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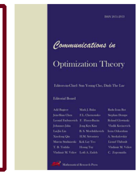

UNIQUENESS AND EXISTENCE RESULTS FOR A PARTIAL DIFFERENTIAL EQUATION IN IMAGE INPAINTING

FATMA ZOHRA NOURI

Mathematical Modeling and Numerical Simulation Laboratory, Faculty of Sciences,

Badji Mokhtar University P.O Box 12 Annaba 23000, Algeria

Abstract. In this paper, we present existence and uniqueness results of entropic solutions for nonlinear problems. First, we introduce the notion of good solutions and their relationships with weak solutions. Then we study a parabolic problem with the space variable dependent absorption, for which we demonstrate the existence and uniqueness of entropic solutions. We also present a particular application to show the competitive results for image inpainting.

Keywords. Elliptic-parabolic-hyperbolic problems; Image inpainting; Parabolic problem; Weak solution.

1. Introduction

Partial differential equations allow to approach mathematically observed phenomena in many scientific disciplines, such as, physics and chemistry. Time-dependent situations are usually translated by evolutionary equations taking into account possible interactions between objects and events. The main concern of the mathematicians confronted with such equations is to provide their meaning in correct functional spaces, to demonstrate their well posedness and wether or not they have solutions. Here we study equations involving the divergent operator of the form

$$Au = -divg(u, \nabla u),$$

with $g : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a field verifying hypotheses of the Leray-Lions type [1] and ∇ is the gradient operator (i.e. in $\mathbb{R}^2, \nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})^T$). Let us first illustrate some of the difficulties that may arise when we study these equations by considering the following degenerate elliptic-parabolic model

$$\begin{cases} \alpha(\frac{\partial u}{\partial t}) - divg(u, \nabla u) = f \;\; in \;\; (0,T) \times \Omega, \\ \alpha(u)(0,.) = \alpha(u_0) \;\; in \;\; \Omega, \\ u = 0 \quad on \;\; \partial\Omega, \end{cases} \tag{1.1}$$

E-mail address: tassili.nan09@gmail.com.

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and the problem of hyperbolic type

$$\begin{cases} \frac{\partial u}{\partial t} + div\Phi(u) = 0 \;\; in \;\; (0,T)\times\mathbb{R}^N, \\ u(0,.) = u_0 \;\; in \; \mathbb{R}^N, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^N$ with Lipchitz boundary $\partial\Omega$ with some appropriate conditions for f,Φ and u_0.

These equations are generally ill posed in the distribution's sense. Many authors studied the well posedness of (1.1). In particular, Boccardo and Gallouët [2] in 1996. It is readily shown that when f is a bounded measure, the weak solution of the elliptic case (when $\alpha = 0$ in the 1^{st} equation of (1.1)) is in the sobolev space $W_0^{p,q}(\Omega)\left(p > 2-\frac{1}{N}, 1 < q < \frac{N(p-1)}{N-1}\right)$, whereas for the parabolic version (the case when $\alpha = Id$), it is in $L^q\left(0,T;W_0^{p,q}(\Omega)\right)$ $(p > 2-\frac{1}{N+1}, 1 < q < \frac{(N+1)(p-1)+1}{N+1})$. However these formulations are weak because in general we do not get the uniqueness of solutions; see, e.g., [3–5]. Such problems occur in the modeling of fluid flows (Newtonians and non-Newtonians) through a porous medium [6] and [7]. The main difficulty in their resolution comes from degeneracy of the positive increasing function α in the first equation of (1.1), corresponding in physics to saturated or dry areas in the medium. This results in absence of the solution's time-regularity that do not allow us to establish weak solutions of the degenerate problem and even their uniqueness. The primary objective sought then is to make assumptions in order to show the existence of unique solutions. The non-uniqueness of weak solutions of problem (1.1) or (1.2) is a classic difficulty, the typical example is that of the Burgers equation (see [8]). It then appears necessary to find a physical criterion that allows us to select from all weak solutions the physically acceptable one, so-called entropic solution. For more details on this issue, readers are referred to [8–11]. Formulations, more appropriate than the usual framework of weak solutions, were then born, with the hope of arriving at a definition of solutions that would make it possible to obtain the uniqueness. Three concepts of solutions have been introduced: solutions called SOLA by Dall'Aglio [12], the entropic solutions by Bénilan et al. [13] and finally the renormalized solutions originating from the results by DiPerna and Lions [14] on the Boltzmann equations.

This paper is focussed on the study of elliptic-parabolic equations of the type diffusion-convection: $-div\; g(u,\nabla u) = f$, where g is a field verifying the Leray-Lions hypotheses. Our motivation is based on some good results for image inpainting by partial differential equations of this type, since the work of Perona-Malik [15]. Their principle depends on successive locally oriented diffusion of the image pixels intensity, with the performance to give an accurate estimation of image structure orientations, particularly when edges are oriented uniformly (see, for example, [16–20]. Most of these papers have shown the efficiency of these models to produce competetive results in image processing in general and particularly in inpainting, however, no theoretical results were developed. Here we first consider this type of equations with mathematical hypotheses, under which conditions they can have solutions and wether or not these solutions are unique. Let us consider the following parabolic problem with absorption

$$\begin{cases} \frac{\partial u}{\partial t} - divg(u,\nabla u) + \beta(.,u) \ni f \;\; in \;\; (0,T)\times\Omega, \quad T>0, \\ u(0,.) = u_0 \;\;\; in \;\; \Omega, \\ u = 0 \;\;\; on \;\; (0,T)\times\partial\Omega. \end{cases} \tag{2.1}$$

The term $\beta(x,.) = \partial F(x,.), F : \Omega \times \mathbb{R} \longrightarrow [0,+\infty[$ is a measurable convex function in $x \in \Omega$ satisfying $F(.,0) = 0$ and $f \in L^1((0,T)\times\Omega)$. Here g is a field verifying the following conditions

(H1) Condition of monotony:

$for\, x \in \mathbb{R}^N, (g(r,x)-g(r,y)).(x-y) \geq 0, \ \ \forall r \in \mathbb{R}, \forall x,y \in \mathbb{R}^N.$

(H2) Coercivity:

$\exists\, \lambda_0 > 0, p > 1$ such that $(g(r,x)-g(r,0)).x \geq \lambda_0 \, |x|^p, \ \ \forall r \in \mathbb{R}, \forall x \in \mathbb{R}^N.$

(H3) Increasing condition: there exists an increasing continuous function

$A : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $|g(r,x)| \leq A(|r|)(1+|x|^{p-1}), \ \ \forall r \in \mathbb{R}, \forall x \in \mathbb{R}^N.$

(H4) Continuous function: $\forall x \in \mathbb{R}^N$, there exists a continuous function

$C : \mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}$ s.t. $|g(r,x)-g(s,x)| \leq C(r,s)\,|r-s|\,(1+|x|^{p-1}), \forall r,s \in \mathbb{R}.$

The study of this parabolic problem is difficult because of difficulties related to the dependence in x of the graph β. Hence we give some results which allow us to affirm that the associated Cauchy problem admits a unique solution.

