Commun. Optim. Theory 2020 (2020), Article ID 4 https://doi.org/10.23952/cot.2020.4

Communications in Optimization Theory
Available online at http://cot.mathres.org

SEQUENTIALLY COMPACT S^{JS}-METRIC SPACES

KUSHAL ROY[1], ISMAT BEG[2,*], MANTU SAHA[1]

[1]Department of Mathematics, The University of Burdwan, Purba Bardhaman-713104, West Bengal, India
[2]Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore-53200, Pakistan

Abstract. We introduce the notion of sequentially compactness on S^{JS}-metric spaces and study the properties of sequentially compact S^{JS}-metric spaces. As an application, we obtain the results on fixed points of mappings defined on sequentially compact S^{JS}-metric spaces.

Keywords. Sequentially compact S^{JS}-metric space; Fixed point; Contractive type mapping.

1. Introduction

Compactness plays an extremely important role in mathematical analysis. A generalization of compactness is sequentially compact if every infinite sequence of points sampled from the space has an infinite subsequence that converges to some point of the space. Various equivalent notions of compactness, including sequential compactness and limit point compactness, can be developed in general metric spaces. In general topological spaces, however, different notions of compactness are not necessarily equivalent. The most useful notion, which is the standard definition of the unqualified term compactness, is phrased in terms of the existence of finite families of open sets that ”cover” the space in the sense that each point of the space lies in some set contained in the family. This more subtle notion exhibits compact spaces as generalizations of finite sets. In spaces that are compact in this sense, it is often possible to patch together information that holds locally into corresponding statements that hold throughout the space. Thus compactness is a sort of generalization of the notion of finiteness. The power of compactness is that it provides a finite structure for infinite sets in the situation where finiteness makes life easier (such as in optimization problems). Soon after the introduction of the concept of metric spaces by Maurice Fréchet in his seminal paper ”Sur quelques points du calcul fonctionnel”, it was felt by researchers that these conditions of metric are too abstract and unrealistic.

*Corresponding author.
E-mail addresses: kushal.roy93@gmail.com (K. Roy), ibeg@lahoreschool.edu.pk (I. Beg), mantusaha.bu@gmail.com (M. Saha).
Received November 2, 2019; Accepted February 20, 2020.

There are several attempts in the literatures to relax/generalize them by several researchers; see, e.g., fuzzy metric spaces [1], modular metric space with the Fatou property [2], generalized D-metric spaces [3,4], $b-$ metric spaces [5], pseudometric spaces/dislocated metric spaces [6], cone metric spaces [7], partial metric spaces [8], generalized cone metric spaces [9], JS-metric spaces [10] and so on.

Recently, Beg et al. [11] presented a very general notion of S^{JS}- metric (see Definition 2.1) which does not satisfy the triangle inequality and symmetry, and studied its properties with several examples.

In this paper, we continue this line of research and define the concept of sequential compactness on S^{JS}-metric spaces, study their properties and show its applications in fixed point theory. This paper is arranged in following manner. Section 2 presents basic preliminaries about S^{JS}-metric spaces. Section 3 proposes the notion of sequentially compact S^{JS}- metric space and study their properties. In Section 4, the applications to fixed point theory are given. Finally, Sections 5, we conclude the paper and proposes some directions for future work.

2. Preliminaries

First we give some basic preliminaries about S^{JS}-metric spaces due to Beg et al. [11] for subsequent use in Sections 3 and 4.

Let X be a nonempty set and $J : X^3 \to [0,\infty]$ be a function. Let us define the set

$$S(J,X,x) = \{\{x_n\} \subset X : \lim_{n\to\infty} J(x,x,x_n) = 0\}$$

for all $x \in X$.

