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STRONG CONVERGENCE OF A MODIFIED MANN ALGORITHM FOR MULTIVALUED QUASI-NONEXPANSIVE MAPPINGS AND MONOTONE MAPPINGS WITH AN APPLICATION

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Abstract. In this paper, we introduce and study a new iterative method which is a combination of a projection method and a modified Mann method for finding a common element of the set of solutions of variational inequality problems for a monotone mappings and the set of fixed points of multivalued quasi-nonexpansive mappings in an infinite dimensional Hilbert space. We prove that the sequences generated by the proposed algorithm converge strongly to a common element of in the set of common solutions. Finally, we apply our results to the problem of finding a common solution of fixed points problems involving multivalued quasi-nonexpansive mappings and optimization problems.

Keywords. Fixed points; Nonlinear multivalued mappings; Variational inequality, Quasi-nonexpansive mappings.

1. INTRODUCTION

Let H be a real Hilbert space. Let $A : H \to H$ be a mapping. The domain of A, $D(A)$, the image of a subset S of H, $A(S)$, the range of A, $R(A)$ and the graph of A, $G(A)$ are defined as follows:

$$D(A) := \{x \in H : Ax \neq \emptyset\},\ A(S) := \cup\{Ax : x \in S\},$$

$$R(A) := A(H),\ G(A) := \{[x,u] :\ x \in D(A),\ u \in Ax\}.$$

Let K be a nonempty, closed and convex subset of H. An operator $A : K \to H$ is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle_H \geq 0,\ \forall\, x, y \in K,$$

A is said to be *k-strongly monotone* if there exists $k \in (0,1)$ such that

$$\langle Ax - Ay, x - y\rangle_H \geq k\|x - y\|^2,\ \forall\, x, y \in K.$$

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An operator $A : K \to H$ is said to be *strongly positive bounded* if there exists a constant $k > 0$ such that

$$\langle Ax, x\rangle_H \geq k\|x\|^2, \ \forall\, x \in K.$$

From the definition of A, we find that a strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and k-strongly monotone operator.

An operator $A : K \to H$ is said α-*inverse strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle_H \geq \alpha\|Ax - Ay\|^2, \ \forall\, x, y \in K.$$

It is immediate that if A is α-inverse strongly monotone, then A is monotone and Lipschitz continuous. The problem of finding $u \in K$ such that

$$\langle Au, v - u\rangle \geq 0, \ \forall v \in K, \tag{1.1}$$

is called the classical variational inequality problem. We denote the set of solutions of variational inequality problem (1.1) by $VI(K,A)$.

The variational inequality problem was formulated in the late 1960's by Lions and Stampacchia [8]. Since then, it has been extensively studied. In numerous models for solving real-life problems, such as, in signal processing, networking, resource allocation, image recovery, and so on, the constraints can be expressed as variational inequality problems. Consequently, the problem of finding solutions of variational inequality problems has become a flourishing area of contemporary research for researchers working in nonlinear analysis and optimization theory; see, for example, [3, 10, 11] and the references therein. In most of the early results on iterative methods for approximating solutions of the variational inequality problem, the map A was often assumed to be inverse strongly monotone.

A well-known method for solving the variational inequality problem is the projection method which starts with $x_1 \in K$ and generates a sequence $\{x_n\}$ as following

$$x_{n+1} = P_K(x_n - \lambda_n Ax_n), \ n \geq 1, \tag{1.2}$$

where $\{\lambda_n\}$ is a sequence of positive numbers satisfying appropriate conditions. In the case that A is α-inverse strongly monotone, Iiduka, Takahashi and Toyoda [7] proved that the sequence $\{x_n\}$ generated by (1.2) converges weakly to an element of $VI(K,A)$.