2. MULTIVOQUE OPERATORS AND THE NOTION OF GOOD SOLUTIONS

2.1. **Multivoque operators.**

Definition 2.1. Let X be a Banach space. The mapping $A :\ X \longrightarrow P(X)$ is a multivoque operator if its effective domain is given by

$$D(A) = \{x \in X;\ A(x) \neq \emptyset\}$$

with an image defined by

$$R(A) = \{y \in X;\ \exists x \in X,\ y \in A(x)\},$$

If $\forall x \in D(A), A(x)$ is reduced to one element, then we say that A is a univoque operator.

By convention, we identify A with its graph in $X \times X$, defined by $G(A) = \{(x,y) \in X \times X;\ y \in A(x)\}$.

Definition 2.2. Let A be a multivoque operator of X,

1- A is accretif in X if

$$\forall b > 0,\ \forall (x,y),\ (\widetilde{x},\widetilde{y}) \in A \ \ \|x-\widetilde{x}\|_X \leq \|x-\widetilde{x}+b(y-\widetilde{y})\|_X .$$

i.e. if the univoque operator $(I+bA)^{-1} : (R+bI) \longrightarrow X$ is a contraction.

2- A is $m-$accretif in X if A is accretif with $R(I+bA) = X, \forall b > 0$.

3- A is $T-$accretif in X if

$$\forall b>0,\ \forall (x,y),\ (\widetilde{x},\widetilde{y})\in A \quad \left\|(x-\widetilde{x})^{+}\right\|_X \leq \left\|(x-\widetilde{x}+b(y-\widetilde{y}))^{+}\right\|_X .$$

4- A est $m-T-$accretif in X if A is $T-$accretif with $R(I+bA)=X$, $b>0$.

Let us give another caracterisation of accretif operators in $L^1(\Omega)$ by introducing the applications

$$sign(r) = \begin{cases} 1,\ r<0, \\ [-1,1],\ r=0, \\ -1,\ r>0, \end{cases}$$

$$sign^{+}(r) = \begin{cases} 0,\ r<0, \\ [0,1],\ r=0, \\ 1,\ r>0. \end{cases}$$

Proposition 2.3. *Let us consider the operator A in $L^1(\Omega)$,*

1- A is accretif iff

$$\forall (u,v),(u',v')\in A, \exists \kappa \in L^{\infty}(\Omega) \quad with\ \kappa\in sign(u-u')\ \ s.t. \int_{\Omega} \kappa(v-v') \geq 0.$$

2- A is $T-$accretif iff

$$\forall (u,v),(u',v')\in A, \exists \kappa \in L^{\infty}(\Omega) \quad with\ \kappa\in sign^{+}(u-u')\ \ s.t. \int_{\Omega} \kappa(v-v') \geq 0.$$

Here we are interested in a particular class of nonlinear operators, where we can use the following result.

Proposition 2.4. *Let us consider an operator A in $L^1(\Omega)$ such that A is accretif and $R(I+bA)$ is dense in $L^1(\Omega)$, $\forall b>0$. Hence the closure $\overline{A}$ of A in $L^1(\Omega)$ is an $m-$T-accretif operator in $L^1(\Omega)$.*

2.2. **The notion of good solutions.** If we consider the following Cauchy problem

$$(CP)(u_0,F) \begin{cases} \frac{\partial u}{\partial t}+Au \ni F, \\ u(0)=u_0, \end{cases}$$

where A is a nonlinear multivoque operator in X, $u_0\in X$ and $F\in L^1(0,T;X)$. A notion of good solutions for problem $(CP)(u_0,F)$ is the one introduced by Bénilan [13], by considering a time $\varepsilon-$discretisation written in the form

$$u_i^{\varepsilon} = (I+(t_i-t_{i-1})A)^{-1}(u_{i-1}^{\varepsilon}+(t_i-t_{i-1})F_i^{\varepsilon},\ i=1,...,N$$

with $u(0)=u_0$. Note that this discretised problem can be solved if $R(I+bA)=X$, where $b>0$ sufficiently small, and uniquely solved if $(I+bA)^{-1}$ is univoque for small b. This solution u_ε called $\varepsilon-$approched solution is piecewise constant. Hence, when $\varepsilon\to 0$, we require that these solutions converge in one sens to a function u that is a solution of $(CP)(u_0,F)$.

Definition 2.5. A solution of $(CP)(u_0,F)$ is called a good solution if, for every function $u \in C([0,T];X)$ with $u(0)=u_0$ and verifying $\forall \varepsilon>0$, there exists an $\varepsilon-$discretisation, denoted by $D_A^{\varepsilon}(t_0,...,t_N;F\varepsilon_1,...,F\varepsilon_N;u_0)$ and an $\varepsilon-$approched solution u^{ε} such that

$$\sup_{t_0\leq t\leq t_N} \|u^{\varepsilon}(t)-u(t)\| \leq \varepsilon.$$

According to a theorem due to Bénilan if A is an accretif operator in X such that $R(I+bA)=X$, $\forall b>0$, for all $u_0 \in \overline{D(A)}^{\|.\|_X}$ and $\forall F \in L^1(0,T;X)$, there exists a unique good solution of $(CP)(u_0,F)$, i.e. this result is true for $m-$accretif operators.

Theorem 2.6. *Let A be an $m-$accretif operator in X, $u_0 \in \overline{D(A)}^{\|.\|_X}$ and $F\in L^1(0,T;X)$. Hence*
1- A function $u\in C([0,T];X)$ and verifying $u(0)=u_0$ is a good solution of $(CP)(u_0,F)$ if and only if, $\forall\ 0\leq s<t\leq T$,

$$\begin{aligned} \|u(t)-x\|_X &\leq \|u(t)-s\|_X + \int_s^t [u(\tau)-x,F(\tau)-y]_+ \, d\tau,\ \forall (x,y)\in A \\ with\,[x,y]_+ &= \lim_{\mu\longrightarrow 0} \frac{\|x+\mu y\|_X-\|x\|_X}{\mu}. \end{aligned}$$

2- If u is a good solution of $(CP)(u_0,F_1)$ and v a good solution of $(CP)(v_0,F_2)$, where $v_0 \in \overline{D(A)}^{\|.\|_X}$ and $F_2\in L^1(0,T;X)$, then, $\forall t\ 0\leq t\leq T$,

$$\|u(t)-x\|_X \leq \|u_0-v_0\|_X + \int_0^t \|F_1(\tau)-F_2(\tau)\|\, d\tau.$$

This result shows on one hand the uniqueness of the good solutions and their continued dependence on the data, for $m-$accretif operators (see Bénilan [3] for details).

