Commun. Optim. Theory 2020 (2020), Article ID 20 https://doi.org/10.23952/cot.2020.20

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

SOME WEIGHTED TRAPEZOIDAL INEQUALITIES FOR PREQUASIINVEX FUNCTIONS

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Abstract. A new identity and some weighted trapezoidal inequalities via prequasiinvexity are established. Several special cases are also derived.

Keywords. Hermite-Hadamard inequality; Hölder inequality; Prequasiinvex.

1. INTRODUCTION

The most prominently inequality for convex functions is the so-called Hermite-Hadamard inequality, which is stated as follows

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int\limits_a^b f(x)dx \leq \tfrac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is a convex function on the finite interval $[a,b]$. If function f is concave, then (1.1) holds in the reverse direction (see [1]). Since the discovery of this double inequality, many authors have established several inequalities connected to inequality (1.1) and various variants, extensions, generalizations and improvements have been established.

In [2], Alomari, Darus and Kirmaci established the following Hermite-Hadamard type inequalities for quasi-convex functions

$$\begin{aligned}&\left|\tfrac{f(a)+f(b)}{2} - \tfrac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq &\tfrac{b-a}{8}\left(\sup\left\{\left|f'(a)\right| + \left|f'\left(\tfrac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right| + \left|f'(b)\right|\right\}\right),\end{aligned}$$

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Received April 12, 2020; Accepted September 9, 2020.
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and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}$$
$$\times\left(\left(\sup\left\{|f'(a)|^q+\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}+\left(\sup\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+|f'(b)|^q\right\}\right)^{\frac{1}{q}}\right).$$

Latif [3] generalized the above results for prequasiinvex functions

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{8}$$
$$\times\left(\sup\left\{|f'(a)|+\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|\right\}+\sup\left\{\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|+|f'(b)|\right\}\right),$$

and

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}$$
$$\times\left(\left(\sup\left\{|f'(a)|^q+\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}+\right.$$
$$\left.+\left(\sup\left\{\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+|f'(b)|^q\right\}\right)^{\frac{1}{q}}\right).$$

We note that the same results obtained by Latif are also obtained by Özcan [4].

In 2011, Hwang [5] investigated an weighted version of some results established in [2]

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,w(x)\,dx\right| \leq \frac{b-a}{4}\int_0^1\int_{\varphi(t)}^{\Psi(t)} w(x)\,dxdt$$
$$\times\left(\sup\left\{|f'(a)|+\left|f'\left(\tfrac{a+b}{2}\right)\right|\right\}+\sup\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|+|f'(b)|\right\}\right),$$

and

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,w(x)\,dx\right| \leq \frac{b-a}{4}\int_0^1\left(\int_{\varphi(t)}^{\Psi(t)} w(x)\,dx\right)dt$$
$$\times\left(\left(\sup\left\{|f'(a)|^q+\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}+\left(\sup\left\{\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+|f'(b)|^q\right\}\right)^{\frac{1}{q}}\right),$$

where $\Psi(t)=\frac{1+t}{2}a+\frac{1-t}{2}b$ and $\varphi(t)=\frac{1-t}{2}a+\frac{1+t}{2}b$.

Motivated by these results, in this paper, we establish a new identity, and then derive some new weighted trapezoidal inequalities for differentiable prequasiinvex functions. Several known results are derived.

2. Preliminaries

In this section, we recall some definitions known in the literature.

Definition 2.1. [1] A set $I\subseteq\mathbb{R}^n$ is said to be convex if, for any $x,y\in H$, and $\forall t\in[0,1]$,

$$tx+(1-t)y\in I.$$

Definition 2.2. [1] A function $f : I \to \mathbb{R}$ is said to be convex on I, where I is an interval of $\mathbb{R}$, if

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

holds for all $x,y \in I$ and all $t \in [0,1]$.

Definition 2.3. [6] A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex on I if

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\}$$

holds for all $x,y \in I$ and all $t \in [0,1]$.

Definition 2.4. [7] A set $K \subset \mathbb{R}^n$ is said to be invex with respect to the map $\eta : K \times K \to \mathbb{R}^n$ if$x+t\eta(y,x) \in K$ holds for all $x,y \in K$ and $t \in [0,1]$.

Definition 2.5. [7] A function $f : K \subset \mathbb{R} \to \mathbb{R}$ is said to be preinvex with respect to η if

$$f(x+t\eta(y,x)) \leq (1-t)f(x)+tf(y)$$

holds for all $x,y \in K$ and $t \in [0,1]$.

