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ASYMPTOTIC INERTIAL SUBGRADIENT EXTRAGRADIENT APPROACH FOR PSEUDOMONOTONE VARIATIONAL INEQUALITIES WITH FIXED POINT CONSTRAINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce asymptotic inertial subgradient extragradient algorithms with a line-search process for solving a variational inequality problem (VIP) with pseudomonotone and Lipschitz continuous mappings and a common fixed-point problem (CFPP) of an asymptotically nonexpansive mapping and a strictly pseudocontractive mapping in a real Hilbert space. The proposed algorithms are based on the inertial subgradient extragradient method with a line-search process, hybrid steepest-descent methods, viscosity approximation methods, Mann iteration methods and asymptotically nonexpansive mappings. Under mild conditions, we prove strong convergence of the proposed algorithms.

Keywords. Asymptotic inertial subgradient extragradient method with line-search process; Pseudomonotone variational inequality problem; Asymptotically nonexpansive mapping; Strictly pseudocontractive mapping; Sequentially weak continuity

1. Introduction

Let H be a real infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be any convex and closed set in H and P_C be the metric projection from space H onto set C. Let $S: C \to H$ be a nonlinear operator on C. One denotes by $\mathrm{Fix}(S)$ the set of all fixed points of S, i.e., $\mathrm{Fix}(S) = \{x \in C : x = Tx\}$. A mapping $T: C \to C$ is called asymptotically nonexpansive if $\|T^n x - T^n y\| \le (\theta_n + 1)\|x - y\|$, $\forall n \ge 1$, $x, y \in C$, where $\{\theta_n\}$ is a real sequence in $[0, +\infty)$ with $\lim_{n\to\infty} \theta_n = 0$. In particular, if $\theta_n = 0$, then T is nonexpansive. Also, one recalls that a mapping $T: C \to C$ is said to be strictly pseudocontractive if $\|Tx - Ty\|^2 \le \zeta\|(I - T)x - (I - T)y\|^2 + \|x - y\|^2$ $\forall x, y \in C$, where ζ is in $[0, 1)$. If $\zeta = 0$, then T is reduced to a nonexpansive mapping. Let $A: H \to H$ be a mapping. The classical variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \ge 0, \quad \forall x \in C. \tag{1.1}$$

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The solution set of the VIP is denoted by VI(C,A). At present, one of the most effective methods for solving the VIP is the extragradient method introduced by Korpelevich [24] in 1976. For any initial $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ x_{n+1} = P_C(x_n - \tau A y_n) \quad \forall n \geq 0, \end{cases} \tag{1.2}$$

with $\tau \in (0, \frac{1}{L})$. If VI$(C,A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by process (1.2) converges weakly to an element in VI(C,A). The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see e.g., [1-16, 18-23, 25-29, 31-36] and references therein, to name but a few.

Let $A, B : C \to H$ be two inverse-strongly monotone mappings and $T : C \to C$ be a ζ-strictly pseudocontractive mapping. In 2010, Yao et al. [2] introduced an iterative method based on the extragradient method for finding an element in the common solution set Ω of variational inequalities for A and B and the fixed point problem of T, that is, for any initial $x_0 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} z_n = P_C(x_n - \mu B x_n), \\ y_n = \alpha_n f(x_n) + (1-\alpha_n) P_C(z_n - \lambda A z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \lambda A z_n) + \delta_n T y_n \quad \forall n \geq 0, \end{cases} \tag{1.3}$$

where $f : C \to C$ be a δ-contraction with $\delta \in [0, \frac{1}{2})$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0,1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ and $(\gamma_n + \delta_n)\zeta \leq \gamma_n < (1-2\delta)\delta_n \ \forall n \geq 0$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $\liminf_{n\to\infty} \delta_n > 0$;
(iv) $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}) = 0$.

They proved the strong convergence of $\{x_n\}$ to an element $x^* \in \Omega$, which solves the VIP: $\langle (I - f)x^*, p - x^* \rangle \geq 0 \ \forall p \in \Omega$.

On the other hand, let $A, B : C \to H$ be two inverse-strongly monotone mappings and $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_n\}$. Very recently, using a modified extragradient method, Cai et al. [38] introduced a viscosity implicit rule for finding an element in the common solution set Ω of variational inequalities for A and B and the fixed point problem of T, that is, for any initial $x_1 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} u_n = s_n x_n + (1-s_n) y_n, \\ z_n = P_C(u_n - \mu B u_n), \\ y_n = P_C(z_n - \lambda A z_n), \\ x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \rho F) T^n y_n], \end{cases} \tag{1.4}$$

where $f : C \to C$ be a δ-contraction with $\delta \in [0,1)$, and $\{\alpha_n\}, \{s_n\}$ are sequences in $(0,1]$ such that

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$;
(iii) $0 < \varepsilon \leq s_n \leq 1$ and $\sum_{n=1}^{\infty} |s_{n+1} - s_n| < \infty$;

(iv) $\sum_{n=1}^{\infty} \|T^{n+1}y_n - T^n y_n\| < \infty$.

They proved the strong convergence of $\{x_n\}$ to an element $x^* \in \Omega$, which solves the VIP: $\langle(\rho F - f)x^*, p - x^*\rangle \geq 0\ \forall p \in \Omega$.

In the extragradient method, one needs to compute two projections onto C for each iteration. Without question, the projection onto a closed convex set C is closely related to a minimum distance problem. If C is a general closed and convex set, this might require a prohibitive amount of computation time. In 2011, Censor et al. [5] modified Korpelevich's extragradient method and first introduced the subgradient extragradient method, in which the second projection onto C is replaced by a projection onto a half-space:

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ C_n = \{x \in H : \langle x_n - \tau A x_n - y_n, x - y_n\rangle \leq 0\}, \\ x_{n+1} = P_{C_n}(x_n - \tau A y_n) \quad \forall n \geq 0, \end{cases}$$

with $\tau \in (0, \frac{1}{L})$. In 2014, Kraikaew and Saejung [22] introduced the Halpern subgradient extragradient method for solving the VIP (1.1), and proved strong convergence of the proposed method to a solution of VIP (1.1).

In 2018, by virtue of the inertial technique, Thong and Hieu [31] first introduced the inertial subgradient extragradient method, and proved weak convergence of the proposed method to a solution of VIP (1.1). Very recently, Thong and Hieu [25] introduced two inertial subgradient extragradient algorithms with linear-search process for solving the VIP (1.1) with monotone and Lipschitz continuous mapping A and the fixed-point problem (FPP) of a quasi-nonexpansive mapping T with a demiclosedness property in a real Hilbert space.

Algorithm 1.1 (see [25, Algorithm 1]). **Initialization:** Given $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying $\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\|$.

Step 2. Compute $z_n = P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \leq 0\}$.

Step 3. Compute $x_{n+1} = (1-\beta_n)w_n + \beta_n T z_n$. If $w_n = z_n = x_{n+1}$ then $w_n \in \mathrm{Fix}(T) \cap \mathrm{VI}(C,A)$. Set $n := n+1$ and go to Step 1.

Algorithm 1.2 (see [25, Algorithm 2]). **Initialization:** Given $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary.

Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Set $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying $\tau\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\|$.

Step 2. Compute $z_n = P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \leq 0\}$.

Step 3. Compute $x_{n+1} = (1-\beta_n)x_n + \beta_n T z_n$. If $w_n = z_n = x_n = x_{n+1}$ then $x_n \in \mathrm{Fix}(T) \cap \mathrm{VI}(C,A)$. Set $n := n+1$ and go to Step 1.

Under mild conditions, they proved weak convergence of the proposed algorithms to an element of $\mathrm{Fix}(T) \cap \mathrm{VI}(C,A)$. Inspired by the research work of [25], we introduce two asymptotic inertial subgradient extragradient algorithms with line-search process for solving VIP (1.1) with pseudomonotone and Lipschitz continuous mapping and the CFPP of an asymptotically nonexpansive mapping and a strictly pseudocontractive mapping in H. The proposed algorithms are based on inertial subgradient extragradient method with line-search process, hybrid steepest-descent method, viscosity approximation method, Mann iteration method and asymptotically nonexpansive mapping. Under suitable conditions, we prove strong convergence of the proposed algorithms to an element in the common solution set of the VIP and CFPP, which solves a certain hierarchical VIP defined on this common solution set. Finally, our main results are applied to solve the VIP and CFPP in an illustrated example.

This paper is organized as follows. In Sect. 2, we recall some definitions and preliminary results for further use. Sect. 3 deals with the convergence analysis of the proposed algorithms. Finally, in Sect. 4, our main results are applied to solve the VIP and CFPP in an illustrated example. Our algorithms are more advantageous and more flexible than Algorithms 1 and 2 in [25] because they involve solving VIP (1.1) with pseudomonotone and Lipschitz continuous mapping and the CFPP of an asymptotically nonexpansive mapping and a strictly pseudocontractive mapping. Our results improve and extend the corresponding results announced in Kraikaew and Saejung [22], Thong and Hieu [25, 31], Yao et al. [2], and Cai et al. [38].

2. PRELIMINARIES

Suppose that $\{x_n\}$ is a sequence in H. Then we denote by $x_n \to x$ (resp., $x_n \rightharpoonup x$) the strong (resp., weak) convergence of $\{x_n\}$ to x. A mapping $T : C \to H$ is called

(i) L-Lipschitz continuous (or L-Lipschitzian) if $\exists L > 0$ such that $\|Tx - Ty\| \le L\|x - y\|\ \forall x, y \in C$;

(ii) monotone if $\langle Tx - Ty, x - y\rangle \ge 0\ \forall x, y \in C$;

(iii) pseudomonotone if $\langle Tx, y - x\rangle \ge 0 \Rightarrow \langle Ty, y - x\rangle \ge 0\ \forall x, y \in C$;

(iv) α-strongly monotone if $\exists \alpha > 0$ such that $\langle Tx - Ty, x - y\rangle \ge \alpha\|x - y\|^2\ \forall x, y \in C$;

(v) sequentially weakly continuous if $\forall \{x_n\} \subset C$, the relation holds: $x_n \rightharpoonup x \Rightarrow Tx_n \rightharpoonup Tx$.