2.3. **Weak solutions.** Let us consider the following problem

$$\frac{\partial u}{\partial t} + \mathbb{A}_{m,n}^{\lambda} u \ni f, \quad u(0)=u_0, \tag{2.2}$$

where the operator $\mathbb{A}_{m,n}^{\lambda}$ is defined by $(u,f)\in \mathbb{A}_{m,n}^{\lambda}$ if and if only $u\in W_0^{1,p}(\Omega)\cap L^{\infty}(\Omega)$ and $f\in L^1(\Omega)$. Ammar et al in [21] shown that there exists a unique good solution if the closure of $A_{m,n}^{\lambda}$ is m-T-accretif with a dense domain in $L^1(\Omega)$. By the nonlinear semi groups theory, this ensures that there exists a unique good solution $u\in C([0,T];L^1(\Omega))$ of (2.2). The question is wether or not this good solution is a weak solution to the corresponding evolution problem

$$\left\{ \begin{array}{c} \frac{\partial u}{\partial t} - divg(u,\nabla u)+\beta_{\lambda(.,u)}+\omega_{m,n}(u)\ni f,\ \ in\ \ (0,T)\times\Omega, \\ u(0,.)=u_0\ \ in\ \ \Omega, \\ u=0, on\ \ (0,T)\times\partial\Omega, \end{array} \right. \tag{2.3}$$

where $\omega_{m,n}=\frac{1}{m}r^+-\frac{1}{n}r^-$ $(r\in\mathbb{R},\ m,n$ are fixed) and $\beta_\lambda(.,u)$ is the regularised's Yosida of $\beta(.,u)$, $x\in\Omega$ defined by $\beta_\lambda(x,.)=\frac{1}{\lambda}(I-(I+\lambda\beta(x,.))^{-1}$, almost everywhere for x. We can also use $\beta_{\lambda_1,\lambda_2}(.,u)=\beta_{\lambda_1}(r^+)+\beta_{\lambda_2}(-r^-)$.

For all $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega g(u,\nabla u).\nabla(u-\phi)+\beta_\lambda(.,u)(u-\phi)+\omega_{m,n}(u)(u-\phi) \le \int_\Omega f(u-\phi). \tag{2.4}$$

Note that β_λ is lipchitzian. The resorts to fact that the perturbations makes it possible to derive estimates of the solutions.

Definition 2.7. A measurable function u: $Q \to \mathbb{R}$ is a weak solution of (2.3) if $u \in C([0,T];L^1(\Omega)) \cap L^p([0,T];W_0^{1,p}(\Omega)) \cap L^\infty(Q))$ $(Q=(0,T)\times\Omega)$ and satisfies

$$-\int_Q (u_\lambda - u_0)\varphi_t + \int_Q g(u_\lambda,\nabla u_\lambda).\nabla\varphi + \int_Q \omega_{m,n}(u_\lambda)\varphi \le \int_Q (f \ -\beta_{\lambda,\lambda}(.,u_\lambda))\varphi,$$

$\forall \varphi \in L^p([0,T];W_0^{1,p}(\Omega)) \cap L^\infty(Q))$ with $\varphi_t \in L^{p'}([0,T];W_0^{-1,p'}(\Omega))$ and $\varphi(T)=0$.

Proposition 2.8. *If $f \in L^\infty(Q)$ and $u_0 \in L^\infty(\Omega)$, then the good solution of (2.2) is a weak solution of (2.3).*

Proof. Let us consider the time-discretised problem associated to (2.2)

$$\frac{u_i^\varepsilon - u_{i-1}^\varepsilon}{t_i - t_{i-1}} + A_{m,n}^\lambda\, u_i^\varepsilon \ni f_i^\varepsilon,$$

with $0=t_0<t_1<...<t_l\le T$ such that $t_i-t_{i-1}<\varepsilon$, $\forall i=1,...,l$, and $T-t_l<\varepsilon$

$$\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i} \|f(t)-f_i^\varepsilon\|\,dt \le \varepsilon,\ \|f_i^\varepsilon\|_{L^\infty(Q)} \le \|f\|_{L^\infty(Q)},\ \forall i=1,...,l$$

and

$$\sum_{i=1}^{l}(t_i-t_{i-1})\,\|f_i^\varepsilon\|_{L^\infty(\Omega)} \le \int_0^T \|f(t,.)\|_{L^\infty(\Omega)}\,dt.$$

If we take $A_{m,n}^\lambda\, u_i^\varepsilon = div\ g(u_i^\varepsilon,\nabla u_i^\varepsilon)+\omega_{m,n}(u_i^\varepsilon)$, for all $f_i^\varepsilon \in L^\infty(\Omega)$, then there exists $u_i^\varepsilon \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ satisfying

$$\int_\Omega \frac{u_i^\varepsilon - u_{i-1}^\varepsilon}{t_i-t_{i-1}}\varphi + \int_\Omega g(u_i^\varepsilon,\nabla u_i^\varepsilon)\nabla\varphi + \int_\Omega \omega_{m,n}(u_i^\varepsilon)\varphi = \int_\Omega (f \ -\beta_{\lambda,\lambda}(.,u_i^\varepsilon)\varphi, \tag{2.5}$$

for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. If we consider a piecewise function defined by $u_\varepsilon(t)=u_i^\varepsilon$, for $t=]t_{i-1},t_i[$, $u_\varepsilon(0)=u_0$. In view of the general theory of nonlinear semigroups, we have that u_ε converges in $L^\infty(0,T;L^1(\Omega))$ to the good solution u of Cauchy problem (2.3). Furthermore we have the following estimation

$$\left\|u_i^\varepsilon\right\|_\infty \le \|u_0\|_\infty + \sum_{j=1}^{i}(t_j-t_{j-1})\left\|f_j^\varepsilon\right\|_\infty + C_{m,n},\forall i,$$

where $C_{m,n}$ is a constant depending on m and n. Hence

$$\|u_\varepsilon\|_{L^\infty(Q)} \le C(\|f\|_{L^\infty(Q)},\|u_0\|_{L^\infty(\Omega)},m,n). \tag{2.6}$$

