Commun. Optim. Theory 2020 (2020), Article ID 23 https://doi.org/10.23952/cot.2020.23

Communications in Optimization Theory
Available online at http://cot.mathres.org

CYCLIC CONTRACTIONS AND COINCIDENCE POINTS VIA COMPARISON AND (w)-COMPARISON FUNCTIONS ON PARTIAL METRIC SPACES

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Abstract. In this paper, we use the notion of the cyclic representation of a nonempty set with respect to a pair of mappings to obtain coincidence points and common fixed points of a pair of self mappings satisfying a new type of contraction condition involving comparison functions and (w)-comparison functions in partial metric spaces. We also provide some examples to justify the validity of our main results.

Keywords. Partial metric; Comparison function; 0-completeness; Coincidence point.

1. Introduction

In 2015, Radenović [1] raised an open question about fixed points of cyclic φ-contractions in complete metric spaces, where φ-is a comparison function. Later on, He and Chen [2] gave an answer to this question and further extended their result to the setting of generalized metric spaces when the number of cyclic sets is odd. In this paper, our intention is to establish these results in the framework of partial metric spaces. In 1994, Matthews [3] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks. In fact, a complete partial metric space is a useful framework to model several complex problems in theory of computation. The works of [4–11] are viable and have opened new avenues for applications in different fields of mathematics and applied sciences. Recently, many authors studied fixed points of cyclic mappings in several spaces. In 2003, Kirk, Srinivasan and Veeramani [12] introduced the notion of cyclic mappings and proved some fixed point theorems for these mappings. Some results for cyclic contractions in partial metric spaces were obtained in [13–17]. In 2013, Shatanawi and Postolache [18] proved some common fixed point theorems with the help of control functions, namely, altering distance functions due to Khan, Swaleh and Sessa [19]. After that, several generalized control functions were used to obtain fixed point

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Received August 20, 2020; Accepted October 19, 2020.

results in various spaces. The results of [1, 2, 20–22] have become the source of the motivation of this study. In this paper, we introduce the concept of the cyclic representation of a nonempty set with respect to a pair of mappings and use it to prove coincidence point and common fixed point results of a pair of self mappings satisfying a new contraction conditions involving comparison functions and (w)-comparison functions in partial metric spaces. We also obtain several important fixed point results in metric spaces and partial metric spaces. Finally, we give some examples to support our results.

2. Some Basic Concepts

In this section, we present some definitions and results in a partial metric space.

Definition 2.1. [3] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$,

(p_1) $p(x,x) = p(y,y) = p(x,y) \Longleftrightarrow x = y$,
(p_2) $p(x,x) \leq p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is called a partial metric space.

It is obvious that if $p(x,y) = 0$, then it follows from (p_1) and (p_2) that $x = y$. However, $x = y$ does not imply $p(x,y) = 0$.

Example 2.2. [3] Let $X = [0,\infty)$ and $p(x,y) = max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a partial metric space.

Example 2.3. [3] Let $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$ and $p([a,b],[c,d]) = (max\{b, d\} - min\{a, c\})$. Then (X, p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Theorem 2.4. *If $U \in \tau_p$ and $x \in U$, then there exists $r > 0$ such that $B_p(x,r) \subseteq U$.*

Proof. Since U is an open set containing x, there exists an open p-ball, say $B_p(y,\varepsilon)$, such that $x \in B_p(y,\varepsilon) \subseteq U$. Then $p(x,y) < p(y,y) + \varepsilon$. Let us choose $0 < r < p(y,y) - p(x,y) + \varepsilon$ and consider the open p-ball $B_p(x,r)$. Then it is easy to verify that $B_p(x,r) \subseteq B_p(y,\varepsilon) \subseteq U$. □

Remark 2.5. Let (X, p) be a partial metric space, (x_n) a sequence in X and $x \in X$. Then (x_n) converges to x with respect to(w.r.t.) τ_p if and only if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

Let $x_n \to x$ w.r.t. τ_p and $\varepsilon > 0$. Then there exists a natural number n_0 such that $x_n \in B_p(x,\varepsilon)$ for all $n \geq n_0$. This gives that $p(x_n,x) - p(x,x) < \varepsilon$ for all $n \geq n_0$. Since $p(x_n,x) - p(x,x) \geq 0$, it follows that $| p(x_n,x) - p(x,x) | < \varepsilon$ for all $n \geq n_0$. This proves that $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.

Conversely, suppose that $\lim_{n\to\infty} p(x_n,x) = p(x,x)$. We show that $x_n \to x$ w.r.t. τ_p. Let $U \in \tau_p$ and $x \in U$. Then there exists $\varepsilon > 0$ such that $x \in B_p(x,\varepsilon) \subseteq U$. By hypotheses, it follows that

$$\lim_{n\to\infty} (p(x_n,x) - p(x,x)) = 0.$$

So, there exists $n_0 \in \mathbb{N}$ such that $p(x_n,x) - p(x,x) < \varepsilon$ for all $n \geq n_0$. This ensures that $x_n \in B_p(x,\varepsilon)$ for all $n \geq n_0$ and hence $x_n \in U$ for all $n \geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_p on X.

Definition 2.6. [3] Let (X,p) be a partial metric space and let (x_n) be a sequence in X. Then

(i) (x_n) converges to a point $x \in X$ if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$. This will be denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(ii) (x_n) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(iii) (X,p) is said to be complete if every Cauchy sequence (x_n) in X converges to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Definition 2.7. [23] A sequence (x_n) in (X,p) is said to be 0-Cauchy if

$$\lim_{n,m\to\infty} p(x_n,x_m) = 0.$$

The space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x,x) = 0$.

It is easy to verify that every closed subset of a 0-complete partial metric space is 0-complete.

Lemma 2.8. *Let (X,p) be a partial metric space.*

(a) *(see [24, 25]) If $p(x_n,z) \to p(z,z) = 0$ as $n \to \infty$, then $p(x_n,y) \to p(z,y)$ as $n \to \infty$ for each $y \in X$.*

(b) *(see [23]) If (X,p) is complete, then it is 0-complete.*

The converse assertion of (b) may not hold, in general. The following example supports the above remark.

Example 2.9. [23] The space $X = [0,\infty) \cap \mathbb{Q}$ with the partial metric $p(x,y) = max\{x, y\}$ is 0-complete, but it is not complete. Moreover, the sequence (x_n) with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X,p), but it is not a 0-Cauchy sequence.

Definition 2.10. [12] Let X be a nonempty set, $m \in \mathbb{N}$, and let $f : X \to X$ be a self mapping. Then $X = \cup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f if

(a) A_i, $i = 1, 2, \cdots, m$ are nonempty subsets of X;

(b) $f(A_1) \subseteq A_2$, $f(A_2) \subseteq A_3, \cdots, f(A_{m-1}) \subseteq A_m$, $f(A_m) \subseteq A_1$.

Definition 2.11. [26] Let f and g be self mappings of a set X. If $y = fx = gx$ for some x in X, then x is called a coincidence point of f and g and y is called a point of coincidence of f and g.

Definition 2.12. [25] The mappings $f, g : X \to X$ are weakly compatible if, for every $x \in X$, the following holds:

$$f(gx) = g(fx) \textit{ whenever } gx = fx.$$

Proposition 2.13. *[26] Let f and g be weakly compatible self maps of a nonempty set X. If f and g have a unique point of coincidence $y = fx = gx$, then y is the unique common fixed point of f and g.*

We now recall the definition of a comparison function.