It is easy to see that every monotone operator is pseudomonotone but the converse is not true. Also, recall that the mapping $T : C \to C$ is a ζ-strict pseudocontraction for some $\zeta \in [0, 1)$ if and only if the inequality holds $\langle Tx - Ty, x - y\rangle \le \|x - y\|^2 - \frac{1-\zeta}{2}\|(I - T)x - (I - T)y\|^2\ \forall x, y \in C$. From [30] we know that if T is a ζ-strictly pseudocontractive mapping, then T satisfies Lipschitz condition $\|Tx - Ty\| \le \frac{1+\zeta}{1-\zeta}\|x - y\|\ \forall x, y \in C$. For each point $x \in H$, we know that there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|\ \forall y \in C$. The mapping P_C is called the metric projection of H onto C.

Lemma 2.1 (see [17]). The following hold:
(i) $\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2\ \forall x, y \in H$;
(ii) $\langle x - P_C x, y - P_C x\rangle \le 0\ \forall x \in H, y \in C$;
(iii) $\|x - y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2\ \forall x \in H, y \in C$;
(iv) $\|x - y\|^2 + 2\langle x - y, y\rangle = \|x\|^2 - \|y\|^2,\ \forall x, y \in H$;
(v) $\|\lambda x + \mu y\|^2 + \lambda\mu\|x - y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2,\ \forall x, y \in H,\ \forall \lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$.

The following lemma is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$.

Lemma 2.2. There holds the inequality

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle \quad \forall x,y \in H.$$

Lemma 2.3 (see [5, Lemma 2.1]). Let $A : C \to H$ be pseudomonotone and continuous. Then $x^* \in C$ is a solution to the VIP $\langle Ax^*, x-x^*\rangle \ge 0\ \forall x \in C$, if and only if $\langle Ax, x-x^*\rangle \ge 0\ \forall x \in C$.

Lemma 2.4 (see [37]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the conditions: $a_{n+1} \le (1-\lambda_n)a_n + \lambda_n\gamma_n\ \forall n \ge 1$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that (i) $\{\lambda_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$, and (ii) $\limsup_{n\to\infty}\gamma_n \le 0$ or $\sum_{n=1}^{\infty}|\lambda_n\gamma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 (see [30]). Let $T : C \to C$ be a ζ-strict pseudocontraction. Then $I-T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $(I-T)x_n \to 0$, then $(I-T)x = 0$, where I is the identity mapping of H.

Lemma 2.6 (see [2]). Let $T : C \to C$ be a ζ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers. Assume $(\gamma+\delta)\zeta \le \gamma$. Then $\|\gamma(x-y) + \delta(Tx-Ty)\| \le (\gamma+\delta)\|x-y\|\ \forall x,y \in C$.

Lemma 2.7 (see [37, Lemma 3.1]). Let $\lambda \in (0,1]$, $T : C \to H$ be a nonexpansive mapping, and the mapping $T^\lambda : C \to H$ be defined by $T^\lambda x := Tx - \lambda\mu F(Tx)\ \forall x \in C$, where $F : H \to H$ is κ-Lipschitzian and η-strongly monotone. Then T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$, i.e., $\|T^\lambda x - T^\lambda y\| \le (1-\lambda\tau)\|x-y\|\ \forall x,y \in C$, where $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1]$.

Lemma 2.8 (see [39]). Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$. Then $I-T$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $(I-T)x_n \to 0$, then $(I-T)x = 0$, where I is the identity mapping of X.

3. Main results

In this section, let the feasible set C be a nonempty closed convex subset of a real Hilbert space H, and assume always that the following hold.

$T : H \to H$ is an asymptotically nonexpansive mapping with $\{\theta_n\}$ and $S : H \to H$ is a ζ-strictly pseudocontractive mapping.

$A : H \to H$ is L-Lipschitz continuous, pseudomonotone on H, and sequentially weakly continuous on C, such that $\Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C,A) \ne \emptyset$.

$f : H \to H$ is a contraction with constant $\delta \in [0,1)$, and $F : H \to H$ is η-strongly monotone and κ-Lipschitzian such that $\delta < \tau := 1-\sqrt{1-\rho(2\eta-\rho\kappa^2)}$ for $\rho \in (0, \frac{2\eta}{\kappa^2})$.

$\{\sigma_n\} \subset [0,1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$ such that

(i) $\sup_{n\ge1}\frac{\sigma_n}{\alpha_n} < \infty$ and $\beta_n + \gamma_n + \delta_n = 1\ \forall n \ge 1$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(iii) $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n} = 0$ and $(\gamma_n+\delta_n)\zeta \le \gamma_n \ \forall n \ge 1$;
(iv) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ and $\liminf_{n\to\infty}\delta_n > 0$.

Algorithm 3.1. Initialization: Given $\gamma > 0,\ l \in (0,1),\ \mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Set $w_n = T^n x_n + \sigma_n(T^n x_n - T^n x_{n-1})$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \le \mu\|w_n - y_n\|. \tag{3.1}$$

Step 2. Compute $z_n = \alpha_n f(x_n) + (I - \alpha_n\rho F)T^n P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \le 0\}$.
Step 3. Compute

$$x_{n+1} = \beta_n x_n + \gamma_n z_n + \delta_n S z_n. \tag{3.2}$$

Again set $n := n+1$ and go to Step 1.

Lemma 3.1. The Armijo-like search rule (3.1) is well defined, and the inequality holds: $\min\{\gamma, \frac{\mu l}{L}\} \le \tau_n \le \gamma$.

Proof. From the L-Lipschitz continuity of A we get $\frac{\mu}{L}\|Aw_n - AP_C(w_n - \gamma l^m A w_n)\| \le \mu\|w_n - P_C(w_n - \gamma l^m A w_n)\|$. Thus, (3.1) holds for all $\gamma l^m \le \frac{\mu}{L}$ and so τ_n is well defined. Obviously, $\tau_n \le \gamma$. If $\tau_n = \gamma$, then the inequality is true. If $\tau_n < \gamma$, then from (3.1) we get $\|Aw_n - AP_C(w_n - \frac{\tau_n}{l}Aw_n)\| > \frac{\mu}{\frac{\tau_n}{l}}\|w_n - P_C(w_n - \frac{\tau_n}{l}Aw_n)\|$. Again from the L-Lipschitz continuity of A we obtain $\tau_n > \frac{\mu l}{L}$. Hence the inequality is valid. □

Lemma 3.2. Let $\{w_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by Algorithm 3.1. Then

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)\|w_n - p\|^2 - (1-\alpha_n\tau)(1+\theta_n)(1-\mu)\times \\ &\quad \times[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + 2\alpha_n\langle(f-\rho F)p, z_n - p\rangle \ \forall p \in \Omega, n \ge n_0,\end{aligned} \tag{3.3}$$

for some $n_0 \ge 1$, where $u_n := P_{C_n}(w_n - \tau_n A y_n)$.

Proof. First, take an arbitrary $p \in \Omega \subset C \subset C_n$. We note that

$$\begin{aligned}\|u_n - p\|^2 &= \|P_{C_n}(w_n - \tau_n A y_n) - P_{C_n}p\|^2 \le \langle u_n - p, w_n - \tau_n A y_n - p\rangle \\ &= \tfrac{1}{2}\|u_n - p\|^2 + \tfrac{1}{2}\|w_n - p\|^2 - \tfrac{1}{2}\|u_n - w_n\|^2 - \langle u_n - p, \tau_n A y_n\rangle.\end{aligned}$$

So, it follows that $\|u_n - p\|^2 \le \|w_n - p\|^2 - \|u_n - w_n\|^2 - 2\langle u_n - p, \tau_n A y_n\rangle$, which together with (3.1) and the pseudomonotonicity of A, we deduce that $\langle Ay_n, p - y_n\rangle \le 0$ and

$$\begin{aligned}\|u_n - p\|^2 &\le \|w_n - p\|^2 - \|u_n - w_n\|^2 + 2\tau_n(\langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - u_n\rangle) \\ &\le \|w_n - p\|^2 - \|u_n - w_n\|^2 + 2\tau_n\langle Ay_n, y_n - u_n\rangle \\ &= \|w_n - p\|^2 - \|u_n - y_n\|^2 - \|y_n - w_n\|^2 + 2\langle w_n - \tau_n A y_n - y_n, u_n - y_n\rangle.\end{aligned} \tag{3.4}$$

Since $u_n = P_{C_n}(w_n - \tau_n Ay_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n Aw_n - y_n, x - y_n\rangle \le 0\}$, we have $\langle w_n - \tau_n Aw_n - y_n, u_n - y_n\rangle \le 0$, which together with (3.1), implies that

$$\begin{aligned} 2\langle w_n - \tau_n Ay_n - y_n, u_n - y_n\rangle &= 2\langle w_n - \tau_n Aw_n - y_n, u_n - y_n\rangle + 2\tau_n\langle Aw_n - Ay_n, u_n - y_n\rangle \\ &\le 2\mu\|w_n - y_n\|\|u_n - y_n\| \le \mu(\|w_n - y_n\|^2 + \|u_n - y_n\|^2). \end{aligned}$$