Let (X,d) be a metric space, K a nonempty subset of X and $T : K \to 2^K$ a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$. Let D be a nonempty subset of a normed space E. The set D is said to be *proximinal* (see, e.g., [16]) if, for each $x \in E$, there exists $u \in D$ such that

$$d(x,u) = \inf\{\|x-y\| : y \in D\} = d(x,D),$$

where $d(x,y) = \|x-y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $K(D)$ and $P(D)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D, respectively. The Pompeiu *Hausdorff metric* on $CB(K)$ is defined by

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for all $A,B \in CB(K)$ (see, Berinde [2]). A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is said to be *L- Lipschitzian* if there exists $L > 0$ such that

$$H(Tx,Ty) \le L\|x-y\|, \ \forall x,y \in D(T). \tag{1.3}$$

If $L \in (0,1)$, we say that T is a *contraction*, and T is said to be *nonexpansive* if $L = 1$. A multivalued map T is said to be quasi-nonexpansive if $H(Tx,Tp) \le \|x-p\|$ holds for all $x \in D(T)$ and $p \in F(T)$.

It is easy to see that the class of mulivalued quasi-nonexpansive mappings properly includes that of multivalued nonexpansive maps with fixed points. Many problems arising in different areas of mathematics such as, optimization, differential equations, mathematical economics, and game theory can be modeled as fixed point equations of the form $x \in Tx$, where T is a multivalued nonexpansive mapping. There are many effective algorithms for solving fixed point problems, see, for example, [13–15] and the references therein.

Historically, one of the most investigated methods for approximating fixed points of single-valued nonexpansive mappings dates back to 1953 and is known as Mann's method, in light of Mann [9]. Let C be a nonempty, closed and convex subset of a Banach space X, Mann's scheme is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. It is known that Mann's iteration process has only weak convergence, even in Hilbert space setting. Therefore, many authors try to modify Mann's iteration to have strong convergence for various nonlinear operators.

Recently, Zeng and Yao [4] introduced a new extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. They obtained the following strong convergence theorem.

Theorem 1.1 (Zeng and Yao [4])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone k-Lipschitz continuous mapping, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \cap VI(C,A) \neq \emptyset$. Let the sequences $\{x_n\}$, $\{y_n\}$ be generated by*

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.5}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the following conditions:
(a) $\{\lambda_n k\} \subset (0,1-\delta)$ for some $\delta \in (0,1)$,
(b) $\{\alpha_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to the same point* $P_F(T) \cap VI(C,A)(x_0)$ *provided that*

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{1.6}$$

We remark that the iterative scheme (1.5) is strongly convergent with the aid of assumption (1.6) on the sequence $\{x_n\}$.

The above results naturally bring us to the following question.

Question Can we construct an iterative algorithm based on a modified Mann method for finding a common element of the set of fixed points of multivalued quasi-nonexpansive mapping and the set of solutions of a variational inequality problem without imposing rigid conditions like (1.6)?

The main aim of this paper is to give an affirmative answer to the question raised. We introduce and study a new approximation method for finding a common element of the set of solutions of variational inequality problem for an inverse strongly monotone mappings and the set of fixed points of multivalued quasi-nonexpansive mappings. We prove the strong convergence of the proposed algorithm without imposing any compactness conditions on either the operators or the space considered in Hilbert spaces. The results presented in the paper extend and improve some recent results announced in the current literatures.

2. Preliminaries

Let us recall the following definitions and results, which will be used in the sequel.

Let H be a real Hilbert space. Let $\{x_n\}$ be a sequence in H, and let $x \in H$. Weak convergence of x_n to x is denoted by $x_n \rightharpoonup x$ and strong convergence by $x_n \to x$. Let K be a nonempty, closed convex subset of H. The nearest point projection from H to K, denoted by P_K, assigns each $x \in H$ to the unique $P_K x$ with the property

$$\|x - P_K x\| \le \|y - x\|$$

for all $y \in K$. It is well known that P_K satisfies

$$\langle x - y, P_K x - P_K y\rangle \ge \|P_K x - P_K y\|^2 \tag{2.1}$$

and

$$\langle P_K z - y,\ z - P_K z\rangle \ge 0 \tag{2.2}$$

for all $z \in K$ and $y \in H$.