Definition 2.14. [22] A function $\varphi:[0,\infty)\to[0,\infty)$ is called a comparison function if it satisfies the following conditions:

$(i)_\varphi$ φ is increasing;
$(ii)_\varphi$ $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n\to\infty$, for all $t\in(0,\infty)$.

Let Φ denote the class of all comparison functions φ.

Remark 2.15. For each $\varphi\in\Phi$, the following assertions hold:

(i) $\varphi(t)<t$ for all $t>0$;
(ii) $\varphi(0)=0$.

Proof. (i) Let $\varphi\in\Phi$. By $(ii)_\varphi$, we have

$$\lim_{n\to\infty}\varphi^n(t)=0,\quad \forall t>0. \tag{2.1}$$

If needed, suppose there exists $t>0$ such that $\varphi(t)\geq t$. Since φ is increasing, we have $\varphi^n(t)\geq t$, for all $n\geq 1$. Taking limit as $n\to\infty$ and using condition (2.1), it follows that $t=0$, a contradiction. Thus, $\varphi(t)<t$ for all $t>0$.

(ii) Suppose that $\varphi(0)=t>0$. Then, by (i), $\varphi(t)<t\Longrightarrow\varphi(\varphi(0))<\varphi(0)\Longrightarrow\varphi(0)<0$, since φ is increasing. This contradicts our supposition that $\varphi(0)>0$. Therefore, $\varphi(0)=0$. □

Remark 2.16. If condition $(ii)_\varphi$ is replaced by

$(iii)_\varphi$ $\sum_{k=0}^{\infty}\varphi^k(t)<\infty$, for all $t\in(0,\infty)$,

then φ is called a strong comparison function.

It deserve mentioning that a strong comparison function is a comparison function, but the converse is not true. The following example supports the above remark.

Example 2.17. [22] Let $\varphi:[0,\infty)\to[0,\infty)$ be defined by $\varphi(t)=\frac{t}{1+t}$. Then φ is a comparison function, but it is not a strong comparison function. In fact, $\varphi^n(t)=\frac{t}{1+nt}$, for all $t>0$. Consequently, for every $t>0$, $(\varphi^n(t))$ converges to 0 as $n\to\infty$, but $\sum_{k=0}^{\infty}\varphi^k(t)=\infty$.

3. Coincidence points via comparison functions

In this section, we use the following notations. Let (X,p) be a partial metric space and $f,g:X\to X$ be self mappings. Then, for $x,y\in X$,

$$M_f(gx,gy)=\max\{p(gx,gy),p(gx,fx),p(gy,fy),\frac{p(gx,fy)}{2},\frac{p(gy,fx)}{2}\},$$

and

$$M_f(x,y)=\max\{p(x,y),p(x,fx),p(y,fy),\frac{p(x,fy)}{2},\frac{p(y,fx)}{2}\}.$$

We begin with the following definition.

Definition 3.1. Let X be a nonempty set, $q\in\mathbb{N}$, and $f,g:X\to X$ self mappings. Then $X=\cup_{i=1}^{q}A_i$ is a cyclic representation of X with respect to the pair (f,g) if

(a) A_i, $i = 1, 2, \cdots, q$ are nonempty subsets of X;
(b) $f(A_i) \subseteq g(A_{i+1})$ for $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$.

Theorem 3.2. *Let (X, p) be a 0-complete partial metric space, $q \in \mathbb{N}$, and $A_1, A_2, \cdots, A_q$ nonempty subsets of X, and $Y = \cup_{i=1}^{q} A_i$. Suppose that $f, g : Y \to Y$ are self mappings, $g(A_1)$, $g(A_2)$, $\cdots, g(A_q)$ are closed subsets of (X, p) and $Y = \cup_{i=1}^{q} A_i$ is a cyclic representation of Y with respect to the pair (f, g). If there exists $\varphi \in \Phi$ such that*

$$p(fx, fy) \leq \varphi(M_f(gx, gy)) \tag{3.1}$$

for all $x, y \in Y$ with $(gx, gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$, then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u, u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. Let x_0 be an arbitrary element of Y. Then there exists $i_0 \in \{1, 2, \cdots, q\}$ such that $x_0 \in A_{i_0}$. Since $f(A_{i_0}) \subseteq g(A_{i_0+1})$, there exists $x_1 \in A_{i_0+1}$ such that $gx_1 = fx_0$. Continuing this process, we can construct a sequence (x_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \cdots$, where $x_n \in A_{i_0+n}$ and $A_{q+k} = A_k$. If $p(gx_n, gx_{n+1}) = 0$ for some $n \in \mathbb{N} \cup \{0\}$, then $gx_n = gx_{n+1} = fx_n$. Hence, gx_{n+1} is a point of coincidence of f and g. Without loss of generality, we may assume that

$$p(gx_n, gx_{n+1}) > 0, \ \forall n \in \mathbb{N} \cup \{0\}.$$

We note that, for all $n \in \mathbb{N}$, there exists $i \in \{1, 2, \cdots, q\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$ and so, $(gx_n, gx_{n+1}) \in g(A_i) \times g(A_{i+1})$. We first compute $M_f(gx_{n-1}, gx_n)$. Note that

$$\begin{aligned} M_f(gx_{n-1}, gx_n) &= \max \left\{ \begin{array}{l} p(gx_{n-1}, gx_n), p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1}), \\ \frac{p(gx_{n-1}, gx_{n+1})}{2}, \frac{p(gx_n, gx_n)}{2} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} p(gx_{n-1}, gx_n), p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1}), \\ \frac{p(gx_{n-1}, gx_{n+1}) + p(gx_n, gx_n)}{2} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1}), \\ \frac{p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})}{2} \end{array} \right\} \\ &= \max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\}. \end{aligned}$$

By $(i)_\varphi$, it follows that

$$\varphi(M_f(gx_{n-1}, gx_n)) \leq \varphi(\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\}). \tag{3.2}$$

For any natural number n, we have by applying conditions (3.1) and (3.2) that

$$\begin{aligned} p(gx_n, gx_{n+1}) &= p(fx_{n-1}, fx_n) \\ &\leq \varphi(M_f(gx_{n-1}, gx_n)) \\ &\leq \varphi(\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\}). \end{aligned} \tag{3.3}$$

If $\max\{p(gx_{n-1},gx_n),p(gx_n,gx_{n+1})\}=p(gx_n,gx_{n+1})$, then we obtain from condition (3.3) and $\varphi(t)<t$ for each $t>0$ that

$$p(gx_n,gx_{n+1})\leq\varphi(p(gx_n,gx_{n+1}))<p(gx_n,gx_{n+1}),$$

which is a contradiction. Therefore,

$$\max\{p(gx_{n-1},gx_n),p(gx_n,gx_{n+1})\}=p(gx_{n-1},gx_n).$$

Thus, we obtain from condition (3.3) that

$$p(gx_n,gx_{n+1})\leq\varphi(p(gx_{n-1},gx_n)),\quad\forall n\in\mathbb{N}. \tag{3.4}$$

By using condition (3.4) and $(i)_\varphi$, we get

$$p(gx_n,gx_{n+1})\leq\varphi^n(p(gx_1,gx_0)),\quad\forall n\geq 0.$$

Using $(ii)_\varphi$, we get

$$\lim_{n\to\infty}p(gx_n,gx_{n+1})=0.$$

It follows from (p_4) that

$$p(gx_n,gx_{n+k})\leq p(gx_n,gx_{n+1})+p(gx_{n+1},gx_{n+2})+\cdots+p(gx_{n+k-1},gx_{n+k}),$$

for $k=2,3,\cdots,q$. This ensures that

$$\lim_{n\to\infty}p(gx_n,gx_{n+k})=0,\quad\forall k=1,2,\cdots,q. \tag{3.5}$$

We now prove the following claim.

