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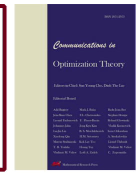

# ON A CERTAIN SUBCLASS OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS INVOLVING POLYLOGARITHM FUNCTIONS

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**Abstract.** In this paper, we introduce and study a new subclass of meromorphic functions with positive coefficients involving the polylogaritham function defined by a new operator $\mathfrak{D}_c f(z)$ and obtain coefficient estimates, growth and distortion theorems, the radius of convexity, integral transforms, convex linear combinations, convolution properties and $\delta$-neighborhoods for the class $\sigma_{c,p}(\alpha,\beta)$.

**Keywords.** Meromorphic function, Polylogarithm function, Coefficient estimates.

## 1. Introduction

The classical polylogarithm function was introduced by Leibnitz and Bernoulli in 1969 as mentioned in [1]. For $|z| < 1$ and $c$ a natural number with $c \geq 2$, the polylogarithm function (which is also known as Jonquiere's function) is defined by the absolutely convergent series:

$$Li_c(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^c}. \tag{1.1}$$

Since then, many mathematicians studied the polylogarithm function, such as, Euler, Spence, Abel, Lobachevsky, Rogers, Ramanujan and many others, and many functional identities by using polylogarithm function were discovered in [2]. Since then, there are few results based on the polylogarithm for many decades. Recently, the work using polylogarithm has again been intensified vividly due to its importance in many fields of mathematics, such as, complex analysis, algebra, geometry, topology, and mathematical physics (quantum field theory) [3–5] and the references therein. In [6], Ponnusamy and Sabapathy discussed the geometric mapping properties of the generalized polylogarithm. In [7], Al-Shaqsi and Darus generalized Ruscheweyh and

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Salagean operators using polylogarithm functions on class $A$ of analytic functions in the open unit disk $U=\{z:|z|<1\}$. By making use of the generalized operator, they introduced certain new subclasses of $A$ and investigated many related polylogarithm function to define a multiplier transformation on the class $A$ in $U$; see [8]. To the best of our knowledge, no research work has been discussed on the polylogarithm function conjunction with meromorphic functions. Thus, in this paper, we redefine the polylogarithm function to be on meromorphic type. Let $\sum$ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{m=0}^{\infty}a_m z^m, \tag{1.2}$$

which are analytic in the punctured open unit disk

$$U^*:=\{z:z\in C,\ 0<|z|<1\}=U\setminus\{0\}. \tag{1.3}$$

A function $f$ in $\sum$ is said to be meromorphically starlike of order $\delta$ if and only if

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\delta;\ (z\in U^*), \tag{1.4}$$

for some $\delta$ $(0\leq\delta<1)$. We denote by $\sum(\delta)$ the class of all meromorphically starlike order $\delta$. Furthermore, a function $f$ in $\sum$ is said to be meromorphically convex of order $\delta$ if and only if

$$\Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\delta;\ (z\in U^*), \tag{1.5}$$

for some $\delta,\ (0\leq\delta<1)$. We denote by $\sum_k(\delta)$ the class of all meromorphically convex order $\delta$. For functions $f\in\sum$ given by (1.2) and $g\in\sum$,

$$g(z)=\frac{1}{z}+\sum_{m=0}^{\infty}b_m z^m, \tag{1.6}$$

we define the Hadamard product (or convolution) of $f$ and $g$ by

$$(f*g)(z)=\frac{1}{z}+\sum_{m=0}^{\infty}a_m b_m z^m. \tag{1.7}$$

Let $\sum_p$ be the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{m=0}^{\infty}a_m z^m;\ a_m\geq 0, \tag{1.8}$$

which are analytic and univalent in $U^*$. Liu and Srivastava [9] defined a function

$$h_p(\alpha_1,\cdots,\alpha_q;\beta_1,\cdots,\beta_s;z)$$

by multiplying the well known generalized hypergeometric function ${}_qF_s$, with $z^{-p}$ as follows:

$$h_p(\alpha_1,\cdots,\alpha_q;\beta_1,\cdots,\beta_s;z)=z^{-p}\ {}_qF_s(\alpha_1,\cdots,\alpha_q;\beta_1,\cdots,\beta_s;z), \tag{1.9}$$

where $\alpha_1,\cdots,\alpha_q;\beta_1,\cdots,\beta_s$ are complex parameters and $q\leq s+1,\ p\in N$. Analogous to Liu and Srivastava results [9] and corresponding to a function $\phi_c(z)$ given by

$$\phi_c(z)=z^{-2}Li_c(z)=\frac{1}{z}+\sum_{m=0}^{\infty}\frac{1}{(m+2)^c}z^m, \tag{1.10}$$

We consider a linear operator $\Omega_c f(z) : \sum \to \sum$, which is defined by the following Hadamard product (or Convolution):

$$\begin{aligned}\Omega_c f(z) &= \phi_c(z) * f(z)\\ &= \frac{1}{z} + \sum_{m=0}^{\infty} \frac{1}{(m+2)^c} a_m z^m. \end{aligned} \tag{1.11}$$

Next, we define the linear operator $\mathfrak{D}_c f(z) : \sum \to \sum$ as follows:

$$\begin{aligned}\mathfrak{D}_c f(z) &= \left\{\Omega_c f(z) - \frac{1}{2^c} a_0\right\}\\ &= \frac{1}{z} + \sum_{m=1}^{\infty} \frac{1}{(m+2)^c} a_m z^m. \end{aligned} \tag{1.12}$$

Now, by making use of operator $\mathfrak{D}_c f(z)$, we define a new subclass of functions in $\sum_p$ as follows.

**Definition 1.1.** For $-1 \leq \alpha < 1$ and $\beta \geq 1$, we let $\sigma_{c,p}(\alpha, \beta)$ be the subclass of $\sum_p$ consisting of functions of the form (1.8) and satisfying the analytic criterion

$$-Re\left\{\frac{z(\mathfrak{D}_c f(z))'}{\mathfrak{D}_c f(z)} + \alpha\right\} > \beta \left|\frac{z(\mathfrak{D}_c f(z))'}{\mathfrak{D}_c f(z)} + 1\right|, \tag{1.13}$$

where $\mathfrak{D}_c f(z)$ is given by (1.12).

The main object of the paper is to study some usual properties of the geometric function theory, such as, coefficient bounds, growth and distortion properties, radius of convexity, convex linear combination and convolution properties, integral operators and $\delta-$neighborhoods for the class $\sigma_{c,p}(\alpha, \beta)$.

## 2. COEFFICIENT INEQUALITY

**Theorem 2.1.** *A function $f$ of the form* (1.8) *is in $\sigma_{c,p}(\alpha, \beta)$ if*

$$\sum_{m=1}^{\infty} \frac{[(1+\beta)(m+1) + 1 - \alpha]}{(m+2)^c} |a_m| \leq 1 - \alpha, \ -1 \leq \alpha < 1 \text{ and } \beta \geq 1. \tag{2.1}$$

*Proof.* It is sufficient to show that

$$\beta\left|\frac{z(\mathfrak{D}_c f(z))'}{\mathfrak{D}_c f(z)} + 1\right| + Re\left\{\frac{z(\mathfrak{D}_c f(z))'}{\mathfrak{D}_c f(z)} + 1\right\} \leq 1 - \alpha.$$

Observe that

$$\begin{aligned}&\beta\Big|\frac{z(\mathfrak{D}_cf(z))'}{\mathfrak{D}_cf(z)}+1\Big|+Re\Big\{\frac{z(\mathfrak{D}_cf(z))'}{\mathfrak{D}_cf(z)}+1\Big\}\\ \leq&(1+\beta)\Big|\frac{z(\mathfrak{D}_cf(z))'}{\mathfrak{D}_cf(z)}+1\Big|\\ \leq&\frac{(1+\beta)\sum\limits_{m=1}^{\infty}\frac{1}{(m+2)^c}(m+1)|a_m||z|^m}{\frac{1}{|z|}-\sum\limits_{m=1}^{\infty}\frac{1}{(m+2)^c}|a_m||z|^m}.\end{aligned}$$