Definition 2.1. Let X be a nonempty set and $J : X^3 \to [0,\infty]$ satisfy the following conditions:
(J_1) $J(x,y,z) = 0$ implies $x = y = z$ for any $x,y,z \in X$;
(J_2) there exists some $b > 0$ such that, for any $(x,y,z) \in X^3$ and $\{z_n\} \in S(J,X,z)$,

$$J(x,y,z) \leq b \limsup_{n\to\infty}(J(x,x,z_n) + J(y,y,z_n))$$

Then the pair (X,J) is called an S^{JS}-metric space.

Additionally if J also satisfies (J_3) $J(x,x,y) = J(y,y,x)$ for all $x,y \in X$, then we call it a symmetric S^{JS}-metric space.

Definition 2.2. Let (X,J) be an S^{JS}-metric space. Then a sequence $\{x_n\} \subset X$ is said to be convergent to an element $x \in X$ if $\{x_n\} \in S(J,X,x)$.

Proposition 2.3. *[11] In an S^{JS}-metric space (X,J), if $\{x_n\}$ converges to both x and y, then $x = y$.*

Next we give some topological properties of an S^{JS}-metric space.

Definition 2.4. Let (X,J) be an S^{JS}-metric space. The open and closed ball of center $x \in X$ and radius $r > 0$ in X are defined as follows

$$\begin{aligned} B_J(x,r) &= \{y \in X : J(x,x,y) < r\}; \\ B_J[x,r] &= \{y \in X : J(x,x,y) \leq r\} \end{aligned} \tag{2.1}$$

Theorem 2.5. *[11] Let (X,J) be an S^{JS}-metric space. Let $\tau = \{\emptyset\} \cup \{U(\neq \emptyset) \subset X :$ for any $x \in U$ there exists $r > 0$ such that $B_J(x,r) \subset U\}$. Then τ forms a topology on X.*

Definition 2.6. Let (X,J) be an S^{JS}-metric space and $F \subset X$. Then F is said to be closed if there exists an open set $U \subset X$ such that $F = U^c$.

Proposition 2.7. *[11] Let (X,J) be an S^{JS}-metric space and $F \subset X$ be closed. Let $\{x_n\} \subset F$ be such that $\{x_n\} \in S(J,X,x)$. Then $x \in F$.*

3. The sequentially compact S^{JS}-metric space

Now we define the sequentially compact S^{JS}-metric space and study their topological properties.

Definition 3.1. Let (X,J) and (Y,J') be two S^{JS}-metric spaces and $T : X \to Y$ be a mapping. Then T is called continuous at $a \in X$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $x \in X$, $J'(Ta,Ta,Tx) < \varepsilon$ whenever $J(a,a,x) < \delta$.

Definition 3.2. Let (X,J) be an S^{JS}-metric space. A family $\{A_\alpha\}_{\alpha\in\Lambda}$, where Λ is an indexing set, of nonempty subsets of X is said to have the finite intersection property if for any finite sub-collection of subsets from $\{A_\alpha\}_{\alpha\in\Lambda}$ has nonempty intersection.

Definition 3.3. An S^{JS}-metric space (X,J) is said to be sequentially compact if every sequence $\{x_n\}$ in X has a convergent subsequence.

Theorem 3.4. *If (X,J) is a sequentially compact S^{JS}-metric space, then every countable family of closed sets with the finite intersection property has non-empty intersection.*

Proof. Let $\{F_i\}_{i=1}^{\infty}$ be a countable family of closed sets with the finite intersection property. Also, let $x_i \in F_1 \cap F_2 \cap \ldots \cap F_i$ for all $i \in \mathbb{N}$. Then $\{x_n\}$ is a sequence in X. Now since X is sequentially compact, we have that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, which converges to some $x \in X$.