Claim. For every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that if $n>m>n_0$ with $n-m\equiv 1\ mod\,q$, then $p(gx_n,gx_m)<\varepsilon$.

Suppose that the claim is not true. Then there exists $\varepsilon_0>0$ such that, for any $N\in\mathbb{N}$, we can find $n>m>n_0$ with $n-m\equiv 1\ mod\,q$ satisfying $p(gx_n,gx_m)\geq\varepsilon_0$. By virtue of condition (3.5), corresponding to this ε_0, there exists $n_0\in\mathbb{N}$ such that, for all $n>n_0$,

$$p(gx_n,gx_{n+k})<\varepsilon_0,\quad\forall k=1,2,\cdots,q. \tag{3.6}$$

For $N=n_0$, we can find $n_1'>m_1>n_0$ with $n_1'-m_1\equiv 1\ mod\,q$ such that $p(gx_{n_1'},gx_{m_1})\geq\varepsilon_0$. In view of condition (3.9), we can choose $n_1\in\{m_1+q+1,m_1+2q+1,\cdots,n_1'\}$ in such a way that it is the smallest integer satisfying $p(gx_{n_1},gx_{m_1})\geq\varepsilon_0$. Thus, we obtain

$$p(gx_{n_1},gx_{m_1})\geq\varepsilon_0,\ p(gx_{n_1-p},gx_{m_1})<\varepsilon_0\ and\ n_1-m_1\equiv 1\ mod\,q.$$

Again, for $N=n_1$, we can find $n_2'>m_2>n_1$ with $n_2'-m_2\equiv 1\ mod\,q$ such that $p(gx_{n_2'},gx_{m_2})\geq\varepsilon_0$. Proceeding as above, we can choose $n_2\in\{m_2+q+1,m_2+2q+1,\cdots,n_2'\}$ such that

$$p(gx_{n_2},gx_{m_2})\geq\varepsilon_0,\ p(gx_{n_2-q},gx_{m_2})<\varepsilon_0\ and\ n_2-m_2\equiv 1\ mod\,q.$$

Continuing in this way, we obtain two subsequences (gx_{m_k}) and (gx_{n_k}) of (gx_n) such that

$$p(gx_{n_k},gx_{m_k})\geq\varepsilon_0,\ p(gx_{n_k-q},gx_{m_k})<\varepsilon_0\ and\ n_k-m_k\equiv 1\ mod\,q. \tag{3.7}$$

We now compute $M_f(gx_{n_k-q}, gx_{m_k})$. We have

$$\begin{aligned}
M_f(gx_{n_k-q}, gx_{m_k}) &= \max\left\{\begin{array}{l} p(gx_{n_k-q}, gx_{m_k}), p(gx_{n_k-q}, fx_{n_k-q}), \\ \\ p(gx_{m_k}, fx_{m_k}), \frac{p(gx_{n_k-q}, fx_{m_k})}{2}, \frac{p(gx_{m_k}, fx_{n_k-q})}{2} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} p(gx_{n_k-q}, gx_{m_k}), p(gx_{n_k-q}, gx_{n_k-q+1}), \\ \\ p(gx_{m_k}, gx_{m_k+1}), \frac{p(gx_{n_k-q}, gx_{m_k+1})}{2}, \\ \\ \frac{p(gx_{m_k}, gx_{n_k-q+1})}{2} \end{array}\right\} \\
&\leq \max\left\{\begin{array}{l} p(gx_{n_k-q}, gx_{m_k}), p(gx_{n_k-q}, gx_{n_k-q+1}), \\ \\ p(gx_{m_k}, gx_{m_k+1}), \frac{p(gx_{n_k-q}, gx_{m_k})+p(gx_{m_k}, gx_{m_k+1})}{2}, \\ \\ \frac{p(gx_{m_k}, gx_{n_k-q})+p(gx_{n_k-q}, gx_{n_k-q+1})}{2} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} p(gx_{n_k-q}, gx_{m_k}), p(gx_{n_k-q}, gx_{n_k-q+1}), \\ \\ p(gx_{m_k}, gx_{m_k+1}) \end{array}\right\}. \qquad (3.8)
\end{aligned}$$

From condition (3.5), corresponding to $\varepsilon_0 > 0$, there exists $N_0 \in \mathbb{N}$ such that, for all $k > N_0$,

$$p(gx_{n_k-q}, gx_{n_k-q+1}) < \varepsilon_0 \textit{ and } p(gx_{m_k}, gx_{m_k+1}) < \varepsilon_0. \qquad (3.9)$$

By using conditions (3.7) and (3.9), we obtain from condition (3.8) that

$$M_f(gx_{n_k-q}, gx_{m_k}) < \varepsilon_0, \ \forall k > N_0. \qquad (3.10)$$

Using condition (3.7) and (p_4), we get

$$\begin{aligned}
\varepsilon_0 \leq p(gx_{n_k}, gx_{m_k}) &\leq p(gx_{n_k}, gx_{n_k-q}) + p(gx_{n_k-q}, gx_{m_k}) \\
&< p(gx_{n_k}, gx_{n_k-q}) + \varepsilon_0.
\end{aligned}$$

Passing to the limit as $k \to \infty$ and using condition (3.5), we obtain

$$\lim_{k\to\infty} p(gx_{n_k}, gx_{m_k}) = \varepsilon_0. \qquad (3.11)$$

On the other hand, by (p_4), we have

$$p(gx_{n_k-q+1}, gx_{m_k+1}) \leq p(gx_{n_k-q+1}, gx_{n_k}) + p(gx_{n_k}, gx_{m_k}) + p(gx_{m_k}, gx_{m_k+1})$$

and

$$p(gx_{n_k-q+1}, gx_{m_k+1}) \geq p(gx_{n_k}, gx_{m_k}) - p(gx_{n_k}, gx_{n_k-q+1}) - p(gx_{m_k+1}, gx_{m_k}).$$

Taking limit as $k \to \infty$ in the above two inequalities, and using conditions (3.5) and (3.11), we obtain

$$\lim_{k\to\infty} p(gx_{n_k-q+1}, gx_{m_k+1}) = \varepsilon_0. \qquad (3.12)$$

By conditions (3.1) and (3.10), for all $k > N_0$, we have

$$p(gx_{n_k-q+1}, gx_{m_k+1}) = p(fx_{n_k-q}, fx_{m_k}) \leq \varphi(M_f(gx_{n_k-q}, gx_{m_k})) \leq \varphi(\varepsilon_0).$$

Passing to the limit as $k \to \infty$ and using condition (3.12), we get $\varepsilon_0 \leq \varphi(\varepsilon_0)$, which contradicts the fact that $\varphi(\varepsilon_0) < \varepsilon_0$. Therefore, our claim is justified.

We now prove that (gx_n) is a 0-Cauchy sequence in $g(Y)$.