Therefore, substituting the last inequality for (3.4), we infer that

$$\|u_n - p\|^2 \le \|w_n - p\|^2 - (1-\mu)\|w_n - y_n\|^2 - (1-\mu)\|u_n - y_n\|^2 \quad \forall p \in \Omega. \tag{3.5}$$

In addition, from Algorithm 3.1 we have

$$\begin{aligned} z_n - p &= \alpha_n f(x_n) + (I - \alpha_n \rho F)T^n P_{C_n}(w_n - \tau_n Ay_n) - p \\ &= \alpha_n(f(x_n) - f(p)) + (I - \alpha_n\rho F)T^n u_n - (I - \alpha_n \rho F)p + \alpha_n(f - \rho F)p. \end{aligned}$$

Taking into account $\lim_{n\to\infty} \frac{\theta_n(2+\theta_n)}{\alpha_n(1-\beta_n)} = 0$, we know that $\theta_n(2+\theta_n) \le \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}$ $\forall n \ge n_0$ for some $n_0 \ge 1$. Hence we have that for all $n \ge n_0$,

$$\begin{aligned} \alpha_n\delta + (1-\alpha_n\tau)(1+\theta_n) &= 1 - \alpha_n(\tau-\delta) + (1-\alpha_n\tau)\theta_n \\ &\le 1 - \alpha_n(\tau-\delta) + \theta_n \le 1 - \tfrac{\alpha_n(\tau-\delta)}{2} \le 1. \end{aligned}$$

Using Lemma 2.2, Lemma 2.7, and the convexity of the function $h(t) = t^2$ $\forall t \in \mathbf{R}$, from (3.5) we obtain that for all $n \ge n_0$,

$$\begin{aligned} &\|z_n - p\|^2 \\ &\le \|\alpha_n(f(x_n) - f(p)) + (I-\alpha_n\rho F)T^n u_n - (I - \alpha_n\rho F)p\|^2 + 2\alpha_n\langle (f-\rho F)p, z_n - p\rangle \\ &\le [\alpha_n\delta\|x_n - p\| + (1-\alpha_n\tau)(1+\theta_n)\|u_n - p\|]^2 + 2\alpha_n\langle (f-\rho F)p, z_n - p\rangle \\ &\le \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)\|u_n - p\|^2 + 2\alpha_n\langle (f-\rho F)p, z_n - p\rangle \\ &\le \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)[\|w_n - p\|^2 - (1-\mu)\|w_n - y_n\|^2 \\ &\quad -(1-\mu)\|u_n - y_n\|^2] + 2\alpha_n\langle (f-\rho F)p, z_n - p\rangle \\ &= \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)\|w_n - p\|^2 - (1-\alpha_n\tau)(1+\theta_n)(1-\mu)\times \\ &\quad \times[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + 2\alpha_n\langle (f-\rho F)p, z_n - p\rangle. \end{aligned}$$

This completes the proof. □

Lemma 3.3. Let $\{w_n\}, \{x_n\}, \{y_n\}, \{z_n\}$ be bounded sequences generated by Algorithm 3.1. If $T^n x_n - T^{n+1}x_n \to 0$, $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$, $w_n - z_n \to 0$ and $\exists \{w_{n_k}\} \subset \{w_n\}$ such that $w_{n_k} \rightharpoonup z \in H$, then $z \in \Omega$.

Proof. From Algorithm 3.1, we get $w_n - x_n = T^n x_n - x_n + \alpha_n(T^n x_n - T^n x_{n-1})$ $\forall n \ge 1$, and hence $\|T^n x_n - x_n\| \le \|w_n - x_n\| + \alpha_n\|T^n x_n - T^n x_{n-1}\| \le \|w_n - x_n\| + (1+\theta_n)\|x_n - x_{n-1}\|$. Utilizing the assumptions $x_n - x_{n+1} \to 0$ and $w_n - x_n \to 0$, we have from $\theta_n \to 0$ that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{3.6}$$

Combining the assumptions $w_n - x_n \to 0$ and $w_n - z_n \to 0$ implies that as $n \to \infty$,

$$\|z_n - x_n\| \le \|w_n - z_n\| + \|w_n - x_n\| \to 0.$$

Note that for each $p \in \Omega$,

$$\begin{aligned}\|w_n - p\|^2 &= \|T^n x_n - p + \sigma_n(T^n x_n - T^n x_{n-1})\|^2 \\ &\le (\|T^n x_n - p\| + \sigma_n \|T^n x_n - T^n x_{n-1}\|)^2 \\ &\le (1+\theta_n)^2(\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|)^2 \\ &= [1+\theta_n(2+\theta_n)][\|x_n - p\|^2 + \sigma_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|)] \\ &= \|x_n - p\|^2 + \Gamma_n + \theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n),\end{aligned}$$

where $\Gamma_n = \sigma_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \sigma_n\|x_n - x_{n-1}\|)$. So it follows from (3.3) that for all $n \ge n_0$,

$$\begin{aligned}&(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ &\le \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)\|w_n - p\|^2 - \|z_n - p\|^2 + 2\alpha_n\langle(f-\rho F)p, z_n - p\rangle \\ &\le \alpha_n\delta\|x_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)[\|x_n - p\|^2 + \Gamma_n \\ &\quad +\theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n)] - \|z_n - p\|^2 + 2\alpha_n\|(f-\rho F)p\|\|z_n - p\| \\ &= [\alpha_n\delta + (1-\alpha_n\tau)(1+\theta_n)]\|x_n - p\|^2 - \|z_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)[\Gamma_n \\ &\quad +\theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n)] + 2\alpha_n\|(f-\rho F)p\|\|z_n - p\| \\ &\le [1 - \tfrac{\alpha_n(\tau-\delta)}{2}]\|x_n - p\|^2 - \|z_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)[\Gamma_n \\ &\quad +\theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n)] + 2\alpha_n\|(f-\rho F)p\|\|z_n - p\| \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 + (1-\alpha_n\tau)(1+\theta_n)[\Gamma_n \\ &\quad +\theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n)] + 2\alpha_n\|(f-\rho F)p\|\|z_n - p\| \\ &\le \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|) + (1+\theta_n)[\Gamma_n \\ &\quad +\theta_n(2+\theta_n)(\|x_n - p\|^2 + \Gamma_n)] + 2\alpha_n\|(f-\rho F)p\|\|z_n - p\|.\end{aligned}$$

Since $\alpha_n \to 0,\ \theta_n \to 0,\ \Gamma_n \to 0$ and $x_n - z_n \to 0$, from the boundedness of $\{x_n\}, \{z_n\}$ we get

$$\lim_{n\to\infty}\|w_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|u_n - y_n\| = 0.$$

Thus we deduce that as $n \to \infty$,

$$\|w_n - u_n\| \le \|w_n - y_n\| + \|y_n - u_n\| \to 0 \quad \text{and} \quad \|x_n - u_n\| \le \|x_n - w_n\| + \|w_n - u_n\| \to 0.$$

Furthermore, by Algorithm 3.1 we get $x_{n+1} - z_n = \beta_n(x_n - z_n) + \delta_n(Sz_n - z_n)$, which immediately yields

$$\delta_n\|Sz_n - z_n\| = \|x_{n+1} - x_n + (1-\beta_n)(x_n - z_n)\| \le \|x_{n+1} - x_n\| + \|x_n - z_n\|.$$

Since $x_n - x_{n+1} \to 0,\ z_n - x_n \to 0$ and $\liminf_{n\to\infty}\delta_n > 0$, we obtain

$$\lim_{n\to\infty}\|z_n - Sz_n\| = 0. \tag{3.7}$$

Noticing $y_n = P_C(w_n - \tau_n A w_n)$, we have $\langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \le 0\ \forall x \in C$, and hence

$$\frac{1}{\tau_n}\langle w_n - y_n, x - y_n\rangle + \langle Aw_n, y_n - w_n\rangle \le \langle Aw_n, x - w_n\rangle \quad \forall x \in C. \tag{3.8}$$

Being weakly convergent, $\{w_{n_k}\}$ is bounded. Then, according to the Lipschitz continuity of A, $\{Aw_{n_k}\}$ is bounded. Since $w_n - y_n \to 0$, $\{y_{n_k}\}$ is bounded as well. Note that $\tau_n \ge \min\{\gamma, \frac{\mu l}{L}\}$. So, from (3.8) we get $\liminf_{k\to\infty}\langle Aw_{n_k}, x - w_{n_k}\rangle \ge 0\ \forall x \in C$. Meantime, observe that $\langle Ay_n, x - y_n\rangle = \langle Ay_n - Aw_n, x - w_n\rangle + \langle Aw_n, x - w_n\rangle + \langle Ay_n, w_n - y_n\rangle$. Since $w_n - y_n \to 0$, from L-Lipschitz continuity of A we obtain $Aw_n - Ay_n \to 0$, which together with (3.8) yields $\liminf_{k\to\infty}\langle Ay_{n_k}, x - y_{n_k}\rangle \ge 0\ \forall x \in C$.

