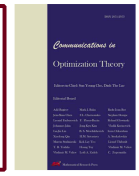
Commun. Optim. Theory 2020 (2020), Article ID 21 https://doi.org/10.23952/cot.2020.21

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org



NEW EXACT SOLUTIONS OF FRACTIONAL BOUSSINESQ-LIKE EQUATIONS

BAOMING ZHONG[1], JUN JIANG[1], YUQIANG FENG[1,2,*]

[1]School of Science, Wuhan University of Science and Technology, Wuhan, China
[2]State Key Laboratory of Environment-Friendly Energy Materials, Mianyang 621010, Sichuan, China

Abstract. Based on the improved sub-equation method, new exact solutions to a class of fractional Boussinesq-like equations in the sense of modified Riemann-Liouville derivative are obtained. The exact solutions include traveling wave solutions, soliton solutions and complex solutions.

Keywords. Boussinesq-like equation; Modified Riemann-Liouville derivative; Improved sub-equation method; Exact solution.

1. INTRODUCTION

Fractional derivatives and integrals for functions are considered to be a generalization of the derivative and integral of traditional integer order. They are important in the real world. For example, fractional derivatives and integrals for functions are extensively used in the fields of mathematics, pharmacology, biotechnology, materials science and electrodynamics, in particular, in the fields of fluid mechanics; see, e.g., [1, 2]. There many nonlinear phenomena, which can be described by fractional mathematics and equations. Therefore, it is important to develop new methods to solve fractional equations. For many years, researchers have focused on finding new methods to solve fractional differential equations, and they have found a number of new methods, which are very effective to solve fractional equation questions; see, e.g., [3, 4, 5, 6]. For example, the exponential rational function method [7] was used to obtain the new exact solutions of mathematical, physical and engineering problems. The first integral method was used in Boussinesq-like equations [8]. For other methods, we refer to Ansatz method, modified Kudryashov method, generalized Kudryashov method, generalized Mittag-Leffler function method, Jacobi elliptic equation method, fractional series expansion method, Chebyshev wavelet method, Taylor expansion method, discontinuous Galerkin method, boundary particle method, collocation method, monotone iterative method, homotopy perturbation method,

*Corresponding author.

E-mail addresses: 1019279634@qq.com (B. Zhong), jiangjun@wust.edu.cn (J. Jiang), yqfeng6@126.com (Y. Feng).

Received August 1, 2020; Accepted September 23, 2020.

modified simple equation method, and local discontinuous Galerkin method. Among all these methods, the improved sub-equation method has more advantages than the numerical, approximate analytical and semi numerical methods. By using the improved sub-equation method, we can obtain exact solutions for a wide range of nonlinear problems of fractional and integer order equations. The Boussinesq type equation (Boussinesq type equation) is often used to simulate the movement of water waves on shallow water coasts or ports. The Boussinesq-Like equations usually model a physical problem as a long wave equation. They are applied in nonlinear water model of wave and ocean engineering.

There are four different types of Boussinesq-Like equations [9, 10, 11]. They are the fractional variable-coefficient Boussinesq-Like equation (1.1), the fractional Boussinesq-Like equation with spatio-temporal dispersion (1.2), (1.3) and the fractional Boussinesq-Like equation with linear instability (1.4)

$$D_{tt}^{2\alpha}u - D_{xx}^{2\beta}u + \theta D_{xx}^{2\beta}(u^2) + \omega D_{xxxx}^{4\beta}u = 0, \tag{1.1}$$

$$D_{tt}^{2\alpha}u - D_{xx}^{2\beta}u - D_{x}^{\beta}(6u^2 D_x^{\beta}u) - D_{xxtt}^{2\alpha+2\beta}u = 0, \tag{1.2}$$

$$D_{tt}^{2\alpha}u - D_{xt}^{\alpha+\beta}u - D_{x}^{\beta}(6u^2 D_x^{\beta}u) - D_{xxxt}^{\alpha+3\beta}u = 0, \tag{1.3}$$

and

$$D_{tt}^{2\alpha}u - D_{x}^{\beta}(6u^2 D_x^{\beta}u) - D_{xxxx}^{4\beta}u = 0, \tag{1.4}$$

where $0 < \alpha, \beta \leq 1, D^{\alpha}(u)$ represents α order modified Riemann-Liouville fractional derivation of the function, which depends on the depth of the fluid and the velocity of the long wave, $D_{tt}^{2\alpha}u = D_t^{\alpha}(D_t^{\alpha}u)$, $D_{xx}^{2\beta}u = D_x^{\beta}(D_x^{\beta}u)$, $D_{xx}^{2\beta}(u^2) = D_x^{\beta}(D_x^{\beta}(u^2))$, $D_{xxxx}^{4\beta}u = D_x^{\beta}(D_x^{\beta}(D_x^{\beta}(D_x^{\beta}u)))$, $D_{xxtt}^{2\alpha+2\beta}u = D_t^{\alpha}(D_t^{\alpha}(D_x^{\beta}(D_x^{\beta}u)))$. In this paper, by using the improved sub-equation method, we obtain some new exact solutions for these four different types of Boussinesq-Like equations.

2. The Improved Sub-equation Method

2.1. **The modified Riemann-Liouville derivative.** Recently, Jumarie [12] proposed a modification of the Riemann-Liouville definition to deal with non-differentiable functions. The modified Riemann-Liouville derivative satisfies the chain rule, and the derivative of order α for a function $f(x)$ is defined as

$$D_x^{\alpha} f(x) \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)]\,d\xi, & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)]\,d\xi, & 0 < \alpha < 1 \\ \left[f^{(\alpha-n)}(x)\right]^{(n)}, & n \leq \alpha < n+1, n \geq 1 \end{cases}$$

where the function $f(x)$ is continuous and derivable, $n \in Z$.

Some useful properties of the modified Riemann-Liouville derivative are presented as follows

$$D_x^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha},$$

$$D_x^{\alpha}[f(x)g(x)] = g(x)D_x^{\alpha}f(x) + f(x)D_x^{\alpha}g(x),$$

$$D_x^{\alpha} f[g(x)] = f'_g[g(x)]\,D_x^{\alpha}g(x) = D_g^{\alpha}f[g(x)]\,(g'_x)^{\alpha},$$

where $0 < \alpha \leq 1$, $\gamma > 0$.

2.2. **Improved sub-equation method for fractional differential equations.** Consider the following fractional partial differential equation

$$P\left(u, D_t^{\alpha} u, D_x^{\beta} u, D_y^{\gamma} u, \cdots\right) = 0 \tag{2.1}$$

where x, y, t are independent variables, $0 < \alpha, \beta, \gamma < 1$, $u = u(x, y, t)$ represents an unknown function and polynomial P includes $u, u^2, \cdots$ and their partial fractional derivatives, and $D^{\alpha}(u)$ represents the modified Riemann-Liouville fractional order derivation.

First, we use the appropriate fractional complex transformation

$$u(x, y, t) = u(\xi), \tag{2.2}$$

$$\xi = \xi(x, y, t), \tag{2.3}$$

and

$$\xi = k\frac{x^{\beta}}{\Gamma(\beta+1)} - l\frac{t^{\alpha}}{\Gamma(\alpha+1)} \tag{2.4}$$

Equation (2.1) is transformed by (2.2)-(2.4) into the linear ordinary differential equation (ODE) as follows $F(u, u', u'', u''', \ldots) = 0$.

Next, we suppose that the solution of the equation is in the following form

$$u(\xi) = \sum_{k=0}^{m} a_k \Phi^k + \sum_{k=1}^{m} b_k \Phi^{-k}. \tag{2.5}$$

In this process, constants $a_k, b_k (k \in N)$ will be determined. m is a positive integer and it is obtained by using the homogeneous balance of the highest order derivative and the nonlinear term. Function $\Phi = \Phi(\xi)$ is the solution of Riccati equation

$$\Phi'(\xi) = r + p\Phi(\xi) + q\Phi^2(\xi). \tag{2.6}$$

Next, we use equations (2.5) and (2.6) to obtain a new equation in terms of Φ. The new equation is arranged as a power of $\Phi^k (k \in Z)$, and next set all the coefficients of Φ^k to zero.

Now, we are in a position to solve the algebraic equation. Substitute the $a_k, b_k (k \in N)$ to equation (2.5). We use the improved sub-equation method and the chain rule of the modified Riemann-Liouville derivative to obtain the following particular solution.

The solutions of the Riccati equation (2.6) are the follows.

1. If p, r and $q \neq 0$, then 2.6) has the following solutions

$$\begin{aligned}
\Phi(\xi) &= -\tfrac{p}{2q} + \tfrac{i\bar{\Delta}}{2q}\tan\left(\tfrac{i\bar{\Delta}\xi}{2} + C\right),\\
\Phi(\xi) &= -\tfrac{p}{2q} - \tfrac{i\bar{\Delta}}{2q}\cot\left(\tfrac{i\bar{\Delta}\xi}{2} + C\right),\\
\Phi(\xi) &= -\tfrac{p}{2q} - \tfrac{\bar{\Delta}}{2q}\tanh\left(\tfrac{\bar{\Delta}\xi}{2} + C\right),\\
\Phi(\xi) &= -\tfrac{p}{2q} - \tfrac{\bar{\Delta}}{2q}\coth\left(\tfrac{\bar{\Delta}\xi}{2} + C\right),
\end{aligned}$$

where $\bar{\Delta} = \begin{cases} \sqrt{p^2 - 4qr}, & p^2 - 4qr \geq 0 \\ i\sqrt{-p^2 + 4qr}, & p^2 - 4qr < 0 \end{cases}$ and C is constant.

2. If $p \neq 0$, $r \neq 0$ and $q = 0$, then 2.6) has the following solution

$$\Phi(\xi) = Ce^{p\xi} - \frac{r}{p}.$$

3. If $p=0$, $r\neq 0$ and $q\neq 0$, then 2.6) has the following solution

$$\Phi(\xi)=\sqrt{\frac{r}{q}}\tan(\sqrt{rq}\xi+C).$$

4. If $p\neq 0$, $r=0$ and $q\neq 0$, then 2.6) has the following solution

$$\Phi(\xi)=\frac{p}{-q+Ce^{-p\xi}}.$$

3. Applications

In this section, we describe the applications of the improved sub-equation method to fractional Boussinesq-Like equations.

