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INTRINSIC DECAY RATES FOR THE ENERGY OF A NONLINEAR VISCOELASTIC EQUATION WITH KIRCHHOFF TYPE DAMPING

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Abstract. In this paper, the intrinsic decay rates for the energy of a nonlinear viscoelastic equation with the kirchhoff type damping of relaxation kernels described by the inequality $g'(t) \leq -H(g(t))$ for all $t \geq 0$, with H convex is proved.

Keywords. Exponential decay; Polynomial decay; Viscoelastic damping; Viscoelastic damping; Intrinsic decay rates.

1. Introduction

In this paper, we study the intrinsic decay rates for the energy of a nonlinear viscoelastic equation with the kirchhoff type damping

$$|u_t|^{\rho} u_{tt} - \left(\xi_0 + \xi_1 \|\nabla u(t)\|^2_{L^2(\Omega)}\right) \Delta u + \int_0^t g(t-s) \Delta u(s)\, ds = 0, \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

with initial data

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x), \quad \text{for } x \in \Omega, \tag{1.2}$$

and boundary conditions

$$u(x,t) = 0, \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, $0 \leq \rho \leq \dfrac{4}{N-2}$ for $N \geq 3$, or $\rho \geq 0$ in $N = 1,2$. It is assumed that the kernels $g(.) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ meet certain conditions to be determined later, and $u_0(x)$, $u_1(x)$ and $u_2(x)$ are given functions. The convolution term $\int_0^t g(t-s) \Delta u(s)\, ds$ reflects the memory effects of materials due to the viscoelasticity. Here the convolution kernel g satisfies proper conditions exhibiting "memory character" which will

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be explained later. The viscoelastic structural damping terms $\xi_0+\xi_1\left\|\nabla u(t)\right\|^2_{L^2(\Omega)}$ are the non-linear stiffness of the membrane and ξ_0 and ξ_1 are positive constants. For more details on using the Kirchhoff type, we refer to [1, 2, 3]. Also, the result of the local existence for problem (1.1)-(1.3) for $\xi_1=0$ was proved in [1] for $\xi_1\neq 0$. We get the same basic results for the local existence of problem (1.1)-(1.3) with a slight change in some calculations that do not affect the basic results.

Cavalcanti et al. [4] studied the intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density

$$\begin{cases} |u_t|^{\rho} u_{tt}-\Delta u_{tt}-\Delta u-\gamma\Delta u_t+\int_0^t g(t-s)\Delta u(s)\,ds=0, & \text{in } \Omega\times(0,\infty), \\ u(x,t)=v(x,t)=0, & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & \text{for } x\in\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n\geq 1)$ with a smooth boundary $\partial\Omega$, ρ is a real number such that $0<\rho\leq\dfrac{2}{n-2}$ if $n\geq 3$ or $\rho>0$ if $n=1,2$ and g represents the kernel of the memory term, which will be assumed to intrinsic decay rates for the energy; see [4].

Cavalcanti, Cavalcanti and Ferreirastudied [5] investigated the well-posedness and the optimal decay rate estimates of the energy associated with the following non-linear viscoelastic equation with strong damping

$$\begin{cases} |u_t|^{\gamma} u_{tt}-\Delta u-\Delta u_{tt}+\int_0^t g(t-s)\Delta u(s)\,ds-\gamma\Delta u_t=0,, & \text{in } \Omega\times(0,\infty), \\ u(x,t)=v(x,t)=0, & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & \text{for } x\in\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n\geq 1)$ with a smooth boundary $\partial\Omega$, γ is a real number such that $0<\gamma\leq\dfrac{2}{n-2}$ if $n\geq 3$ or $\gamma>0$ if $n=1,2$ and g represents the kernel of the memory term which will be assumed to decay exponentially. The proof of the existence of solutions was investigated under the assumption that $\gamma\geq 0$ and the uniform decay rates of the energy was obtained in the case of $\gamma>0$; see [5].

Wu [6] studied the general decay of the energy for a viscoelastic equation with damping and source terms

$$\begin{cases} |u_t|^{\gamma} u_{tt}-\Delta u-\Delta u_{tt}+\int_0^t g(t-s)\Delta u(s)\,ds+u_t=|u|^{p-2}u,, & \text{in } \Omega\times(0,\infty), \\ u(x,t)=v(x,t)=0, & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & \text{for } x\in\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n\geq 1)$ with a smooth boundary $\partial\Omega$, γ is a real number such that $0<\gamma\leq\dfrac{2}{n-2}$ if $n\geq 3$ or $\gamma>0$ if $n=1,2$ and g represents the kernel of the memory term which will be assumed to decay exponentially.