Let $\varepsilon > 0$ be given. By the above claim, there exists $r_1 \in \mathbb{N}$ such that if $n > m > r_1$ with $n - m \equiv 1 \; mod\, q$ then $p(gx_n, gx_m) < \frac{\varepsilon}{q}$. Further, by using condition (3.5), corresponding to this $\varepsilon > 0$, there exists $r_2 \in \mathbb{N}$ such that for all $n > r_2$, $p(gx_n, gx_{n+1}) < \frac{\varepsilon}{q}$. Let $r = max\{r_1, r_2\} \in \mathbb{N}$. Let $m, n \in \mathbb{N}$, $n, m > r$ with $n > m$. Then, there exists $s \in \{0, 1, 2, \cdots, q-1\}$ such that $n - (m+s) \equiv 1 \; mod\, q$. By using (p_4) repeatedly, we have

$$\begin{aligned}
p(gx_m, gx_n) &\leq p(gx_m, gx_{m+1}) + p(gx_{m+1}, gx_{m+2}) + \cdots + p(gx_{m+s}, gx_n) \\
&< \frac{\varepsilon}{q} + \frac{\varepsilon}{q} + \cdots + \frac{\varepsilon}{q} \\
&= (s+1).\frac{\varepsilon}{q} \\
&\leq \varepsilon.
\end{aligned}$$

This proves that (gx_n) is a 0-Cauchy sequence in $g(Y)$. Since $g(Y) = \cup_{i=1}^{q} g(A_i)$, it follows that $g(Y)$ is a closed subset of the 0-complete partial metric space (X, p) and hence $g(Y)$ is 0-complete. So, (gx_n) converges to some point $u \in g(Y)$ such that $p(u, u) = 0$. Therefore,

$$\lim_{n \to \infty} p(gx_n, u) = p(u, u) = 0. \tag{3.13}$$

We next prove that $u \in \cap_{i=1}^{q} g(A_i)$.

As $x_0 \in A_{i_0}$, it follows that the sequence $(gx_{nq})_{n \geq 0} \subseteq g(A_{i_0})$. Since $g(A_{i_0})$ is closed, condition (3.13) ensures that $u \in g(A_{i_0})$. Again, we get $(gx_{nq+1})_{n \geq 0} \subseteq g(A_{i_0+1})$, where $A_{q+k} = A_k$. Proceeding as above, we obtain that $u \in g(A_{i_0+1})$. Continuing in this way, we get

$$u \in \cap_{i=1}^{q} g(A_i). \tag{3.14}$$

Now we show that u is a point of coincidence of f and g. We note that, for all $n \in \mathbb{N}$, there exists $i \in \{1, 2, \cdots, q\}$ such that $x_n \in A_i$. In view of condition (3.14), it follows that $u \in g(A_{i+1})$ where $A_{q+1} = A_1$. So, there exists $t \in A_{i+1}$ such that $u = gt$. Therefore, $(gx_n, gt) \in g(A_i) \times g(A_{i+1})$. By applying (3.1), we obtain that, for all $n \in \mathbb{N}$,

$$p(gx_{n+1}, ft) = p(fx_n, ft) \leq \varphi(M_f(gx_n, gt)), \tag{3.15}$$

where

$$\begin{aligned}
M_f(gx_n, gt) &= max\left\{\begin{array}{l} p(gx_n, gt), p(gx_n, fx_n), p(gt, ft), \\ \frac{p(gx_n, ft)}{2}, \frac{p(gt, fx_n)}{2} \end{array}\right\} \\
&= max\left\{\begin{array}{l} p(gx_n, gt), p(gx_n, gx_{n+1}), p(gt, ft), \\ \frac{p(gx_n, ft)}{2}, \frac{p(gt, gx_{n+1})}{2} \end{array}\right\}.
\end{aligned}$$

Suppose that $p(gt,ft)\neq 0$. Let $\varepsilon=\frac{p(gt,ft)}{2}>0$. Since $\lim_{n\to\infty}p(gx_n,gt)=0$, there exists $k\in\mathbb{N}$ such that

$$p(gx_n,gt)<\varepsilon,\ for\ each\ n\geq k. \tag{3.16}$$

Then, for each $n\geq k$

$$\begin{aligned} p(gx_n,ft) &\leq p(gx_n,gt)+p(gt,ft)-p(gt,gt)\\ &\leq p(gx_n,gt)+p(gt,ft)\\ &< 3\varepsilon. \end{aligned} \tag{3.17}$$

Moreover, for $n\geq k$, we have

$$p(gx_n,gx_{n+1})\leq p(gx_n,gt)+p(gt,gx_{n+1})<2\varepsilon. \tag{3.18}$$

Thus, for $n\geq k$, it follows from conditions (3.16), (3.17) and (3.18) that

$$max\left\{\begin{array}{l} p(gx_n,gt),p(gx_n,gx_{n+1}),p(gt,ft),\\ \frac{p(gx_n,ft)}{2},\frac{p(gt,gx_{n+1})}{2} \end{array}\right\}=2\varepsilon=p(gt,ft).$$

Therefore, we obtain from (3.15) that

$$p(gx_{n+1},ft)\leq\varphi(p(gt,ft)),\ for\ each\ n\geq k. \tag{3.19}$$

By using condition (3.19), for $n\geq k$, we have

$$\begin{aligned} p(gt,ft) &\leq p(gt,gx_{n+1})+p(gx_{n+1},ft)-p(gx_{n+1},gx_{n+1})\\ &\leq p(gt,gx_{n+1})+\varphi(p(gt,ft)). \end{aligned}$$

Passing to the limit as $n\to\infty$, we get

$$p(gt,ft)\leq\varphi(p(gt,ft)),$$

which is a contradiction, since $\varphi(t)<t$ for each $t>0$. Therefore, $p(gt,ft)=0$ and hence $ft=gt=u$. Therefore, u is a point of coincidence of f and g such that $u\in\cap_{i=1}^{q}g(A_i)$ and $p(u,u)=0$.

For the uniqueness, we assume that there is another point of coincidence v of f and g such that $v\in\cap_{i=1}^{q}g(A_i)$ and $p(v,v)=0$. By supposition, there exists $x\in Y$ satisfying $v=gx=fx$. Taking $u\in g(A_i)$, $v\in g(A_{i+1})$ and applying (3.1), we have

$$\begin{aligned} p(u,v) &= p(ft,fx)\\ &\leq \varphi(max\{p(gt,gx),p(gt,ft),p(gx,fx),\frac{p(gt,fx)}{2},\frac{p(gx,ft)}{2}\})\\ &= \varphi(max\{p(u,v),p(u,u),p(v,v),\frac{p(u,v)}{2},\frac{p(v,u)}{2}\})\\ &= \varphi(p(u,v)). \end{aligned} \tag{3.20}$$

If $p(u,v)>0$, then we get from condition (3.20) that

$$0<p(u,v)\leq\varphi(p(u,v)),$$

which is a contradiction, since $\varphi(t) < t$ for each $t > 0$. So, it must be the case that $p(u,v) = 0$ and hence $u = v$. Thus, f and g have a unique point of coincidence $u \in \cap_{i=1}^{q} g(A_i)$ and $p(u,u) = 0$. If f and g are weakly compatible, then we conclude from Proposition 2.13 that f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$. □

Corollary 3.3. *Let (X,p) be a 0-complete partial metric space and let $f,g : X \to X$ be such that $f(X) \subseteq g(X)$ and $g(X)$ a closed subset of (X,p). If there exists $\varphi \in \Phi$ such that*

$$p(fx,fy) \leq \varphi(M_f(gx,gy))$$

for all $x,y \in X$, then f and g have a unique point of coincidence u in $g(X)$ such that $p(u,u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof follows from Theorem 3.2 immediately by taking $A_1 = A_2 = \cdots = A_q = X$. □

Corollary 3.4. *Let (X,p) be a 0-complete partial metric space and let $f : X \to X$ be a mapping. Suppose there exists $\varphi \in \Phi$ such that*

$$p(fx,fy) \leq \varphi(M_f(x,y))$$

for all $x,y \in X$. Then f has a unique fixed point u in X such that $p(u,u) = 0$.