Remark 2.1. In the context of the variational inequality problem, this implies

$$u \in VI(K,A) \Longleftrightarrow u = P_K(I - \theta A)u,\ \ \theta > 0. \tag{2.3}$$

Lemma 2.2 (Demiclosedness Principle [6]). *Let H be a real Hilbert space, and K a nonempty closed and convex subset of H. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values. Then $I - T$ is demi-closed at zero.*

Lemma 2.3 ([5]). *Let H be a real Hilbert space. Then, for any $x, y \in H$, the following inequalities hold:* $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$, *and*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - (1-\lambda)\lambda\|x-y\|^2,\ \lambda \in (0,1).$$

Lemma 2.4 (Xu [19])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n$ for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that*
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |\sigma_n\alpha_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *[12] Let $\{t_n\}$ be a sequence of real numbers that does not decrease at infinity in a sense that there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $\{t_{n_i}\}$ such that $t_{n_i} \leq t_{n_{i+1}}$ for all $i \geq 0$. For sufficiently large numbers $n \in \mathbb{N}$, an integer sequence $\{\tau(n)\}$ is defined as follows:*

$$\tau(n) = \max\{k \leq n : \ t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \to \infty$ as $n \to \infty$ and

$$\max\{t_{\tau(n)}, \ t_n\} \leq t_{\tau(n)+1}.$$

Lemma 2.6. *(Rockafellar [17]) Let C be a nonempty closed and convex subset of a real Hilbert space H and A be a monotone, hemicontinuous map of C into H. Let $B \subset H \times H$ be an operator defined as follows:*

$$Bz = \begin{cases} Az + N_K(z) & si \quad z \in K, \\ \emptyset & si \quad z \notin K, \end{cases} \tag{2.4}$$

where $N_K(z)$ is the normal K at z and is defined as follows:

$$N_K(z) = \{w \in H \ : \ \langle w, z-v \rangle \geq 0 \ \ \forall\, v \in K\}.$$

Then, B is maximal monotone and $B^{-1}(0) = VI(C,A)$.

Lemma 2.7. *Let H be a real Hilbert space and K be a nonempty, closed convex subset of H. Let $A : K \to H$ be an α-inverse strongly monotone mapping. Then, $I - \theta A$ is nonexpansive mapping for all $x, y \in K$ and $\theta \in [0, 2\alpha]$.*

Proof. For all $x, y \in K$, we have

$$\begin{aligned} \|(I-\theta A)x - (I-\theta A)y\|^2 &= \|(x-y) - \theta(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\theta\langle Ax - Ay, x-y\rangle + \|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 + \theta(\theta - 2\alpha)\|Ax-Ay\|^2. \end{aligned}$$

This completes the short proof. □

3. Main Results

We now prove the following result.

Theorem 3.1. *Let K be a nonempty, closed convex cone of a real Hilbert space H and $A : K \to H$ be an α-inverse strongly monotone operator. Let $T : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping such that $G := F(T) \cap VI(K,A) \neq \emptyset$ and $Tp = \{p\}$ $\forall p \in G$. Let $\{x_n\}$ be* *a sequence defined as follows:*

$$\begin{cases} x_0 \in K, \\ z_n = P_K(I - \theta_n A)x_n, \\ y_n = \beta_n z_n + (1-\beta_n)v_n,\ v_n \in Tz_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \end{cases} \tag{3.1}$$

where $\{\beta_n\}$, $\{\lambda_n\}$, $\{\theta_n\}$ *and* $\{\alpha_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*
(i) $\lim\limits_{n\to\infty} \alpha_n = 0$, (ii) $\liminf\limits_{n\to\infty} \beta_n(1-\beta_n) > 0$ *and* $\theta_n \in [a,b] \subset \big(0, min\{1, 2\alpha\}\big)$,
(iii) $\lim\limits_{n\to\infty} \lambda_n = 1$ *and* $\sum\limits_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.
Assume that $I - T$ *are demiclosed at origin. Then, the sequences* $\{x_n\}$ *and* $\{z_n\}$ *generated by* (3.1) *converge strongly to* $x^* \in G$*, where* $x^* = P_G(0)$*, with* P_G *the the metric projection from* K *onto* G*.*

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Let $p \in G$. Using (3.1), the fact that T is quasi-nonexpansive, inequality (3.15) and Lemma 2.7, we have

$$\begin{aligned} \|y_n - p\| &\leq \beta_n\|z_n - p\| + (1-\beta_n)\|v_n - p\| \\ &\leq \beta_n\|z_n - p\| + (1-\beta_n)H(Tz_n, Tp) \\ &\leq \beta_n\|z_n - p\| + (1-\beta_n)\|z_n - p\| \\ &= \|P_K(I-\theta_n A)x_n - P_K(I-\theta_n A)p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