Now, if $x \notin \cap_{i=1}^{\infty} F_i$, then $x \notin F_j$ for some $j \in \mathbb{N}$. Therefore $x \in X \backslash F_j$ and so there exists some $r > 0$ such that $B_J(x,r) \subset X \backslash F_j$. Since $J(x,x,x_{n_k}) \to 0$ as $k \to \infty$, we have that there exists $N \in \mathbb{N}$ such that $J(x,x,x_{n_k}) < r$ for all $k \geq N$. Thus $x_{n_k} \in B_J(x,r) \subset X \backslash F_j$ for all $k \geq N$ but $x_m \in F_j$ for any $m \geq j$. If we choose $k_0 = \max\{N, j\}$, then $x_{n_{k_0}} \in F_j \cap (X \backslash F_j)$, which is a contradiction. Hence $x \in \cap_{i=1}^{\infty} F_i$ and thus $\cap_{i=1}^{\infty} F_i \neq \emptyset$. □

Proposition 3.5. *Let (X,J) be a sequentially compact S^{JS}-metric space and $F \subset X$ be closed. Then F is also sequentially compact.*

Proof. Let $\{x_n\} \subset F$ be an arbitrary sequence. Since X is sequentially compact , we have that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \in S(J,X,x)$ for some $x \in X$. As F is closed, therefore by Proposition 2.7, we have $x \in F$. Hence F is also sequentially compact. □

Proposition 3.6. *Let (X,J) be an S^{JS}-metric space and $A \subset X$ be sequentially compact. Then A is closed.*

Proof. To prove that A is closed, we have to show $X\backslash A$ is open. Without loss of generality, let us take $X\backslash A \neq \emptyset$. Let $x \in X\backslash A$ and let us assume that $B_J(x, \frac{1}{n}) \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. Also let $x_n \in B_J(x, \frac{1}{n}) \cap A$ for all $n \geq 1$. Since $J(x,x,x_n) < \frac{1}{n} \to 0$ as $n \to \infty$, we have that $\{x_n\}$ converges to x. Now A is sequentially compact. So $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\} \in S(J,X,y)$ for some $y \in A$. Therefore, by Proposition 2.3, we have $x = y$, a contradiction. So there exists some $N \in \mathbb{N}$ such that $B_J(x, \frac{1}{N}) \subset X\backslash A$. Hence A is closed. □

Proposition 3.7. (a) *The union of a finite collection of sequentially compact subsets of an S^{JS}-metric space is sequentially compact.*

(b) *The intersection of an arbitrary family of sequentially compact subsets of an S^{JS}-metric space is sequentially compact.*

Proof. (a) Let $\{A_1, A_2, ..., A_n\}$ be a finite collection of sequentially compact subsets of an S^{JS}-metric space (X,J).

Let $A = \cup_{i=1}^{n} A_i$ and $\{x_n\} \subset A$. Therefore, there exists at least one A_i, $1 \leq i \leq n$ such that A_i contains infinitely many terms of $\{x_n\}$. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}\mid_{A_i}$, which converges to some $x \in A_i$. Then x also belongs to A and hence A is also sequentially compact.

(b) Let $\{B_\gamma\}_{\gamma\in\Lambda}$ be an arbitrary collection of sequentially compact subsets of (X,J). Let $\{x_n\} \subset B = \cap_{\gamma\in\Lambda} B_\gamma$. Then $\{x_n\} \subset B_\gamma$ for all $\gamma \in \Lambda$. In particular, since B_γ is sequentially compact, we have that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, which converges to some $x \in B_\gamma$. Since each B_γ is closed, by Proposition 3.6, we get that B is closed in X. So $x \in B$ and this proves that B is sequentially compact. □

Proposition 3.8. *Let (X,J) and (Y,J') be two S^{JS}-metric spaces and $T : X \to Y$ be a continuous mapping. If X is sequentially compact, then $T(X)$ is also sequentially compact.*

Proof. Let $\{y_n\} \subset T(X)$ be an arbitrary sequence. Then there exists a sequence $\{x_n\}$ in X such that $y_n = Tx_n$ for all $n \in \mathbb{N}$. Since X is sequentially compact, we have that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in X, which converges to some $x \in X$. Since T is continuous, therefore, it can be easily seen that $\{Tx_{n_k}\} \in S(J', Y, Tx)$. Hence $T(X)$ is also sequentially compact. □

4. Applications to fixed points

Following the literatures [12–14], we now prove some contractive fixed point theorems on a sequentially compact S^{JS}- metric space.