Letting $z\to 1$ along the real axis, we obtain

$$\begin{aligned}&\beta\Big|\frac{z(\mathfrak{D}_cf(z))'}{\mathfrak{D}_cf(z)}+1\Big|+Re\Big\{\frac{z(\mathfrak{D}_cf(z))'}{\mathfrak{D}_cf(z)}+1\Big\}\\ \leq&\frac{(1+\beta)\sum\limits_{m=1}^{\infty}\frac{1}{(m+2)^c}(m+1)|a_m|}{1-\sum\limits_{m=1}^{\infty}\frac{1}{(m+2)^c}|a_m|}.\end{aligned}$$

The last expression is bounded by $(1-\alpha)$ if

$$\sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c}|a_m|\leq 1-\alpha.$$

Hence the theorem is proved. □

**Corollary 2.2.** *Let the function $f$ defined by (1.8) be in the class $\sigma_{c,p}(\alpha,\beta)$. Then*

$$a_m\leq\sum_{m=1}^{\infty}\frac{(m+2)^c(1-\alpha)}{(1+\beta)(m+1)+1-\alpha},\ (m\geq 1)\tag{2.2}$$

*holds for the functions of the form*

$$f_m(z)=\frac{1}{z}+\sum_{m=1}^{\infty}\frac{(m+2)^c(1-\alpha)}{(1+\beta)(m+1)+1-\alpha}z^m.\tag{2.3}$$

## 3. Distortion Theorems

**Theorem 3.1.** *Let the function $f$ be defined by (1.8) in the class $\sigma_{c,p}(\alpha,\beta)$. Then, for $0<|z|=r<1$,*

$$\frac{1}{r}-\frac{3^c(1-\alpha)}{(3+2\beta-\alpha)}r\leq|f(z)|\leq\frac{1}{r}+\frac{3^c(1-\alpha)}{(3+2\beta-\alpha)}r\tag{3.1}$$

*with equality for the function*

$$f(z)=\frac{1}{z}+\frac{3^c(1-\alpha)}{(3+2\beta-\alpha)}.\tag{3.2}$$

*Proof.* Suppose that $f$ is in $\sigma_{c,p}(\alpha,\beta)$. In view of Theorem 2.1, we have

$$\frac{(3+2\beta-\alpha)}{3^c}\sum_{m=1}^{\infty} a_m \leq \sum_{m=1}^{\infty} \frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c} \leq (1-\alpha),$$

which evidently yields

$$\sum_{m=1}^{\infty} a_m \leq \frac{3^c(1-\alpha)}{(3+2\beta-\alpha)}.$$

Consequently, we obtain $|f(z)| = \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right|$. Thus,

$$\begin{aligned}\sum_{m=1}^{\infty} a_m &\leq |\frac{1}{z}| + \sum_{m=1}^{\infty} a_m |z|^m \leq \frac{1}{r} + r\sum_{m=1}^{\infty} a_m \\ &\leq \frac{1}{r} + \frac{3^c(1-\alpha)}{(3+2\beta-\alpha)} r.\end{aligned}$$

We also have

$$\begin{aligned}|f(z)| &= \left|\frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m\right| \\ &\geq |\frac{1}{z}| - \sum_{m=1}^{\infty} a_m |z|^m \geq \frac{1}{r} - r\sum_{m=1}^{\infty} a_m \\ &\geq \frac{1}{r} - \frac{3^c(1-\alpha)}{(3+2\beta-\alpha)} r.\end{aligned}$$

Hence, we obtain (3.1). □

**Theorem 3.2.** *Let the function $f$ be defined by (1.8) in the class $\sigma_c, p(\alpha,\beta)$. Then, for $0 < |z| = r < 1$,*

$$\frac{1}{r^2} - \frac{3^c(1-\alpha)}{(3+2\beta-\alpha)} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{3^c(1-\alpha)}{(3+2\beta-\alpha)}.$$