Definition 2.6. [8] A function $f : K \subseteq \mathbb{R} \to \mathbb{R}$ is said to be prequasiinvex function with respect to the bifunction $\eta(.,.)$ if

$$f(x+t\eta(y,x)) \leq \max\{f(y), f(x)\}$$

holds for all $x,y \in K$ and $t \in [0,1]$.

3. Main results

Lemma 3.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I°, where $a,b \in I^\circ$ with $\eta(b,a) > 0$, and let $w : [a,a+\eta(b,a)] \to [0,+\infty)$ be continuous function and symmetric to $\frac{2a+\eta(b,a)}{2}$. If $f' \in L([a,a+\eta(b,a)])$, then one has the following equality*

$$\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)w(x)\,dx$$
$$= \frac{\eta(b,a)}{4}\int_0^1 \left[\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right]\left[f'(\varphi(t))-f'(\psi(t))\right]dt, \tag{3.1}$$

where

$$\varphi(t) = a+\tfrac{1+t}{2}\eta(b,a) \quad and \quad \psi(t) = a+\tfrac{1-t}{2}\eta(b,a).$$

Proof. Integrating by parts the right side of (3.1), we get

$$\frac{\eta(b,a)}{4}\int_0^1 \left[\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right]\left[f'(\varphi(t))-f'(\psi(t))\right]dt$$
$$= \frac{\eta(b,a)}{4}\left\{\frac{2}{\eta(b,a)}\left[\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right][f(\varphi(t))+f(\psi(t))]\Big|_0^1 - \int_0^1 [w(\varphi(t))+w(\psi(t))][f(\varphi(t))+f(\psi(t))]\,dt\right\}$$
$$= \left[\int_a^{a+\eta(b,a)} w(x)\,dx\right]\frac{f(a+\eta(b,a))+f(a)}{2} - \frac{\eta(b,a)}{4}\int_0^1 [w(\varphi(t))+w(\psi(t))]f(\varphi(t))\,dt$$
$$- \frac{\eta(b,a)}{4}\int_0^1 [w(\varphi(t))+w(\psi(t))]f(\psi(t))\,dt. \tag{3.2}$$

Using the fact that $w(x)$ is symmetric to $\frac{2a+\eta(b,a)}{2}$, and making the appropriate change of variable, (3.2) gives

$$\begin{aligned}
&\tfrac{\eta(b,a)}{4}\left\{\int_0^1\left[\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right]\left[f'(\varphi(t))-f'(\psi(t))\right]dt\right\}\\
=&\tfrac{f(a+\eta(b,a))+f(a)}{2}\int_a^{a+\eta(b,a)}w(x)\,dx\\
&-\left(\tfrac{\eta(b,a)}{2}\int_0^1 w(\varphi(t))f(\varphi(t))\,dt+\tfrac{\eta(b,a)}{2}\int_0^1 w(\psi(t))f(\psi(t))\,dt\right)\\
=&\tfrac{f(a+\eta(b,a))+f(a)}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\left(\int_{a+\frac{\eta(b,a)}{2}}^{a+\eta(b,a)}w(x)f(x)\,dx-\int_{a+\frac{\eta(b,a)}{2}}^{a}w(x)f(x)\,dx\right)\\
=&\tfrac{f(a+\eta(b,a))+f(a)}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}w(x)f(x)\,dx,
\end{aligned}$$

which is the desired result. □

Theorem 3.2. *Let $f:K=[a,a+\eta(b,a)]\to\mathbb{R}$ be differentiable on K° with $f'\in L([a,a+\eta(b,a)])$, where $a,b\in K^\circ$ and $\eta(b,a)>0$, and let $w:K\to[0,+\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $|f'|$ is prequasiinvex, then*

$$\begin{aligned}
&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right|\\
\leq&\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\right)\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,\left|f'(a+\eta(b,a))\right|\right\}\right.\\
&\left.+\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,\left|f'(a)\right|\right\}\right).
\end{aligned}$$

Proof. Using Lemma 3.1, the properties of modulus, and the prequasiinvexity of $|f'|$, we get

$$\begin{aligned}
&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)}w(x)\,dx-\int_a^{a+\eta(b,a)}f(x)w(x)\,dx\right|\\
\leq&\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)\left|f'(\varphi(t))\right|dt\right.\\
&\left.+\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)\left|f'(\psi(t))\right|dt\right)\right)\\
\leq&\tfrac{\eta(b,a)}{4}\left(\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\ \max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,\left|f'(a+\eta(b,a))\right|\right\}\right)\right.\\
&\left.+\left(\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\ \max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,\left|f'(a)\right|\right\}\right)\right)\right)\\
=&\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\right)\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,\left|f'(a+\eta(b,a))\right|\right\}\right.\\
&\left.+\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|,\left|f'(a)\right|\right\}\right).
\end{aligned}$$

The proof is completed. □

Remark 3.3. If $\eta(b,a)=b-a$, then Theorem 3.2 is be reduced to [5, Theorem 2.8].

