R})$. If it is not bounded, then we divide the above inequality by $a := \|u\|$ and let $a \to \infty$ to obtain

$$\begin{aligned} 1 &\leq \sum_{i=1}^{3} \left[\frac{T^{\beta+\gamma}}{|\Lambda|} \omega_i + \varpi_i \right] \lim_{a\to\infty} \frac{Ma^p + N}{a} \\ &= 0, \end{aligned}$$

which is a contradiction. Thus the set Θ is bounded and the operator $\mathscr{F}$ has at least one fixed point which represent the solution of problem (1.1). This completes the proof. □

Now, we give an existence and uniqueness result by Banach contraction principle.

Theorem 3.6. *Under hypothesis (H1), problem* (1.1) *has a unique solution if*

$$\sum_{i=1}^{3} \left[\frac{T^{\beta+\gamma}}{|\Lambda|} \omega_i + \varpi_i \right] L_i < 1. \tag{3.16}$$

Proof. Letting $u, v \in C(\mathrm{J},\mathbb{R})$ and $t \in \mathrm{J}$, we have

$$
\begin{aligned}
|\mathscr{F}u(t)-\mathscr{F}v(t)| \leq\ & \mathscr{I}_q^{\gamma}|h_u-h_v|(t)+\mathscr{I}_q^{\beta+\gamma}|g_u-g_v|(t)+\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u-f_v|(t) \\
& +\frac{t^{\beta+\gamma}}{\Lambda}\Bigg\{\sum_{i=0}^{m}\lambda_i\mathscr{I}_q^{\gamma+\sigma_i}|h_u-h_v|(\xi_i) \\
& +\sum_{i=0}^{m}\lambda_i\mathscr{I}_q^{\beta+\gamma+\sigma_i}|g_u-g_v|(\xi_i)+\sum_{i=0}^{m}\lambda_i\mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i}|f_u-f_v|(\xi_i) \\
& -a\mathscr{I}_q^{\gamma}|h_u-h_v|(\eta)-a\mathscr{I}_q^{\beta+\gamma}|g_u-g_v|(\eta)-a\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u-f_v|(\eta) \\
& -b\mathscr{I}_q^{\gamma}|h_u-h_v|(T)-b\mathscr{I}_q^{\beta+\gamma}|g_u-g_v|(T)-b\mathscr{I}_q^{\alpha+\beta+\gamma}|f_u-f_v|(T)\Bigg\} \\
& \leq L_3\|u-v\|\Bigg\{\mathscr{I}_q^{\gamma}(1)(T) \\
& +\frac{t^{\beta+\gamma}}{|\Lambda|}\Bigg\{\Bigg(\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\gamma+\sigma_i}(1)(\xi_i)+|a|\mathscr{I}_q^{\gamma}(1)(\eta)+|b|\mathscr{I}_q^{\gamma}(1)(T)\Bigg)\Bigg\}\Bigg\} \\
& +L_2\|u-v\|\Bigg\{\mathscr{I}_q^{\beta+\gamma}(1)(T)+\frac{t^{\beta+\gamma}}{|\Lambda|} \\
& \times\Bigg\{\Bigg(\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\beta+\gamma+\sigma_i}(1)(\xi_i)+|a|\mathscr{I}_q^{\beta+\gamma}(1)(\eta)+|b|\mathscr{I}_q^{\beta+\gamma}(1)(T)\Bigg)\Bigg\}\Bigg\} \\
& +L_1\|u-v\|\Bigg\{\mathscr{I}_q^{\alpha+\beta+\gamma}(1)(T)+\frac{t^{\beta+\gamma}}{|\Lambda|} \\
& \times\Bigg\{\Bigg(\sum_{i=0}^{m}|\lambda_i|\mathscr{I}_q^{\alpha+\beta+\gamma+\sigma_i}(1)(\xi_i)+|a|\mathscr{I}_q^{\alpha+\beta+\gamma}(1)(\eta)+|b|\mathscr{I}_q^{\alpha+\beta+\gamma}(1)(T)\Bigg)\Bigg\}\Bigg\} \\
& =\sum_{i=1}^{3}\left[\frac{T^{\beta+\gamma}}{|\Lambda|}\omega_i+\varpi_i\right]L_i\|u-v\|.
\end{aligned}
$$

where ω_i and ϖ_i are given by (3.14) and (3.13), respectively for $i=1,2,3$. In view of the given condition $\sum_{i=1}^{3}\left[\frac{T^{\beta+\gamma}}{|\Lambda|}\omega_i+\varpi_i\right]L_i<1$, it follows that the mapping $\mathscr{F}$ is a contraction. Hence, by the Banach fixed point theorem, $\mathscr{F}$ has a unique fixed point, which is a unique solution of problem (1.1). The proof is completed. □

Remark 3.7. If the growth condition (H2) is formulated for $p=1$, then the conclusions of Theorem 3.5 remain valid provided that

$$\sum_{i=1}^{3}\left[\frac{T^{\beta+\gamma}}{|\Lambda|}\omega_i+\varpi_i\right]M<1.$$

4. Examples

In this section, in order to illustrate our results, we consider the following example.