*The result is sharp, and the extremal function is of the form (2.3).*

*Proof.* From Theorem 2.1, we have

$$\frac{(3+2\beta-\alpha)}{3^c}\sum_{m=1}^{\infty} m a_m \leq \sum_{m=1}^{\infty} \frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c} \leq (1-\alpha),$$

which evidently yields

$$\sum_{m=1}^{\infty} m a_m \leq \frac{3^c(1-\alpha)}{(3+2\beta-\alpha)}.$$

Consequently , we obtain

$$\begin{aligned}|f(z)| &= \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m r^{m-1} \leq \frac{1}{r^2} + \sum_{m=1}^{\infty} m a_m \\ &\leq \frac{1}{r^2} + \frac{3^c(1-\alpha)}{(3+2\beta-\alpha)}\end{aligned}$$

and then

$$|f(z)| \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m r^{m-1} \geq \frac{1}{r^2} - \sum_{m=1}^{\infty} m a_m$$
$$\geq \frac{1}{r^2} - \frac{3^c(1-\alpha)}{(3+2\beta-\alpha)}.$$

This completes the proof. □

## 4. Class Preserving Integral Operators

In this section, we consider the class preserving integral operators of the form (1.8).

**Theorem 4.1.** *Let the function $f$ be defined by (1.8) in the class $\sigma_{c,p}(\alpha,\beta)$. Then*

$$F(z) = \mu z^{-\mu-1} \int_0^z t^\mu f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\mu}{\mu+m+1} a_m z^m, \ \mu > 0 \tag{4.1}$$

*belongs to the class $\sigma[\delta(\alpha,\beta,m,\mu)]$, where*

$$\delta(\alpha,\beta,m,\mu) = \frac{3^c(3+2\beta-\alpha)(\mu+2)-(1-\alpha)\mu}{3^c(3+2\beta-\alpha)(\mu+2)+(1-\alpha)\mu}. \tag{4.2}$$

*The result is sharp for $f(z) = \frac{1}{z} + \frac{3^c(1-\alpha)}{(3+2\beta-\alpha)} z$.*

*Proof.* Letting

$$f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$$

be in $\sigma_{c,p}(\alpha,\beta)$, we have

$$F(z) = \mu z^{-\mu-1} \int_0^z t^\mu f(t)dt = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{\mu}{\mu+m+1} a_m z^m, \ \mu > 0.$$

It is sufficient to show that

$$\sum_{m=1}^{\infty} \frac{m+\delta}{1-\delta} \frac{\mu a_m}{m+\mu+1} \leq 1. \tag{4.3}$$

Since $f(z)$ is in $\sigma_{c,p}(\alpha,\beta)$, we have

$$\sum_{m=1}^{\infty} \frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)} |a_m| \leq 1. \tag{4.4}$$

Thus (4.3) is satisfied if

$$\frac{(m+\delta)\mu}{(1-\delta)(m+\mu+1)} \leq \frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)}, \quad \text{for each } m$$

or

$$\delta \leq \frac{(m+2)^c[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) - m\mu(1-\alpha)}{(m+2)^c[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) + m\mu(1-\alpha)} \tag{4.5}$$

and

$$G(m) = \frac{(m+2)^c[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) - m\mu(1-\alpha)}{(m+2)^c[(1+\beta)(m+1)+(1-\alpha)](\mu+m+1) + m\mu(1-\alpha)}.$$

Then $G(m+1)-G(m)>0$, for each $m$. Hence $G(m)$ is increasing function of $m$. Since

$$G(1)=\frac{3^c(3+2\beta-\alpha)(\mu+2)-\mu(1-\alpha)}{3^c(3+2\beta-\alpha)(\mu+2)+\mu(1-\alpha)}.$$

The desired result follows. □

## 5. Convex Linear Combinations and Convolution Properties

**Theorem 5.1.** *If the function $f$ be in $\sigma_{c,p}(\alpha,\beta)$, then $f(z)$ is meromorphically convex of order $\delta$ $(0\leq\delta<1)$ in $|z|<r=r(\alpha,\beta,\delta)$, where*

$$r(\alpha,\beta,\delta)=\inf_{n\geq 1}\left\{\frac{(1-\delta)[(1+\beta)(1+m)+1-\alpha]}{(m+2)^c(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}$$