Remark 3.4. If $w(x)=\frac{1}{\eta(b,a)}$, then Theorem 3.2 is reduced to [3, Theorem 2.1].

Remark 3.5. If $w(x)=\frac{1}{\eta(b,a)}$ and $\eta(b,a)=b-a$, then Theorem 3.2 is be reduced to [2, Theorem 2.2].

Corollary 3.6. *Let the assumptions of Theorem 3.2 hold.*
(1) If $|f'|$ is increasing, then

$$\begin{aligned}&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|\\ \leq&\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)dt\right)\\ &\times\left(\left|f'(a+\eta(b,a))\right|+\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right).\end{aligned}$$

(2) If $|f'|$ is decreasing, then

$$\begin{aligned}&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|\\ \leq&\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)dt\right)\left(\left|f'(a)\right|+\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right).\end{aligned}$$

Remark 3.7. If $w(x)=\frac{1}{\eta(b,a)}$, then Corollary 3.6 recaptures [3, Corollary 2.1]. Moreover, if $\eta(b,a)=b-a$, we obtain [2, Corollary 2.1].

Theorem 3.8. *Let $f:K=[a,a+\eta(b,a)]\to\mathbb{R}$ be differentiable on K° with $f'\in L([a,a+\eta(b,a)])$, where $a,b\in K^\circ$ and $\eta(b,a)>0$, and let $w:K\to[0,+\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $|f'|^q$, where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, is prequasiinvex function, then*

$$\begin{aligned}&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|\\ \leq&\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)^p\right)^{\frac{1}{p}}\\ &\times\left(\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Proof. Using Lemma 3.1, the properties of modulus, the Hölder's inequality, and the prequasi-invexity, we have

$$\begin{aligned}
&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|\\
&\le\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)\left|f'(\varphi(t))\right|dt+\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)\left|f'(\psi(t))\right|dt\right)\\
&\le\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)^p\right)^{\frac{1}{p}}\left(\left(\int_0^1\left|f'(\varphi(t))\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'(\psi(t))\right|^q dt\right)^{\frac{1}{q}}\right)\\
&\le\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)^p\right)^{\frac{1}{p}}\\
&\quad\times\left(\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\int_0^1 dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a)\right|^q\right\}\int_0^1 1dt\right)^{\frac{1}{q}}\right)\\
&=\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)^p\right)^{\frac{1}{p}}\left(\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a+\eta(b,a))\right|^q\right\}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right),
\end{aligned}$$

which is the desired result. □

Remark 3.9. If $w(x)=\frac{1}{\eta(b,a)}$, then Theorem 3.8 is be reduced to [3, Theorem 2.2]. Moreover, if $\eta(b,a)=b-a$, then we obtain [2, Theorem 2.3].

Corollary 3.10. *Let the assumptions of Theorem 3.8 hold.*

(1) If $|f'|$ is increasing, then

$$\begin{aligned}
&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|\\
&\le\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)^p\right)^{\frac{1}{p}}\\
&\quad\times\left(\left|f'(a+\eta(b,a))\right|+\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right).
\end{aligned}$$

(2) If $|f'|$ is decreasing, then

$$\begin{aligned}
&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx-\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|\\
&\le\tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)^p\right)^{\frac{1}{p}}\left(\left|f'(a)\right|+\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right).
\end{aligned}$$

Remark 3.11. If $w(x)=\frac{1}{\eta(b,a)}$, then Corollary 3.10 recaptures [3, Corollary 2.2]. Moreover, $\eta(b,a)=b-a$ we obtain [2, Corollary 2.2].

Theorem 3.12. *Let $f:K=[a,a+\eta(b,a)]\to\mathbb{R}$ be differentiable on K° with $f'\in L([a,a+\eta(b,a)])$, where $a,b\in K^{\circ}$ and $\eta(b,a)>0$, and let $w:K\to[0,+\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$. If $|f'|^{q}$, where $q\geq 1$, is prequasiinvex, then*

$$\begin{aligned}&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)}w(x)\,dx-\int_{a}^{a+\eta(b,a)}f(x)\,w(x)\,dx\right|\\&\leq\tfrac{\eta(b,a)}{4}\left(\int_{0}^{1}\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\right)\left(\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{q},\left|f'\left(a+\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{q},\left|f'(a)\right|^{q}\right\}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Proof. Using Lemma 3.1, the properties of modulus, the power mean inequality, and the prequasiinvexity of $|f'|^{q}$, we get