Proof. The proof follows from Theorem 3.2 immediately by taking $A_1 = A_2 = \cdots = A_q = X$ and $g = I$, the identity map on X. □

Corollary 3.5. *Let (X,p) be a 0-complete partial metric space, $q \in \mathbb{N}$, and let $A_1, A_2, \cdots, A_q$ be nonempty subsets of X, and $Y = \cup_{i=1}^{q} A_i$. Suppose that $f,g : Y \to Y$ are self mappings, $g(A_1), g(A_2), \cdots, g(A_q)$ are closed subsets of (X,p) and $Y = \cup_{i=1}^{q} A_i$ is a cyclic representation of Y with respect to the pair (f,g). Also, assume that*

$$p(fx,fy) \leq \frac{M_f(gx,gy)}{1 + M_f(gx,gy)}$$

for all $x,y \in Y$ with $(gx,gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$. Then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u,u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. The proof follows from Theorem 3.2 immediately by taking $\varphi(t) = \frac{t}{1+t}$ for each $t \geq 0$. □

Corollary 3.6. *Let (X,p) be a 0-complete partial metric space, $q \in \mathbb{N}$, and let $A_1, A_2, \cdots, A_q$ be nonempty subsets of X, and $Y = \cup_{i=1}^{q} A_i$. Suppose that $f,g : Y \to Y$ are self mappings, $g(A_1), g(A_2), \cdots, g(A_q)$ are closed subsets of (X,p) and $Y = \cup_{i=1}^{q} A_i$ is a cyclic representation of Y with respect to the pair (f,g). If there exists $k \in [0,1)$ such that*

$$p(fx,fy) \leq k\max\{p(gx,gy), p(gx,fx), p(gy,fy), \frac{p(gx,fy)}{2}, \frac{p(gy,fx)}{2}\}$$

for all $x,y \in Y$ with $(gx,gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$, where $A_{q+1} = A_1$, then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u,u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. The proof follows from Theorem 3.2 immediately by taking $\varphi(t) = kt$ for each $t \geq 0$, where $k \in [0,1)$ is a fixed number. □

Corollary 3.7. *Let (X,p) be a 0-complete partial metric space and $f : X \to X$ be a mapping. If there exists $k \in [0,1)$ such that*

$$p(fx,fy) \leq k\,max\{p(x,y), p(x,fx), p(y,fy), \frac{p(x,fy)}{2}, \frac{p(y,fx)}{2}\}$$

for all $x, y \in X$, then f has a unique fixed point u in X with $p(u,u) = 0$.

Proof. The result follows from Theorem 3.2 immediately by taking $A_1 = A_2 = \cdots = A_q = X$, $g = I$ and $\varphi(t) = kt$ for each $t \geq 0$, where $k \in [0,1)$ is a fixed number. □

Corollary 3.8. *Let (X,p) be a 0-complete partial metric space, $q \in \mathbb{N}$, and let $A_1, A_2, \cdots, A_q$ be nonempty subsets of X, $Y = \cup_{i=1}^{q} A_i$. Suppose that $f, g : Y \to Y$ are self mappings, $g(A_1)$, $g(A_2)$, $\cdots$, $g(A_q)$ are closed subsets of (X,p) and $Y = \cup_{i=1}^{q} A_i$ is a cyclic representation of Y with respect to the pair (f,g). If there exists α, β, γ, δ, $\nu \geq 0$ with $\alpha+\beta+\gamma+2\delta+2\nu < 1$ such that*

$$p(fx,fy) \leq \alpha\, p(gx,gy) + \beta\, p(gx,fx) + \gamma\, p(gy,fy) + \delta\, p(gx,fy) + \nu\, p(gy,fx) \tag{3.21}$$

for any $(gx,gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \cdots, q$ with $A_{q+1} = A_1$, then f and g have a unique point of coincidence u in $\cap_{i=1}^{q} g(A_i)$ with $p(u,u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$.

Proof. From condition (3.21), we obtain

$$\begin{aligned} p(fx,fy) &\leq \alpha\, p(gx,gy) + \beta\, p(gx,fx) + \gamma\, p(gy,fy) + \delta\, p(gx,fy) + \nu\, p(gy,fx) \\ &\leq (\alpha+\beta+\gamma+2\delta+2\nu)\, M_f(gx,gy) \\ &= k M_f(gx,gy), \end{aligned}$$

where $k = (\alpha+\beta+\gamma+2\delta+2\nu) \in [0,1)$. Now, Corollary 3.6 can be applied to obtain our desired result. □

Remark 3.9. It is worth mentioning that Theorem 2.2 [2] can be obtained as a special case of Theorem 3.2. Moreover, we obtain various important fixed point results in metric spaces and partial metric spaces including the Matthews version of Banach contraction theorem [3] as a special case of Corollary 3.8.

We give an example to justify the validity of our main result.

Example 3.10. Let $X = \{[2-3^{-n}, 2] : n \in \mathbb{N}\} \cup \{[2, 2+3^{-n}] : n \in \mathbb{N}\} \cup \{\{2\}\}$, where $\{2\} = [2,2]$. We define $p : X \times X \to \mathbb{R}^+$ by $p([a,b],[c,d]) = max\{b,d\} - min\{a,c\}$. Then (X,p) is a 0-complete partial metric space. Let $A_1 = \{[2-3^{-n}, 2] : n \in \mathbb{N}\} \cup \{\{2\}\}$ and $A_2 = \{[2, 2+3^{-n}] : n \in \mathbb{N}\} \cup \{\{2\}\}$. Obviously, $X = A_1 \cup A_2$. Define mappings $f, g : X \to X$ by

$$fx = \begin{cases} [2, 2+3^{-(n+2)}], \ if\ x = [2-3^{-n}, 2], \\ [2-3^{-(n+2)}, 2], \ if\ x = [2, 2+3^{-n}], \\ \{2\}, \ if\ x = \{2\} \end{cases}$$

and

$$gx = \begin{cases} [2-3^{-(n+1)}, 2], \ if \ x = [2-3^{-n}, 2], \\ \\ [2, 2+3^{-(n+1)}], \ if \ x = [2, 2+3^{-n}], \\ \\ \{2\}, \ if \ x = \{2\}. \end{cases}$$

Then, $f(A_1) \subseteq g(A_2)$, $f(A_2) \subseteq g(A_1)$ and so $X = A_1 \cup A_2$ is a cyclic representation of X with respect to the pair (f,g). Moreover, $g(A_1)$, $g(A_2)$ are closed subsets of (X,p). We now verify the condition (3.1) with the comparison function $\varphi : [0,\infty) \to [0,\infty)$ given by $\varphi(t) = \frac{t}{1+t}$. We now consider the following cases:

Case-I: $x = [2-3^{-n}, 2] \in A_1$, $y = [2, 2+3^{-k}] \in A_2$, $n, k \in \mathbb{N}$ with $n < k$.