If we choose a test function $\varphi = u_i^\varepsilon$ in (2.5) and sum over i=1,2,...,l, by using hypothesis **(H$_2$)**, Gauss-Green's theorem and the monotony of $\omega_{m,n}$, we obtain

$$\int\limits_\Omega \zeta(u_\varepsilon(t)) + \lambda_0 \int\limits_Q |\nabla u_\varepsilon|^p \leq \int\limits_Q f_\varepsilon u_\varepsilon + \int\limits_\Omega \zeta(u_0).$$

However u_ε is bounded in ε, $u_0 \in L^1(\Omega)$. Therefore the ε-uniform estimation

$$\|u_\varepsilon\|_{L^p(0,T,W^{1,p}(\Omega)} \leq C, \;\; C \textit{ is} \text{ a constant independent of } \varepsilon. \tag{2.7}$$

There exists a function u (measurable) such that, for a subsequence denoted by $(u_\varepsilon)_\varepsilon$, $u_\varepsilon \longrightarrow u$, weakly in $L^p(0,T;W_0^{1,p}(\Omega))$. Let the function

$$\widehat{u_\varepsilon}(t) = u_{i-1}^\varepsilon + \frac{t-t_{i-1}}{t_i - t_{i-1}}(u_i^\varepsilon - u_{i-1}^\varepsilon), t \in [t_{i-1}, t_i]_{i=1}^l,$$

be continuous, piecewise linear verifying $\forall\, t \in [t_{i-1},t_i]_{i=1}^l$, $(\widehat{u_\varepsilon}(t))_t = \frac{u_i^\varepsilon - u_{i-1}^\varepsilon}{t_i - t_{i-1}}$ and $u_\varepsilon \longrightarrow u$ in $L^\infty(0,T;L^1(\Omega))$. Hence$u \in C([0,T];L^1(\Omega))$. If we choose the test function $\varphi = \frac{1}{k}T_k(u_i^\varepsilon)$ in equation (2.5), integrate over $(0,T)$, the limit when k tends to 0 is

$$\int\limits_Q \left|\beta_{\lambda,\lambda}(.,u_\varepsilon)\right| \leq C.$$

From estimation (2.6) and the fact that $u_\varepsilon \to u$, almost everywhere in Q, we have $\beta_{\lambda,\lambda}(.,u_\varepsilon) \to \beta_{\lambda,\lambda}(.,u)$ *in* $L^1(Q)$ when $\varepsilon \to 0$. On the other hand $(g(u_\varepsilon,\nabla u_\varepsilon))_\varepsilon$ is bounded in $(L^{p'}(Q))^N$, hence we can suppose that $g(u_\varepsilon,\nabla u_\varepsilon)$ converges weakly to χ in $(L^{p'}(Q))^N$ when $\varepsilon \to 0$. Let us show that $g(u,\nabla u) = div\chi$. The sufficient requirement is to show

$$\limsup_{\varepsilon\to 0} \int\limits_Q g(u_\varepsilon,\nabla u_\varepsilon).\nabla(u_\varepsilon - u) \leq 0, \tag{2.8}$$

and use the classic argument of monotony. By considering in (2.5) the function test $\varphi = (u_i^\varepsilon - u)$ and sum over $i = 1,..,l$, we get

$$\begin{aligned} &\int\limits_Q g(u_\varepsilon,\nabla u_\varepsilon).\nabla(u_\varepsilon - u) \\ \leq\; &\int\limits_Q f_\varepsilon(u_\varepsilon - u) - \int\limits_Q (u_\varepsilon - u)\beta_{\lambda,\lambda}(.,u_\varepsilon) - \int\limits_Q \omega_{m,n}(u_\varepsilon)(u_\varepsilon - u) \\ &-\int_0^T \langle (u_\varepsilon)_t - u_t, (u_\varepsilon - u)\rangle - \int_0^T \langle u_t, (u_\varepsilon - u)\rangle \\ \leq\; &\int\limits_Q f_\varepsilon(u_\varepsilon - u) - \int\limits_Q (u_\varepsilon - u)\beta_{\lambda,\lambda}(.,u_\varepsilon) - \int\limits_Q \omega_{m,n}(u_\varepsilon)(u_\varepsilon - u) \\ &+\frac{1}{2}\int\limits_\Omega |u_\varepsilon - u|^2(0) - \frac{1}{2}\int\limits_\Omega |u_\varepsilon - u|^2(T) - \int_0^T \langle u_t, (u_\varepsilon - u)\rangle. \end{aligned}$$

Note that we have used integration by part, the monotony of $\beta_{\lambda,\lambda}$ and $\omega_{m,n}$. By using the theorem of the dominated convergence, (2.8) is proved. Now let $\varphi \in D(\Omega)$ and $a \in \mathbb{R}$

$$\begin{aligned}
a\int_Q \chi.\nabla\varphi &\geq \lim_{\varepsilon\to 0} a\int_Q g(u_\varepsilon,\nabla u_\varepsilon).\nabla\varphi \\
&\geq \lim_{\varepsilon\to 0}\sup\int_Q g(u_\varepsilon,\nabla u_\varepsilon).\nabla(u_\varepsilon - u + a\varphi) \\
&\geq \lim_{\varepsilon\to 0}\sup\int_Q g(u_\varepsilon,\nabla(u-\alpha\varphi).\nabla(u_\varepsilon - u + a\varphi) \\
&\geq a\int_Q g(u,\nabla(u-a\varphi).\nabla\varphi.
\end{aligned}$$

If we divide by $a > 0$, respectively by $a < 0$, then $g(u,\nabla u) = div\chi$ in $D'(\Omega)$ when $a \to 0$. Therefore as $\varepsilon \to 0$, we obtain a weak solution $u^{\lambda}_{m,n}$ for (2.3). □

Remark 2.9. By convenience if u is solution of (2.3), then $-u$ is a solution of

$$\frac{\partial u}{\partial t} + div\widehat{g}(u,\nabla u) + \widehat{\beta}_\lambda(.,u) + \widehat{\omega}_{m,n}(u) = \widehat{f}$$

with $\widehat{g}(r,\zeta) = -g(-r,-\zeta), \widehat{\beta}_\lambda(.,r) = -\beta(.,-r), \widehat{\omega}_{m,n}(r) = -\omega_{m,n}(-r)$ and $\widehat{f} = -f$.