Hence,

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{3.2}$$

Using (3.1) and inequality (3.2), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\| \\ &\leq \alpha_n\lambda_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq \alpha_n\lambda_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq [1-(1-\lambda_n)\alpha_n]\|x_n - p\| + (1-\lambda_n)\alpha_n\|p\| \\ &\leq \max\{\|x_n - p\|,\ \|p\|\}. \end{aligned}$$

By induction, it is easy to see that

$$\|x_n - p\| \leq \max\{\|x_0 - p\|,\ \|p\|\},\ \ n \geq 1.$$

Hence $\{x_n\}$ is bounded, so are $\{z_n\}$, and $\{Tx_n\}$. Let $p \in G$. From (3.1), inequality (3.2) and Lemma 2.3, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_n z_n + (1-\beta_n)v_n - p\|^2 \\
&= (1-\beta_n)\|v_n - p\|^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|v_n - z_n\|^2 \\
&\leq (1-\beta_n)H(Tz_n, Tp)^2 + \beta_n\|z_n - p\|^2 - \beta_n(1-\beta_n)\|z_n - v_n\|^2 \\
&\leq (1-\beta_n)\|x_n - p\|^2 + \beta_n\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - z_n\|^2.
\end{aligned}$$

Hence,

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \beta_n(1-\beta_n)\|z_n - v_n\|^2. \tag{3.3}$$

Therefore, by Lemma 2.3 and inequality (3.3), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n - p\|^2 \\
&= \|\alpha_n\lambda_n\big(x_n - p\big) + (1-\alpha_n)\big(y_n - p\big) - (1-\lambda_n)\alpha_n p\|^2 \\
&\leq \|\alpha_n\big(\lambda_n x_n - \lambda_n p\big) + (1-\alpha_n)\big(y_n - p\big)\|^2 + 2(1-\lambda_n)\alpha_n\langle p, p - x_{n+1}\rangle \\
&\leq \alpha_n\lambda_n^2\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 + 2(1-\lambda_n)\alpha_n\langle p, p - x_{n+1}\rangle \\
&\leq \alpha_n\lambda_n\|x_n - p\|^2 + (1-\alpha_n)\Big[\|x_n - p\|^2 - \beta_n(1-\beta_n)\|z_n - v_n\|^2\Big] \\
&\quad + 2(1-\lambda_n)\alpha_n\langle p, p - x_{n+1}\rangle \\
&\leq [1 - (1-\lambda_n)\alpha_n]\|x_n - p\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|z_n - v_n\|^2 \\
&\quad + 2(1-\lambda_n)\alpha_n\langle p, p - x_{n+1}\rangle.
\end{aligned}$$

Therefore,

$$(1-\alpha_n)\beta_n(1-\beta_n)\|z_n - v_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2(1-\lambda_n)\alpha_n\langle p, p - x_{n+1}\rangle. \tag{3.4}$$

Since $\{x_n\}$ is bounded, there exists a constant $B > 0$ such that

$$(1-\lambda_n)\langle p, p - x_{n+1}\rangle \leq B, \ \text{ for all } n \geq 0.$$

Hence,

$$(1-\alpha_n)\beta_n(1-\beta_n)\|z_n - v_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n B. \tag{3.5}$$

Now we prove that $\{x_n\}$ converges strongly to x^*.

We divide the proof into two cases.

Case 1. Assume that the sequence $\{\|x_n - p\|\}$ is monotonically decreasing sequence. Then $\{\|x_n - p\|\}$ is convergent. Clearly, we have

$$\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \to 0. \tag{3.6}$$

It then implies from (3.5) that

$$\lim_{n\to\infty}(1-\alpha_n)\beta_n(1-\beta_n)\|z_n - v_n\|^2 = 0. \tag{3.7}$$