Next we show that $x_n - Tx_n \to 0$. Indeed, note that

$$\begin{aligned}\|Tx_n - x_n\| &\le \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - x_n\| \\ &\le (1+\theta_1)\|x_n - T^n x_n\| + \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - x_n\| \\ &= (2+\theta_1)\|x_n - T^n x_n\| + \|T^{n+1}x_n - T^n x_n\|.\end{aligned}$$

Hence from (3.6) and the assumption $T^n x_n - T^{n+1}x_n \to 0$ we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.9}$$

We now take a sequence $\{\varepsilon_k\} \subset (0,1)$ satisfying $\varepsilon_k \downarrow 0$ as $k \to \infty$. For all $k \ge 1$, we denote by m_k the smallest positive integer such that

$$\langle Ay_{n_j}, x - y_{n_j}\rangle + \varepsilon_k \ge 0 \quad \forall j \ge m_k. \tag{3.10}$$

Since $\{\varepsilon_k\}$ is decreasing, it is clear that $\{m_k\}$ is increasing. Noticing that $\{y_{m_k}\} \subset C$ guarantees $Ay_{m_k} \ne 0$ $\forall k \ge 1$, we set $\mu_{m_k} = \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}$, we get $\langle Ay_{m_k}, \mu_{m_k}\rangle = 1$ $\forall k \ge 1$. So, from (3.10) we get $\langle Ay_{m_k}, x + \varepsilon_k \mu_{m_k} - y_{m_k}\rangle \ge 0$ $\forall k \ge 1$. Again from the pseudomonotonicity of A we have $\langle A(x + \varepsilon_k \mu_{m_k}), x + \varepsilon_k \mu_{m_k} - y_{m_k}\rangle \ge 0$ $\forall k \ge 1$. This immediately leads to

$$\langle Ax, x - y_{m_k}\rangle \ge \langle Ax - A(x + \varepsilon_k \mu_{m_k}), x + \varepsilon_k \mu_{m_k} - y_{m_k}\rangle - \varepsilon_k \langle Ax, \mu_{m_k}\rangle \quad \forall k \ge 1. \tag{3.11}$$

We claim that $\lim_{k\to\infty} \varepsilon_k \mu_{m_k} = 0$. Indeed, from $w_{n_k} \rightharpoonup z$ and $w_n - y_n \to 0$, we obtain $y_{n_k} \rightharpoonup z$. So, $\{y_n\} \subset C$ guarantees $z \in C$. Again from the sequentially weak continuity of A, we know that $Ay_{n_k} \rightharpoonup Az$. Thus, we have $Az \ne 0$ (otherwise, z is a solution). Taking into account the sequentially weak lower semicontinuity of the norm $\|\cdot\|$, we get $0 < \|Az\| \le \liminf_{k\to\infty} \|Ay_{n_k}\|$. Note that $\{y_{m_k}\} \subset \{y_{n_k}\}$ and $\varepsilon_k \downarrow 0$ as $k \to \infty$. So it follows that $0 \le \limsup_{k\to\infty} \|\varepsilon_k \mu_{m_k}\| = \limsup_{k\to\infty} \frac{\varepsilon_k}{\|Ay_{m_k}\|} \le \frac{\limsup_{k\to\infty} \varepsilon_k}{\liminf_{k\to\infty} \|Ay_{n_k}\|} = 0$. Hence we get $\varepsilon_k \mu_{m_k} \to 0$.

Next we show that $z \in \Omega$. Indeed, from $w_n - x_n \to 0$ and $w_{n_k} \rightharpoonup z$, we get $x_{n_k} \rightharpoonup z$. From (3.9) we have $x_{n_k} - Tx_{n_k} \to 0$. Note that Lemma 2.8 guarantees the demiclosedness of $I - T$ at zero. Thus $z \in \mathrm{Fix}(T)$. Meantime, from $w_n - z_n \to 0$ and $w_{n_k} \rightharpoonup z$, we get $z_{n_k} \rightharpoonup z$. From (3.7) we have $z_{n_k} - Sz_{n_k} \to 0$. From Lemma 2.5 it follows that $I - S$ is demiclosed at zero, and hence we get $(I - S)z = 0$, i.e., $z \in \mathrm{Fix}(S)$. On the other hand, letting $k \to \infty$, we deduce that the right hand side of (3.11) tends to zero by the uniform continuity of A, the boundedness of $\{y_{m_k}\}, \{\mu_{m_k}\}$ and the limit $\lim_{k\to\infty} \varepsilon_k \mu_{m_k} = 0$. Thus, we get $\langle Ax, x - z\rangle = \liminf_{k\to\infty} \langle Ax, x - y_{m_k}\rangle \ge 0$ $\forall x \in C$. By Lemma 2.3 we have $z \in \mathrm{VI}(C, A)$. Therefore, $z \in \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C, A) = \Omega$. This completes the proof. □

Theorem 3.1. Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. Assume that $T^n x_n - T^{n+1}x_n \to 0$. Then

$$x_n \to x^* \in \Omega \iff \begin{cases} x_n - x_{n+1} \to 0, \\ x_n - y_n \to 0 \end{cases}$$

where $x^* \in \Omega$ is a unique solution to the VIP: $\langle (\rho F - f)x^*, p - x^*\rangle \ge 0$ $\forall p \in \Omega$.

Proof. First of all, since $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [a, b] \subset (0,1)$. We claim that $P_\Omega(f + I - \rho F)$ is a contraction. Indeed, by Lemma 2.7 we have

$$\begin{aligned}\|P_\Omega(f+I-\rho F)x-P_\Omega(f+I-\rho F)y\| &\le \|f(x)-f(y)\|+\|(I-\rho F)x-(I-\rho F)y\|\\ &\le \delta\|x-y\|+(1-\tau)\|x-y\|=[1-(\tau-\delta)]\|x-y\| \quad \forall x,y\in H,\end{aligned}$$

which implies that $P_\Omega(f+I-\rho F)$ is a contraction. Banach's Contraction Mapping Principle guarantees that $P_\Omega(f+I-\rho F)$ has a unique fixed point. Say $x^*\in H$, that is, $x^*=P_\Omega(f+I-\rho F)x^*$. Thus, there exists a unique solution $x^*\in\Omega=\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ to the VIP

$$\langle(\rho F-f)x^*,p-x^*\rangle\ge 0 \quad \forall p\in\Omega. \tag{3.12}$$

It is now easy to see that the necessity of the theorem is valid. Indeed, if $x_n\to x^*\in\Omega=\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$, then $x^*=Tx^*,\ x^*=Sx^*$ and $x^*=P_C(x^*-\tau_nAx^*)$, which together with Algorithm 3.1, imply that

$$\begin{aligned}\|w_n-x^*\| &= \|T^nx_n-x^*+\sigma_n(T^nx_n-T^nx_{n-1})\|\\ &\le (1+\theta_n)(\|x_n-x^*\|+\sigma_n\|x_n-x_{n-1}\|)\to 0\ (n\to\infty),\end{aligned}$$

and hence

$$\begin{aligned}\|y_n-x_n\| &\le \|y_n-x^*\|+\|x_n-x^*\|\\ &= \|P_C(w_n-\tau_nAw_n)-P_C(x^*-\tau_nAx^*)\|+\|x_n-x^*\|\\ &\le \|w_n-x^*\|+\tau_n\|Aw_n-Ax^*\|+\|x_n-x^*\|\\ &\le (1+\gamma L)\|w_n-x^*\|+\|x_n-x^*\|\to 0\ (n\to\infty).\end{aligned}$$

In addition, it is clear that

$$\|x_n-x_{n+1}\|\le\|x_n-x^*\|+\|x_{n+1}-x^*\|\to 0\ (n\to\infty).$$

Next we show the sufficiency of the theorem. To the aim, we assume $\lim_{n\to\infty}(\|x_n-x_{n+1}\|+\|x_n-y_n\|)=0$ and divide the proof of the sufficiency into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, take an arbitrary $p\in\Omega=\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$. Then $Tp=p,\ Sp=p$, and (3.5) holds, i.e.,

$$\|u_n-p\|^2\le\|w_n-p\|^2-(1-\mu)\|w_n-y_n\|^2-(1-\mu)\|u_n-y_n\|^2. \tag{3.13}$$

This immediately implies that

$$\|u_n-p\|\le\|w_n-p\| \quad \forall n\ge 1. \tag{3.14}$$

From the definition of w_n, we get

$$\begin{aligned}\|w_n-p\| &\le \|T^nx_n-p\|+\sigma_n\|T^nx_n-T^nx_{n-1}\|\\ &\le (1+\theta_n)(\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|)\\ &= (1+\theta_n)(\|x_n-p\|+\alpha_n\cdot\tfrac{\sigma_n}{\alpha_n}\|x_n-x_{n-1}\|).\end{aligned} \tag{3.15}$$

Since $\sup_{n\ge1}\frac{\sigma_n}{\alpha_n}<\infty$ and $\sup_{n\ge1}\|x_n-x_{n-1}\|<\infty$, we know that $\sup_{n\ge1}\frac{\sigma_n}{\alpha_n}\|x_n-x_{n-1}\|<\infty$, which hence implies that there exists a constant $M_1>0$ such that

$$\frac{\sigma_n}{\alpha_n}\|x_n-x_{n-1}\|\le M_1 \quad \forall n\ge 1. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we obtain

$$\|u_n-p\|\le\|w_n-p\|\le(1+\theta_n)(\|x_n-p\|+\alpha_nM_1) \quad \forall n\ge 1. \tag{3.17}$$

So, from Algorithm 3.1, Lemma 2.7 and (3.17) it follows that for all $n \geq n_0$,

$$\begin{aligned} \|z_n - p\| &= \|\alpha_n(f(x_n) - f(p)) + (I - \alpha_n\rho F)T^n u_n - (I - \alpha_n\rho F)p + \alpha_n(f - \rho F)p\| \\ &\leq \alpha_n\delta\|x_n - p\| + (1 - \alpha_n\tau)(1 + \theta_n)\|u_n - p\| + \alpha_n\|(f - \rho F)p\| \\ &\leq [\alpha_n\delta + (1 - \alpha_n\tau)(1 + \theta_n)^2](\|x_n - p\| + \alpha_n M_1) + \alpha_n\|(f - \rho F)p\| \\ &\leq [\alpha_n\delta + 1 - \alpha_n\tau + \theta_n(2 + \theta_n)](\|x_n - p\| + \alpha_n M_1) + \alpha_n\|(f - \rho F)p\| \\ &\leq (1 - \tfrac{\alpha_n(\tau-\delta)}{2})(\|x_n - p\| + \alpha_n M_1) + \alpha_n\|(f - \rho F)p\| \\ &\leq (1 - \tfrac{\alpha_n(\tau-\delta)}{2})\|x_n - p\| + \alpha_n(M_1 + \|(f - \rho F)p\|), \end{aligned}$$

which together with Lemma 2.6 and $(\gamma_n + \delta_n)\zeta \leq \gamma_n$, implies that for all $n \geq n_0$,