3.1. **Fractional variable-coefficient Boussinesq-like equation.** Fractional variable-coefficient Boussinesq-Like equation of the form (1.1) is converted into the following ODE by the formula (2.2) to (2.4)

$$l^2u''-k^2u''+\theta k^2(u^2)''+\omega k^4u^{(4)}=0.$$

We integrate the equation twice (nelecting constant of integration) and obtain

$$(l^2-k^2)u+\theta k^2u^2+\omega k^4u''=0. \tag{3.1}$$

The solution of equation (3.1) is as (2.5)

$$u(\xi)=a_0+a_1\Phi+a_2\Phi^2+\frac{b_1}{\Phi}+\frac{b_2}{\Phi^2}. \tag{3.2}$$

We substitute equation (3.2) together with its derivatives into equation (3.1). The algebraic equation is arranged according to the powers of the function Φ^k. Thus, we obtain the following coefficients of Φ^k

$$\begin{aligned}
&\Phi^0: a_1pr\omega k^4+2a_2r^2\omega k^4+b_1pq\omega k^4+2b_2q^2\omega k^4+\theta k^2{a_0}^2+2\theta k^2a_1b_1+2\theta k^2a_2b_2\\
&\quad -k^2a_0+l^2a_0,\\
&\Phi^1: \omega k^4a_1p^2+2\omega k^4a_1qr+6a_2pr\omega k^4+2\theta k^2a_0a_1+2\theta k^2a_2b_1-k^2a_1+l^2a_1,\\
&\Phi^2: 3a_1pq\omega k^4+4\omega k^4a_2p^2+8\omega k^4a_2qr+2\theta k^2a_0a_2+\theta k^2{a_1}^2-k^2a_2+l^2a_2,\\
&\Phi^3: 2\omega k^4a_1q^2+10a_2pq\omega k^4+2\theta k^2a_1a_2,\\
&\Phi^4: 6\omega k^4a_2q^2+\theta k^2{a_2}^2,\\
&\Phi^{-1}: \omega k^4b_1p^2+2\omega k^4b_1rp+6b_2pq\omega k^4+2\theta k^2a_0b_1+2\theta k^2a_1b_2-k^2b_1+l^2b_1,\\
&\Phi^{-2}: 3b_1pr\omega k^4+4\omega k^4b_2p^2+8\omega k^4b_2rq+2\theta k^2a_0b_2+\theta k^2{b_1}^2-k^2b_2+l^2b_2,\\
&\Phi^{-3}: 2b_1r^2\omega k^4+10b_2pr\omega k^4+2\theta k^2b_2b_1,\\
&\Phi^{-4}: 6b_2r^2\omega k^4+\theta k^2{b_2}^2.
\end{aligned}$$

Letting the coefficients of Φ^k be zero, we have that $a_0,\ a_1,\ b_1,\ p,\ q,\ r,\ k,\ l$ can be solved and the solutions of equation (1.1) can be obtained.

1. $a_0=a_0,\ a_1=0,\ a_2=0, b_1=\pm\frac{2r\omega\sqrt{3\sigma}}{\theta}, b_2=-\frac{6\omega r^2k^2}{\theta}, k=k, l=l, p=\mp\frac{\sqrt{3\sigma}}{3\omega k^2}, q=-\frac{\eta}{6r\omega k^4}, r=r$, where $\sigma=\ l^2-2a_0\theta k^2-k^2,\ \eta=\ l^2+a_0\theta k^2-k^2,\ \bar{\sigma}=\left\{\begin{array}{l}\sqrt{\sigma},\sigma\geq 0\\ i\sqrt{-\sigma},\sigma<0\end{array}\right.$,

$$\bar{\omega}=\left\{\begin{array}{l}\sqrt{2\omega\eta+\sigma},\ 2\omega\eta+\sigma\geq 0\\ i\sqrt{-2\omega\eta-\sigma},2\omega\eta+\sigma<0\end{array}\right..$$

It follows that

$$u_{1,1}(x,t)=\mp\frac{2\eta\bar{\sigma}}{\theta k^2}(\pm\frac{\bar{\sigma}}{\omega}+\frac{i\bar{\omega}}{|\omega|}\tan(\frac{\sqrt{3}i\bar{\omega}\xi}{6|\omega|k^2}+C))^{-1}-\frac{2\omega\eta^2}{\theta k^2}(\mp\bar{\sigma}-\frac{i\omega\bar{\omega}}{|\omega|}\tan(\frac{\sqrt{3}i\bar{\omega}\xi}{6|\omega|k^2}+C))^{-2},$$

$$u_{1,2}(x,t)=\mp\frac{2\eta\bar{\sigma}}{\theta k^2}(\pm\frac{\bar{\sigma}}{\omega}-\frac{i\bar{\omega}}{|\omega|}\cot(\frac{\sqrt{3}i\bar{\omega}\xi}{6|\omega|k^2}+C))^{-1}-\frac{2\omega\eta^2}{\theta k^2}(\mp\bar{\sigma}+\frac{i\omega\bar{\omega}}{|\omega|}\cot(\frac{\sqrt{3}i\bar{\omega}\xi}{6|\omega|k^2}+C))^{-2},$$

$$u_{1,3}(x,t)=\mp\frac{2\eta\bar{\sigma}}{\theta k^2}(\pm\frac{\bar{\sigma}}{\omega}-\frac{\bar{\omega}}{|\omega|}\tanh(\frac{\sqrt{3}\bar{\omega}\xi}{6|\omega|k^2}+C))^{-1}-\frac{2\omega\eta^2}{\theta k^2}(\mp\bar{\sigma}+\frac{\omega\bar{\omega}}{|\omega|}\tanh(\frac{\sqrt{3}\bar{\omega}\xi}{6|\omega|k^2}+C))^{-2},$$

and

$$u_{1,4}(x,t)=\mp\frac{2\eta\bar{\sigma}}{\theta k^2}(\pm\frac{\bar{\sigma}}{\omega}-\frac{\bar{\omega}}{|\omega|}\coth(\frac{\sqrt{3}\bar{\omega}\xi}{6|\omega|k^2}+C))^{-1}-\frac{2\omega\eta^2}{\theta k^2}(\mp\bar{\sigma}+\frac{\omega\bar{\omega}}{|\omega|}\coth(\frac{\sqrt{3}\bar{\omega}\xi}{6|\omega|k^2}+C))^{-2}.$$

2. $a_0=a_0$, $a_1=0, a_2=0, b_1=\pm\frac{2r\omega\sqrt{3\sigma}}{\theta}, b_2=-\frac{6\omega r^2k^2}{\theta}, k=k, l=l, p=\mp\frac{\sqrt{3\sigma}}{3\omega k^2}, q=-\frac{a_0\theta}{6r\omega k^2}, r=r$, where $\sigma=3k^2-3l^2-2a_0\theta k^2$, $\eta=\frac{2a_0\theta\omega k^2-\sigma}{\omega^2}$, $\bar{\sigma}=\begin{cases}\sqrt{\sigma}, \sigma\geq 0\\ i\sqrt{-\sigma}, \sigma<0\end{cases}$, and

$$\bar{\omega}=\begin{cases}\sqrt{\eta},\ 2\omega\eta+\sigma\geq 0\\ i\sqrt{-\eta}, 2\omega\eta+\sigma<0\end{cases}.$$

Hence,

$$u_{2,1}(x,t)=\mp 2a_0\bar{\sigma}(\pm\frac{\bar{\sigma}}{\omega}+i\bar{\omega}\tan(\frac{\sqrt{3}i\bar{\omega}\xi}{6k^2}+C))^{-1}-\frac{a_0^2k^2\omega\theta}{2}(\mp\bar{\sigma}-i\omega\bar{\omega}\tan(\frac{\sqrt{3}i\bar{\omega}\xi}{6k^2}+C))^{-2},$$

$$u_{2,2}(x,t)=\mp 2a_0\bar{\sigma}(\pm\frac{\bar{\sigma}}{\omega}-i\bar{\omega}\cot(\frac{\sqrt{3}i\bar{\omega}\xi}{6k^2}+C))^{-1}-\frac{a_0^2k^2\omega\theta}{2}(\mp\bar{\sigma}+i\omega\bar{\omega}\cot(\frac{\sqrt{3}i\bar{\omega}\xi}{6k^2}+C))^{-2},$$

$$u_{2,3}(x,t)=\mp 2a_0\bar{\sigma}(\pm\frac{\bar{\sigma}}{\omega}-\bar{\omega}\tanh(\frac{\sqrt{3}\bar{\omega}\xi}{6k^2}+C))^{-1}-\frac{a_0^2k^2\omega\theta}{2}(\mp\bar{\sigma}+\omega\bar{\omega}\tanh(\frac{\sqrt{3}\bar{\omega}\xi}{6k^2}+C))^{-2},$$

and

$$u_{2,4}(x,t)=\mp 2a_0\bar{\sigma}(\pm\frac{\bar{\sigma}}{\omega}-\bar{\omega}\coth(\frac{\sqrt{3}\bar{\omega}\xi}{6k^2}+C))^{-1}-\frac{a_0^2k^2\omega\theta}{2}(\mp\bar{\sigma}+\omega\bar{\omega}\coth(\frac{\sqrt{3}\bar{\omega}\xi}{6k^2}+C))^{-2}.$$

3. $a_0=\frac{k^2-l^2}{4\theta k^2}$, $a_1=0, a_2=-\frac{6\omega k^4q^2}{\theta k^2}, b_1=0, b_2=-\frac{3(k^4-2k^2l^2+l^4)}{128\theta\omega q^2k^6}, k=k, l=l, p=0, q=q, r=-\frac{k^2-l^2}{16\omega k^4q}$.