*The result is sharp.*

*Proof.* Let $f(z)$ be in $\sigma_{c,p}(\alpha,\beta)$. Then, by Theorem 2.1, we have

$$\sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c}|a_m|\leq(1-\alpha). \tag{5.1}$$

It is sufficient to show that

$$\left|2+\frac{zf''(z)}{f'(z)}\right|\leq 1-\delta,$$

for $|z|<r=r(\alpha,\beta,\delta)$, where $r(\alpha,\beta,\delta)$ is specified in the statement of the theorem. Then

$$\left|2+\frac{zf''(z)}{f'(z)}\right|=\left|\frac{\sum\limits_{m=1}^{\infty}m(m+1)a_mz^{m-1}}{\frac{-1}{z^2}+\sum\limits_{m=1}^{\infty}ma_mz^{m-1}}\right|\leq\sum_{m=1}^{\infty}\frac{m(m+1)a_m|z|^{m+1}}{1-\sum\limits_{m=1}^{\infty}ma_m|z|^{m+1}}.$$

This is bounded by $(1-\delta)$ if

$$\sum_{m=1}^{\infty}\frac{m(m+2-\delta)}{1-\delta}a_m|z|^{m+1}\leq 1. \tag{5.2}$$

By (5.1), it follow that (5.2) is true if

$$\frac{m(m+2-\delta)}{1-\delta}|z|^{m+1}\leq\frac{(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)},\ m\geq 1$$

or

$$|z|\leq\left\{\frac{(1-\delta)[(1+\beta)(1+m)+1-\alpha]}{(m+2)^c(1-\alpha)m(m+2-\delta)}\right\}^{\frac{1}{m+1}}. \tag{5.3}$$

Setting $|z|=r(\alpha,\beta,\delta)$ in (5.3), the result follows. The result is sharp for the function

$$f_m(z)=\frac{1}{z}+\frac{(m+2)^c(1-\alpha)}{[(1+\beta)(m+1)+1-\alpha]}z^m,\ (m\geq 1).$$

□

**Theorem 5.2.** *Let* $f_0(z) = \frac{1}{z}$ *and*

$$f_m(z) = \frac{1}{z} + \frac{(m+2)^c(1-\alpha)}{[(1+\beta)(m+1)+1-\alpha]}z^m,\ (m \geq 1).$$

*Then* $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ *is in the class* $\sigma_{c,p}(\alpha,\beta)$ *if and only if it can be expressed in the form* $f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$, *where* $\lambda_0 \geq 0, \lambda_m \geq 0\ (m \geq 1)$ *and* $\lambda_0 + \sum_{m=1}^{\infty} \lambda_m = 1$.

*Proof.* Let

$$f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$$

with $\lambda_0 \geq 0, \lambda_m \geq 0\ (m \geq 1)$ and

$$\lambda_0 + \sum_{m=1}^{\infty} \lambda_m = 1.$$

Then

$$\begin{aligned} f(z) &= \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z) \\ &= \frac{1}{z} + \sum_{m=1}^{\infty} \lambda_m \frac{(m+2)^c(1-\alpha)}{[(1+\beta)(m+1)+1-\alpha]}z^m. \end{aligned}$$

Since

$$\sum_{m=1}^{\infty} \frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)}\lambda_m \frac{(m+2)^c(1-\alpha)}{[(1+\beta)(m+1)+1-\alpha]} = \sum_{m=1}^{\infty} \lambda_m = 1-\lambda_0 \leq 1,$$

we obtain from Theorem 2.1 that $f$ is in the class $\sigma_{c,p}(\alpha,\beta)$.