Mu and Ma [7] studied the system of nonlinear wave equations with Balakrishnan–Taylor damping

$$\begin{cases} u_{tt} - \left(\xi_0 + \xi_1 \|\nabla u\|^2 + \sigma (\nabla u, \nabla u_t)\right) \Delta u + \int_0^t g_1 (t-s) \Delta u (s)\, ds = f_1, \ (x,t) \in Q_T, \\ v_{tt} - \left(\xi_0 + \xi_1 \|\nabla u\|^2 + \sigma (\nabla u, \nabla u_t)\right) \Delta v + \int_0^t g_2 (t-s) \Delta v (s)\, ds = f_2, \ (x,t) \in Q_T, \\ u(x,t) = v(x,t) = 0, \qquad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1 \qquad \text{for } x \in \Omega, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \qquad \text{for } x \in \Omega, \end{cases}$$

where $Q_T := \Omega \times (0,T)$, Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$ and $T < \infty$. Here, g_i $(i = 1,2)$ represents the kernel of the memory term (see [19]). All the parameters ξ_0, ξ_1 and σ are assumed to be positive constants, $f_1 = |v|^{q+1} |u|^{p-1} u$ and $f_2 = |u|^{p+1} |v|^{q-1} v$. Cavalcanti et al. [8] studied the intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional dissipative effects.

Lasiecka, Messaoudi and Mustafa [9] studied the note on intrinsic decay rates for abstract wave equations with memory. Lasiecka and Wang [10] further studied the intrinsic decay rate estimates for the semilinear abstract second order equations with memory. Hao and Cai [11] studied the uniform decay of solutions for the coupled viscoelastic wave equations. Cavalcanti et al. [12] studied the existence and the uniform decay rates for viscoelastic problems with the nonlinear boundary damping. For more results in this direction, we refer to [13, 14, 15, 16].

However, the above results and results presented in [17]-[26] do not involve the intrinsic decay rates for the energy of problem (1.1)-(1.3) for $\xi_1 \neq 0$. Motivated by the above research, we consider the intrinsic decay rates for the energy of relaxation kernels described by the inequality $g'(t) \leq -H(g(t))$ for all $t \geq 0$ and for $\xi_1 \neq 0$ of (1.1)-(1.3) in this paper. The outline of the paper is as follows. In the second section, Section 2, we present some basic concepts, establish some useful inequalities, and define the energy $E(t)$ associated to (1.1)-(1.3). We show that it is a non-increasing function of t. Finally, in Section 3, we prove the intrinsic decay rates for the energy of the posed problem.

2. Preliminaries

Let $L^p(\Omega)$ be the weighted Banach space equipped with the norm

$$\|u(t)\|_{L^p(\Omega)} := \left[\int_\Omega |u(t)|^p \, dx\right]^{\frac{1}{p}}.$$

In particular, the Hilbert space of square integral functions having the finite norm

$$\|u(t)\|_{L^2(\Omega)} := \left[\int_\Omega u^2(t) \, dx\right]^{\frac{1}{2}}.$$

We list some useful inequalities.

- *Holder's inequality.* If $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $\frac{1}{p}+\frac{1}{q}=1$, then
$$\int_\Omega f(t)\,g(t)\,dt \le \|f(t)\|_{L^p(\Omega)} \times \|g(t)\|_{L^q(\Omega)}.$$
- *Cauchy–Schwarz inequality*. If $f,\ g \in L^2(\Omega)$, then
$$\left(\int_\Omega f(t)\,g(t)\,dt\right)^2 \le \|f(t)\|^2_{L^2(\Omega)} \times \|g(t)\|^2_{L^2(\Omega)}.$$
- *ε-Cauchy inequality*. For all $\alpha,\ \beta \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^*_+$, we have
$$|\alpha\beta| \le \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2.$$
- *Jensen's inequality*. Let F be a convex increasing function on $[\alpha,b]$, let $f:\Omega\longmapsto[\alpha,b]$, and let h be an integrable function such that $h(x)\ge 0$ and $\int_\Omega h(x)\,dx = h_0 > 0$. Then,
$$\int_\Omega F^{-1}(f(x))\,h(x)\,dx \le h_0F^{-1}\left[h_0^{-1}\int_\Omega f(x)\,h(x)\,dx\right].$$
- Since $0\le\rho\le\frac{4}{N-2}$, if $N\ge 3$, then
$$H_0^1(\Omega)\hookrightarrow L^{(\rho+2)}(\Omega),$$
and the same occurs for $N=1,2$ when $\rho\ge 0$.