Using the fact that $\lim_{n\to\infty}\inf\beta_n(1-\beta_n) > 0$, we have

$$\lim_{n\to\infty}\|z_n - v_n\| = 0. \tag{3.8}$$

Hence,

$$\lim_{n\to\infty} d(z_n, Tz_n) = 0. \tag{3.9}$$

From (3.1), convexity of $\|.\|^2$ and Lemma 2.7, it follows that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(\lambda_n x_n)+(1-\alpha_n)y_n-p\|^2 \\
&\le \alpha_n\|(\lambda_n x_n)-p\|^2+(1-\alpha_n)\|y_n-p\|^2 \\
&\le \alpha_n\|(\lambda_n x_n)-p\|^2+(1-\alpha_n)\|z_n-p\|^2 \\
&= \alpha_n\|(\lambda_n x_n)-p\|^2+(1-\alpha_n)\|P_K(I-\theta_n A)x_n-P_K(I-\theta_n A)p\|^2 \\
&\le \alpha_n\|(\lambda_n x_n)-p\|^2+(1-\alpha_n)\Big[\|x_n-p\|^2+\theta_n(\theta_n-2\alpha)\|Ax_n-Ap\|^2\Big] \\
&\le \alpha_n\|(\lambda_n x_n)-p\|^2+(1-\alpha_n)\|x_n-p\|^2+(1-\alpha_n)a(b-2\alpha)\|Ax_n-Ap\|^2.
\end{aligned}$$

Therefore, we have

$$(1-\alpha_n)a(2\alpha-b)\|Ax_n-Ap\|^2 \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|(\lambda_n x_n)-p\|^2.$$

Since, $\alpha_n \to 0$ as $n\to\infty$, inequality (3.6) and $\{x_n\}$ is bounded, we obtain

$$\lim_{n\to\infty}\|Ax_n-Ap\|^2 = 0. \tag{3.10}$$

By using inequality (2.1) and (3.1), we have

$$\begin{aligned}
\|z_n-p\|^2 &= \|P_K(I-\theta_n A)x_n-P_K(I-\theta_n A)p\|^2 \\
&\le \langle z_n-p, P_K(I-\theta_n A)x_n-P_K(I-\theta_n A)p\rangle \\
&= \frac{1}{2}\Big[\|(I-\theta_n A)x_n-(I-\theta_n A)p\|^2+\|z_n-p\|^2 \\
&- \|(I-\theta_n A)x_n-(I-\theta_n A)p-(z_n-p)\|^2\Big] \\
&\le \frac{1}{2}\Big[\|x_n-p\|^2+\|z_n-p\|^2-\|x_n-z_n\|^2+2\theta_n\langle z_n-p, Ax_n-Ap\rangle \\
&- {\theta_n}^2\|Ax_n-Ap\|^2\Big].
\end{aligned}$$

So, we obtain

$$\|z_n-p\|^2 \le \|x_n-p\|^2-\|x_n-z_n\|^2+2\theta_n\langle z_n-p, Ax_n-Ap\rangle-{\theta_n}^2\|Ax_n-Ap\|^2,$$

and thus

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n\|(\lambda_n x_n)-p\|^2+(1-\alpha_n)\|y_n-p\|^2 \\
&\le \alpha_n\|(\lambda_n x_n)-p\|^2+(1-\alpha_n)\|z_n-p\|^2 \\
&\le \alpha_n\|(\lambda_n x_n)-p\|^2+\|x_n-p\|^2-(1-\alpha_n)\|x_n-z_n\|^2-(1-\alpha_n){\theta_n}^2\|Ax_n-Ap\|^2 \\
&+ 2\theta_n(1-\alpha_n)\langle z_n-p, Ax_n-Ap\rangle.
\end{aligned}$$

Since, $\alpha_n \to 0$ as $n\to\infty$, inequalities (3.6) and (3.10), we obtain

$$\lim_{n\to\infty}\|x_n-z_n\|^2 = 0. \tag{3.11}$$

Since H is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to a in K and

$$\limsup_{n\to+\infty}\langle x^*, x^*-x_n\rangle = \lim_{k\to+\infty}\langle x^*, x^*-x_{n_k}\rangle.$$