Example 4.1. Let us consider the system (1.1) with specific data:

$$\begin{aligned} &T=1, && m=1, && a=1, && b=1,\\ &\alpha=\frac{3}{4}, && \beta=\frac{2}{3}, && \gamma=\frac{2}{5}, && \sigma_1=\frac{3}{2},\\ &\lambda_1=1, && \xi_1=\frac{3}{4}, && \eta=\frac{3}{4}, && q=\frac{3}{4}. \end{aligned} \tag{4.1}$$

In order to illustrate Theorem 3.5, we take

$$\begin{aligned} f(t,u(t)) &= \frac{1}{e^t+9}\left(\frac{|u(t)|}{1+|u(t)|}\right),\\ g(t,u(t)) &= \frac{1}{2(t+2)^2}\left(u+\sqrt{1+u^2}\right),\\ h(t,u(t)) &= \frac{1}{8}+\frac{\sin\sqrt{|u(t)|}}{24} \end{aligned} \tag{4.2}$$

in (1.1) and note that

$$\begin{aligned} |f(t,u)-f(t,v)| &= \frac{1}{e^t+9}\left(\left|\frac{|u|}{1+|u|}-\frac{|v|}{1+|v|}\right|\right)\\ &\le \frac{1}{e^t+9}\left(\frac{|u-v|}{(1+|u|)(1+|v|)}\right)\\ &\le \frac{1}{10}|u-v|, \end{aligned}$$

and

$$\begin{aligned} |g_1(t,u)-g(t,v)| &\le \frac{1}{8}|u-v|,\\ |h(t,u)-h(t,v)| &\le \frac{1}{24}|u-v| \end{aligned} \tag{4.3}$$

Use the given values of the parameters in (3.4), (3.14) and (3.13). Hence condition (H1) holds with $L_1=\frac{1}{10}$, $L_2=\frac{1}{4}$ and $L_3=\frac{1}{24}$. We check that condition (3.16) is satisfied. Indeed, we can find

$$\sum_{i=1}^{3}\left[\frac{T^{\beta+\gamma}}{|\Lambda|}\omega_i+\varpi_i\right]L_i=0.4332<1,$$

In view of Theorem 3.6, we have that boundary value problem (1.1) has a unique solution.

Example 4.2. Let us consider the problem (1.1) with the data:

$$\begin{aligned} &T=1, && m=1, && a=1, && b=1,\\ &\alpha=\frac{1}{4}, && \beta=\frac{1}{2}, && \gamma=\frac{4}{5}, && \sigma_1=\frac{5}{2},\\ &\lambda_1=1, && \xi_1=\frac{2}{3}, && \eta=\frac{1}{4}, && q=\frac{1}{2}. \end{aligned} \tag{4.4}$$

Using the given values of the parameters in (3.4) and (3.3), we can find that

$$\sum_{i=1}^{3}\left[\frac{T^{\beta+\gamma}}{|\Lambda|}\omega_i+\varpi_i\right]=2.7244. \tag{4.5}$$

In order to illustrate Theorem 3.5, we take

$$\begin{aligned} f(t,u(t)) &= \frac{1}{e^{(t-1)}+9}\left(\frac{|u(t)|}{1+|u(t)|}\right)+e^t, \\ g(t,u(t)) &= \frac{1}{8}+\frac{\sin\sqrt{|u(t)|}}{12}+cos(t), \\ h(t,u(t)) &= \frac{cos\sqrt{|u(t)|}}{15}+2(1+t) \end{aligned} \tag{4.6}$$

in (1.1) and note that

$$\begin{aligned} |f(t,u)-f(t,v)| &\leq \frac{1}{10}|u-v|, \\ |g(t,u)-g(t,v)| &\leq \frac{1}{12}|u-v|, \\ |h(t,u)-h(t,v)| &\leq \frac{1}{15}|u-v|. \end{aligned}$$

Hence, the condition (H1) holds with $L_1=\frac{1}{10}$, $L_2=\frac{1}{12}$, $L_3=\frac{1}{15}$. Further, from the above given data, it is easy to calculate

$$\sum_{i=1}^{3} L_i\varpi_i = 0.09554.$$

On the other hand, for any $t\in \mathrm{J}, u\in\mathbb{R}$ we have

$$\begin{aligned} |f(t,u)| &\leq \frac{1}{10}|u|+1, \\ |g(t,u)| &\leq \frac{1}{12}|u|+1, \\ |h(t,u)| &\leq \frac{1}{15}|u|+2. \end{aligned}$$

Hence, condition (H2) holds with $M=MaxM_1,M_2,M_3=\frac{1}{10}$, $N=MaxN_1,N_2,N_3=2$, and $p=1$. In view of Theorem 3.5,

$$\Theta=\{u\in\mathscr{C}:\ \text{there exist } \varsigma\in[0,1] \text{ such that } x=\varsigma\mathscr{F}u\},$$

is the solution set. Hence

$$\begin{aligned} \|u\| &\leq \xi(\|\mathscr{T}u\|+\|\mathscr{S}u\|) \\ &\leq \sum_{i=1}^{3}\left[\frac{T^{\beta+\gamma}}{|\Lambda|}\omega_i+\varpi_i\right](M\|u\|+N), \end{aligned}$$

from which we have

$$\|u\| \leq \frac{\sum_{i=1}^{3}\left[\frac{T^{\beta+\gamma}}{|\Lambda|}\omega_i+\varpi_i\right]N}{1-\sum_{i=1}^{3}\left[\frac{T^{\beta+\gamma}}{|\Lambda|}\omega_i+\varpi_i\right]M}=7.4891.$$

From Theorem 3.5, the problem (1.1) with the data (4.1) and (4.2) has at least a solution u in $C(\mathrm{J}\times\mathbb{R},\mathbb{R})$. Furthermore, $\sum_{i=1}^{3}\left[\frac{T^{\beta+\gamma}}{|\Lambda|}\omega_i+\varpi_i\right]L_i=0.1775<1$. Using Theorem 3.6, we have that the boundary value problem (1.1) with the data (4.4) and (4.6) has a unique solution.

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