3. Entropic solutions

Definition 3.1. A solution of (1.2) is said to be entropic if $u_0 \in L^\infty((0,T)\times\mathbb{R}^N)$, for all $k \in \mathbb{R}$ and $\forall\phi \in C_c^\infty([0,T]\times\mathbb{R}^N), \phi \geq 0$, we have

$$\int_0^T\int_{\mathbb{R}^N} |u-k|\,\phi_t + \int_0^T\int_{\mathbb{R}^N} sign(u-k)(\phi(u)-\phi(k)).\nabla\phi + \int_{\mathbb{R}^N} |u_0-k|\,\phi(0) \geq 0.$$

For example, for the associated elliptic problem ($\alpha = 0$ *in* (1.1)), the entropic solution u is a measurable function satisfying

$$\begin{cases} T_k(u) \in W_0^{1,P}(\Omega),\ \forall k > 0 \\ \int_\Omega g(u,\nabla u).\nabla T_k(u-\phi) \leq \int_\Omega f\, T_k(u-\phi),\ \forall\phi \in W_0^{1,p}(\Omega) \end{cases}$$

where T_k is a truncation function at k, defined by $T_k(z) = min(k, max(z,k))$.

In parallel with this definition, Murat [22] developed the concept of renormalized solutions for the elliptic problems, and proved for many problems besides, equivalent to that of entropic solutions, and both lead to a single solution. Effectively these renormalized solutions verify

$$\begin{cases} T_k(u) \in W_0^{1,P}(\Omega),\ \lim \int_{\{h\leq|u|\leq h+k\}} |\nabla u|^p = 0, \quad \forall k > 0, \\ \int_\Omega g(u,\nabla u).\nabla(S(u)\phi) \leq \int_\Omega f\, S(u)\phi),\ \forall\phi \in W_0^{1,p}(\Omega), \end{cases}$$

and for all S, a regular function with a compact support.

3.1. **Existence of entropic solutions.** At this stage we study the perturbed problem (2.3) and show how to get an entropic solution of problem (2.2) when m,n tend to ∞.

Theorem 3.2. *Problem (2.3) admits an entropic solution for $f \in L^1(\Omega)$ and*

$$u_0 \in \overline{\{v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)\,;\ \gamma_-(x) \le \widetilde{v}(x) \le \gamma_+(x)\ \ a.e.\ x \in \Omega\}}^{L^1(\Omega)}.$$

Proof. Step 1. A priori estimation.

We denote by $\mathscr{C}$ the space defined by $\mathscr{C} = \{u \in L^\infty(\Omega)\ s.t.\ v_-(x) \le u(x) \le v_+(x)\}$. If $v_\pm$ are the upper and lower solutions, then we can approach $f \in L^1(\Omega)$ and $u_0 \in \overline{\mathscr{C}}^{L^1(\Omega)}$ by bi-monotone subsequences $(f_{m,n})_{m,n} \subset L^\infty(\Omega)$ and $(u^0_{m,n})_{m,n} \subset \mathscr{C}$. Hence from the above results, there exists $(u_{m,n}) \in L^p(\Omega)(0,T;W^{1,p}(\Omega)) \cap L^\infty(\Omega)$ verifying

$$\int_0^T\int_\Omega (u_{m,n})_t\varphi + \int_0^T\int_\Omega g(u_{m,n},\nabla u_{m,n})\nabla\varphi + \int_0^T\int_\Omega \omega_{m,n}(u_{m,n})\varphi \le \int_0^T\int_\Omega f\,\varphi. \tag{3.1}$$

For a fixed $k > 0$, if we take $\phi = T_k(u_{m,n})$ in (3.1), by using hypothesis (H_2) the monotony of $\omega_{m,n}$, the integration by parts gives

$$\int_\Omega\int_0^{u_{m,n}(t)} T_k(r)dr + \int_\Omega\int_0^{u^0_{m,n}} T_k(r)dr + \lambda_0\int_Q |\nabla T_k(u_{m,n})|^P \le \int_0^T\int_\Omega f_{m,n}T_k(u_{m,n}) - g(u_{m,n},0).\nabla T_k(u_{m,n}), \tag{3.2}$$

which yields

$$\int_\Omega\int_0^{u_{m,n}(t)} T_k(r)dr + \int_\Omega\int_0^{u^0_{m,n}} T_k(r)dr + \lambda_0\int_Q |\nabla T_k(u_{m,n})|^P \le Ck. \tag{3.3}$$

Hence $(\nabla T_k(u_{m,n}))_{m,n}$ is bounded in $(L^p(Q))^N$. Thus $(T_k(u_{m,n}))_{m,n}$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and $T_k(u_{m,n}) \longrightarrow v_k$ in $(L^p\left(0,T;W_0^{1,p}(\Omega)\right)$ as $(m,n) \longrightarrow \infty$. It can be implied that $(u_{m,n})_{m,n}$ is bounded in $L^\infty\left(0,T;L^1(\Omega)\right)$. Note that C in (3.3) is a constant independent of m and n.

Step 2. Strong convergence of $(u_{m,n})_{m,n}$.

Let $(u^\lambda_{m,n})$ be a weak solution of (3.1). We have, for $(m,n) \longrightarrow \infty$ and $\lambda \longrightarrow 0$, $u^\lambda_{m,n} \longrightarrow u_{m,n}$ strongly in $L^1(\Omega)$. It remains to demonstrate the strong convergence of $u_{m,n}$. Let $\overline{m} > m$ and $\overline{n} > n$. Therefore, we can write

$$u^\lambda_{m,\overline{n}} \le u^\lambda_{m,n} \le u^\lambda_{\overline{m},n} \quad a.e\ \ in\ Q.$$

As $\lambda \longrightarrow 0$, we have

$$u_{m,\overline{n}} \le u_{m,n} \le u_{\overline{m},n} \quad a.e\ \ in\ Q.$$

By the theorem of monotone convergence, we get

$$\lim_{m\longrightarrow\infty} u_{m,n} = u_n \ \ in\ L^1(Q) \ \ and \ \ \lim_{n\longrightarrow\infty} u_n = u \ \ in\ L^1(Q).$$

This implies that $u_{m,n}$ converges strongly *in* $L^1(Q)$.

Step 3. $(u_{m,n})_{m,n} \in C(0,T;L^1(\Omega))$.