From (3.9) and $I-T$ is demiclosed, we obtain $a\in F(T)$. Let us show $a\in VI(K,A)$. Now, let us introduce the multivalued map $B: H\to 2^H$ defined by:

$$Bz = \begin{cases} Az+N_K(z), & \text{if } \ z\in K,\\ \emptyset, & \text{if } \ z\notin K, \end{cases} \tag{3.12}$$

where $N_K(z)$ is the normal K at z and is defined as follows:

$$N_K(z) = \{w\in H \ : \ \langle w, z-v\rangle \geq 0 \ \forall\, v\in K\}.$$

From Lemma 2.6, we have that B is maximal monotone and $B^{-1}(0) = VI(K,A)$. Let $(u,v)\in G(A)$. Since $v-Au\in N_K(u)$ and $z_n\in K$, we have

$$\langle u-z_n, v-Au\rangle \geq 0.$$

On other hand, from $z_n = P_K(I-\theta_n A)x_n$, we have $\langle u-z_n, z_n-(I-\theta_n A)x_n\rangle \geq 0$. Hence,

$$\langle u-z_n, \frac{z_n-x_n}{\theta_n}+Ax_n\rangle \geq 0.$$

Therefore,

$$\begin{aligned}
\langle u-z_{n_k}, v\rangle &\geq \langle u-z_{n_k}, Au\rangle\\
&\geq \langle u-z_{n_k}, Au\rangle - \langle u-z_{n_k}, \frac{z_{n_k}-x_{n_k}}{\theta_{n_k}}+Ax_{n_k}\rangle\\
&\geq \langle u-z_{n_k}, Au-Az_{n_k}\rangle + \langle u-z_{n_k}, Az_{n_k}-Ax_{n_k}\rangle - \langle u-z_{n_k}, \frac{z_{n_k}-x_{n_k}}{\theta_{n_k}}\rangle\\
&\geq \langle u-z_{n_k}, Az_{n_k}-Ax_{n_k}\rangle - \langle u-z_{n_k}, \frac{z_{n_k}-x_{n_k}}{\theta_{n_k}}\rangle.
\end{aligned}$$

By using the fact that A is $\dfrac{1}{\alpha}$ Lipschitz, we have

$$\langle u-z_{n_k}, v\rangle \geq -M\Big(\frac{\|z_{n_k}-x_{n_k}\|}{\alpha}+\frac{\|z_{n_k}-x_{n_k}\|}{a}\Big),$$

where M is a positive constant such that $\sup_{k\geq 1}\{\|u-z_{n_k}\|\}\leq M$. Since $z_{n_k}\rightharpoonup a$, it follows from (3.11) that $\langle u-a, v\rangle \geq 0$ as $k\to\infty$. Since B is maximal monotone, we have $a\in B^{-1}(0)$ and $a\in VI(K,A)$. Therefore, $a\in G$. On other hand, using the fact that $x^* = P_G(0)$, we have

$$\begin{aligned}
\limsup_{n\to+\infty}\langle x^*, x^*-x_n\rangle &= \lim_{k\to+\infty}\langle x^*, x^*-x_{n_k}\rangle\\
&= \langle x^*, x^*-a)\rangle \leq 0.
\end{aligned}$$

Finally, we show that $x_n \to x^*$. From (3.1), we get that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \langle x_{n+1}-x^*, x_{n+1}-x^*\rangle \\
&= \alpha_n\lambda_n\langle x_n-x^*, x_{n+1}-x^*\rangle + (1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle \\
&+ (1-\alpha_n)\langle y_n-x^*, x_{n+1}-x^*\rangle \\
&\leq \alpha_n\lambda_n\|x_n-x^*\|\|x_{n+1}-x^*\| + (1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle \\
&+ (1-\alpha_n)\|x_n-x^*\|\|x_{n+1}-x^*\| \\
&\leq [1-(1-\lambda_n)\alpha_n]\|x_n-x^*\|\|x_{n+1}-x^*\| + (1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle \\
&\leq \frac{1-(1-\lambda_n)\alpha_n}{2}(\|x_n-x^*\|^2 + \|x_{n+1}-x^*\|^2) + (1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1}-x^*\|^2 \leq [1-(1-\lambda_n)\alpha_n]\|x_n-x^*\| + 2(1-\lambda_n)\alpha_n\langle x^*, x^*-x_{n+1}\rangle.$$

We can check that all the assumptions of Lemma 2.4 are satisfied. Therefore, $x_n \to x^*$.
Case 2. Assume that the sequence $\{\|x_n-x^*\|\}$ is not monotonically decreasing sequence.