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n(x_n - p) + \gamma_n(z_n - p) + \delta_n(Sz_n - p)\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(Sz_n - p)]\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)\|z_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1 - \beta_n)[(1 - \tfrac{\alpha_n(\tau-\delta)}{2})\|x_n - p\| + \alpha_n(M_1 + \|(f - \rho F)p\|)] \\ &= [1 - \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - p\| + \alpha_n(1 - \beta_n)(M_1 + \|(f - \rho F)p\|) \\ &= [1 - \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - p\| + \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2} \cdot \tfrac{2(M_1+\|(f-\rho F)p\|)}{\tau-\delta} \\ &\leq \max\{\|x_n - p\|, \tfrac{2(M_1+\|(f-\rho F)p\|)}{\tau-\delta}\}. \end{aligned}$$

By induction, we obtain $\|x_n - p\| \leq \max\{\|x_{n_0} - p\|, \frac{2(M_1+\|(\rho F-f)p\|)}{\tau-\delta}\}\ \forall n \geq n_0$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}, \{w_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}, \{Sz_n\}, \{T^n u_n\}, \{T^n x_n\}$.

Step 2. We show that for all $n \geq n_0$,

$$(1 - \alpha_n\tau)(1 - \beta_n)(1 + \theta_n)(1 - \mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4,$$

with constant $M_4 > 0$. Indeed, utilizing Lemma 2.6, Lemma 3.2 and the convexity of $\|\cdot\|^2$, from $(\gamma_n + \delta_n)\zeta \leq \gamma_n$ we obtain that for all $n \geq n_0$,

$$\begin{aligned} &\|x_{n+1} - p\|^2 = \|\beta_n(x_n - p) + \gamma_n(z_n - p) + \delta_n(Tz_n - p)\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n - p) + \delta_n(Tz_n - p)]\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\{\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)(1 + \theta_n)\|w_n - p\|^2 \\ &\quad -(1 - \alpha_n\tau)(1 + \theta_n)(1 - \mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + 2\alpha_n\langle(f - \rho F)p, z_n - p\rangle\} \\ &\leq \beta_n\|x_n - p\|^2 + (1 - \beta_n)\{\alpha_n\delta\|x_n - p\|^2 + (1 - \alpha_n\tau)(1 + \theta_n)\|w_n - p\|^2 \\ &\quad -(1 - \alpha_n\tau)(1 + \theta_n)(1 - \mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] + \alpha_n M_2\}, \end{aligned} \tag{3.18}$$

where $\sup_{n\geq 1} 2\|(f - \rho F)p\|\|z_n - p\| \leq M_2$ for some $M_2 > 0$. Also, from (3.17) we have

$$\begin{aligned} \|w_n - p\|^2 &\leq (1 + \theta_n)^2(\|x_n - p\| + \alpha_n M_1)^2 \\ &= [1 + \theta_n(2 + \theta_n)][\|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2)] \\ &= \|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2) \\ &\quad +\theta_n(2 + \theta_n)[\|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2)] \\ &\leq \|x_n - p\|^2 + \alpha_n M_3, \end{aligned} \tag{3.19}$$

where $\sup_{n\geq 1}\{2M_1\|x_n - p\| + \alpha_n M_1^2 + \frac{\theta_n}{\alpha_n}(2 + \theta_n)[\|x_n - p\|^2 + \alpha_n(2M_1\|x_n - p\| + \alpha_n M_1^2)]\} \leq M_3$ for some $M_3 > 0$. Note that $\alpha_n\delta + (1 - \alpha_n\tau)(1 + \theta_n) \leq 1 - \frac{\alpha_n(\tau-\delta)}{2}$ for all $n \geq n_0$. Substituting

(3.19) for (3.18), we deduce that for all $n \geq n_0$,

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)[\|x_n-p\|^2+\alpha_nM_3] \\
&\ -(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+\alpha_nM_2\} \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\{(1-\tfrac{\alpha_n(\tau-\delta)}{2})\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)\alpha_nM_3 \\
&\ -(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+\alpha_nM_2\} \\
&= [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+(1-\beta_n)(1-\alpha_n\tau)(1+\theta_n)\alpha_nM_3 \\
&\ -(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+(1-\beta_n)\alpha_nM_2 \\
&\leq \|x_n-p\|^2-(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+\alpha_nM_4,
\end{aligned} \tag{3.20}$$

where $\sup_{n\geq 1}(M_2+(1+\theta_n)M_3) \leq M_4$ for some $M_4>0$. This immediately implies that for all $n \geq n_0$,

$$(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2] \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nM_4. \tag{3.21}$$

Step 3. We show that for all $n \geq n_0$,

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}[\tfrac{4}{\tau-\delta}\langle(f-\rho F)p,z_n-p\rangle \\
&\ +\tfrac{4M}{\tau-\delta}\cdot\tfrac{\sigma_n}{\alpha_n}\cdot\|x_n-x_{n-1}\|+\tfrac{4M^2}{\tau-\delta}\cdot\tfrac{\theta_n}{\alpha_n}],
\end{aligned}$$

with constant $M>0$. Indeed, we have

$$\begin{aligned}
\|w_n-p\|^2 &\leq (1+\theta_n)^2(\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|)^2 \\
&= \|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|(2\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|) \\
&\ +\theta_n(2+\theta_n)(\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|)^2 \\
&\leq \|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|M+\theta_nM^2,
\end{aligned} \tag{3.22}$$

where $\sup_{n\geq 1}(2+\theta_n)(\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|) \leq M$ for some $M>0$. Note that $\alpha_n\delta+(1-\alpha_n\tau)(1+\theta_n) \leq 1-\frac{\alpha_n(\tau-\delta)}{2}$ for all $n \geq n_0$. Thus, combining (3.18) and (3.22), we have that for all $n \geq n_0$,

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \beta_n\|x_n-p\|^2+(1-\beta_n)[\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)\|w_n-p\|^2 \\
&\ +2\alpha_n\langle(f-\rho F)p,z_n-p\rangle] \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)[\|x_n-p\|^2 \\
&\ +\sigma_n\|x_n-x_{n-1}\|M+\theta_nM^2]+2\alpha_n\langle(f-\rho F)p,z_n-p\rangle\} \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\{(1-\tfrac{\alpha_n(\tau-\delta)}{2})\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)\times \\
&\ \times[\sigma_n\|x_n-x_{n-1}\|M+\theta_nM^2]+2\alpha_n\langle(f-\rho F)p,z_n-p\rangle\} \\
&\leq [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+(1-\beta_n)[\sigma_n\|x_n-x_{n-1}\|2M+\theta_n2M^2] \\
&\ +2\alpha_n(1-\beta_n)\langle(f-\rho F)p,z_n-p\rangle \\
&= [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}[\tfrac{4}{\tau-\delta}\langle(f-\rho F)p,z_n-p\rangle \\
&\ +\tfrac{4M}{\tau-\delta}\cdot\tfrac{\sigma_n}{\alpha_n}\cdot\|x_n-x_{n-1}\|+\tfrac{4M^2}{\tau-\delta}\cdot\tfrac{\theta_n}{\alpha_n}].
\end{aligned} \tag{3.23}$$

Step 4. We show that $\{x_n\}$ converges strongly to a unique solution $x^* \in \Omega$ to the VIP (3.12). Indeed, putting $p = x^*$, we deduce from (3.23) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le [1 - \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n - x^*\|^2 + \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2} \times \\ &\quad \times [\tfrac{4}{\tau-\delta}\langle (f-\rho F)x^*, z_n - x^*\rangle + \tfrac{4M}{\tau-\delta}\cdot\tfrac{\sigma_n}{\alpha_n}\cdot\|x_n - x_{n-1}\| + \tfrac{4M^2}{\tau-\delta}\cdot\tfrac{\theta_n}{\alpha_n}]. \end{aligned} \tag{3.24}$$

By Lemma 2.4, it suffices to show that $\limsup_{n\to\infty}\langle (f-\rho F)x^*, z_n - x^*\rangle \le 0$. From (3.21), $x_n - x_{n+1} \to 0$, $\alpha_n \to 0$, $\theta_n \to 0$ and $\{\beta_n\} \subset [a,b] \subset (0,1)$, we obtain

$$\begin{aligned} &\limsup_{n\to\infty}(1-\alpha_n\tau)(1-b)(1+\theta_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ &\le \limsup_{n\to\infty}(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n - y_n\|^2 + \|u_n - y_n\|^2] \\ &\le \limsup_{n\to\infty}[\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_4] \\ &\le \limsup_{n\to\infty}[\|x_n - p\|^2 - \|x_{n+1} - p\|^2] + \limsup_{n\to\infty}\alpha_n M_4 \\ &\le \limsup_{n\to\infty}(\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| = 0. \end{aligned}$$

This immediately implies that

$$\lim_{n\to\infty}\|w_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.25}$$