In this case, we have $3^{-k} < 3^{-n}$ and $3^{-k} \leq 3^{-(n+1)}$. Then,

$$p(fx, fy) = p([2, 2+3^{-(n+2)}], [2-3^{-(k+2)}, 2]) = \frac{1}{9}(3^{-n} + 3^{-k}) < \frac{2}{9}.3^{-n},$$

$$\begin{aligned} p(gx, gy) &= p([2-3^{-(n+1)}, 2], [2, 2+3^{-(k+1)}]) = 3^{-(k+1)} + 3^{-(n+1)} \\ &= \frac{1}{3}.3^{-k} + 3^{-(n+1)} \leq (\frac{1}{3}+1)3^{-(n+1)} = \frac{4}{9}.3^{-n}, \end{aligned}$$

$$p(gx, fx) = p([2-3^{-(n+1)}, 2], [2, 2+3^{-(n+2)}]) = 3^{-(n+2)} + 3^{-(n+1)} = \frac{4}{9}.3^{-n},$$

$$p(gy, fy) = p([2, 2+3^{-(k+1)}], [2-3^{-(k+2)}, 2]) = 3^{-(k+1)} + 3^{-(k+2)} < \frac{4}{9}.3^{-n},$$

$$p(gx, fy) = p([2-3^{-(n+1)}, 2], [2-3^{-(k+2)}, 2]) = 3^{-(n+1)} = \frac{1}{3}.3^{-n},$$

$$p(fx, gy) = p([2, 2+3^{-(n+2)}], [2, 2+3^{-(k+1)}]) = 3^{-(n+2)} = \frac{1}{9}.3^{-n}.$$

Thus, $M(gx, gy) = \frac{4}{9}.3^{-n}$ and

$$\begin{aligned} \varphi(M(gx, gy)) = \frac{M(gx, gy)}{1+M(gx, gy)} &= \frac{\frac{4.3^{-n}}{9}}{1+\frac{4.3^{-n}}{9}} \\ &= \frac{4.3^n}{9.3^n+4}.3^{-n} \\ &= \frac{2}{9}.\frac{18.3^n}{4+9.3^n}.3^{-n} \\ &> \frac{2}{9}.3^{-n}. \end{aligned}$$

Therefore, $p(fx, fy) < \frac{2}{9}.3^{-n} < \varphi(M(gx, gy))$.

Case-II: $x = [2-3^{-n}, 2] \in A_1$, $y = [2, 2+3^{-k}] \in A_2$, $n, k \in \mathbb{N}$ with $n > k$.

In this case, we have $3^{-k} > 3^{-n}$ and $3^{-n} \leq 3^{-(k+1)}$. Then,
$p(fx, fy) < \frac{2}{9}.3^{-k}$, $p(gx, gy) = \frac{1}{3}(3^{-k} + 3^{-n}) \leq \frac{4}{9}.3^{-k}$, $p(gx, fx) = \frac{4}{9}.3^{-n}$, $p(gy, fy) = \frac{4}{9}.3^{-k}$ and $p(gx, fy) = 3^{-(k+2)} = \frac{1}{9}.3^{-k}$, $p(fx, gy) = 3^{-(k+1)} = \frac{1}{3}.3^{-k}$.

Thus, $M(gx,gy)=\frac{4}{9}.3^{-k}$ ans so $\varphi(M(gx,gy))>\frac{2}{9}.3^{-k}$. Therefore,

$$p(fx,fy)<\frac{2}{9}.3^{-k}<\varphi(M(gx,gy)).$$

Case-III: $x=[2-3^{-n},2]\in A_1,\ y=[2,2+3^{-k}]\in A_2,\ n,k\in\mathbb{N}$ with $n=k$.
Then, $p(fx,fy)=\frac{2}{9}.3^{-n}$, $p(gx,gy)=\frac{2}{3}.3^{-n}$, $p(gx,fx)=\frac{4}{9}.3^{-n}$, $p(gy,fy)=\frac{4}{9}.3^{-n}$ and $p(gx,fy)=\frac{1}{3}.3^{-n}$, $p(fx,gy)=\frac{1}{3}.3^{-n}$. Thus, $M(gx,gy)=\frac{4}{9}.3^{-n}$ and

$$p(fx,fy)=\frac{2}{9}.3^{-n}<\varphi(M(gx,gy)).$$

Case-IV: $x=[2-3^{-n},2]\in A_1, n\in\mathbb{N}, y=\{2\}\in A_2$.
Then,

$$p(fx,fy)=p([2,2+3^{-(n+2)}],\{2\})=3^{-(n+2)}=\frac{1}{9}.3^{-n},$$

$$p(gx,gy)=p([2-3^{-(n+1)},2],\{2\})=3^{-(n+1)}=\frac{1}{3}.3^{-n},$$

$$p(gx,fx)=p([2-3^{-(n+1)},2],[2,2+3^{-(n+2)}])=3^{-(n+2)}+3^{-(n+1)}=\frac{4}{9}.3^{-n},$$

$p(gy,fy)=p(\{2\},\{2\})=0$, $p(gx,fy)=p([2-3^{-(n+1)},2],\{2\})=\frac{1}{3}.3^{-n}$, $p(fx,gy)=p([2,2+3^{-(n+2)}],\{2\})=\frac{1}{9}.3^{-n}$. Thus, $M(gx,gy)=\frac{4}{9}.3^{-n}$. Therefore,

$$p(fx,fy)=\frac{1}{9}.3^{-n}<\frac{2}{9}.3^{-n}<\varphi(M(gx,gy)).$$

Case-V: $x=\{2\}\in A_1, y=[2,2+3^{-n}]\in A_2, n\in\mathbb{N}$.
In this case, we have

$$p(fx,fy)=\frac{1}{9}.3^{-n}, M(gx,gy)=\frac{4}{9}.3^{-n}.$$

Therefore,

$$p(fx,fy)=\frac{1}{9}.3^{-n}<\frac{2}{9}.3^{-n}<\varphi(M(gx,gy)).$$

Case-VI: $x=y=\{2\}$ is trivial.

The other possibility is treated similarly. Moreover, f and g are weakly compatible. Thus, all the conditions of Theorem 3.2 are fulfilled and $\{2\}$ is the unique common fixed point of f and g in $g(A_1)\cap g(A_2)$ with $p(\{2\},\{2\})=0$.

4. Coincidence points via (w)-comparison functions

This section begins with the following notations. Let (X,p) be a partial metric space and $f,g:X\to X$ be self mappings. Then, for $x,y\in X$,

$$N_f(gx,gy)=\max\{p(gx,gy),p(gx,fx),p(gy,fy)\},$$

and

$$N_f(x,y)=\max\{p(x,y),p(x,fx),p(y,fy)\}.$$

We now recall the definition of a (w)-comparison function.

Definition 4.1. [2] A function $\psi:[0,\infty)\to[0,\infty)$ is called a (w)-comparison function if it satisfies:

$(i)_\psi$ $\psi(0)=0$;
$(ii)_\psi$ $\psi(t)<t$, for all $t\in(0,\infty)$;
$(iii)_\psi$ the function $\phi(t):=t-\psi(t)$ is increasing, i.e., $t_1\leq t_2$ implies $\phi(t_1)\leq\phi(t_2)$, for $t_1,t_2\in[0,\infty)$.