We suppose that the kernel $g(t)$ is a $C^1(\mathbb{R}_+,\mathbb{R}_+)$ function satisfying
(**A1**) $g(0)>0$ and $\int_0^\infty g(s)\,ds < \xi_0$.
(**A2**) $g'(t)\le -H(g(t))$ for all $t\ge 0$, where $H\in C^1(\mathbb{R}_+)$ with $H(0)=0$ a given strictly increasing and convex function. Moreover,
$$H\in C^2(0,\infty) \text{ and } \lim\inf_{x\to 0^+}\left\{x^2H''(x)-xH'(x)+H(x)\right\}\ge 0.$$
(**A3**) With reference to the function H introduced above, let $y(t)$ be the solution of the ODE
$$y'(t)+H(y(t))=0,\quad y(0)=g(0)>0.$$
(**A4**) We assume that there exists $\alpha_0\in[0,1)$ such that $y^{1-\alpha_0}\in L_1(1,\infty)$.

Remark 2.1. The assumption $\xi_0-\int_0^t g(s)\,ds>0$ is necessary to guarantee the hyperbolicity of problem (1.1)-(1.3).

We define the binary notations
$$\begin{cases} (g\circ w)(t) := \int_0^t g(t-s)\,|w(x,s)-w(x,t)|^2\,ds, \\ (g\diamond w)(t) := \int_0^t g(t-s)\,(w(t)-w(s))\,ds. \end{cases} \tag{2.1}$$
We define the corresponding energy functional by
$$\begin{aligned} E(t) \ :\ &= \frac{1}{\rho+2}\|u_t(t)\|^{\rho+2}_{\rho+2} + \frac{1}{2}\left(\xi_0-\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)} \\ &+\frac{\xi_1}{4}\|\nabla u(t)\|^4_{L^2(\Omega)} + \frac{1}{2}\int_\Omega (g\circ\nabla u)(t)\,dx. \end{aligned} \tag{2.2}$$

Lemma 2.2. $\frac{d}{dt}\{E(t)\} = \frac{1}{2}\int_\Omega (g' \circ \nabla u)(t)\,dx - \frac{1}{2}g(t)\|\nabla u(t)\|^2_{L^2(\Omega)} \leq 0.$

Proof. Multiplying (1.1) by $u_t(t)$ and integrating over Ω, we have

$$\int_\Omega |u_t|^\rho u_{tt}u_t dx - \int_\Omega \left(\xi_0 + \xi_1 \|\nabla u(t)\|^2_{L^2(\Omega)}\right)\Delta u(x,t)\,u_t(x,t)\,dx$$
$$+ \int_\Omega u_t(x,t)\int_0^t g(t-s)\,\Delta u(x,s)\,ds dx = 0. \tag{2.3}$$

It follows that

$$\int_\Omega |u_t|^\rho u_{tt}u_t dx = \frac{1}{\rho+2}\frac{d}{dt}\left\{\|u_t(t)\|^{\rho+2}_{\rho+2}\right\}. \tag{2.4}$$

This further implies

$$-\int_\Omega \left(\xi_0 + \xi_1 \|\nabla u(t)\|^2_{L^2(\Omega)}\right)\Delta u(x,t)\,u_t(x,t)\,dx$$
$$= \frac{d}{dt}\left\{\frac{1}{2}\left(\xi_0 + \frac{\xi_1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}, \tag{2.5}$$

and

$$\int_\Omega u_t(x,t)\int_0^t g(t-s)\,\Delta u(x,s)\,ds dx$$
$$= \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (g\circ\nabla u)(t)\,dx - \left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}$$
$$-\frac{1}{2}\int_\Omega \left(g'\circ\nabla u\right)(t)\,dx + \frac{1}{2}g(t)\|\nabla u(t)\|^2_{L^2(\Omega)}. \tag{2.6}$$

From the above equalities, we find that

$$\frac{1}{\rho+2}\frac{d}{dt}\left\{\|u_t(t)\|^{\rho+2}_{\rho+2}\right\} + \frac{d}{dt}\left\{\frac{1}{2}\left(\xi_0 + \frac{\xi_1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}$$
$$+\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (g\circ\nabla u)(t)\,dx - \left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}$$
$$-\frac{1}{2}\int_\Omega \left(g'\circ\nabla u\right)(t)\,dx + \frac{1}{2}g(t)\|\nabla u(t)\|^2_{L^2(\Omega)}$$
$$= 0,$$

that is,

$$\frac{d}{dt}\left\{\frac{1}{\rho+2}\|u_t(t)\|^{\rho+2}_{\rho+2} + \frac{1}{2}\left(\xi_0 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right.$$
$$\left.+\frac{\xi_1}{4}\|\nabla u(t)\|^4_{L^2(\Omega)} + \frac{1}{2}\int_\Omega (g\circ\nabla u)(t)\,dx\right\}$$
$$= \frac{1}{2}\int_\Omega \left(g'\circ\nabla u\right)(t)\,dx - \frac{1}{2}g(t)\|\nabla u(t)\|^2_{L^2(\Omega)}.$$

This completes the proof. □

3. The Decay of Solutions

In this section, we prove the intrinsic decay rates for the energy of the posed problem. Now, we are in a position to state our main result.