Theorem 4.1. *Let (X,J) be a sequentially compact S^{JS}-metric space such that, for any continuous map $G : X \to X$, $J(x,x,Gx)$ is a continuous function on X. Suppose that T is a continuous map on X such that, for all $x,y \in X$ with $J(x,x,y) > 0$,*

$$J(Tx,Tx,Ty) < \max\{J(x,x,Tx), J(y,y,Ty), J(x,x,y)\}. \tag{4.1}$$

If, for some $z \in X$, $J(z,z,Tz) < \infty$, then T has a fixed point u in X with $d_J(u,u) = 0$. Moreover if u' is another fixed point of T such that $d_J(u',u') < \infty$ and $d_J(u,u') < \infty$, then $u = u'$.

Proof. Let us denote $d_J(x,y) = J(x,x,y)$ for all $x,y \in X$ and $z_n = T^n z$ for all $n \geq 0$. If $d_J(z_n, z_{n+1}) = 0$ for some $n \in \mathbb{N} \cup \{0\}$, then $z_n = z_{n+1}$ and we have $z_n = z_{n+k}$ for all $k \geq 1$. Therefore $d_J(z_n, z_{n+k}) =$

0 for all $k \in \mathbb{N}$. Clearly, $\{z_m\}$ converges to $u = z_n$. Since T is continuous, it follows that $Tu = u$. So we assume that $d_j(z_n, z_{n+1}) > 0$ for all $n \geq 0$. Also let $V(y) = d_J(y, Ty)$. Then by our assumption V is continuous. Now,

$$V(z_{n+1}) = d_j(z_{n+1}, z_{n+2}) < \max\{d_j(z_n, z_{n+1}), d_j(z_{n+1}, z_{n+2}), d_j(z_n, z_{n+1})\}$$

implies $V(z_{n+1}) < V(z_n)$ for all $n \in \mathbb{N} \cup \{0\}$, that is, for any $n \geq 0$,

$$V(z_{n+1}) < V(z_n) < V(z_0) = V(z) < \infty.$$

Thus there exists some $r \geq 0$ such that $\lim_{n\to\infty} V(z_n) = r$. Since the sequence $\{z_n\}$ has a convergent subsequence, which converges to some $u \in X$ and V is continuous, it follows that $V(u) = r$. Also since T is continuous we get $V(Tu) = r$.

Now if $r > 0$, then

$$r = V(Tu) = d_J(Tu, T^2u) < \max\{d_J(u, Tu), d_J(Tu, T^2u), d_J(u, Tu)\} = r,$$

which is a contradiction. Therefore $r = 0$ and $d_J(u, Tu) = d_J(Tu, T^2u) = 0$. So u is a fixed point of T. Now we prove that the sequence $\{z_n\}$ converges to u. For any given $\varepsilon > 0$, there exists a positive integer M such that $\{V(z_M), d_J(z_M, u)\} < \varepsilon$. Thus

$$\begin{aligned} d_J(z_n, u) &= d_J(Tz_{n-1}, Tu) \\ &< \max\{d_J(z_{n-1}, u), d_J(z_{n-1}, z_n), d_J(u, Tu)\} \\ &= \max\{V(z_{n-1}), d_J(z_{n-1}, u))\} \\ &< \max\{V(z_{n-1}), V(z_{n-2}), d_J(z_{n-2}, u)\} \\ &= \max\{V(z_{n-2}), d_J(z_{n-2}, u)\} \\ &\quad \dots \\ &< \max\{V(z_M), d_J(z_M, u)\} < \varepsilon \end{aligned} \tag{4.2}$$

for all $n > M$. Hence $\lim_{n\to\infty} z_n = u$. Clearly $d_J(u, u) = 0$.