4. $a_0=\frac{3(k^2-l^2)}{4\theta k^2}, a_1=0, a_2=-\frac{6\omega k^4q^2}{\theta k^2}, b_1=0, b_2=-\frac{3(k^4-2k^2l^2+l^4)}{128\theta\omega q^2k^6}, k=k, l=l, p=0, q=q, r=-\frac{k^2-l^2}{16\omega k^4q}$. Let $\sigma=-\frac{k^2-l^2}{\omega}$, and $\bar{\sigma}=\begin{cases}\sqrt{\sigma}, \sigma\geq 0\\ i\sqrt{-\sigma}, \sigma<0\end{cases}$. It follows that

$$u_{3,1}(x,t)=\frac{k^2-l^2}{4\theta k^2}+\frac{3(k^2-l^2)}{8\theta k^2}(\tan(\frac{\bar{\sigma}\xi}{4k^2}+C))^2+\frac{3(k^2-l^2)}{8\theta k^2}(\tan(\frac{\bar{\sigma}\xi}{4k^2}+C))^{-2},$$

and

$$u_{4,1}(x,t)=\frac{3(k^2-l^2)}{4\theta k^2}+\frac{3(k^2-l^2)}{8\theta k^2}(\tan(\frac{\bar{\sigma}\xi}{4k^2}+C))^2+\frac{3(k^2-l^2)}{8\theta k^2}(\tan(\frac{\bar{\sigma}\xi}{4k^2}+C))^{-2}.$$

5. $a_0=-\frac{3\omega k^4p^2+k^2-l^2}{2\theta k^2}$, $a_1=-\frac{6\omega pqk^2}{\theta}, a_2=-\frac{6\omega q^2k^2}{\theta}, b_1=0, b_2=0, k=k, l=l, p=p, q=q, r=\frac{\omega k^4q^2+k^2-l^2}{4\omega k^4q}$.

6. $a_0=\frac{3(-\omega k^4p^2+k^2-l^2)}{2\theta k^2}$, $a_1=-\frac{6\omega pqk^2}{\theta}, a_2=-\frac{6\omega q^2k^2}{\theta}, b_1=0, b_2=0, k=k, l=l, p=p, q=q, r=\frac{\omega k^4q^2-k^2+l^2}{4\omega k^4q}$. Let $\sigma=\frac{\omega k^4q^2+k^2-l^2}{\omega k^4}-p^2$, and $\bar{\sigma}=\begin{cases}\sqrt{\sigma}, \sigma\geq 0\\ i\sqrt{-\sigma}, \sigma<0\end{cases}$. It follows that

$u_{5,1}(x,t)=-\frac{3\omega k^4p^2+k^2-l^2}{2\theta k^2}-\frac{3\omega pk^2}{\theta}(-p+\bar{\sigma}\tan(\frac{\bar{\sigma}\xi}{2}+C))-\frac{3\omega k^2}{2\theta}(-p+\bar{\sigma}\tan(\frac{\bar{\sigma}\xi}{2}+C))^2,$

$u_{5,2}(x,t)=-\frac{3\omega k^4p^2+k^2-l^2}{2\theta k^2}-\frac{3\omega pk^2}{\theta}(-p-\bar{\sigma}\cot(\frac{\bar{\sigma}\xi}{2}+C))-\frac{3\omega k^2}{2\theta}(-p-\bar{\sigma}\cot(\frac{\bar{\sigma}\xi}{2}+C))^2,$

$u_{5,3}(x,t)=-\frac{3\omega k^4p^2+k^2-l^2}{2\theta k^2}-\frac{3\omega pk^2}{\theta}(-p-i\bar{\sigma}\tanh(\frac{i\bar{\sigma}\xi}{2}+C))-\frac{3\omega k^2}{2\theta}(-p-i\bar{\sigma}\tanh(\frac{i\bar{\sigma}\xi}{2}+C))^2,$

$u_{5,4}(x,t)=-\frac{3\omega k^4p^2+k^2-l^2}{2\theta k^2}-\frac{3\omega pk^2}{\theta}(-p-i\bar{\sigma}\coth(\frac{i\bar{\sigma}\xi}{2}+C))-\frac{3\omega k^2}{2\theta}(-p-i\bar{\sigma}\coth(\frac{i\bar{\sigma}\xi}{2}+C))^2,$

$u_{6,1}(x,t)=\frac{3(-\omega k^4p^2+k^2-l^2)}{2\theta k^2}-\frac{3\omega pk^2}{\theta}(-p+\bar{\sigma}\tan(\frac{\bar{\sigma}\xi}{2}+C))-\frac{3\omega k^2}{2\theta}(-p+\bar{\sigma}\tan(\frac{\bar{\sigma}\xi}{2}+C))^2,$

$u_{6,2}(x,t)=\frac{3(-\omega k^4p^2+k^2-l^2)}{2\theta k^2}-\frac{3\omega pk^2}{\theta}(-p-\bar{\sigma}\cot(\frac{\bar{\sigma}\xi}{2}+C))-\frac{3\omega k^2}{2\theta}(-p-\bar{\sigma}\cot(\frac{\bar{\sigma}\xi}{2}+C))^2,$

$u_{6,3}(x,t)=\frac{3(-\omega k^4p^2+k^2-l^2)}{2\theta k^2}-\frac{3\omega pk^2}{\theta}(-p-i\bar{\sigma}tanh(\frac{i\bar{\sigma}\xi}{2}+C))-\frac{3\omega k^2}{2\theta}(-p-i\bar{\sigma}tanh(\frac{i\bar{\sigma}\xi}{2}+C))^2,$

$u_{6,4}(x,t)=\frac{3(-\omega k^4p^2+k^2-l^2)}{2\theta k^2}-\frac{3\omega pk^2}{\theta}(-p-i\bar{\sigma}\coth(\frac{i\bar{\sigma}\xi}{2}+C))-\frac{3\omega k^2}{2\theta}(-p-i\bar{\sigma}\coth(\frac{i\bar{\sigma}\xi}{2}+C))^2,$

which $\xi=\frac{kx^\beta}{\Gamma(\beta+1)}-\frac{lt^\alpha}{\Gamma(\alpha+1)}$, $\theta r\omega k\neq 0$. We obtain eighteen forms of solutions in this section, and all of them are exact traveling wave solutions and complex solutions. The multiple soliton solutions include $u_{i,3}, u_{i,4}, i=1,2,5,6$.

3.2. **The first fractional Boussinesq-Like equation with spatio-temporal dispersion.** The first fractional Boussinesq-Like equation with spatio-temporal dispersion of form (1.2) is converted into the following ODE by the formula (2.2) to (2.4)

$$l^2u''-k^2(u''+12uu'^2+6u^2u'')-k^2l^2u^{(4)}=0. \quad (3.3)$$

The solution of equation (3.3) is the form (2.5). We obtain the solution of equation (3.3) as

$$u(\xi)=a_0+a_1\Phi+\frac{b_1}{\Phi}. \quad (3.4)$$

We substitute equation (3.4) together with its necessary derivatives into equation (3.3). The equation is arranged according to the powers of the function Φ^k. Thus, we obtain the following coefficients of Φ^k

$\Phi^0: -a_1k^2l^2p^3r-8a_1k^2l^2pqr^2-b_1k^2l^2p^3q-8\,b_1k^2l^2pq^2r-6a_0{}^2a_1k^2pr-6a_0{}^2b_1k^2pq$
$-12a_0a_1{}^2k^2r^2-12a_0b_1{}^2k^2q^2-6a_1{}^2b_1k^2pr-6a_1b_1{}^2k^2pq-a_1k^2pr+a_1l^2pr-b_1k^2pq$
$+b_1l^2pq,$

$\Phi^1: -a_1k^2l^2p^4-22\,a_1k^2l^2p^2qr-16a_1k^2l^2q^2r^2-6\,a_0{}^2a_1k^2p^2-12\,a_0{}^2a_1k^2qr$
$-36a_0a_1{}^2k^2pr-12a_1{}^3k^2r^2-6a_1{}^2b_1k^2p^2-12a_1{}^2b_1k^2qr-a_1k^2p^2-2a_1k^2qr+a_1l^2p^2$
$+2a_1l^2qr,$

$\Phi^2: -15a_1k^2l^2p^3q-60\,a_1k^2l^2q^2r-18a_0{}^2a_1k^2pq-24\,a_0a_1{}^2k^2p^2-48\,a_0a_1{}^2k^2qr$
$-30a_1{}^3k^2pr-18a_1{}^2b_1k^2pq-3a_1k^2pq+3l^2a_1pq,$

$\Phi^3: -50a_1k^2l^2p^2q^2-40\,a_1k^2l^2q^3r-12a_0{}^2a_1k^2q^2-60\,a_0a_1{}^2k^2pq-18\,a_1{}^3k^2p^2$
$-36a_1{}^3k^2qr-12a_1{}^2b_1k^2q^2-2a_1k^2q^2+2l^2a_1q^2,$

$\Phi^4: -60k^2l^2a_1pq^3-36a_0a_1{}^2k^2q^2-42a_1{}^3k^2pq,$

$\Phi^5: -24k^2l^2a_1q^4-24k^2a_1{}^3q^2,$

$\Phi^{-1}: -b_1k^2l^2p^4-22\,b_1k^2l^2p^2qr-16b_1k^2l^2q^2r^2-6\,a_0{}^2b_1k^2p^2-12\,a_0{}^2b_1k^2qr$
$-36a_0b_1{}^2k^2pq-6a_1b_1{}^2k^2p^2-12a_1b_1{}^2k^2qr-12b_1{}^3k^2q^2-b_1k^2p^2-2b_1k^2qr+b_1l^2p^2$
$+2b_1l^2qr,$

$\Phi^{-2}: -15b_1k^2l^2p^3r - 60\, b_1k^2l^2pqr^2 - 18a_0{}^2b_1k^2pr - 24\, a_0b_1{}^2k^2p^2 - 48\, a_0b_1{}^2k^2qr$
$\quad - 18a_1b_1{}^2k^2pr - 30b_1{}^3k^2pq - 3b_1k^2pr + 3l^2b_1rp,$
$\Phi^{-3}: -50b_1k^2l^2p^2r^2 - 40\, b_1k^2l^2qr^3 - 12a_0{}^2b_1k^2r^2 - 60\, a_0b_1{}^2k^2pr - 12a_1b_1{}^2k^2r^2$
$\quad - 18b_1{}^3k^2p^2 - 36b_1{}^3k^2qr - 2b_1k^2r^2 + 2l^2b_1r^2,$
$\Phi^{-4}: -60k^2l^2b_1r^3p - 36a_0b_1{}^2k^2r^2 - 42b_1{}^3k^2pr,$
$\Phi^{-5}: -24k^2l^2b_1r^4 - 24\, k^2b_1{}^3l^2.$

Letting the coefficients of Φ^k be zero, we have that $a_0,\ a_1,\ b_1,\ p,\ q,\ r,\ k,$ is solvable, and the solutions of equation (1.2) are obtainable.