Lemma 3.1. *Let us assume that* $(\mathbf{A1})-(\mathbf{A4})$ *are the place. Then, there exists a positive constant* $T_0>0$ *such that*

$$E((n+1)T)+\tilde{H}\left(C_9^{-1}E((n+1)T)\right)\leq E(nT),\quad n=1,2,3...$$

for all $T>T_0$ *and all* $n\in N$.

Proof. For this purpose, multiplying (1.1) by the viscoelastic multiplier $(g\diamond u)(t)$ and integrating over $\Omega\times(nT,(n+1)T)$, we infer that

$$\begin{aligned}&\int_{nT}^{(n+1)T}\left(|u_t(t)|^{\rho}u_{tt}(t),(g\diamond u)(t)\right)_{\Omega}dt\\&-\int_{nT}^{(n+1)T}\left(\left(\xi_0+\xi_1\|\nabla u(t)\|^2_{L^2(\Omega)}\right)\Delta u(t),(g\diamond u)(t)\right)_{\Omega}dt\\&+\int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)\Delta u(s)ds,(g\diamond u)(t)\right)_{\Omega}dt=0.\end{aligned}\tag{3.1}$$

We next analyze the above terms separately. For the first term, by using

$$|u_t(t)|^{\rho}u_{tt}(t)=\frac{1}{\rho+1}\frac{d}{dt}\left\{|u_t(t)|^{\rho}u_t(t)\right\},$$

we get

$$|u_t(t)|^{\rho}u_{tt}(t)(g\diamond u)(t)=\frac{1}{\rho+1}\frac{d}{dt}\left\{|u_t(t)|^{\rho}u_t(t)\right\}\left(\int_0^t g(t-s)(u(t)-u(s))ds\right).$$

Then

$$\begin{aligned}|u_t(t)|^{\rho}u_{tt}(t)(g\diamond u)(t)\;=\;&\frac{1}{\rho+1}\frac{d}{dt}\left\{|u_t(t)|^{\rho}u_t(t)\left(\int_0^t g(t-s)(u(t)-u(s))ds\right)\right\}\\&-\frac{1}{\rho+1}|u_t(t)|^{\rho}u_t(t)\left(\int_0^t g'(t-s)(u(t)-u(s))ds\right)\\&-\frac{1}{\rho+1}|u_t(t)|^{\rho}u_t(t)\left(\int_0^t g(t-s)u_t(t)ds\right),\end{aligned}\tag{3.2}$$

By integrating (3.2) over $\Omega\times(nT,(n+1)T)$, we have

$$\begin{aligned}&\int_{nT}^{(n+1)T}\left(|u_t(t)|^{\rho}u_{tt}(t),(g\diamond u)(t)\right)_{\Omega}dt\\=\;&\frac{1}{\rho+1}\left(|u_t(t)|^{\rho}u_t(t),\int_0^t g(t-s)(u(t)-u(s))ds\right)_{\Omega}\Bigg|_{nT}^{(n+1)T}\\&-\frac{1}{\rho+1}\int_{nT}^{(n+1)T}\left(|u_t(t)|^{\rho}u_t(t),\int_0^t g'(t-s)(u(t)-u(s))ds\right)_{\Omega}dt\\&-\frac{1}{\rho+1}\int_{nT}^{(n+1)T}\left(\int_0^t g(s)ds\right)\int_{\Omega}|u_t(t)|^{\rho+2}dxdt.\end{aligned}\tag{3.3}$$

For the second term, by using integrating by parts, we have

$$\begin{aligned}
& -\int_{nT}^{(n+1)T}\left(\left(\xi_0+\xi_1\|\nabla u(t)\|_{L^2(\Omega)}^2\right)\Delta u(t),(g\diamond u)(t)\right)_\Omega dt \\
= & -\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u(t)\|_{L^2(\Omega)}^2\right)\left(\Delta u(t),\int_0^t g(t-s)(u(t)-u(s))\,ds\right)_\Omega dt \\
= & \int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u(t)\|_{L^2(\Omega)}^2\right)\left(\nabla u(t),\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_\Omega dt.
\end{aligned} \tag{3.4}$$