Let $B_n = \|x_n-x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ B_k \leq B_{k+1}\}$. We have that τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. From (3.5), we have

$$(1-\alpha_{\tau(n)})\beta_{\tau(n)}(1-\beta_{\tau(n)})\|z_{\tau(n)} - v_{\tau(n)}\|^2 \leq 2\alpha_{\tau(n)}B \to 0 \text{ as } n \to \infty.$$

Furthermore, we have

$$\|z_{\tau(n)} - v_{\tau(n)}\| \to 0 \text{ as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty} d\Big(z_{\tau(n)}, Tz_{\tau(n)}\Big) = 0. \tag{3.13}$$

By same argument as in Case 1, we can show that $x_{\tau(n)}$ converges weakly in H and $\limsup_{n\to+\infty}\langle x^*, x^* - x_{\tau(n)}\rangle \leq 0$. We have for all $n \geq n_0$,

$$0 \leq \|x_{\tau(n)+1}-x^*\|^2 - \|x_{\tau(n)}-x^*\|^2 \leq \Big(1-\lambda_{\tau(n)}\Big)\alpha_{\tau(n)}[-\|x_{\tau(n)}-x^*\|^2 + 2\langle x^*, x^*-x_{\tau(n)+1}\rangle],$$

which implies that

$$\|x_{\tau(n)}-x^*\|^2 \leq 2\langle x^*, x^*-x_{\tau(n)+1}\rangle.$$

Then,

$$\lim_{n\to\infty}\|x_{\tau(n)}-x^*\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} B_{\tau(n)} = \lim_{n\to\infty} B_{\tau(n)+1} = 0.$$

Thus, by Lemma 2.5, we conclude that

$$0 \leq B_n \leq \max\{B_{\tau(n)},\ B_{\tau(n)+1}\} = B_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} B_n = 0$, that is $\{x_n\}$ converges strongly to x^*. This completes the proof. □

We now apply Theorem 3.1 for finding a common element of the fixed point problem involving multivalued nonexpansive and the set of solutions of variational inequality problem without the demiclosedness assumption.

Theorem 3.2. *Let K be a nonempty, closed convex cone of a real Hilbert space H and $A : K \to H$ be an α-inverse strongly monotone operator. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values such that $G := F(T) \cap VI(K,A) \neq \emptyset$ and $Tp = \{p\}$ $\forall p \in G$. Let $\{x_n\}$ be a sequence defined as follows:*

$$\begin{cases} x_0 \in K, \\ z_n = P_K(I - \theta_n A)x_n, \\ y_n = \beta_n z_n + (1-\beta_n)v_n, \ v_n \in Tz_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \end{cases} \tag{3.14}$$

where $\{\beta_n\}$, $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\quad (ii)$ $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$ and $\theta_n \in [a,b] \subset \big(0, min\{1, 2\alpha\}\big)$,
(iii) $\lim_{n\to\infty} \lambda_n = 1$ and $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.
Then, the sequences $\{x_n\}$ and $\{z_n\}$ generated by (3.14) converge strongly to $x^* \in G$, where $x^* = P_G(0)$, with P_G the the metric projection from K onto G.

Proof. Since every multivalued nonexpansive mapping is multivalued quasi-nonexpansive, we can obtain from Lemma 2.2 and Theorem 3.1 the desired theorem immediately. □

Remark 3.3. We assume that K is a cone. But, in some cases, for example, if K is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in K$ for all $\lambda \in (0,1)$ and $x \in K$. Therefore, our results can be used to approximate a common element of the set of fixed points of multivalued quasi-nonexpansive mapping and the set of solutions of a variational inequality problem from the closed unit ball to itself.