1. $a_0 = \mp\frac{p}{2i},\ a_1 = 0, b_1 = \pm irl, k = k, l = l, p = p, q = \frac{l^2k^2p^2-2k^2+2l^2}{4l^2rk^2}, r = r,$ where

$$\sigma = \frac{il^2k^2p^2 - 2k^2 + 2l^2}{2lk^2},$$

and

$$\bar{\sigma} = \begin{cases} \sqrt{\frac{l^2-k^2}{l^2k^2}}, \frac{l^2-k^2}{l^2k^2} \geq 0, \\ i\sqrt{-\frac{l^2-k^2}{l^2k^2}}, \frac{l^2-k^2}{l^2k^2} < 0. \end{cases}$$

It follows that

$$u_{1,1}(x,t) = \mp\frac{p}{2i} \pm \sigma(-p + \sqrt{2}\bar{\sigma}tan(\frac{\sqrt{2}\bar{\sigma}\xi}{2} + C))^{-1},$$

$$u_{1,2}(x,t) = \mp\frac{p}{2i} \pm \sigma(-p - \sqrt{2}\bar{\sigma}\cot(\frac{\sqrt{2}\bar{\sigma}\xi}{2} + C))^{-1},$$

$$u_{1,3}(x,t) = \mp\frac{p}{2i} \pm \sigma(-p - \sqrt{2}i\bar{\sigma}\tanh(\frac{\sqrt{2}i\bar{\sigma}\xi}{2} + C))^{-1},$$

and

$$u_{1,4}(x,t) = \mp\frac{p}{2i} \pm \sigma(-p - \sqrt{2}i\bar{\sigma}\coth(\frac{\sqrt{2}i\bar{\sigma}\xi}{2} + C))^{-1}.$$

2. $a_0 = \mp\frac{p}{2i},\ a_1 = \pm iql, b_1 = 0, k = k, l = l, p = p, q = q, r = \frac{l^2k^2p^2-2k^2+2l^2}{4l^2qk^2}$, where

$$\bar{\sigma} = \begin{cases} \sqrt{\frac{l^2-k^2}{l^2k^2}}, \frac{l^2-k^2}{l^2k^2} \geq 0, \\ i\sqrt{-\frac{l^2-k^2}{l^2k^2}}, \frac{l^2-k^2}{l^2k^2} < 0. \end{cases}$$

It follows that

$$u_{2,1}(x,t) = \mp\frac{p-pl}{2i} \pm \frac{\sqrt{2}il\bar{\sigma}}{2}tan(\frac{\sqrt{2}\bar{\sigma}\xi}{2} + C),$$

$$u_{2,2}(x,t) = \mp\frac{p-pl}{2i} \mp \frac{\sqrt{2}il\bar{\sigma}}{2}\cot(\frac{\sqrt{2}\bar{\sigma}\xi}{2} + C),$$

$$u_{2,3}(x,t) = \mp\frac{p-pl}{2i} \pm \frac{\sqrt{2}l\bar{\sigma}}{2}\tanh(\frac{\sqrt{2}i\bar{\sigma}\xi}{2} + C),$$

and

$$u_{2,4}(x,t) = \mp\frac{p-pl}{2i} \pm \frac{\sqrt{2}l\bar{\sigma}}{2}\coth(\frac{\sqrt{2}i\bar{\sigma}\xi}{2} + C).$$

3. $a_0 = \mp\frac{p}{2i},\ a_1 = 0, b_1 = \pm irl, k = \pm l\sqrt{\frac{2}{2-l^2p^2}}, l = l, p = p, q = 0, r = r,\quad 2 > l^2p^2$. Hence, $u_{3,1}(x,t) = \mp\frac{p}{2i} \pm irl(Ce^{p\,\xi} - \frac{r}{p})^{-1} = \pm\frac{p}{2i} \pm irl(Ce^{p\,\xi} - \frac{r}{p})^{-1}.$

4. $a_0=\mp\frac{p}{2i}$, $a_1=\pm iql, b_1=0, k=\pm l\sqrt{\frac{2}{2-l^2p^2}}, l=l, p=p, q=q, r=0,\ \ 2>l^2p^2$. On the other hand, we have $u_{4,1}(x,t)=\mp\frac{p}{2i}\pm iql\frac{p}{-q+Ce^{-p\xi}}=\mp\frac{p}{2i}\pm\frac{ilqp}{-q+Ce^{-p\xi}}$.

5. $a_0=0$, $a_1=\pm iql, b_1=\pm\frac{l^2-k^2}{8iqlk^2}, k=k, l=l, p=0, q=q, r=\frac{l^2-k^2}{8qk^2l^2}.\sqrt{b^2-4ac}$, where $\sigma{=}l^2-k^2, \bar{\sigma}=\begin{cases}\sqrt{\sigma},\sigma\geq 0,\\ i\sqrt{-\sigma},\sigma<0.\end{cases}$ We also have

$$u_{5,1}(x,t)=\pm\frac{\sqrt{2}i\bar{\sigma}}{4k}\tan(\frac{\sqrt{2}\bar{\sigma}\xi}{4\left|kl\right|}+C)\pm\frac{\sqrt{2}\bar{\sigma}}{4ik}\cot(\frac{\sqrt{2}\bar{\sigma}\xi}{4\left|kl\right|}+C)\ \ .$$

6. $a_0=\mp\frac{p}{2i}$, $a_1=\pm iql, b_1=\pm\frac{l^2k^2p^2-2k^2+2l^2}{8iqlk^2}, k=k, l=l, p=p, q=q, r=\frac{2k^2-2l^2-l^2k^2p^2}{8l^2qk^2}$, where $\sigma{=}\frac{2k^2-2l^2-3l^2k^2p^2}{2l^2k^2}$, $\bar{\sigma}=\begin{cases}\sqrt{\sigma},\sigma\geq 0,\\ i\sqrt{-\sigma},\sigma<0.\end{cases}$ Note that

$$u_{6,1}(x,t)=\pm\frac{pl-p}{2i}\pm\frac{il\bar{\sigma}}{2}\tan(\frac{\bar{\sigma}\xi}{2}+C)\pm\frac{l^2k^2p^2-2k^2+2l^2}{4ilk^2}(-p+\bar{\sigma}\tan(\frac{\bar{\sigma}\xi}{2}+C))^{-1},$$

$$u_{6,2}(x,t)=\pm\frac{pl-p}{2i}\mp\frac{il\bar{\sigma}}{2}\cot(\frac{\bar{\sigma}\xi}{2}+C)\pm\frac{l^2k^2p^2-2k^2+2l^2}{4ilk^2}(-p-\bar{\sigma}\cot(\frac{\bar{\sigma}\xi}{2}+C))^{-1},$$

$$u_{6,3}(x,t)=\pm\frac{pl-p}{2i}\pm\frac{l\bar{\sigma}}{2}\tanh(\frac{i\bar{\sigma}\xi}{2}+C)\pm\frac{l^2k^2p^2-2k^2+2l^2}{4ilk^2}(-p-i\bar{\sigma}\tanh(\frac{i\bar{\sigma}\xi}{2}+C))^{-1},$$

and

$$u_{6,4}(x,t)=\pm\frac{pl-p}{2i}\pm\frac{l\bar{\sigma}}{2}\coth(\frac{i\bar{\sigma}\xi}{2}+C)\pm\frac{l^2k^2p^2-2k^2+2l^2}{4ilk^2}(-p-i\bar{\sigma}\coth(\frac{i\bar{\sigma}\xi}{2}+C))^{-1},$$

where $\xi=\frac{kx^\beta}{\Gamma(\beta+1)}-\frac{lt^\alpha}{\Gamma(\alpha+1)},\ \ lrkq\neq 0$. We obtain fifteen forms of solutions in this section, and all of them are exact traveling wave solutions and complex solutions. The soliton solutions include $u_{i,3}, u_{i,4}, i=1,2,6$.

3.3. **The second fractional Boussinesq-like equation with spatio-temporal dispersion.** The second fractional Boussinesq-Like equation with spatio-temporal dispersion of form (1.3) is converted into the following ODE by the formula (2.2) to (2.4)

$$l^2u''+lku''-(12uu'^2+6u^2u'')k^2+k^3lu^{(4)}=0. \tag{3.5}$$

The solution of equation (3.5) is the form (2.5)

$$u(\xi){=}a_0{+}a_1\Phi{+}\frac{b_1}{\Phi}. \tag{3.6}$$

We substitute equation (3.6) together with its necessary derivatives into equation (3.5), and the equation is arranged according to the powers of the function Φ^k. Thus, we obtain the following coefficients of Φ^k

$\Phi^0 : a_1k^3lp^3r+8a_1k^3lpqr^2+ b_1k^3lp^3q+8\, b_1k^3lpq^2r-6a_0{}^2a_1k^2pr-6a_0{}^2b_1k^2pq-12a_0$ $a_1{}^2k^2r^2-12a_0b_1{}^2k^2q^2-6a_1{}^2b_1k^2pr-6a_1b_1{}^2k^2pq+a_1klpr+a_1l^2pr+b_1klpq+b_1l^2pq$

$\Phi^1 : a_1k^3lp^4+22\,a_1k^3lp^2qr+ 16a_1k^3lq^2r^2-6\, a_0{}^2a_1k^2p^2-12\,a_0{}^2a_1k^2qr-36a_0a_1{}^2k^2pr$ $-12\,a_1{}^3k^2r^2-6a_1{}^2b_1k^2p^2-12a_1{}^2b_1k^2qr+a_1klp^2+2a_1klqr+a_1l^2p^2+2a_1l^2qr$