Let us consider the test function $p_\varepsilon(u_{m,n} - u_{m',n'})$. By using the hypotheses (H_1) and (H_4), the monotony of $\omega_{m,n}$, for $\varepsilon \longrightarrow 0$ the integration by parts gives

$$\int_\Omega |u_{m,n} - u_{m',n'}| \leq \int_Q |f_{m,n} - f_{m',n'}| + \int_\Omega \left|u^0_{m,n} - u^0_{m',n'}\right|, \ \forall t,$$

$f_{m,n},\ f_{m',n'} \to f$ and $u^0_{m,n},\ u^0_{m',n'} \to u_0$ *in* $L^1(\Omega)$. Hence $(u_{m,n})_{m,n}$ is a Cauchy sequence in $C\left(0,T;L^1(\Omega)\right)$ and converges to $u \in C\left(0,T;L^1(\Omega)\right)$.

Step 4. Argument of the pseudo-monotonicity. Since $(T_k(u_{m,n}))_{m,n}$ is bounded in $L^p(0,T;W^{1,p}_0(\Omega))$, from (H_3), $g(T_k(u_{m,n}),\nabla T_k(u_{m,n}))$ is bounded in $(L^{p'}\Omega))^N$. At this stage, we need to prove that $div\chi_k = divg(T_k(u_{m,n}),\nabla T_k(u))$ (χ_k is a subsequence). We use the regularisation method due to Landes [23]. let $\mu > 0$ and $(u^0_\mu\ )_\mu$ be a sequence such that

$$\begin{cases} (u^0_\mu\ )_\mu \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \\ \left\|u^0_\mu\ \right\|_{L^\infty(\Omega)} \leq k \\ u^0_\mu \ \longrightarrow T_k(u_0\ ) \ \ a.e\ on\ \Omega \ \ when\ \mu \longrightarrow \infty \\ \frac{1}{\mu}\left\|u^0_\mu\ \right\|_{W^{1,p}_0(\Omega)} \longrightarrow 0 \qquad when\ \mu \longrightarrow \infty. \end{cases}$$

Therefore for $\mu, k > 0$, the regularised of Landes denoted by $(T_k(u\ ))_\mu$ is the unique solution of

$$\begin{cases} \frac{\partial (T_k(u\ ))_\mu}{\partial t} = \mu(T_k(u\ ) - (T_k(u\ ))_\mu \ \ on\ Q, \\ (T_k(u\ ))_\mu(0,.) = u^0_\mu \ \ on\ \ \Omega, \end{cases}$$

which satisfies

$$(T_k(u\ ))_\mu,\ \frac{\partial (T_k(u\ ))_\mu}{\partial t} \in L^p\left(0,T;W^{1,p}_0(\Omega)\right) \cap L^\infty(Q).$$

Let us assume that $(T_k(u\ ))_\mu \longrightarrow T_k(u)$, strongly in $L^p(0,T;W^{1,p}_0(\Omega)) \cap L^\infty(Q)$ *a.e. on* Q, $(T_k(u\ ))_\mu(t) \longrightarrow T_k(u)(t)$ *a.e. on*$Q, \forall t$ and $\|(T_k(u\ ))_\mu\|_{L^\infty(\Omega)} \leq k, \forall \mu > 0$. If we take the test function $k\Phi(u_{m,n})(T_k(u_{m,n}) - (T_k(u))_\mu)$ in (3.1), then

$$\lim_{\mu\longrightarrow\infty} \inf \lim_{m,n\longrightarrow\infty} \int_0^T \prec (u_{m,n})_t, k\,\Phi(u_{m,n})(T_k(u_{m,n}) - (T_k(u))_\mu) \succ \geq 0, \tag{3.4}$$

$$\lim_{\mu\longrightarrow\infty} \lim_{m,n\longrightarrow\infty} \int_0^T k\ \omega_{m,n}(u_{m,n})\Phi(u_{m,n})(T_k(u_{m,n}) - (T_k(u))_\mu) = 0 \tag{3.5}$$

and

$$\lim_{\mu\longrightarrow\infty} \lim_{m,n\longrightarrow\infty} \int_0^T k\ f_{m,n}\Phi(u_{m,n})(T_k(u_{m,n}) - (T_k(u))_\mu) = 0. \tag{3.6}$$

It remains to treat the term

$$\lim_{\mu\longrightarrow\infty}\lim_{m,n\longrightarrow\infty}\int_0^T g(u_{m,n},\nabla u_{m,n})\nabla\varphi.$$

Since $u_{m,n}(t)$ and $(T_k(u_{m,n}))_\mu(t)$ converge almost everywhere to $u(t)$ *and* $T_k(u)(t)$, respectively $\forall t$ when $\mu\longrightarrow\infty$, from ((3.4)-(3.6)), we have

$$\lim_{\mu\longrightarrow\infty}\sup\lim_{m,n\longrightarrow\infty}\int_0^T g(u_{m,n},\nabla u_{m,n})\ k.\nabla\Phi(u_{m,n})(T_k(u_{m,n})-(T_k(u))_\mu)\leq 0 \tag{3.7}$$

It results that

$$\lim_{\mu\longrightarrow\infty}\sup\lim_{m,n\longrightarrow\infty}\int_0^T kg(T_k(u_{m,n}),\nabla T_k(u_{m,n})).\nabla(T_k(u_{m,n})-(T_k(u))_\mu)\leq 0 \tag{3.8}$$

From this limit, we prove that $div\ \chi_k = divg(T_k(u_{m,n}),\nabla T_k(u)),\ \forall t>0$. For $\Psi\in\mathscr{D}(Q),\ \alpha\in\mathbb{R}$, we have

$$\begin{aligned}\alpha\int_Q k\chi_k.\nabla\Psi &= \lim_{n\longrightarrow\infty}\int_Q \alpha k\ g(T_k(u_{m,n}),\nabla T_k(u_{m,n}).\nabla\Psi\\ &\geq \lim_{\mu\longrightarrow\infty}\sup\lim_{m,n\longrightarrow\infty}\sup\int_Q kg(T_k(u_{m,n}),\nabla T_k(u_{m,n})).\nabla(DT_k+\alpha\Psi)\\ &\geq \lim_{\mu\longrightarrow\infty}\sup\lim_{m,n\longrightarrow\infty}\sup\int_Q kg(T_k(u_{m,n}),\nabla\left[T_k(u))_\mu-\alpha\Psi\right].\nabla(DT_k+\alpha\Psi)\\ &\geq \int_Q kg(T_k(u),\nabla\left[T_k(u))_\mu-\alpha\Psi\right].\nabla\Psi\end{aligned}$$

with $DT_k = T_k(u_{m,n})-(T_k(u))_\mu$. If we divide by $\alpha>0$, respectively $\alpha<0$, going through the limit $\alpha\longrightarrow 0$, we get $div\chi_k = divg(T_k(u_{m,n}),\nabla T_k(u)),\ \forall k>0$.