Lemma 4.2. *[2] If $\psi:[0,\infty)\to[0,\infty)$ is a (w)-comparison function, then the following hold:*
(1) *$\psi(t)\leq t$, for any $t\in[0,\infty)$;*
(2) *for $k\geq 1$, $\psi^k(t)<t$, for any $t\in(0,\infty)$;*
(3) *$(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n\to\infty$, for all $t\in(0,\infty)$.*

He and Chen [2] studied via examples that the comparison function and the (w)-comparison function are independent in the sense that one does not imply the other.

Theorem 4.3. *Let (X,p) be a 0-complete partial metric space, $q\in\mathbb{N}$, and let $A_1,A_2,\cdots,A_q$ be nonempty subsets of X, and $Y=\cup_{i=1}^q A_i$. Suppose that $f,g:Y\to Y$ are self mappings, $g(A_1)$, $g(A_2)$, $\cdots$, $g(A_q)$ are closed subsets of (X,p) and $Y=\cup_{i=1}^q A_i$ is a cyclic representation of Y with respect to the pair (f,g). If there exists a (w)-comparison function ψ such that*

$$p(fx,fy)\leq\psi(N_f(gx,gy)) \tag{4.1}$$

for all $x,y\in Y$ with $(gx,gy)\in g(A_i)\times g(A_{i+1})$, $i=1,2,\cdots,q$, where $A_{q+1}=A_1$, then f and g have a unique point of coincidence u in $\cap_{i=1}^q g(A_i)$ with $p(u,u)=0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\cap_{i=1}^q g(A_i)$.

Proof. Let $x_0\in Y$ be arbitrary. Then there exists $i_0\in\{1,2,\cdots,q\}$ such that $x_0\in A_{i_0}$. Since $f(A_{i_0})\subseteq g(A_{i_0+1})$, there exists $x_1\in A_{i_0+1}$ such that $gx_1=fx_0$. Continuing this process, we can construct a sequence (x_n) such that $gx_n=fx_{n-1}$, $n=1,2,3,\cdots$, where $x_n\in A_{i_0+n}$ and $A_{q+k}=A_k$. If $p(gx_n,gx_{n+1})=0$ for some $n\in\mathbb{N}\cup\{0\}$, then $gx_n=gx_{n+1}=fx_n$ and hence gx_{n+1} is a point of coincidence of f and g. Without loss of generality, we may assume that

$$p(gx_n,gx_{n+1})>0,\ \forall n\in\mathbb{N}\cup\{0\}.$$

We note that, for all $n\in\mathbb{N}$, there exists $i\in\{1,2,\cdots,q\}$ such that $(x_n,x_{n+1})\in A_i\times A_{i+1}$. So, $(gx_n,gx_{n+1})\in g(A_i)\times g(A_{i+1})$. We first compute $N_f(gx_{n-1},gx_n)$. Observe that

$$\begin{aligned}N_f(gx_{n-1},gx_n) &= max\{p(gx_{n-1},gx_n),p(gx_{n-1},fx_{n-1}),p(gx_n,fx_n)\}\\ &= max\{p(gx_{n-1},gx_n),p(gx_{n-1},gx_n),p(gx_n,gx_{n+1})\}\\ &= max\{p(gx_{n-1},gx_n),p(gx_n,gx_{n+1})\}.\end{aligned}$$

Thus, it follows that

$$\psi(N_f(gx_{n-1},gx_n))=\psi(max\{p(gx_{n-1},gx_n),p(gx_n,gx_{n+1})\}). \tag{4.2}$$

For any natural number n, we have from conditions (4.1) and (4.2) that

$$\begin{aligned}p(gx_n,gx_{n+1}) &= p(fx_{n-1},fx_n)\\ &\leq \psi(N_f(gx_{n-1},gx_n))\\ &= \psi(max\{p(gx_{n-1},gx_n),p(gx_n,gx_{n+1})\}). \end{aligned}\tag{4.3}$$

If $max\{p(gx_{n-1},gx_n),p(gx_n,gx_{n+1})\}=p(gx_n,gx_{n+1})$, then we obtain from condition (4.3) and the fact that $\psi(t)<t$ for each $t>0$ that

$$p(gx_n,gx_{n+1})\leq\psi(p(gx_n,gx_{n+1}))<p(gx_n,gx_{n+1}),$$

which is a contradiction. Therefore,

$$max\{p(gx_{n-1},gx_n),p(gx_n,gx_{n+1})\}=p(gx_{n-1},gx_n).$$

Thus, we obtain from condition (4.3) that

$$p(gx_n,gx_{n+1})\leq\psi(p(gx_{n-1},gx_n))<p(gx_{n-1},gx_n),\ for\ all\ n\in\mathbb{N}. \tag{4.4}$$

This shows that $(p(gx_n,gx_{n+1}))$ is a decreasing sequence, which is bounded below. So there exists $r\geq 0$ such that $\lim_{n\to\infty}p(gx_n,gx_{n+1})=r$. Suppose that $r>0$. Then, we have $\psi(r)<r$. As $p(gx_n,gx_{n+1})\geq r$, by using $(iii)_\psi$, we obtain

$$r-\psi(r)\leq p(gx_n,gx_{n+1})-\psi(p(gx_n,gx_{n+1})),\ \forall n\in\mathbb{N}.$$

Using condition (4.4), we obtain

$$r-\psi(r)\leq p(gx_n,gx_{n+1})-p(gx_{n+1},gx_{n+2}),\ \forall n\in\mathbb{N}.$$

Passing to the limit as $n\to\infty$, we get $r\leq\psi(r)$, a contradiction since $\psi(r)<r$. Therefore, $r=0$ and

$$\lim_{n\to\infty}p(gx_n,gx_{n+1})=0. \tag{4.5}$$

From (p_4), we have

$$p(gx_n,gx_{n+k})\leq p(gx_n,gx_{n+1})+p(gx_{n+1},gx_{n+2})+\cdots+p(gx_{n+k-1},gx_{n+k}),$$

for $k=2,3,\cdots,q$. This ensures that

$$\lim_{n\to\infty}p(gx_n,gx_{n+k})=0,\ for\ k=1,2,\cdots,q. \tag{4.6}$$

We now prove the following claim.

Claim. For every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that if $n>m>n_0$ with $n-m\equiv 1\ mod\, q$ then $p(gx_n,gx_m)<\varepsilon$.

Suppose that the above claim is not true. Then there exists $\varepsilon_0>0$ such that, for any $N\in\mathbb{N}$, we can find $n>m>n_0$ with $n-m\equiv 1\ mod\, q$ satisfying $p(gx_n,gx_m)\geq\varepsilon_0$. This gives that $N_f(gx_n,gx_m)\geq p(gx_n,gx_m)\geq\varepsilon_0$. By $(iii)_\psi$, we have

$$\varepsilon_0-\psi(\varepsilon_0)\leq N_f(gx_n,gx_m)-\psi(N_f(gx_n,gx_m)). \tag{4.7}$$