$\Phi^2 : 15a_1k^3lp^3q+60\,a_1k^3lpq^2r- 18a_0{}^2a_1k^2pq-24\, a_0a_1{}^2k^2p^2-48\,a_0a_1{}^2k^2qr-30a_1{}^3$ $k^2pr-18a_1{}^2b_1k^2pq-3a_1lkpq+3l^2a_1pq$

$\Phi^3\ : 50a_1k^3lp^2q^2+40\,a_1k^3lq^3r- 12a_0{}^2a_1k^2q^2-60\, a_0a_1{}^2k^2pq-18\,a_1{}^3k^2p^2-36a_1{}^3k^2$ $qr-12a_1{}^2b_1k^2q^2+2lka_0q^2+2l^2a_1q^2$

$\Phi^4\ : 60k^3la_1pq^3-36a_0a_1{}^2k^2q^2-42a_1{}^3k^2pq$

$\Phi^5\ : 24k^3la_1q^4-24k^2a_1{}^3q^2$

$\Phi^{-1} : b_1k^3lp^4+22\,b_1k^3lp^2qr+ 16b_1k^3lq^2r^2-6\, a_0{}^2b_1k^2p^2-12\,a_0{}^2b_1k^2qr-36a_0b_1{}^2k^2pq$ $-6a_1b_1{}^2k^2p^2-12a_1b_1{}^2k^2qr-12b_1{}^3k^2q^2+b_1klp^2+2b_1klqr+b_1l^2p^2+2b_1l^2qr$

$\Phi^{-2} : 15b_1k^3lp^3r+60\,b_1k^3lpqr^2- 18a_0{}^2b_1k^2pr-24\,a_0b_1{}^2k^2p^2-48a_0b_1{}^2k^2qr$ $-18a_1b_1{}^2k^2pr-30b_1{}^3k^2pq+3b_1lkpr+3l^2b_1rp$

$\Phi^{-3} : 50b_1k^3lp^2r^2+40\,b_1k^3lqr^3- 12a_0{}^2b_1k^2r^2-60\,a_0b_1{}^2k^2pr-12a_1b_1{}^2k^2r^2-18b_1{}^3k^2$ $p^2-36b_1{}^3k^2qr+2b_1lkr^2+2l^2b_1r^2$

$\Phi^{-4} : 60k^3lb_1r^3p-36a_0b_1{}^2k^2r^2- 42b_1{}^3k^2pr$

$\Phi^{-5} : 24b_1k^3lr^4-24\,k^2b_1{}^3l^2.$

Let the coefficients of Φ^k be zero. Then $a_0,\ a_1,\ b_1,\ p,\ q,\ r,\ k,\ l$ are solvable and the solutions of equation (1.3) can be obtained.

1. $a_0=\pm\frac{p\sqrt{lk}}{2}$, $a_1=0, b_1=\pm|r|\sqrt{lk}, k=k, l=l, p=p, q=\frac{-p^2k^3+2k+2l}{4k^3r}, r=r,$ where $\sigma=\frac{k+l-p^2k^3}{k^3}$, $\eta=\frac{2k+2l-p^2k^3\sqrt{lk}}{2k^3}$, $\bar\sigma=\begin{cases}\sqrt{\sigma},\sigma\geq 0\\ i\sqrt{-\sigma},\sigma<0\end{cases}$.

Hence, we have

$$u_{1,1}(x,t)=\pm\frac{p\sqrt{lk}}{2}\pm\eta(-p+\sqrt{2}\bar\sigma\tan(\frac{\sqrt{2}\bar\sigma\xi}{2}+C))^{-1},$$

$$u_{1,2}(x,t)=\pm\frac{p\sqrt{lk}}{2}\pm\eta(-p-\sqrt{2}\bar\sigma\cot(\frac{\sqrt{2}\bar\sigma\xi}{2}+C))^{-1},$$

$$u_{1,3}(x,t)=\pm\frac{p\sqrt{lk}}{2}\pm\eta(-p-\sqrt{2}i\bar\sigma\tanh(\frac{\sqrt{2}i\bar\sigma\xi}{2}+C))^{-1},$$

$$u_{1,4}(x,t)=\pm\frac{p\sqrt{lk}}{2}\pm\eta(-p-\sqrt{2}i\bar\sigma\coth(\frac{\sqrt{2}i\bar\sigma\xi}{2}+C))^{-1}.$$

2. $a_0=\pm\frac{p\sqrt{lk}}{2}$, $a_1=\pm|q|\sqrt{lk}, b_1=0, k=k, l=l, p=p, q=q, r=\frac{-p^2k^4q^2+2k^2q^2+2a_1{}^2}{4k^4q^3}$, where $\sigma=\frac{k+l-p^2k^3}{k^3}$, $\bar\sigma=\begin{cases}\sqrt{\sigma},\sigma\geq 0\\ i\sqrt{-\sigma},\sigma<0\end{cases}$.

If $q>0$, then

$$u_{2,1}(x,t)=\pm\frac{\sqrt{2lk}\bar\sigma}{2}\tan(\frac{\sqrt{2}\bar\sigma\xi}{2}+C),$$

$$u_{2,2}(x,t)=\mp\frac{\sqrt{2lk}\bar\sigma}{2}\cot(\frac{\sqrt{2}\bar\sigma\xi}{2}+C),$$

$$u_{2,3}(x,t)=\mp\frac{\sqrt{2lk}i\bar{\sigma}}{2}\tanh(\frac{\sqrt{2}i\bar{\sigma}\xi}{2}+C),$$

and

$$u_{2,4}(x,t)=\mp\frac{\sqrt{2lk}i\bar{\sigma}}{2}\coth(\frac{\sqrt{2}i\bar{\sigma}\xi}{2}+C).$$

If $q<0$, then

$$u_{2,5}(x,t)=\pm p\sqrt{lk}\mp\frac{\sqrt{2lk}\bar{\sigma}}{2}\tan(\frac{\sqrt{2}\bar{\sigma}\xi}{2}+C),$$

$$u_{2,6}(x,t)=\pm p\sqrt{lk}\pm\frac{\sqrt{2lk}\bar{\sigma}}{2}\cot(\frac{\sqrt{2}\bar{\sigma}\xi}{2}+C),$$

$$u_{2,7}(x,t)=\pm p\sqrt{lk}\pm\frac{\sqrt{2lk}i\bar{\sigma}}{2}\tanh(\frac{\sqrt{2}i\bar{\sigma}\xi}{2}+C),$$

and

$$u_{2,8}(x,t)=\pm p\sqrt{lk}\pm\frac{\sqrt{2lk}i\bar{\sigma}}{2}\coth(\frac{\sqrt{2}i\bar{\sigma}\xi}{2}+C).$$

3. $a_0=\pm\frac{p\sqrt{lk}}{2}$, $a_1=\pm|q|\sqrt{lk}, b_1=\pm\frac{\sqrt{lk}2k+2l-p^2k^3}{8k^3q}, k=k, l=l, p=p, q=q, r=\frac{-p^2k^4q^2+2k^2q^2+2a_1{}^2}{8k^4q^3}$,
where $\sigma=\frac{2k+2l-3p^2k^3}{2k^3}$, $\bar{\sigma}=\begin{cases}\sqrt{\sigma}, \sigma\geq 0\\ i\sqrt{-\sigma}, \sigma<0\end{cases}$.

If $q>0$, then

$$u_{3,1}(x,t)=\pm\frac{\sqrt{lk}\bar{\sigma}}{2}\tan(\frac{\bar{\sigma}\xi}{2}+C)\pm\frac{\sqrt{lk}2k+2l-p^2k^3}{4k^3}\left(-p+\bar{\sigma}\tan(\frac{\bar{\sigma}\xi}{2}+C)\right)^{-1},$$

$$u_{3,2}(x,t)=\mp\frac{\sqrt{lk}\bar{\sigma}}{2}\cot(\frac{\bar{\sigma}\xi}{2}+C)\pm\frac{\sqrt{lk}2k+2l-p^2k^3}{4k^3}\left(-p-\bar{\sigma}\cot(\frac{\bar{\sigma}\xi}{2}+C)\right)^{-1},$$

$$u_{3,3}(x,t)=\mp\frac{\sqrt{lk}i\bar{\sigma}}{2}\tanh(\frac{i\bar{\sigma}\xi}{2}+C)\pm\frac{\sqrt{lk}2k+2l-p^2k^3}{4k^3}\left(-p-i\bar{\sigma}\tanh(\frac{i\bar{\sigma}\xi}{2}+C)\right)^{-1},$$

and

$$u_{3,4}(x,t)=\mp\frac{\sqrt{lk}i\bar{\sigma}}{2}\coth(\frac{i\bar{\sigma}\xi}{2}+C)\pm\frac{\sqrt{lk}2k+2l-p^2k^3}{4k^3}\left(-p-i\bar{\sigma}\coth(\frac{i\bar{\sigma}\xi}{2}+C)\right)^{-1}.$$

If $q<0$, then

$$u_{3,5}(x,t)=\pm p\sqrt{lk}\mp\frac{\sqrt{lk}\bar{\sigma}}{2}\tan(\frac{\bar{\sigma}\xi}{2}+C)\pm\frac{\sqrt{lk}2k+2l-p^2k^3}{4k^3}\left(-p+\bar{\sigma}\tan(\frac{\bar{\sigma}\xi}{2}+C)\right)^{-1},$$

$$u_{3,6}(x,t)=\pm p\sqrt{lk}\pm\frac{\sqrt{lk}\bar{\sigma}}{2}\cot(\frac{\bar{\sigma}\xi}{2}+C)\pm\frac{\sqrt{lk}2k+2l-p^2k^3}{4k^3}\left(-p-\bar{\sigma}\cot(\frac{\bar{\sigma}\xi}{2}+C)\right)^{-1},$$

$$u_{3,7}(x,t)=\pm p\sqrt{lk}\pm\frac{\sqrt{lk}i\bar{\sigma}}{2}\tanh(\frac{i\bar{\sigma}\xi}{2}+C)\pm\frac{\sqrt{lk}2k+2l-p^2k^3}{4k^3}\left(-p-i\bar{\sigma}\tanh(\frac{i\bar{\sigma}\xi}{2}+C)\right)^{-1},$$

and

$$u_{3,8}(x,t) = \pm p\sqrt{lk} \pm \frac{\sqrt{lk}i\bar{\sigma}}{2}\coth(\frac{i\bar{\sigma}\xi}{2}+C) \pm \frac{\sqrt{lk}2k+2l-p^2k^3}{4k^3}(-p-i\bar{\sigma}\coth(\frac{i\bar{\sigma}\xi}{2}+C))^{-1}.$$

4. $a_0 = 0$, $a_1 = \pm|q|\sqrt{lk}, b_1 = \pm\frac{\sqrt{lk}k+l}{8k^3q}, k = k, l = l, p = 0, q = q, r = -\frac{k+l}{8k^3q}$, where $\sigma = \begin{cases} \sqrt{\frac{k+l}{k}}, \frac{k+l}{k} \geq 0 \\ i\sqrt{-\frac{k+l}{k}}, \frac{k+l}{k} < 0 \end{cases}$.