Step 5. Limit in the equation. If we consider a test function of the form $\Phi=\Psi.S(u_{m,n}-\phi)$ in (3.1) with $S\in\left\{p\in C^1(\mathbb{R}),\ p(0)=0,\ 0\leq p'\leq 1,\ supp(p')\ compact\right\},\ \phi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $\Psi\in\mathscr{D}[0,T)$. Let us define $l:=\|\phi\|_\infty+\max\{|z|,\ z\in\ supp(S')\}$. Taking the limit of each term by the dominated convergence and hypothesis (H_4), we obtain

$$\begin{aligned}\int_Q g(u_{m,n},\nabla(u_{m,n})).\nabla S(U_{m,n}) \geq &\int_Q g(T_l(u_{m,n}),\nabla T_l(u_{m,n})).\nabla T_l(u)S'(U_{m,n})\\ &+\int_Q g(T_l(u_{m,n}),\nabla T_l(u)).\nabla(T_l(u_{m,n})-T_k(u)).S'(U_{m,n})\\ &-\int_Q g(T_l(u_{m,n}),\nabla T_l(u_{m,n}).\nabla\phi.S'(U_{m,n}).\end{aligned} \tag{3.9}$$

with $U_{m,n}=(u_{m,n}-\phi)$. This yields $u_{m,n}\longrightarrow u$, strongly in $L^1(Q)$.

□

3.2. **Uniqueness of the solution.** In the following theorem, we prove that the entropic solution of $(CP)(u_0, f)$ is unique.

Theorem 3.3. *Let u and v be two entropic solutions of the problems $(CP)(u_0, f)$ and $(CP)(v_0, g)$ respectively. Then*

$$\int_{\Omega} |u(t) - v(t)| \leq \int_{\Omega} |u_0 - v_0| + \int_{Q} |f - g| \ \ \textit{almost everywhere } t \in [0, T].$$

Proof. To prove this theorem, we need the two following results:

1- If u and v are two weak solutions of

$$\begin{aligned} \frac{\partial u}{\partial t} + A_{m,n} u &\ni f_{m,n}, \ \ u(0) = u^0_{m,n}, \\ \frac{\partial v}{\partial t} + A_{m,n} v &\ni g_{m,n}, \ \ v(0) = v^0_{m,n}, \end{aligned}$$

then they satisfy the contraction's principle

$$\int_{\Omega} |u_{m,n}(t) - v_{m,n}(t)| \leq \int_{\Omega} \left|u^0_{m,n} - v^0_{m,n}\right| + \int_{Q} |f_{m,n} - g_{m,n}| \ \ \text{almost everywhere } t \in [0, T].$$

This is immediate from the nonlinear semi group's theory. Effectively $u_{m,n}$ is a good solution of $\frac{\partial u}{\partial t} + A_{m,n} u \ni f_{m,n}, u(0) = u^0_{m,n} \in \mathscr{C}$ and $A_{m,n}$ is m-T-accretif in $L^1(\Omega)$.

2- If u is an entropic solution of $(CP)(u_0, f)$. If we denote problem (2.3) by $(P_{m,n})(u_0, f)$ and consider and $u_{m,n}$ a solution of $(P_{m,n})(u^0_{m,n}, f_{m,n})$, then

$$\lim_{m,n \longrightarrow \infty} \|u_{m,n}(t) - u(t)\|_{L^1(\Omega)} = 0, \text{ a.e. } t \in [0, T].$$

Let us consider two entropic solutions u and v. Note that $u(t) = \lim_{m,n \longrightarrow \infty} u_{m,n}(t)$ and $v(t) = \lim_{m,n \longrightarrow \infty} v_{m,n}(t)$ such that $u_{m,n}(t)$ and $v_{m,n}(t)$ verify

$$\int_{\Omega} |u_{m,n}(t) - v_{m,n}(t)| \leq \int_{\Omega} \left|u^0_{m,n} - v^0_{m,n}\right| + \int_{Q} |f_{m,n} - g_{m,n}| \ \ \text{a.e. } t \in [0, T].$$

Besides, we have

$$\int_{\Omega} |u(t) - v(t)| \leq \int_{\Omega} |u(t) - u_{m,n}(t)| + |u_{m,n}(t) - v_{m,n}(t)| + |v_{m,n}(t) - v(t)|, \quad \text{a.e. } t \in [0, T],$$

$f_{m,n} \longrightarrow f, \ f_{m,n} \longrightarrow f$ in $L^1(Q)$ and $u^0_{m,n} \longrightarrow u^0, v^0_{m,n} \longrightarrow v^0$ in $L^1(\Omega)$ when $m, n \longrightarrow \infty$. By the dominated convergence's theorem, we obtain

$$\int_{\Omega} |u(t) - v(t)| \leq \int_{\Omega} \left|u^0 - v^0\right| + \int_{Q} |f - g| \quad \forall\, t,$$

which proves the well posedness and uniqueness of the solution. □

4. THE APPLICATION: IMAGE INPAINTING

From the work of Perona-Malik [15], the anisotropic diffusion partial differential equations (PDEs) were largely developed for multiple applications in image processing, see, for example, [20,24,25]. Their principle is based on the application of successive locally oriented diffusion of the image pixels intensity, which is controlled to preserve important structures. Diffusion based inpainting mainly uses nonlinear PDEs and variational approaches to propagate the information in direction of level lines, by minimizing an energy functional. Most of PDEs-based inpainting techniques use the gradient vector to estimate edges orientation. Recently, the most popular estimator is the structure tensor, also known as second moment matrix; that is frequently used in oriented diffusion filtering [17, 26, 27], with the performance to give an accurate estimation of image structure orientations, particularly when edges are oriented uniformly.