Conversely, suppose that the function $f$ is in the class $\sigma_{c,p}(\alpha,\beta)$. Since

$$\begin{aligned} a_m &\leq \frac{(m+2)^c(1-\alpha)}{[(1+\beta)(m+1)+1-\alpha]},\ (m \geq 1) \\ \lambda_m &= \frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)}a_m, \end{aligned}$$

and $\lambda_0 = 1 - \sum_{m=1}^{\infty} \lambda_m$, it follows that $f(z) = \lambda_0 f_0(z) + \sum_{m=1}^{\infty} \lambda_m f_m(z)$. This completes the proof of the theorem. □

For the functions $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ and $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ belongs to $\sum_p$, we denote by $(f*g)(z)$ the convolution of $f(z)$ and $g(z)$ or

$$(f*g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m.$$

**Theorem 5.3.** *If the functions* $f(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m z^m$ *and* $g(z) = \frac{1}{z} + \sum_{m=1}^{\infty} b_m z^m$ *be in the class* $\sigma_{c,p}(\alpha,\beta)$, *then*

$$(f*g)(z) = \frac{1}{z} + \sum_{m=1}^{\infty} a_m b_m z^m$$

*is in the class* $\sigma_{c,p}(\alpha,\beta)$.

*Proof.* Suppose that $f(z)$ and $g(z)$ are in $\sigma_{c,p}(\alpha,\beta)$. By Theorem 2.1, we have

$$\sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)}a_m\le 1,$$
$$\sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)}b_m\le 1.$$

Since $f(z)$ and $g(z)$ are regular are in $E$, so is $(f*g)(z)$. Furthermore,

$$\sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)}a_mb_m\le\left\{\frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)}\right\}^2a_mb_m$$
$$\le\Big(\sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)}a_m\Big)\Big(\sum_{m=1}^{\infty}\frac{[(1+\beta)(m+1)+1-\alpha]}{(m+2)^c(1-\alpha)}b_m\Big)$$
$$\le 1.$$

Hence by Theorem 2.1, $(f*g)(z)$ is in the class $\sigma_{c,p}(\alpha,\beta)$. □

## 6. NEIGHBORHOODS FOR THE CLASS $\sigma_{c,p}(\alpha,\beta)$

**Definition 6.1.** A function $f\in\sum_p$ is said to in the class $\sigma_{c,p}(\alpha,\beta,\gamma)$ if there exists a function $g\in\sigma_{c,p}(\alpha,\beta)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\gamma,\ z\in E,\ (0\le\gamma<1). \tag{6.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [10] and Ruschweyh [11], we define the $\delta$-neighborhhod of a function $f\in\sum_p$ by

$$N_\delta(f):=\Big\{g\in\sum_p:\ g(z)=\frac{1}{z}+\sum_{m=1}^{\infty}b_mz^m:\ \sum_{m=1}^{\infty}m|a_m-b_m|\le\delta\Big\}. \tag{6.2}$$

**Theorem 6.2.** *If* $g\in\sigma_{c,p}(\alpha,\beta)$ *and*

$$\gamma=1-\frac{\delta(3+2\beta-\alpha)}{2+2\beta)}, \tag{6.3}$$

*then* $N_\delta(g)\subset\sigma_{c,p}(\alpha,\beta,\gamma)$.

*Proof.* Let $f\in N_\delta(g)$. It follows from (6.2) that

$$\sum_{m=1}^{\infty}m|a_m-b_m|\le\delta, \tag{6.4}$$

which implies the coefficient inequality that

$$\sum_{m=1}^{\infty}|a_m-b_m|\le\delta,\ (m\in N). \tag{6.5}$$

Since $g \in \sigma_{c,p}(\alpha,\beta)$, we have $\sum_{m=1}^{\infty} b_m < \frac{1-\alpha}{3+2\beta-\alpha}$. It follows that

$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\sum_{m=1}^{\infty}|a_m-b_m|}{1-\sum_{m=1}^{\infty} b_m} \leq \frac{\delta(3+2\beta-\alpha)}{2+2\beta} = 1-\gamma$$

provided $\gamma$ is given by (6.3). Hence, by definition $f \in \sigma_{c,p}(\alpha,\beta,\gamma)$ for $\gamma$ given by (6.3), we obtain the desired result immediately. □

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