Hence,

$$u_{4,1}(x,t) = \pm i\sqrt{\frac{l\sigma}{8k}}\tan(\sqrt{\frac{\sigma}{8k^2}}i\xi + C) \pm i\sigma\sqrt{\frac{l}{8(k+l)}}\cot(\sqrt{\frac{\sigma}{8k^2}}i\xi + C),$$

where $\xi = \frac{kx^\beta}{\Gamma(\beta+1)} - \frac{lt^\alpha}{\Gamma(\alpha+1)}$, $krq \neq 0$, $lk > 0$. We obtain twenty-one forms of solutions in this section, all of them are exact traveling wave solutions and complex solutions. The soliton solutions include $u_{1,j}, u_{2,j}, u_{3,j}, j = 3,4,7,8$.

3.4. **Fractional Boussinesq-Like equations with linear instability.** The fractional Boussinesq-Like equation with linear instability of form (1.4) is converted into the following ODE

$$l^2u'' + lku'' - k^2(12uu'^2 + 6u^2u'') - k^4u^{(4)} = 0. \tag{3.7}$$

We obtain the solution of equation (3.7) as

$$u(\xi) = a_0 + a_1\Phi + \frac{b_1}{\Phi}$$

Thus we obtain the following coefficients of Φ^k

$\Phi^0 : a_1k^3lp^3r + 8a_1k^3lpqr^2 + b_1k^3lp^3q + 8\,b_1k^3lpq^2r - 6a_0{}^2a_1k^2pr - 6a_0{}^2b_1k^2pq - 12a_0a_1{}^2 \cdot k^2r^2 - 12a_0b_1{}^2k^2q^2 - 6a_1{}^2b_1k^2pr - 6a_1b_1{}^2k^2pq + a_1klpr + a_1l^2pr + b_1klpq + b_1l^2pq$

$\Phi^1 : a_1k^3lp^4 + 22\,a_1k^3lp^2qr + 16a_1k^3lq^2r^2 - 6\,a_0{}^2a_1k^2p^2 - 12a_0{}^2a_1k^2qr - 36a_0a_1{}^2k^2pr - 12a_1{}^3k^2r^2 - 6a_1{}^2b_1k^2p^2 - 12a_1{}^2b_1k^2qr + a_1klp^2 + 2a_1klqr + a_1l^2p^2 + 2a_1l^2qr$

$\Phi^2 : 15a_1k^3lp^3q + 60\,a_1k^3lpq^2r - 18a_0{}^2a_1k^2pq - 24\,a_0a_1{}^2k^2p^2 - 48\,a_0a_1{}^2k^2qr - 30a_1{}^3k^2pr - 18a_1{}^2b_1k^2pq - 3a_1lkpq + 3l^2a_1pq$

$\Phi^3 : 50a_1k^3lp^2q^2 + 40\,a_1k^3lq^3r - 12a_0{}^2a_1k^2q^2 - 60\,a_0a_1{}^2k^2pq - 18\,a_1{}^3k^2p^2 - 36a_1{}^3k^2qr - 12a_1{}^2b_1k^2q^2 + 2lka_0q^2 + 2l^2a_1q^2$

$\Phi^4 : 60k^3la_1pq^3 - 36a_0a_1{}^2k^2q^2 - 42a_1{}^3k^2pq$

$\Phi^5 : 24k^3la_1q^4 - 24k^2a_1{}^3q^2$

$\Phi^{-1} : b_1k^3lp^4 + 22\,b_1k^3lp^2qr + 16b_1k^3lq^2r^2 - 6\,a_0{}^2b_1k^2p^2 - 12\,a_0{}^2b_1k^2qr - 36a_0b_1{}^2k^2pq - 6a_1b_1{}^2k^2p^2 - 12a_1b_1{}^2k^2qr - 12b_1{}^3k^2q^2 + b_1klp^2 + 2b_1klqr + b_1l^2p^2 + 2b_1l^2qr$

$\Phi^{-2} : 15b_1k^3lp^3r + 60\,b_1k^3lpqr^2 - 18a_0{}^2b_1k^2pr - 24\,a_0b_1{}^2k^2p^2 - 48\,a_0b_1{}^2k^2qr - 18a_1b_1{}^2k$

$\Phi^{-3} : 50b_1k^3lp^2r^2 + 40\,b_1k^3lqr^3 - 12a_0{}^2b_1k^2r^2 - 60\,a_0b_1{}^2k^2pr - 12a_1b_1{}^2k^2r^2 - 18b_1{}^3k^2p^2 - 36b_1{}^3k^2qr + 2b_1lkr^2 + 2l^2b_1r^{2^2}pr - 30b_1{}^3k^2pq + 3b_1lkpr + 3l^2b_1rp$

$\Phi^{-4} : 60k^3lb_1r^3p - 36a_0b_1{}^2k^2r^2 - 42b_1{}^3k^2pr$

$\Phi^{-5} : 24b_1k^3lr^4 - 24\,k^2b_1{}^3l^2.$

Let the coefficients of Φ^k be zero. Then $a_0,\ a_1,\ b_1,\ p,\ q,\ r,\ k,\ l$ are solvable and the solutions of equation (1.4) can be obtainable.

1. $a_0=\mp\frac{kp}{2i}$, $a_1=0, b_1=\pm irk, k=k, l=l, p=p, q=\frac{k^4p^2+2kl+2l^2}{4k^4r}, r=r$, where

$$\sigma=\frac{ik^4p^2+2kl+2l^2}{2k^3}.$$

2. $a_0=\mp\frac{kp}{2i}$, $a_1=\pm iqk, b_1=0, k=k, l=l, p=p, q=q, r=\frac{k^4p^2+2kl+2l^2}{4k^4q}$.

3. $a_0=\mp\frac{kp}{2i}$, $a_1=\pm iqk, b_1=\pm\frac{k^4p^2+2kl+2l^2}{8iqk^3}, k=k, l=l, p=p, q=q, r=-\frac{k^4p^2+2kl+2l^2}{8qk^4}$, where $\eta=-\frac{3k^4p^2+2kl+2l^2}{2}$. Let $\bar{\sigma}=\begin{cases}\sqrt{\eta}, \eta\geq 0\\ i\sqrt{-\eta}, \eta<0\end{cases}$ and $\bar{\omega}=\begin{cases}\sqrt{kl+l^2},\ kl+l^2\geq 0\\ i\sqrt{-(kl+l^2)}, kl+l^2<0\end{cases}$.

If $\eta>0, kl+l^2\geq 0$, then

$$u_{1,1}(x,t)=\mp\frac{kp}{2i}\pm\sigma(-p+\frac{\sqrt{2}\bar{\omega}}{k^2}\tan(\frac{\sqrt{2}\bar{\omega}\xi}{2k^2}+C))^{-1},$$

$$u_{1,2}(x,t)=\mp\frac{kp}{2i}\pm\sigma(-p-\frac{\sqrt{2}\bar{\omega}}{k^2}\cot(\frac{\sqrt{2}\bar{\omega}\xi}{2k^2}+C))^{-1},$$

$$u_{1,3}(x,t)=\mp\frac{kp}{2i}\pm\sigma(-p-\frac{\sqrt{2}i\bar{\omega}}{k^2}\tanh(\frac{\sqrt{2}i\bar{\omega}\xi}{2k^2}+C))^{-1},$$

$$u_{1,4}(x,t)=\mp\frac{kp}{2i}\pm\sigma(-p-\frac{\sqrt{2}i\bar{\omega}}{k^2}\coth(\frac{\sqrt{2}i\bar{\omega}\xi}{2k^2}+C))^{-1},$$

$$u_{2,1}(x,t)=\pm\frac{\sqrt{2}i\bar{\omega}}{2k}\tan(\frac{\sqrt{2}\bar{\omega}\xi}{2k^2}+C),$$

$$u_{2,2}(x,t)=\mp\frac{\sqrt{2}i\bar{\omega}}{2k}\cot(\frac{\sqrt{2}\bar{\omega}\xi}{2k^2}+C),$$

$$u_{2,3}(x,t)=\pm\frac{\sqrt{2}\bar{\omega}}{2k}\tanh(\frac{\sqrt{2}i\bar{\omega}\xi}{2k^2}+C),$$

$$u_{2,4}(x,t)=\pm\frac{\sqrt{2}\bar{\omega}}{2k}\coth(\frac{\sqrt{2}i\bar{\omega}\xi}{2k^2}+C),$$

$$u_{3.1}(x,t)=\pm\frac{i\bar{\sigma}}{2k}\tan(\frac{\bar{\sigma}\xi}{2k^2}+C))\pm\frac{k^4p^2+2kl+2l^2}{4ik^3}-p+\frac{\bar{\sigma}}{k^2}\tan(\frac{\bar{\sigma}}{2k^2}+C))^{-1},$$

$$u_{3.2}(x,t)=\mp\frac{i\bar{\sigma}}{2k}\cot(\frac{\bar{\sigma}\xi}{2k^2}+C))\pm\frac{k^4p^2+2kl+2l^2}{4ik^3}-p-\frac{\bar{\sigma}}{k^2}\cot(\frac{\bar{\sigma}}{2k^2}+C))^{-1},$$

$$u_{3.3}(x,t)=\pm\frac{\bar{\sigma}}{2k}\tanh(\frac{\bar{\sigma}i\xi}{2k^2}+C))\pm\frac{k^4p^2+2kl+2l^2}{4ik^3}-p-\frac{i\bar{\sigma}}{k^2}\tanh(\frac{i\bar{\sigma}}{2k^2}+C))^{-1},$$

and

$$u_{3.4}(x,t)=\pm\frac{\bar{\sigma}}{2k}\coth(\frac{\bar{\sigma}i\xi}{2k^2}+C))\pm\frac{k^4p^2+2kl+2l^2}{4ik^3}-p-\frac{i\bar{\sigma}}{k^2}\coth(\frac{i\bar{\sigma}}{2k^2}+C))^{-1}.$$