Uniqueness Let u' be another fixed point of T such that $d_J(u, u') < \infty$. Since $d_J(u', u') < \infty$, it follows that $d_J(u', u') = 0$. Let $d_J(u, u') > 0$. Then

$$d_J(u, u') = d_J(Tu, Tu') < \max\{d_J(u, u), d_J(u', u'), d_J(u, u')\} = d_J(u, u'),$$

a contradiction. Hence $d_J(u, u') = 0$ and we get $u = u'$. □

Corollary 4.2. *Let (X, J) be a sequentially compact S^{JS}-metric space such that, for any continuous map $G : X \to X$, $J(x, x, Gx)$ is a continuous function on X. Also let T be a mapping on X such that, for all $x, y \in X$ with $J(x, x, y) > 0$,*

$$J(Tx, Tx, Ty) < J(x, x, y). \tag{4.3}$$

Then, for any z in X with $J(z, z, Tz) < \infty$, $\{T^n z\}$ converges to u and u is a fixed point of T.

Proof. Clearly, T satisfies all the conditions of Theorem 4.1. So the result of this Corollary follows immediately. □

Corollary 4.3. *Let (X,J) be a sequentially compact S^{JS}-metric space such that, for any continuous map $G : X \to X$, $J(x,x,Gx)$ is a continuous function on X. Also let T be a continuous mapping on X such that, for all $x,y \in X$ with $J(x,x,y) > 0$,*

$$J(Tx,Tx,Ty) < \frac{1}{2}[J(x,x,Tx)+J(y,y,Ty)]. \tag{4.4}$$

Then, for any z in X with $J(z,z,Tz) < \infty$, $\{T^n z\}$ converges to u and u is a fixed point of T.

Proof. Clearly T satisfies all the conditions of Theorem 4.1. So the result of this Corollary follows immediately. □

Corollary 4.4. *Let (X,J) be a sequentially compact S^{JS}-metric space such that, for any continuous map $G : X \to X$, $J(x,x,Gx)$ is a continuous function on X and T be a continuous mapping on X such that, for all $x,y \in X$ with $J(x,x,y) > 0$,*

$$J(Tx,Tx,Ty) < \max\{J(x,x,Tx),J(y,y,Ty)\}. \tag{4.5}$$

Then, for any z in X with $J(z,z,Tz) < \infty$, $\{T^n z\}$ converges to v and v is a fixed point of T.

Proof. Clearly T satisfies all the conditions of Theorem 4.1. So the result of this Corollary follows immediately. □

Definition 4.5. [15] Let X be a non-empty set and $S : X^3 \to [0,\infty)$ be a function satisfying the following conditions, for each $x,y,z,w \in X$,
(i) $S(x,y,z) = 0$ if and only if $x = y = z$;
(ii) $S(x,y,z) \le S(x,x,w)+S(y,y,w)+S(z,z,w)$.
The function S is called an S-metric and the pair (X,S) is called an S-metric space.

Obviously, every S-metric space is a S^{JS}-metric space. Converse is not always true.

Corollary 4.6. *Let (X,S) be a sequentially compact S-metric space and T be a continuous mapping on X such that, for all $x,y \in X$ with $x \neq y$,*

$$S(Tx,Tx,Ty) < \max\{S(x,x,Tx),S(y,y,Ty),S(x,x,y)\}.$$

Then, for any y in X, $\{T^n y\}$ converges to v and v is a fixed point of T.