As $n-m\equiv 1\ mod\, q$, there exists $j\in\{1,2,\cdots,q\}$ such that $(x_{m+1},x_{n+1})\in A_j\times A_{j+1}$, where $A_{q+1}=A_1$. By applying condition (4.1), we obtain

$$p(gx_{n+1},gx_{m+1})\leq\psi(N_f(gx_n,gx_m)). \tag{4.8}$$

Using conditions (4.7), (4.8) and (p_4), we get

$$\begin{aligned}\varepsilon_0-\psi(\varepsilon_0) &\leq N_f(gx_n,gx_m)-p(gx_{n+1},gx_{m+1})\\ &\leq p(gx_n,gx_m)+p(gx_n,fx_n)+p(gx_m,fx_m)-p(gx_{n+1},gx_{m+1})\\ &= p(gx_n,gx_m)+p(gx_n,gx_{n+1})+p(gx_m,gx_{m+1})-p(gx_{n+1},gx_{m+1})\\ &\leq 2\,p(gx_n,gx_{n+1})+p(gx_{n+1},gx_{m+1})\\ &\quad +2\,p(gx_{m+1},gx_m)-p(gx_{n+1},gx_{m+1})\\ &= 2\,(p(gx_n,gx_{n+1})+p(gx_{m+1},gx_m)).\end{aligned}$$

Since $(p(gx_n,gx_{n+1}))$ is decreasing and $n>m$, it follows that $p(gx_n,gx_{n+1})<p(gx_m,gx_{m+1})$. Thus, $\varepsilon_0-\psi(\varepsilon_0)<4\,p(gx_{m+1},gx_m)$ and hence $p(gx_{m+1},gx_m)>\frac{\varepsilon_0-\psi(\varepsilon_0)}{4}>0$. This ensures that $(p(gx_{m+1},gx_m))$ does not converge to 0, a contradiction. Therefore, our claim is justified. By an argument similar to that used in the proof of Theorem 3.2, we can prove that (gx_n) is a 0-Cauchy sequence in $g(Y)$ which converges to some point $u\in\cap_{i=1}^{q}g(A_i)$ such that $p(u,u)=0$.

Now we prove that u is a point of coincidence of f and g.

We note that, for all $n\in\mathbb{N}$, there exists $i\in\{1,2,\cdots,q\}$ such that $x_n\in A_i$. As $u\in\cap_{i=1}^{q}g(A_i)$, it follows that $u\in g(A_{i+1})$ where $A_{q+1}=A_1$. So, there exists $t\in A_{i+1}$ such that $u=gt$. Therefore, $(gx_n,gt)\in g(A_i)\times g(A_{i+1})$. By applying (4.1), we obtain

$$p(gx_{n+1},ft)=p(fx_n,ft)\leq\psi(N_f(gx_n,gt)),\forall n\in\mathbb{N}, \tag{4.9}$$

where

$$\begin{aligned}N_f(gx_n,gt) &= \max\{p(gx_n,gt),p(gx_n,fx_n),p(gt,ft)\}\\ &= \max\{p(gx_n,gt),p(gx_n,gx_{n+1}),p(gt,ft)\}.\end{aligned}$$

Suppose that $p(gt,ft)\neq 0$. Let $\varepsilon=\frac{p(gt,ft)}{2}>0$. Since $\lim_{n\to\infty}p(gx_n,gt)=0$, there exists $k\in\mathbb{N}$ such that

$$p(gx_n,gt)<\varepsilon,\ for\ each\ n\geq k. \tag{4.10}$$

Then, for each $n\geq k$

$$p(gx_n,gx_{n+1})\leq p(gx_n,gt)+p(gt,gx_{n+1})<2\varepsilon. \tag{4.11}$$

Thus, for $n\geq k$, it follows from conditions (4.10) and (4.11) that

$$\max\{p(gx_n,gt),p(gx_n,gx_{n+1}),p(gt,ft)\}=2\varepsilon=p(gt,ft).$$

Therefore, we obtain from (4.9) that

$$p(gx_{n+1},ft)\leq\psi(p(gt,ft)),\ for\ each\ n\geq k. \tag{4.12}$$

By using condition (4.12), for $n\geq k$, we have

$$\begin{aligned}p(gt,ft) &\leq p(gt,gx_{n+1})+p(gx_{n+1},ft)-p(gx_{n+1},gx_{n+1})\\ &\leq p(gt,gx_{n+1})+\psi(p(gt,ft)).\end{aligned}$$

Passing to the limit as $n\to\infty$, we get

$$p(gt,ft)\leq\psi(p(gt,ft)),$$

which is a contradiction, since $\psi(t) < t$ for each $t > 0$. Therefore, $p(gt, ft) = 0$ and hence $ft = gt = u$. Therefore, u is a point of coincidence of f and g such that $u \in \cap_{i=1}^{q} g(A_i)$ and $p(u,u) = 0$. For the uniqueness, we assume that there is another point of coincidence v of f and g such that $v \in \cap_{i=1}^{q} g(A_i)$ and $p(v,v) = 0$. Note that there exists $x \in Y$ satisfying $v = gx = fx$. Taking $u \in g(A_i)$, $v \in g(A_{i+1})$ and applying (4.1), we have

$$\begin{aligned} p(u,v) &= p(ft, fx) \\ &\le \psi(\max\{p(gt,gx), p(gt,ft), p(gx,fx)\}) \\ &= \psi(\max\{p(u,v), p(u,u), p(v,v)\}) \\ &= \psi(p(u,v)). \end{aligned} \tag{4.13}$$

If $p(u,v) > 0$, then we get from condition (4.13) that

$$0 < p(u,v) \le \psi(p(u,v)),$$

which contradicts the fact that $\psi(t) < t$ for each $t > 0$. So, it must be the case that $p(u,v) = 0$ and hence $u = v$. Thus, f and g have a unique point of coincidence $u \in \cap_{i=1}^{q} g(A_i)$ and $p(u,u) = 0$. If f and g are weakly compatible, then we conclude from Proposition 2.13 that f and g have a unique common fixed point in $\cap_{i=1}^{q} g(A_i)$. □

Corollary 4.4. *Let (X, p) be a 0-complete partial metric space and let $f, g : X \to X$ be such that $f(X) \subseteq g(X)$ and $g(X)$ a closed subset of (X, p). If there exists a (w)-comparison function ψ such that*

$$p(fx, fy) \le \psi(N_f(gx, gy))$$

for all $x, y \in X$, then f and g have a unique point of coincidence u in $g(X)$ such that $p(u,u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof follows from Theorem 4.3 immediately by taking $A_1 = A_2 = \cdots = A_q = X$. □

Corollary 4.5. *Let (X, p) be a 0-complete partial metric space and let $f : X \to X$ be a mapping. Suppose there exists a (w)-comparison function ψ such that*

$$p(fx, fy) \le \psi(N_f(x, y))$$

for all $x, y \in X$. Then f has a unique fixed point u in X such that $p(u,u) = 0$.

Proof. The proof follows from Theorem 4.3 immediately by taking $A_1 = A_2 = \cdots = A_q = X$ and $g = I$. □

The following example supports Theorem 4.3.

Example 4.6. Suppose that X, p, f, g and A_1, A_2 are all same as those of Example 3.10. Define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{t}{2}$. Then, ψ is a (w)-comparison function. Proceeding similarly to that of Example 3.10, it is easy to compute that $p(fx, fy) \le \psi(N_f(gx, gy))$ for all $x, y \in X$ with $(gx, gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2$, where $A_3 = A_1$. Moreover, f and g are weakly compatible. Thus, all the conditions of Theorem 4.3 hold true and $\{2\}$ is the unique common fixed point of f and g in $g(A_1) \cap g(A_2)$ with $p(\{2\}, \{2\}) = 0$.

Remark 4.7. In view of Lemma 2.8 and Example 2.9, it follows that every complete partial metric space is 0-complete, but the converse is not true, in general. Therefore, the results of this paper are obtained under the weaker assumption that the underlying partial metric space is 0-complete. In fact, they are also valid if the space is complete.

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