4. $a_0=\mp\frac{1}{ki}\sqrt{\frac{-l^2-kl}{2}}$, $a_1=0, b_1=\pm irk, k=k, l=l, p=\pm\frac{\sqrt{-2l^2-2kl}}{k^2}, q=0, r=r,\ l^2+kl<0$.
Note that

$$u_{4,1}(x,t)=\pm\frac{1}{ki}\sqrt{\frac{-l^2-kl}{2}}\pm irk(Ce^{\pm\frac{\sqrt{-2l^2-2kl}}{k^2}\xi}-\frac{r}{p})^{-1}.$$

5. $a_0=\mp\frac{\sqrt{-2l^2-2kl}}{2ki}$, $a_1=\pm iqk, b_1=0, k=k, l=l, p=\pm\frac{\sqrt{-2l^2-2kl}}{k^2}, q=q, r=0,\ l^2+kl<0$.

On the other hand, we have

$$u_{5,1}(x,t) = \mp\frac{\sqrt{-2l^2-2kl}}{2ki} \pm \frac{iq\sqrt{-2l^2-2kl}}{k(-q+Ce^{\mp\frac{\sqrt{-2l^2-2kl}}{k^2}\xi})}.$$

6. $a_0 = 0$, $a_1 = \pm iqk, b_1 = \pm\frac{l(k+l)}{8iqk^3}, k = k, l = l, p = 0, q = q, r = \frac{l(k+l)}{8qk^4}$, where $\sigma = \frac{l(k+l)}{8}$, $\bar{\sigma} = \begin{cases} \sqrt{\sigma}, \sigma \geq 0 \\ i\sqrt{-\sigma}, \sigma < 0 \end{cases}$.

Note that

$$u_{6,1}(x,t) = \pm\frac{i}{k}\bar{\sigma}\tan(\frac{\bar{\sigma}\xi}{k^2}+C) \pm \frac{\bar{\sigma}}{ik}\cot(\frac{\bar{\sigma}\xi}{k^2}+C),$$

where $\xi = \frac{kx^\beta}{\Gamma(\beta+1)} - \frac{lt^\alpha}{\Gamma(\alpha+1)}$, $krq \neq 0$, $lk > 0$. We obtain fifteen forms of solutions in this section, all of them are exact traveling wave solutions and complex solutions. The soliton solutions include $u_{1,j}, u_{2,j}, u_{3,j}, j = 3,4.$.

4. Figures of the solutions

In this section, we show some typical visualize solutions as follows.

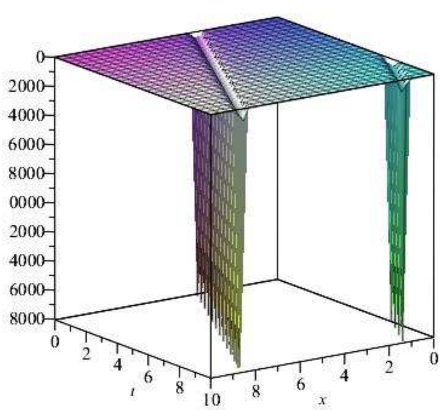


FIGURE 1. $u_{5,1}$ of equation (1.1)

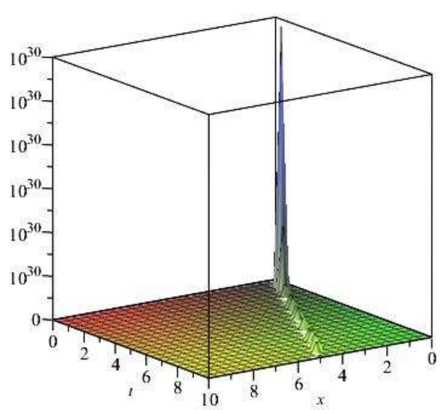


FIGURE 2. $u_{5,2}$ of equation (1.1)

Figure 1 and Figure 2 present the exact traveling wave solutions $u_{5,1}, u_{5,2}$ of fractional variable-coefficient Boussinesq-like equation. The uniform assignment of the variable of solutions $u_{5,1}, u_{5,2}$ is $k = 2, l = \omega = \pi = \alpha = \beta = 1, c = p = 0$. The interval of solutions $u_{5,1}, u_{5,2}$ designated as $x \in [0,10]$, $t \in [0,10]$. From the function of $u_{5,1}, u_{5,2}$, we get that the solutions $u_{5,1}, u_{5,2}$ are periodic functions. Substituting the given value into $u_{5,1}, u_{5,2}$, we get $u_{5,1} = -\frac{3}{8} - \frac{9}{8}\left\{\tan\left[\frac{\sqrt{3}}{8}(\frac{2x-t}{\Gamma(2)})\right]\right\}^2$, $u_{5,2} = -\frac{3}{8} + \frac{9}{8}\left\{\cot\left[\frac{\sqrt{3}}{8}(\frac{2x-t}{\Gamma(2)})\right]\right\}^2$. The period of the functions $u_{5,1}, u_{5,2}$ with respect to the variable x is $\frac{4\sqrt{3}\pi}{3}$. If $x \to \frac{2\sqrt{3}}{3}\pi$, then $u_{5,1} \to \infty$, and $t \to \frac{4\sqrt{3}}{3}\pi$, $u_{5,1} \to \infty$. If $x \in [0, \frac{2\sqrt{3}}{3}\pi]$, then $u_{5,1}, u_{5,2}$ decrease as the increases of x. If $t \in [0, \frac{4\sqrt{3}}{3}\pi]$, then $u_{5,1}$ and $u_{5,2}$ increase as the increases of t. Here, we let the other variable to zero avoid difficult computation.

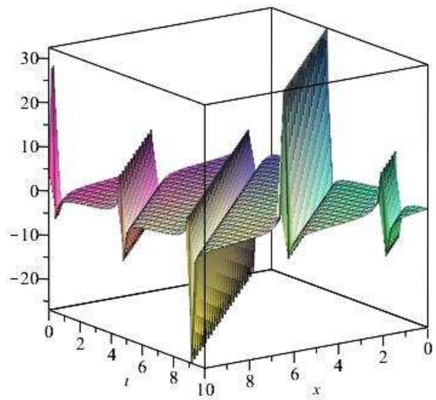


FIGURE 3. $u_{1,1}$ of equation (1.3)

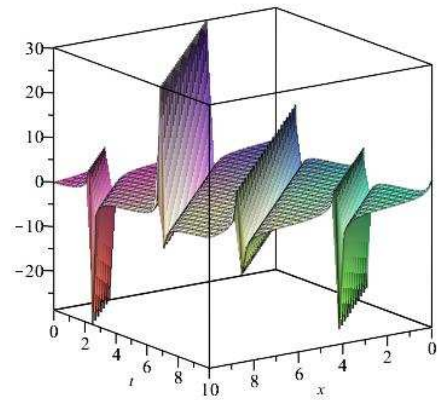


FIGURE 4. $u_{1,2}$ of equation (1.3)

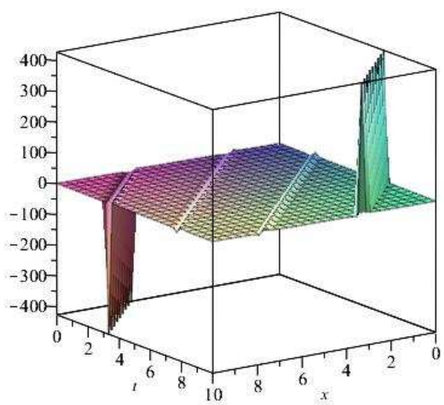


FIGURE 5. $u_{2,1}$ of equation (1.3)

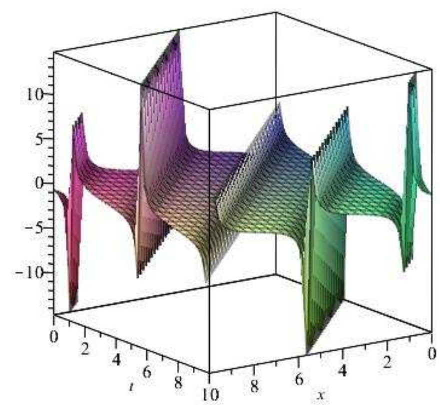


FIGURE 6. $u_{2,2}$ of equation (1.3)

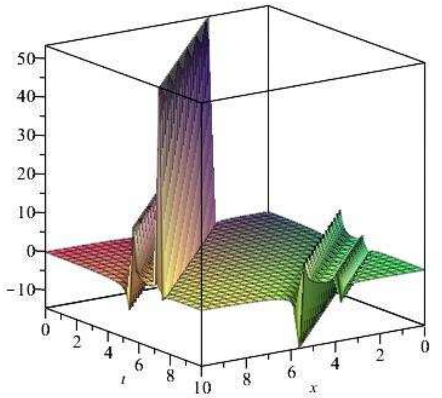


FIGURE 7. $u_{3,1}$ of equation (1.3)

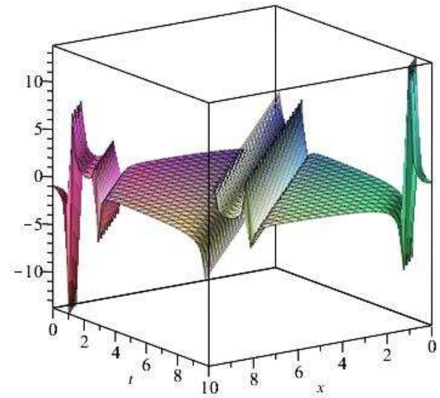


FIGURE 8. $u_{3,2}$ of equation (1.3)