Proof. If G is a continuous mapping, then, for any $x \in X$ and any sequence $\{x_n\}$ in X converging to x,

$$|S(x_n,x_n,Gx_n)-S(x,x,Gx)| \le 2[S(x_n,x_n,x)+S(Gx_n,Gx_n,Gx)] \to 0$$

as $n \to \infty$. Therefore $S(x,x,Gx)$ is a continuous function on X whenever G is continuous. Therefore all the conditions of Theorem 4.1 are satisfied and it follows that v is a fixed point of T. □

Example 4.7. Let $X = [0,\infty)$ and $J : X^3 \to X$ be defined by $J(x,y,z) = |x-y|+|x-z|$ for all $x,y,z \in X$ and $T : X \to X$ be defined by $Tx = x-\tan^{-1} x$ for all $x \in X$. Then T satisfies $J(Tx,Tx,Ty) < J(x,x,y)$ for all $x,y \in X$ and also for $x_0 = 0$ the iterative sequence $\{T^n x_0\}$ converges to 0 and 0 is the unique fixed point of T.

Example 4.8. Let $X=[0,5]$ and $J:X^3\to X$ be defined by $J(x,y,z)=|x-y|+|x-z|$ for all $x,y,z\in X$ and $T:X\to X$ be defined by

$$Tx=\begin{cases}\frac{3x}{4}, & \text{if } x\in[0,4]\\ 10-\frac{7x}{4}, & \text{if } x\in[4,5]\end{cases} \tag{4.6}$$

Then T is continuous and satisfies the contractive condition (4.6) but it does not satisfy $J(Tx,Tx,Ty)<J(x,x,y)$ for all $x,y\in X$ since $J(T4,T4,T5)=\frac{7}{4}\not< J(4,4,5)$. Thus all the conditions of Corollary 4.6 are satisfied. Also for $z_0=0$, the iterative sequence $\{T^n z_0\}$ converges to 0 and 0 is the unique fixed point of T.

5. Conclusion

In this paper, we started the study of sequentially compact S^{JS}-metric spaces, their properties and the applications in fixed point theory. This is a very challenging area of research and there are vast opportunities for further work on sequentially compact S^{JS}-metric spaces. In future, we continue to study the applications in optimization theory, Bolzano Weierstrass theorem, Lebesgue's covering theorem, the notion of totally bounded and Ascoli's theorem on sequentially compact S^{JS}-metric spaces.

Acknowledgements

The first author was financial support by the research fellowship (Grant No. 09/025(0223)/2017-EMR-I) awarded by the Council of Scientific and Industrial Research, New Delhi, India.

References

[1] O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems 12 (1984), 215-229.
[2] W.M. Kozlowski, Moduler Function Spaces, Monographs and Textbooks in Pure and Applied Mathematics, 122 (1988), Dekker, New York.
[3] B.C. Dhage, Generalized metric spaces and mapping with fixed point, Bull. Calcutta Math. Soc. 84 (1992), 329-336
[4] Z. Mustafa, B. Sim, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289-297.
[5] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5-11.
[6] P. Hitzler, A.K. Seda, Dislocated topologies, J. Electr. Eng. 51 (2000), 3-7.
[7] L. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.
[8] M. Bukatin, R. Kopperman, S. Mathews, H. Pajoohesh, Partial metric spaces, Amer. Math. Monthly 116 (2009), 708-718.
[9] I. Beg, M. Abbas, T. Nazir, Generalized cone metric spaces, J. Nonlinear Sci. Appl. 3 (2010), 21-31.
[10] M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, Fixed Point Theory Appl. 2015 (2015), 61.
[11] I. Beg, K. Roy, M. Saha, S^{JS}-metric and topological spaces, submitted.
[12] M. Edelstein, On fixed points and periodic points under contractive mappings, J. London Math. Soc. 37 (1962), 74-79.
[13] W. Kirk, N. Shahzad, Fixed Point Theory in Distance Spaces, Springer Switzerland, 2014.
[14] V.M. Sehgal, On fixed and periodic points of a class of mappings, J. London Math. Soc. 5 (1972), 571-576.
[15] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in $S-$metric spaces, Mat. Vesnik, 64 (2012), 258-266.