Obviously, the assumption $\|x_n - y_n\| \to 0$ together with (3.25), guarantees that $\|w_n - x_n\| \le \|w_n - y_n\| + \|y_n - x_n\| \to 0$ $(n \to \infty)$. So it follows that

$$\begin{aligned} \|T^n x_n - x_n\| &= \|w_n - x_n - \sigma_n(T^n x_n - T^n x_{n-1})\| \\ &\le \|w_n - x_n\| + \sigma_n(1+\theta_n)\|x_n - x_{n-1}\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.26}$$

Since $z_n = \alpha_n f(x_n) + (I - \alpha_n\rho F)T^n u_n$ with $u_n := P_{C_n}(w_n - \tau_n A y_n)$, from (3.25), (3.26) and the boundedness of $\{x_n\}, \{T^n u_n\}$, we conclude that as $n \to \infty$,

$$\begin{aligned} &\|z_n - x_n\| = \|\alpha_n f(x_n) - \alpha_n\rho F T^n u_n + T^n u_n - x_n\| \\ &\le \alpha_n(\|f(x_n)\| + \|\rho F T^n u_n\|) + \|T^n u_n - x_n\| \\ &\le \alpha_n(\|f(x_n)\| + \|\rho F T^n u_n\|) + \|T^n u_n - T^n y_n\| + \|T^n y_n - T^n x_n\| + \|T^n x_n - x_n\| \\ &\le \alpha_n(\|f(x_n)\| + \|\rho F T^n u_n\|) + (1+\theta_n)(\|u_n - y_n\| + \|y_n - x_n\|) + \|T^n x_n - x_n\| \to 0 \end{aligned} \tag{3.27}$$

(due to the assumption $\|x_n - y_n\| \to 0$). Obviously, the limit $\lim_{n\to\infty}\|w_n - x_n\| = 0$ together with (3.27), guarantees that $\|w_n - z_n\| \le \|w_n - x_n\| + \|x_n - z_n\| \to 0$ $(n \to \infty)$. From the boundedness of $\{z_n\}$, it follows that there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle (f-\rho F)x^*, z_n - x^*\rangle = \lim_{k\to\infty}\langle (f-\rho F)x^*, z_{n_k} - x^*\rangle. \tag{3.28}$$

Since H is reflexive and $\{z_n\}$ is bounded, we may assume, without loss of generality, that $z_{n_k} \rightharpoonup \tilde{z}$. Hence from (3.28) we get

$$\limsup_{n\to\infty}\langle (f-\rho F)x^*, z_n - x^*\rangle = \lim_{k\to\infty}\langle (f-\rho F)x^*, z_{n_k} - x^*\rangle = \langle (f-\rho F)x^*, \tilde{z} - x^*\rangle. \tag{3.29}$$

It is easy to see from $w_n - z_n \to 0$ and $z_{n_k} \rightharpoonup \tilde{z}$ that $w_{n_k} \rightharpoonup \tilde{z}$. Since $T^n x_n - T^{n+1}x_n \to 0$, $x_n - x_{n+1} \to 0$, $w_n - x_n \to 0$, $w_n - z_n \to 0$ and $w_{n_k} \rightharpoonup \tilde{z}$, by Lemma 3.3 we infer that $\tilde{z} \in \Omega$. Therefore, from (3.12) and (3.29) we conclude that

$$\limsup_{n\to\infty}\langle (f-\rho F)x^*, z_n - x^*\rangle = \langle (f-\rho F)x^*, \tilde{z} - x^*\rangle \le 0. \tag{3.30}$$

Note that $\{\beta_n\} \subset [a,b] \subset (0,1)$, $\{\frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\} \subset [0,1]$, $\sum_{n=1}^{\infty} \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2} = \infty$, and

$$\limsup_{n\to\infty}[\frac{4}{\tau-\delta}\langle (f-\rho F)x^*, z_n - x^*\rangle + \frac{4M}{\tau-\delta}\cdot\frac{\sigma_n}{\alpha_n}\cdot\|x_n - x_{n-1}\| + \frac{4M^2}{\tau-\delta}\cdot\frac{\theta_n}{\alpha_n}] \le 0. \tag{3.31}$$

Consequently, applying Lemma 2.4 to (3.24), we have $\lim_{n\to 0}\|x_n - x^*\| = 0$. This completes the proof. □

Next, we introduce another asymptotic inertial subgradient extragradient algorithm with line-search process.

Algorithm 3.2. Initialization: Given $\gamma > 0$, $l \in (0,1)$, $\mu \in (0,1)$. Let $x_0, x_1 \in H$ be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Set $w_n = T^n x_n + \sigma_n(T^n x_n - T^n x_{n-1})$ and compute $y_n = P_C(w_n - \tau_n A w_n)$, where τ_n is chosen to be the largest $\tau \in \{\gamma, \gamma l, \gamma l^2, ...\}$ satisfying

$$\tau\|Aw_n - Ay_n\| \le \mu\|w_n - y_n\|. \tag{3.32}$$

Step 2. Compute $z_n = \alpha_n f(x_n) + (I - \alpha_n\rho F)T^n P_{C_n}(w_n - \tau_n A y_n)$ with $C_n := \{x \in H : \langle w_n - \tau_n A w_n - y_n, x - y_n\rangle \le 0\}$.
Step 3. Compute

$$x_{n+1} = \beta_n w_n + \gamma_n z_n + \delta_n S z_n. \tag{3.33}$$

Again set $n := n+1$ and go to Step 1.

It is worth pointing out that Lemmas 3.1, 3.2 and 3.3 are still valid for Algorithm 3.2.

Theorem 3.2. Let the sequence $\{x_n\}$ be generated by Algorithm 3.2. Assume that $T^n x_n - T^{n+1}x_n \to 0$. Then

$$x_n \to x^* \in \Omega \iff \begin{cases} x_n - x_{n+1} \to 0, \\ x_n - y_n \to 0 \end{cases}$$

where $x^* \in \Omega$ is a unique solution to the VIP: $\langle (\rho F - f)x^*, p - x^*\rangle \ge 0\ \forall p \in \Omega$.

Proof. Utilizing the same arguments as in the proof of Theorem 3.1, we deduce that there exists a unique solution $x^* \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C,A)$ to the VIP (3.12), and that the necessity of the theorem is valid.

Next we show the sufficiency of the theorem. To the aim, we assume $\lim_{n\to\infty}(\|x_n - x_{n+1}\| + \|x_n - y_n\|) = 0$ and divide the proof of the sufficiency into several steps.

Step 1. We show that $\{x_n\}$ is bounded. Indeed, utilizing the same arguments as in Step 1 of the proof of Theorem 3.1, we obtain that inequalities (3.13)-(3.17) hold. Taking into account $\lim_{n\to\infty}\frac{\theta_n(2+\theta_n)}{\alpha_n(1-\beta_n)} = 0$, we know that $\theta_n(2+\theta_n) \le \frac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\ \forall n \ge n_0$ for some $n_0 \ge 1$. Hence we deduce that for all $n \geq n_0$,

$$\begin{aligned}&\alpha_n(1-\beta_n)\delta + [1-\alpha_n(1-\beta_n)\tau](1+\theta_n)^2 \\ &= 1-\alpha_n(1-\beta_n)(\tau-\delta) + [1-\alpha_n(1-\beta_n)\tau]\theta_n(2+\theta_n) \\ &\leq 1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}.\end{aligned}$$

Also, from Algorithm 3.2, Lemma 2.7 and (3.17) it follows that

$$\begin{aligned}\|z_n-p\| &= \|\alpha_n(f(x_n)-f(p)) + (I-\alpha_n\rho F)T^n u_n - (I-\alpha_n\rho F)p + \alpha_n(f-\rho F)p\| \\ &\leq \alpha_n\delta\|x_n-p\| + (1-\alpha_n\tau)(1+\theta_n)\|u_n-p\| + \alpha_n\|(f-\rho F)p\| \\ &\leq \alpha_n\delta\|x_n-p\| + (1-\alpha_n\tau)(1+\theta_n)\|w_n-p\| + \alpha_n\|(f-\rho F)p\|,\end{aligned}$$

which together with Lemma 2.6 and $(\gamma_n+\delta_n)\zeta \leq \gamma_n$, implies that for all $n \geq n_0$,

$$\begin{aligned}&\|x_{n+1}-p\| = \|\beta_n(w_n-p) + \gamma_n(z_n-p) + \delta_n(Sz_n-p)\| \\ &\leq \beta_n\|w_n-p\| + (1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n-p) + \delta_n(Tz_n-p)]\| \\ &\leq \beta_n\|w_n-p\| + (1-\beta_n)\|z_n-p\| \\ &\leq \beta_n\|w_n-p\| + (1-\beta_n)[\alpha_n\delta\|x_n-p\| + (1-\alpha_n\tau)(1+\theta_n)\|w_n-p\| + \alpha_n\|(f-\rho F)p\|] \\ &\leq \alpha_n(1-\beta_n)\delta\|x_n-p\| + [1-\alpha_n(1-\beta_n)\tau](1+\theta_n)\|w_n-p\| + \alpha_n(1-\beta_n)\|(f-\rho F)p\| \\ &\leq \alpha_n(1-\beta_n)\delta\|x_n-p\| + [1-\alpha_n(1-\beta_n)\tau](1+\theta_n)^2(\|x_n-p\| + \alpha_n M_1) \\ &\quad +\alpha_n(1-\beta_n)\|(f-\rho F)p\| \\ &\leq [\alpha_n(1-\beta_n)\delta + (1-\alpha_n(1-\beta_n)\tau)(1+\theta_n)^2](\|x_n-p\| + \alpha_n M_1) \\ &\quad +\alpha_n(1-\beta_n)\|(f-\rho F)p\| \\ &\leq [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}](\|x_n-p\| + \alpha_n M_1) + \alpha_n(1-\beta_n)\|(f-\rho F)p\| \\ &\leq [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\| + \tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}\cdot\tfrac{2(\frac{M_1}{1-\beta_n} + \|(f-\rho F)p\|)}{\tau-\delta} \\ &\leq \max\{\|x_n-p\|, \tfrac{2(\frac{M_1}{1-b} + \|(f-\rho F)p\|)}{\tau-\delta}\}.\end{aligned}$$

By induction, we obtain $\|x_n-p\| \leq \max\{\|x_{n_0}-p\|, \frac{2(\frac{M_1}{1-b} + \|(f-\rho F)p\|)}{\tau-\delta}\}\ \forall n \geq n_0$. Thus, $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}, \{w_n\}, \{y_n\}, \{z_n\}, \{f(x_n)\}, \{Sz_n\}, \{T^n u_n\}, \{T^n x_n\}$.