Figures 3-8 present the exact traveling wave solutions from $u_{i,1}, u_{i,2} (i = 1,2,3)$ of the second fractional Boussinesq-like equation with spatio-temporal dispersion. The uniform assignment of the variable of solutions $u_{i,1}, u_{i,2} (i = 1,2,3)$ is $k = l = p = \alpha = \beta = 1, c = 0$. The interval of solutions $u_{i,1}, u_{i,2} (i = 1,2,3)$ designated as $x \in [0,10],\ t \in [0,10]$. From the function of $u_{i,1}, u_{i,2} (i = 1,2,3)$, we get that the solutions are periodic functions. Substituting the given value into $u_{i,1}, u_{i,2} (i = 1,2,3)$, we get $u_{1,1} = \frac{1}{2} + \frac{3}{2\left\{-1+\sqrt{2}\tan\left[\sqrt{\frac{1}{2}}\left(\frac{x-t}{\Gamma(2)}\right)\right]\right\}}$, $u_{1,2} = \frac{1}{2} + \frac{3}{2\left\{-1-\sqrt{2}\cot\left[\sqrt{\frac{1}{2}}\left(\frac{x-t}{\Gamma(2)}\right)\right]\right\}}$. The period of the functions $u_{1,1}, u_{1,2}$ with respect to the variable x is $\sqrt{2}\pi$. When $x \to \frac{\sqrt{2}}{2}\pi, t \to \frac{\sqrt{2}}{2}\pi$ $u_{1,1}, u_{1,2} \to \frac{1}{2}$. When $x \in [0, \frac{\sqrt{2}}{2}\pi]$, $u_{1,1}, u_{1,2}$ decrease as the increases of x. When $t \in [0, \frac{\sqrt{2}}{2}\pi]$, $u_{1,1}, u_{1,2}$ increase as the increases of t . Here, we let the other variable to zero avoid difficult computation.

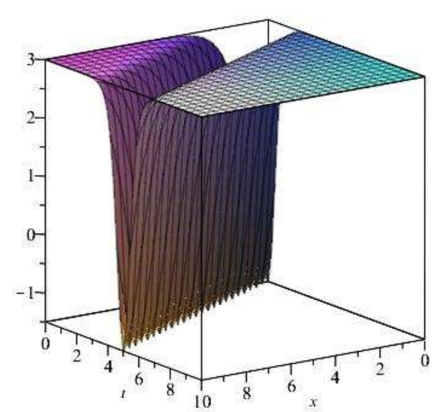


FIGURE 9. $u_{5,3}$ of equation (1.1)

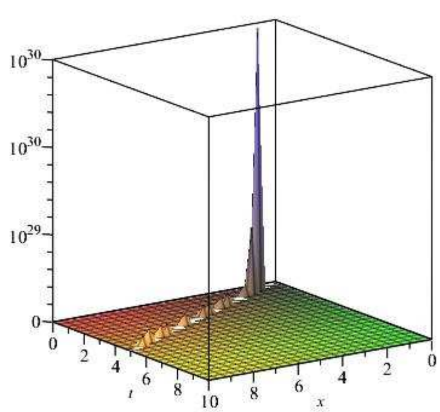


FIGURE 10. $u_{5,4}$ of equation (1.1)

Figure 9 and Figure 10 present the multiple soliton solutions from $u_{5,3}, u_{5,4}$ of the fractional variable-coefficient Boussinesq-like equation. The uniform assignment of the variable of solutions $u_{5,3}, u_{5,4}$ is $l = 2, k = \omega = \pi = \alpha = \beta = 1, c = p = 0$. the interval designated as $x \in [0,10], t \in [0,10]$. Substitute the given value into $u_{5,3}, u_{5,4}$, we get $u_{5,3} = -\frac{3}{2} + \frac{9}{2}\left\{\tanh\left[\frac{\sqrt{3}}{2}(\frac{x-2t}{\Gamma(2)})\right]\right\}^2$, $u_{5,4} = -\frac{3}{2} + \frac{9}{2}\left\{\coth\left[\frac{\sqrt{3}}{2}(\frac{x-2t}{\Gamma(2)})\right]\right\}^2$. When $x \to \infty, t \to \infty$ $u_{5,3}, u_{5,4} \to 3$,When $x \to 0, t \to 0$ $u_{5,4} \to \infty$. When $x \in [0, +\infty)$, $u_{5,3}$ increases as the increases of x . and $u_{5,4}$ decreases as the increases of x. Here, we let the other variable to zero avoid difficult computation. The solutions of $u_{5,3}, u_{5,4}$ are original symmetry.

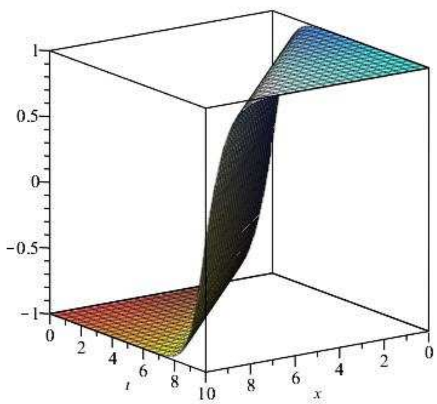


FIGURE 11. $u_{2,7}$ of equation (1.3)

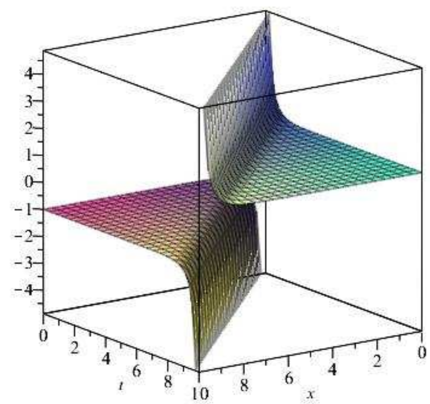


FIGURE 12. $u_{2,8}$ of equation (1.3)

Figure 11 and Figure 12 present the multiple soliton solutions from $u_{2,3}, u_{2,4}$ of the second fractional Boussinesq-Like equation with spatio-temporal dispersion, The uniform assignment of the variable of solutions $u_{2,3}, u_{2,4}$ is $k = l = \alpha = \beta = 1, p = 2, c = 0$. Substitute the given value into $u_{2,3}, u_{2,4}$, we get $u_{2,3} = -\tanh(\frac{x-t}{\Gamma(2)})$, $u_{2,4} = -\coth(\frac{x-t}{\Gamma(2)})$. When $x \to \infty, t \to \infty$ $u_{2,3}, u_{2,4} \to |1|$. ,When $x \to 0, t \to 0$ $u_{2,4} \to \infty$. ,When $x \in [0, +\infty)$, $u_{2,3}$ decreases as the increases of x and $u_{2,8}$ increases as the increases of x . Here, we let the other variable to zero avoid difficult computation. The solutions of $u_{2,3}, u_{2,4}$ are original symmetry.

5. CONCLUDING REMARK

By using the improved sub-equation method, which is based on a complex transform $u(x,y,t) = u(\xi), \xi = \xi(x,y,t)$, and the chain rule, we converted the nonlinear fractional differential equations with the modified Riemann-Liouville derivative into Riccati equations. Solving the correspondence Riccati equations, we can find the exact analytical solutions, including wave solutions, soliton solutions and complex solutions, of some Boussinesq-Like equations.

The improved sub-equation method overcomes the weakness of numerical, approximate analytical and semi numerical methods. It can be applicable to long wave equations such as the generalized regularized long wave equation, time fractional dispersive long-wave equation and the dispersive long wave equation.

Funding

This research was supported by the Open Project of State Key Laboratory of Environment-Friendly Energy Materials (19kfhg08), the Natural Science Foundation (61473338), and Hubei Province Key Laboratory of Systems Science in Metallurgical Process (Y201705).

References

[1] D. Kumar, J. Singh, D. Baleanu, Analysis of regularized long-wave equation associated with a new fractional operator with Mittag-Leffler type kernel, Phys. A 492 (2018), 155-167.

[2] B. Ahmad, S.K. Ntouyas, A. Alsaedi, New existence results for nonlinear fractional differential equations with three point integral boundary conditions, Adv. Diff. Equ. 2011 (2011), 1-11.

[3] H.A. Ghany, A.A. Hyder, M. Zakarya, Exact solutions of stochastic fractional korteweg de-vries equation with conformable derivatives, Chin. Phys B 29 (2020), 80-87.

[4] Y. Zhao, S.Y. Sun, Z.L. Han, Theory of fractional hybrid differential equations, Comput. Math. Appl. 62 (2011), 1312-1324.

[5] E. Nane, E.R. Nwaeze, M.E. Omaba. Asymptotic behavior of solution and non-existence of global solution to a class of conformable time-fractional stochastic equation, Stat Prob. Lett. 163 (2020), 108792.

[6] H.C. Choi, Y.S. Sin, K.S. Jong, Existence results for nonlinear multiorder fractional differential equations with integral and anti-periodic boundary conditions, J. Appl. Math. 2020 (2020), 1-9.

[7] S. Din, S. Bibi, Exact solutions for nonlinear fractional differential equations using exponential rational function method, Opt. Quant Electron. 49 (2017), 1-12.

[8] M. Eslami, M. Mirzazadeh, First integral method to look for exact solutions of a variety of Boussinesq-like equations, Ocean Eng. 83 (2014), 133-137.

[9] J.A. Pava, C. Banquet, J.D. Silva, The regularized Boussinesq equation: instability of periodic traveling waves, J. Differential Equations 254 (2013), 3994-4023.

[10] S. Wang, X. Su, Global existence and asymptotic behavior of solution for the sixth order Boussinesq equation with damped term, Nonlinear Anal. 120 (2015), 171-185.

[11] M. Eslami, M. Mirzazadeh, First integral method to look for exact solutions of a variety of Boussinesq-Like equations, Ocean Eng. 83 (2014), 133-137.

[12] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results, Comput. Math. Appl. 51 (2006), 51. 1367-1376.