Corollary 3.4. *Let H be a real Hilbert space and B be the closed unit ball of H. Let $A : B \to H$ be an α-inverse strongly monotone operator. Let $T : B \to CB(B)$ be a multivalued quasi-nonexpansive mapping such that $G := F(T) \cap VI(K,A) \neq \emptyset$ and $Tp = \{p\}$ $\forall p \in G$. Let $\{x_n\}$ be a sequence defined as follows*

$$\begin{cases} x_0 \in B, \\ z_n = P_K(I - \theta_n A)x_n, \\ y_n = \beta_n z_n + (1-\beta_n)v_n, \ v_n \in Tz_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \end{cases} \tag{3.15}$$

where $\{\beta_n\}$, $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying the following conditions:
$(i)\lim_{n\to\infty}\alpha_n = 0,\quad (ii)\ \lim_{n\to\infty}\inf\beta_n(1-\beta_n) > 0$ *and* $\theta_n \in [a,b] \subset \big(0, min\{1, 2\alpha\}\big)$,
$(iii)\lim_{n\to\infty}\lambda_n = 1$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.
Assume that $I-T$ are demiclosed at origin. Then, the sequences $\{x_n\}$ and $\{z_n\}$ generated by (3.15) converge strongly to $x^* \in G$, where $x^* = P_G(0)$, with P_G the the metric projection from K onto G.

4. The Application

In this section, we apply our main results for finding a common element of fixed points problems involving multivalued quasi-nonexpansive mappings and optimization problems.

Problem 4.1. Let K be a nonempty, closed convex cone of a real Hilbert space H. We consider the following minimization problem :

$$\min_{x\in K} f(x), \tag{4.1}$$

where f be a continuously Fréchet differentiable, convex functional on K.

We denote the set of solutions of Problem (4.1) by Ω_1.

Problem 4.2. Let K be a nonempty, closed convex cone of a real Hilbert space H. We consider the following fixed point problem :

$$\textit{find } x \in K \textit{ such that } x \in Tx, \tag{4.2}$$

where $T : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping.

We denote the set of solutions of Problem (4.2) by Ω_2.

Lemma 4.3. *(Baillon and Haddad [1]) Let H be a real Hilbert space, f a continuously Fréchet differentiable, convex functional on H and ∇f the gradient of f. If ∇f is $\frac{1}{\alpha}$-Lipschitz continuous, then ∇f is α-inverse strongly monotone.*

Remark 4.4. A necessary condition of optimality for a point $x^* \in K$ to be a solution of the minimization problem (4.1) is that x^* solves the following variational inequality problem:

$$\langle \nabla f(x^*), p - x^*\rangle \geq 0$$

for all $p \in K$.

Consequently, the following theorem can be obtained.

Theorem 4.5. *Let K be a nonempty, closed convex cone of a real Hilbert space H and $f : K \to \mathbb{R}$ a continuously Fréchet differentiable, convex functional on K and ∇f is $\frac{1}{\alpha}$-Lipschitz continuous. Let $T : K \to CB(K)$ be a multivalued quasi-nonexpansive mapping such that $G := \Omega_1 \cap \Omega_2 \neq \emptyset$*

and $Tp = \{p\}$ $\forall p \in G$. Let $\{x_n\}$ be a sequence defined as follows:

$$\begin{cases} x_0 \in K, \\ z_n = P_K(I - \theta_n \nabla f)x_n, \\ y_n = \beta_n z_n + (1-\beta_n)v_n,\ v_n \in Tz_n, \\ x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)y_n, \end{cases} \tag{4.3}$$

where $\{\beta_n\}$, $\{\lambda_n\}$, $\{\theta_n\}$ and $\{\alpha_n\}$ be sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\quad (ii)$ $\lim_{n\to\infty} \inf \beta_n(1-\beta_n) > 0$ and $\theta_n \in [a,b] \subset \big(0, min\{1, 2\alpha\}\big)$,
(iii) $\lim_{n\to\infty} \lambda_n = 1$ and $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.
Assume that $I - T$ are demiclosed at origin. Then, the sequences $\{x_n\}$ and $\{z_n\}$ generated by (4.3) converge strongly to common solution of Problem (4.1) and Problem (4.2).

Proof. Using properties of f, it follows by Lemma 4.3 ∇f is α-inverse strongly monotone on K onto H. Using Remark 4.4 the proof follows Theorem (3.1). □

Remark 4.6. By using relationship between inverse strongly monotone operators and strictly pseudo-contractive operators and results of Takahashi [18]. Our results can be applied to the cases : (1) Fixed points of strictly pseudo-contractive operators. (2) Solutions of complementarity problem.

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