Step 2. We show that for all $n \geq n_0$,

$$(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2 + \|u_n-y_n\|^2] \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \alpha_n M_4,$$

with constant $M_4 > 0$. Indeed, utilizing Lemma 2.6, Lemma 3.2 and the convexity of $\|\cdot\|^2$, from $(\gamma_n+\delta_n)\zeta \leq \gamma_n$ we get

$$\begin{aligned}&\|x_{n+1}-p\|^2 \\ &= \|\beta_n(w_n-p) + \gamma_n(z_n-p) + \delta_n(Sz_n-p)\|^2 \\ &\leq \beta_n\|w_n-p\|^2 + (1-\beta_n)\|\tfrac{1}{1-\beta_n}[\gamma_n(z_n-p) + \delta_n(Sz_n-p)]\|^2 \\ &\leq \beta_n\|w_n-p\|^2 + (1-\beta_n)\|z_n-p\|^2 \\ &\leq \beta_n\|w_n-p\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2 + (1-\alpha_n\tau)(1+\theta_n)\|w_n-p\|^2 \\ &\quad -(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2 + \|u_n-y_n\|^2] + 2\alpha_n\langle(f-\rho F)p, z_n-p\rangle\} \\ &\leq \beta_n\|w_n-p\|^2 + (1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2 + (1-\alpha_n\tau)(1+\theta_n)\|w_n-p\|^2 \\ &\quad -(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2 + \|u_n-y_n\|^2] + \alpha_n M_2\},\end{aligned} \tag{3.34}$$

where $\sup_{n\geq 1} 2\|(f-\rho F)p\|\|z_n-p\| \leq M_2$ for some $M_2>0$. Also, from (3.17) we have

$$\begin{aligned}\|w_n-p\|^2 &\leq (1+\theta_n)^2(\|x_n-p\|+\alpha_n M_1)^2\\ &= \|x_n-p\|^2+\alpha_n(2M_1\|x_n-p\|+\alpha_n M_1^2)\\ &\quad +\theta_n(2+\theta_n)[\|x_n-p\|^2+\alpha_n(2M_1\|x_n-p\|+\alpha_n M_1^2)]\\ &\leq \|x_n-p\|^2+\alpha_n M_3,\end{aligned} \tag{3.35}$$

where $\sup_{n\geq 1}\{2M_1\|x_n-p\|+\alpha_n M_1^2+\frac{\theta_n}{\alpha_n}(2+\theta_n)[\|x_n-p\|^2+\alpha_n(2M_1\|x_n-p\|+\alpha_n M_1^2)]\} \leq M_3$ for some $M_3>0$. Note that $\alpha_n\delta+(1-\alpha_n\tau)(1+\theta_n)\leq 1-\frac{\alpha_n(\tau-\delta)}{2}$ for all $n\geq n_0$. Substituting (3.35) for (3.34), we obtain that for all $n\geq n_0$,

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\leq \beta_n\|w_n-p\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)[\|x_n-p\|^2+\alpha_n M_3]\\ &\quad -(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+\alpha_n M_2\}\\ &\leq \beta_n(\|x_n-p\|^2+\alpha_n M_3)+(1-\beta_n)\{(1-\tfrac{\alpha_n(\tau-\delta)}{2})\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)\alpha_n M_3\\ &\quad -(1-\alpha_n\tau)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+\alpha_n M_2\}\\ &= [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+\beta_n\alpha_n M_3+(1-\beta_n)(1-\alpha_n\tau)(1+\theta_n)\alpha_n M_3\\ &\quad -(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+(1-\beta_n)\alpha_n M_2\\ &\leq \|x_n-p\|^2-(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]+\alpha_n M_4,\end{aligned} \tag{3.36}$$

where $\sup_{n\geq 1}(M_2+(1+\theta_n)M_3)\leq M_4$ for some $M_4>0$. This immediately implies that for all $n\geq n_0$,

$$(1-\alpha_n\tau)(1-\beta_n)(1+\theta_n)(1-\mu)[\|w_n-y_n\|^2+\|u_n-y_n\|^2]\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n M_4. \tag{3.37}$$

Step 3. We show that for all $n\geq n_0$,

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}[\tfrac{4}{\tau-\delta}\langle(f-\rho F)p,z_n-p\rangle\\ &\quad +\tfrac{4M}{(\tau-\delta)(1-b)}\cdot\tfrac{\sigma_n}{\alpha_n}\cdot\|x_n-x_{n-1}\|+\tfrac{4M^2}{(\tau-\delta)(1-b)}\cdot\tfrac{\theta_n}{\alpha_n}],\end{aligned}$$

with constant $M>0$. Indeed, we have

$$\|w_n-p\|^2\leq(1+\theta_n)^2(\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|)^2\leq\|x_n-p\|^2+\sigma_n\|x_n-x_{n-1}\|M+\theta_n M^2, \tag{3.38}$$

where $\sup_{n\geq 1}(2+\theta_n)(\|x_n-p\|+\sigma_n\|x_n-x_{n-1}\|)\leq M$ for some $M>0$. Note that $\alpha_n\delta+(1-\alpha_n\tau)(1+\theta_n)\leq 1-\frac{\alpha_n(\tau-\delta)}{2}$ for all $n\geq n_0$. Thus, combining (3.34) and (3.38), we have that for all $n \geq n_0$,

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \beta_n\|w_n-p\|^2+(1-\beta_n)[\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)\|w_n-p\|^2 \\
&\quad +2\alpha_n\langle (f-\rho F)p, z_n-p\rangle] \\
&\leq \beta_n\|w_n-p\|^2+(1-\beta_n)\{\alpha_n\delta\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)[\|x_n-p\|^2 \\
&\quad +\sigma_n\|x_n-x_{n-1}\|M+\theta_n M^2]+2\alpha_n\langle (f-\rho F)p, z_n-p\rangle\} \\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\{(1-\tfrac{\alpha_n(\tau-\delta)}{2})\|x_n-p\|^2+(1-\alpha_n\tau)(1+\theta_n)\times \\
&\quad \times[\sigma_n\|x_n-x_{n-1}\|M+\theta_n M^2]+2\alpha_n\langle (f-\rho F)p, z_n-p\rangle\} \\
&\quad +\beta_n[\sigma_n\|x_n-x_{n-1}\|M+\theta_n M^2] \\
&\leq [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+(1+\theta_n)[\sigma_n\|x_n-x_{n-1}\|M+\theta_n M^2] \\
&\quad +2\alpha_n(1-\beta_n)\langle (f-\rho F)p, z_n-p\rangle \\
&= [1-\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}]\|x_n-p\|^2+\tfrac{\alpha_n(1-\beta_n)(\tau-\delta)}{2}[\tfrac{4}{\tau-\delta}\langle (f-\rho F)p, z_n-p\rangle \\
&\quad +\tfrac{4M}{(\tau-\delta)(1-b)}\cdot\tfrac{\sigma_n}{\alpha_n}\cdot\|x_n-x_{n-1}\|+\tfrac{4M^2}{(\tau-\delta)(1-b)}\cdot\tfrac{\theta_n}{\alpha_n}].
\end{aligned} \tag{3.39}$$

Step 4. We show that $\{x_n\}$ converges strongly to a unique solution $x^* \in \Omega$ to the VIP (3.12). Indeed, utilizing the same argument as in Step 4 of the proof of Theorem 3.1, we obtain the desired assertion. This completes the proof. □

Remark 3.1. Compared with the corresponding results in Kraikaew and Saejung [22], Thong and Hieu [25, 31] and Yao et al. [2] and Cai et al. [38], our results improve and extend them in the following aspects.

(i) The problem of finding an element of $\mathrm{VI}(C,A)$ in [22] is extended to develop our problem of finding an element of $\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ where T is asymptotically nonexpansive and S is strictly pseudocontractive. The Halpern subgradient extragradient method for solving the VIP in [22] is extended to develop our asymptotic inertial subgradient extragradient method with line-search process for solving the VIP and CFPP, which is based on inertial subgradient extragradient method with line-search process, hybrid steepest-descent method, viscosity approximation method, Mann iteration method and asymptotically nonexpansive mapping.

(ii) The problem of finding an element of $\mathrm{VI}(C,A)$ in [31] is extended to develop our problem of finding an element of $\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ where T is asymptotically nonexpansive and S is strictly pseudocontractive. The inertial subgradient extragradient method with weak convergence for solving the VIP in [31] is extended to develop our asymptotic inertial subgradient extragradient method with line-search process (which is strongly convergent) for solving the VIP and CFPP, which is based on inertial subgradient extragradient method with line-search process, hybrid steepest-descent method, viscosity approximation method, Mann iteration method and asymptotically nonexpansive mapping.