$$\begin{aligned}&\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)}w(x)\,dx-\int_{a}^{a+\eta(b,a)}f(x)\,w(x)\,dx\right|\\&\leq\tfrac{\eta(b,a)}{4}\left(\int_{0}^{1}\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\right)^{1-\frac{1}{q}}\\&\times\left(\left(\int_{0}^{1}\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)\left|f'(\varphi(t))\right|^{q}dt\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_{0}^{1}\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)\left|f'(\psi(t))\right|^{q}dt\right)^{\frac{1}{q}}\right)\\&\leq\tfrac{\eta(b,a)}{4}\left(\int_{0}^{1}\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\right)^{1-\frac{1}{q}}\\&\times\left(\left(\left(\int_{0}^{1}\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\right.\right.\right.\\&\left.\left.\times\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{q},\left|f'\left(a+\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\\&+\left(\left(\int_{0}^{1}\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\right.\right.\\&\left.\left.\left.\times\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{q},\left|f'(a)\right|^{q}\right\}\right)^{\frac{1}{q}}\right)\\&=\tfrac{\eta(b,a)}{4}\left(\int_{0}^{1}\left(\int_{\psi(t)}^{\varphi(t)}w(x)\,dx\right)dt\right)\\&\times\left(\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{q},\left|f'\left(a+\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.\\&\left.+\left(\max\left\{\left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|^{q},\left|f'(a)\right|^{q}\right\}\right)^{\frac{1}{q}}\right).\end{aligned}$$

The proof is achieved. □

Remark 3.13. Theorem 3.12 will be reduced to Theorem 2.10 from [5], if we take $\eta(b,a) = b-a$.

Remark 3.14. Theorem 3.12 will be reduced to Theorem 2.3 from [3], if we put $w(x) = \frac{1}{\eta(b,a)}$.

Remark 3.15. Theorem 3.12 will be reduced to Theorem 2.4 from [2], if we choose $w(x) = \frac{1}{\eta(b,a)}$ and $\eta(b,a) = b-a$.

Corollary 3.16. *Under the assumptions of Theorem 3.12*

(1) If $|f'|$ is increasing

$$\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|$$
$$\leq \tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)dt\right)\left(\left|f'(a+\eta(b,a))\right| + \left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right).$$

(2) If $|f'|$ is decreasing

$$\left|\tfrac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(x)\,dx - \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx\right|$$
$$\leq \tfrac{\eta(b,a)}{4}\left(\int_0^1\left(\int_{\psi(t)}^{\varphi(t)} w(x)\,dx\right)dt\right)\left(\left|f'(a)\right| + \left|f'\left(a+\tfrac{1}{2}\eta(b,a)\right)\right|\right).$$

Remark 3.17. Corollary 3.16 recapture Corollary 2.3 from [3], if we take $w(x) = \frac{1}{\eta(b,a)}$. Moreover, if we choose $\eta(b,a) = b-a$, we obtain Corollary 2.3 from [2].

4. Applications to special means

We consider the means for arbitrary real numbers a,b: the Arithmetic mean: $A(a,b) = \frac{a+b}{2}$, the Geometric mean: $G(a,b) = \sqrt{ab}$, $a,b > 0$ and The p-Logarithmic mean:

$$L_p(a,b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}},$$

$a,b > 0, a \neq b$ and $p \in \mathbb{R} \setminus \{0,-1\}$.

Proposition 4.1. *Let $a,b \in \mathbb{R}$ with $0 < a < b$. Then*

$$\left|A\left(a^2,(a+A(a,b))^2\right) - L_2^2(a,a+A(a,b))\right| \leq \tfrac{a}{2}A(a,b) + \tfrac{3}{8}A^2(a,b).$$

Proof. Let $f(x) = x^2$. The assertion follows from Theorem 3.2 with $\eta(b,a) = A(a,b)$ and $w(x) = 1$. □

Proposition 4.2. *Let $a,b \in \mathbb{R}$ with $0 < a < b$. Then*

$$\left|A\left(a^{\frac{1}{2}},b^{\frac{1}{2}}\right) - L_{\frac{1}{2}}^{\frac{1}{2}}(a,b)\right| \leq \frac{(b-a)A\left(a^{\frac{1}{2}},A^{\frac{1}{2}}(a,b)\right)}{4\sqrt{3}G(a,A(a,b))}.$$

Proof. Let function $f(x) = 2\sqrt{x}$. The assertion follows from Theorem 3.8 with $p = 2, \eta(b,a) = b-a$ and $w(x) = \frac{1}{\eta(b,a)}$. □

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