4.1. **Nonlinear structure orientation.** The NLST uses an anisotropic diffusion to reduce smoothing in the presence of edges and enhance the image structure. This process is given by

$$\frac{\partial s_{i,j}}{\partial t} = div[g(|\lambda_{\max} - \lambda_{\min}|)\nabla s_{i,j}],\ \imath, j = 1,2, \tag{3.1}$$

with $g(r) = \frac{1}{\sqrt{\delta^2 + (\frac{r}{k})^2}}$ the diffusivity function, where $\delta > 0$ is a small fixed parameter used to avoid singularities, $k > 0$ and $\lambda_{\max}, \lambda_{\min}$ are the maximum and minimum eigenvalues of S, respectively with $r = |\lambda_{\max} - \lambda_{\min}|$ determining the coherence in the image data. When applying a diffusion process to the matrix-valued data $S = (s_{ij})_{i,j=1,2}$, the positive semi-definiteness of the original data S_0 is preserved and the maximum-minimum principle for the field of the eigenvalues associated with a matrix field is guaranteed (for more details the reader is referred to [17]). By using the eigenvalue decomposition, S can be expressed as follows:

$$S = \begin{pmatrix} V_1 & V_2 \end{pmatrix} \begin{pmatrix} \lambda_{\max} & 0 \\ 0 & \lambda_{\min} \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix}. \tag{3.2}$$

The eigenvector V_1, corresponding to $\lambda_{\max}$, represents the direction of the maximum signal variation while the eigenvector V_2, associated to $\lambda_{\min}$, gives the structure orientation. By analyzing the eigenvalues of the NLST, $r = |\lambda_{\max} - \lambda_{\min}|$ gives very good information about the local image anisotropy so that in the homogeneous regions $r \approx 0$, in neighbourhood of straight edges $r \gg 0$, but in the presence of junctions, corners or a curve $r > 0$. The NLST (3.1) has been highlighted as being able to more accurately estimate the structure orientation. It has several advantages, such as directional smoothing, edge enhancing and it avoids the dislocation of discontinuities. Details on numerical implementation and results can be found in [17] and [28]. Here, we make use of these advantages together with the well posedness of the problem shown previously, to show that this inpainting approach is suitable for real medical images with complex geometries.

4.2. **Implementation.** For numerical investigations, we solve (3.1) by a finite difference scheme using a forward time and a central space discretisations, by the following algorithm:

Algorithm 4.1. 1 - Initialisation: input time step Δt, the number of iterations NT and the contrast parameter k.

2 - Approximate the space partial derivatives
$S_x = \frac{\partial S^n}{\partial x}, S_y = \frac{\partial S^n}{\partial y}, S_{xx} = \frac{\partial^2 S^n}{\partial x^2}, S_{yy} = \frac{\partial^2 S^n}{\partial y^2}$ and $S_{xy} = \frac{\partial^2 S^n}{\partial x \partial y}$, using central finite difference schemes with space steps $\Delta x = \Delta y = 1$.

3 - Compute the structure tensor S_0^n from

$$S_0^n = \left(\nabla S \nabla S^T\right) = \begin{pmatrix} (S_x)^2 & (S_x S_y) \\ (S_x S_y) & (S_y)^2 \end{pmatrix}.$$

4 - Compute the initial eigenvalues $\lambda_{\max}^n$ *and* $\lambda_{\min}^n$ of S_0.

5 - Compute the elements of S^{n+1} by

$$S^{n+1} = S^n + \Delta t . div \left[g(|\lambda_{\max}^n - \lambda_{\min}^n|) \nabla S^n \right].$$

6 - If the number of iterations is less than NT, then repeat Step 5.

7 - Compute $\lambda_{\max}^n, \lambda_{\min}^n$, V_1 and V_2 from (3.2).

8 - Compute the local contrast $r^n = |\lambda_{\max}^n - \lambda_{\min}^n|$ and the corner intensity

$$C^n = |(\nabla \cdot (V_2^n V_2^{n^T}))^T \cdot \nabla s^n|.$$

5. Results and comments

In this section, the obtained results are compared with the ones by Chan et al. [29] (TV), Shao et al. [24] (SH) and Zhang et al. [25] (ZH). These results are analyzed using visual quality, peak signal noise ratio (PSNR) and structural similarity measure (SSIM). All computations are carried out on real images of brain. The parameters are set as: the time step $\Delta t = 0.1$ and $k = 4$. The principle of determining the number of iterations NT is based on our perceptual quality of the inpainted images and the values of PSNR and SSIM; here $NT = 10$ is sufficient to get a satisfactory result. The obtained numerical results are summarized as follows:

TABLE 1. PSNR and SSIM values for the Brain image (Figure 1)

Models	Measures	Values
TV	PSNR	25.68
	SSIM	0.9361
ZH	PSNR	25.38
	SSIM	0.9336
SH	PSNR	25.59
	SSIM	0.9281
Model (3.1)	PSNR	**27.74**
	SSIM	**0.9578**

The comparison of the PSNR and the SSIM, lead us to deduce that our approach is more significant and efficient compare to the others, as shown in Table 1. The brain slices presented in Figure 1, show the performance of our model in reproducing the fidelity of healthy parts, see Figure 1 (c), (f) and (i).

Note that the artificial lesions of fixed volumes ($5 \times 5 \times 5$ voxel spheroids, shown in Figure

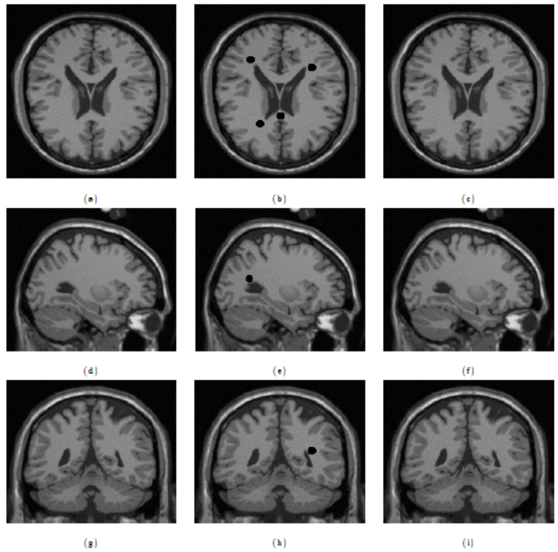

FIGURE 1. Inpainting of MS lesions in MRI image (a)-(d)-(g) : healthy brain slices for axial, sagittal and coronal section, (b)-(e)-(h) synthetic lesions simulation, (c)-(f)-(i), results of lesions filling

1 (b), (e) and (h)), are added to healthy brain and placed at multiple locations (infratentorial, periventricular and juxtacortical white matter locations). Note that the time variable in (3.1) serves as an iteration parameter.

6. CONCLUSION

In this paper, we first proved some results for an elliptic-parabolic type problem. This type of equations is used in many applications and require a deep theoretical study, such as, well posedness, and the existence and uniqueness of their solutions, to ensure effective applications. Here we explored a particular application for an image inpainting approach using oriented diffusion based on the derived NLST. We concluded that this approach is not only able to preserve the geometry of the structures, but also eliminates the artifacts along the contours and in the neighbourhood of structures. The numerical inpainting results, obtained in terms of SSIM, PSNR and visual quality, assert the effectiveness of the proposed algorithm and the satisfactory and competetive results compare to other algorithms, as shown in Table 1 and Figure 1 (c), (f) and (i). Furthermore only small number of iterations are used.

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