(iii) The problem of finding an element of $\mathrm{VI}(C,A)\cap\mathrm{Fix}(T)$ (where A is monotone and T is quasi-nonexpansive) in [25] is extended to develop our problem of finding an element of $\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ where T is asymptotically nonexpansive and S is strictly pseudocontractive. The inertial subgradient extragradient method with line-search (which is weakly convergent) for solving the VIP and FPP in [31] is extended to develop our asymptotic inertial subgradient extragradient method with line-search process (which is strongly convergent) for solving the VIP and CFPP, which is based on inertial subgradient extragradient method with line-search process, hybrid steepest-descent method, viscosity approximation method, Mann iteration method and asymptotically nonexpansive mapping. It is worth pointing out that the inertial subgradient extragradient method with line-search process in [25] combines the inertial subgradient extragradient method [31] with Mann iteration method.

(iv) The problem of finding an element in the common solution set Ω of variational inequalities for inverse-strongly monotone mappings A and B and the fixed point problem of strictly pseudocontractive mapping T in [2], is extended to develop our problem of finding an element of $\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ where T is asymptotically nonexpansive and S is strictly pseudocontractive. The relaxed extragradient method in [2] (i.e., iterative scheme (1.3) in this paper), is extended to develop our asymptotic inertial subgradient extragradient method with line-search process (which is strongly convergent) for solving the VIP and CFPP, which is based on inertial subgradient extragradient method with line-search process, hybrid steepest-descent method, viscosity approximation method, Mann iteration method and asymptotically nonexpansive mapping. Meantime, the restrictions $\delta\in[0,\frac{1}{2}),\ \gamma_n<(1-2\delta)\delta_n$ and $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n})=0$ imposed on (1.3), are dropped, where $\delta\in[0,\frac{1}{2})$ is weakened to the condition $\delta\in[0,1)$.

(v) The problem of finding an element in the common solution set Ω of variational inequalities for inverse-strongly monotone mappings A and B and the fixed point problem of asymptotically nonexpansive mapping T in [38], is extended to develop our problem of finding an element of $\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ where T is asymptotically nonexpansive and S is strictly pseudocontractive. The viscosity implicit rule involving a modified extragradient method in [38] (i.e., iterative scheme (1.4) in this paper), is extended to develop our asymptotic inertial subgradient extragradient method with line-search process for solving the VIP and CFPP, which is based on inertial subgradient extragradient method with line-search process, hybrid steepest-descent method, viscosity approximation method, Mann iteration method and asymptotically nonexpansive mapping. Meantime, the restrictions $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$ and $\sum_{n=1}^{\infty}\|T^{n+1}y_n-T^ny_n\|<\infty$ imposed on (1.4), are dropped, where $\sum_{n=1}^{\infty}\|T^{n+1}y_n-T^ny_n\|<\infty$ is weakened to the condition $\|T^{n+1}x_n-T^nx_n\|\to 0\ (n\to\infty)$.

4. APPLICATIONS

In this section, our main results are applied to solve the VIP and CFPP in an illustrated example. The initial point $x_0=x_1$ is randomly chosen in $\mathbf{R}$. Take $f(x)=F(x)=\frac{1}{2}x,\ \gamma=l=\mu=\frac{1}{2},\ \sigma_n=\alpha_n=\frac{1}{n+1},\ \beta_n=\frac{1}{3},\ \gamma_n=\frac{1}{2},\ \delta_n=\frac{1}{6}$ and $\rho=2$. Then we know that $\delta=\kappa=\eta=\frac{1}{2}$, and

$$\tau=1-\sqrt{1-\rho(2\eta-\rho\kappa^2)}=1-\sqrt{1-2(2\cdot\frac{1}{2}-2(\frac{1}{2})^2)}=1\in(0,1].$$

We first provide an example of Lipschitz continuous and pseudomonotone mapping A, asymptotically nonexpansive mapping T and strictly pseudocontractive mapping S with $\Omega=\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)\neq\emptyset$. Let $C=[-1,1]$ and $H=\mathbf{R}$ with the inner product $\langle a,b\rangle=ab$ and induced norm $\|\cdot\|=|\cdot|$. Let $A,T,S:H\to H$ be defined as $Ax:=\frac{1}{1+|\sin x|}-\frac{1}{1+|x|}$, $Tx:=\frac{2}{3}\sin x$ and $Sx:=\frac{3}{8}x+\frac{1}{2}\sin x$ for all $x\in H$. Now, we first show that A is pseudomonotone and Lipschitz continuous with $L=2$. Indeed, for all $x,y\in H$ we have

$$\begin{aligned}\|Ax-Ay\| &= |\tfrac{1}{1+\|\sin x\|}-\tfrac{1}{1+\|x\|}-\tfrac{1}{1+\|\sin y\|}+\tfrac{1}{1+\|y\|}| \\ &\le |\tfrac{1}{1+\|\sin x\|}-\tfrac{1}{1+\|\sin y\|}|+|\tfrac{1}{1+\|x\|}-\tfrac{1}{1+\|y\|}| \\ &= |\tfrac{1+\|\sin y\|-1-\|\sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)}|+|\tfrac{1+\|y\|-1-\|x\|}{(1+\|x\|)(1+\|y\|)}| \\ &= |\tfrac{\|\sin y\|-\|\sin x\|}{(1+\|\sin x\|)(1+\|\sin y\|)}|+|\tfrac{\|y\|-\|x\|}{(1+\|x\|)(1+\|y\|)}| \\ &\le \tfrac{\|\sin x-\sin y\|}{(1+\|\sin x\|)(1+\|\sin y\|)}+\tfrac{\|x-y\|}{(1+\|x\|)(1+\|y\|)} \\ &\le \|\sin x-\sin y\|+\|x-y\| \\ &\le 2\|x-y\|.\end{aligned}$$

This implies that A is Lipschitz continuous with $L=2$. Next, we show that A is pseudomonotone. For any given $x,y\in H$, it is clear that the relation holds:

$$\langle Ax,y-x\rangle=(\frac{1}{1+|\sin x|}-\frac{1}{1+|x|})(y-x)\ge 0\Rightarrow\langle Ay,y-x\rangle=(\frac{1}{1+|\sin y|}-\frac{1}{1+|y|})(y-x)\ge 0.$$

Furthermore, it is easy to see that T is asymptotically nonexpansive with $\theta_n=(\frac{2}{3})^n\ \forall n\ge 1$, such that $\|T^{n+1}x_n-T^nx_n\|\to 0$ as $n\to\infty$. Indeed, we observe that

$$\|T^nx-T^ny\|\le\frac{2}{3}\|T^{n-1}x-T^{n-1}y\|\le\cdots\le(\frac{2}{3})^n\|x-y\|\le(1+\theta_n)\|x-y\|,$$

and

$$\|T^{n+1}x_n-T^nx_n\|\le(\frac{2}{3})^{n-1}\|T^2x_n-Tx_n\|=(\frac{2}{3})^{n-1}\|\frac{2}{3}\sin(Tx_n)-\frac{2}{3}\sin x_n\|\le 2(\frac{2}{3})^n\to 0\ (n\to\infty).$$

It is clear that $\mathrm{Fix}(T)=\{0\}$ and

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}=\lim_{n\to\infty}\frac{(2/3)^n}{1/(n+1)}=0.$$

In addition, it is clear that S is strictly pseudocontractive with constant $\zeta=\frac{3}{4}$. Indeed, we observe that for all $x,y\in H$,

$$\|Sx-Sy\|^2\le[\frac{3}{8}\|x-y\|+\frac{1}{2}\|\sin x-\sin y\|]^2\le\|x-y\|^2+\frac{3}{4}\|(I-S)x-(I-S)y\|^2.$$

It is clear that $(\gamma_n+\delta_n)\zeta=(\frac{1}{2}+\frac{1}{6})\cdot\frac{3}{4}\le\frac{1}{2}=\gamma_n$ for all $n\ge 1$. Therefore, $\Omega=\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)=\{0\}\ne\emptyset$. In this case, Algorithm 3.1 can be rewritten as follows:

$$\begin{cases} w_n=T^nx_n+\frac{1}{n+1}(T^nx_n-T^nx_{n-1}), \\ y_n=P_C(w_n-\tau_nAw_n), \\ z_n=\frac{1}{n+1}\cdot\frac{1}{2}x_n+\frac{n}{n+1}T^nP_{C_n}(w_n-\tau_nAy_n), \\ x_{n+1}=\frac{1}{3}x_n+\frac{1}{2}z_n+\frac{1}{6}Sz_n\quad\forall n\ge 1, \end{cases}\tag{4.1}$$

where for each $n\ge 1$, C_n and τ_n are chosen as in Algorithm 3.1. Then, by Theorem 3.1, we know that $\{x_n\}$ converges to $0\in\Omega=\mathrm{Fix}(T)\cap\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ if and only if $|x_n-x_{n+1}|+|x_n-y_n|\to 0$ as $n\to\infty$.

On the other hand, Algorithm 3.2 can be rewritten as follows:

$$\begin{cases} w_n = T^n x_n + \frac{1}{n+1}(T^n x_n - T^n x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = \frac{1}{n+1} \cdot \frac{1}{2} x_n + \frac{n}{n+1} T^n P_{C_n}(w_n - \tau_n A y_n), \\ x_{n+1} = \frac{1}{3} w_n + \frac{1}{2} z_n + \frac{1}{6} S z_n \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where for each $n \geq 1$, C_n and τ_n are chosen as in Algorithm 3.2. Then, by Theorem 3.2, we know that $\{x_n\}$ converges to $0 \in \Omega = \mathrm{Fix}(T) \cap \mathrm{Fix}(S) \cap \mathrm{VI}(C,A)$ if and only if $|x_n - x_{n+1}| + |x_n - y_n| \to 0$ as $n \to \infty$.

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