For the third term, by using integrating by parts, we have

$$\begin{aligned}
& \int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)\Delta u(s)\,ds,(g\diamond u)(t)\right)_\Omega dt \\
= & -\int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)\nabla u(s)\,ds,\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_\Omega dt \\
= & \int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds,\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_\Omega dt \\
& -\int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)\nabla u(t)\,ds,\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_\Omega dt.
\end{aligned} \tag{3.5}$$

Substituting (3.3)-(3.5) into (3.1), we arrive at

$$\begin{aligned}
& \frac{1}{\rho+1}\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\int_\Omega |u_t|^{\rho+2}\,dxdt \\
= & \frac{1}{\rho+1}\left(|u_t|^\rho u_t(t),\int_0^t g(t-s)(u(t)-u(s))\,ds\right)_\Omega\Big|_{nT}^{(n+1)T} \\
& -\frac{1}{\rho+1}\int_{nT}^{(n+1)T}\left(|u_t|^\rho u_t(t),\int_0^t g'(t-s)(u(t)-u(s))\,ds\right)_\Omega dt \\
& +\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^2\right)\left(\nabla u,\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_\Omega dt \\
& +\int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds,\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_\Omega dt \\
& -\int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)\nabla u(t)\,ds,\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_\Omega dt \\
= & J_1+J_2+J_3+J_4+J_5.
\end{aligned} \tag{3.6}$$

We next estimate $|J_1|$, where

$$J_1 \ : \ = \frac{1}{\rho+1}\left(|u_t|^\rho u_t((n+1)T), \int_0^{(n+1)T} g((n+1)T-s)(u((n+1)T)-u(s))\,ds\right)_\Omega$$
$$-\frac{1}{\rho+1}\left(|u_t|^\rho u_t(nT), \int_0^{nT} g(nT-s)(u(nT)-u(s))\,ds\right)_\Omega .$$

Now, let $m\in N$ be an arbitrary, natural number. Using the Holder's inequality and the Young's inequality (for $\varepsilon=\varepsilon_1$), we get

$$\begin{aligned}
& \frac{1}{\rho+1}\left(|u_t|^\rho u_t(mT), \int_0^{mT} g(mT-s)(u(mT)-u(s))\,ds\right)_\Omega \\
\leq\ & \frac{1}{\rho+1}\int_0^{mT} g(mT-s)\,\|u_t(mT)\|_{\rho+2}^{\rho+1}\,\|u(mT)-u(s)\|_{\rho+2}\,ds \\
\leq\ & \int_0^{mT} g(mT-s)\left[\frac{\varepsilon_1}{2}\|u_t(mT)\|_{\rho+2}^{2(\rho+1)}+\frac{1}{2\varepsilon_1(\rho+1)^2}\|u(mT)-u(s)\|_{\rho+2}^2\right]ds \\
=\ & \frac{\varepsilon_1}{2}\left(\int_0^{mT} g(mT-s)\,ds\right)\|u_t(mT)\|_{\rho+2}^{2(\rho+1)} \\
& +\frac{1}{2\varepsilon_1(\rho+1)^2}\int_0^{mT} g(mT-s)\left[\|u(mT)-u(s)\|_{\rho+2}^2\right]ds \\
=\ & \frac{\varepsilon_1}{2}\left(\int_0^{mT} g(s)\,ds\right)\|u_t(mT)\|_{\rho+2}^{\rho+2}\,\|u_t(mT)\|_{\rho+2}^{\rho} \\
& +\frac{1}{2\varepsilon_1(\rho+1)^2}\int_0^{mT} g(mT-s)\left[\|u(mT)-u(s)\|_{\rho+2}^2\right]ds. \qquad (3.7)
\end{aligned}$$

Using (2.2) and (2.3), we get

$$\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2}\leq E(t)\leq E(0),$$

which in turn implies that

$$\|u_t(t)\|_{\rho+2}^{\rho}\leq[(\rho+2)E(0)]^{\frac{\rho}{(\rho+2)}}. \qquad (3.8)$$

Using the embedding $H^1\hookrightarrow L^{2(\rho+1)}$ and (2.1), we get

$$\begin{aligned}
& \frac{1}{2\varepsilon_1(\rho+1)^2}\int_0^{mT} g(mT-s)\left[\|u(mT)-u(s)\|_{\rho+2}^2\right]ds \\
\leq\ & \frac{C_0}{2\varepsilon_1(\rho+1)^2}\int_0^{mT} g(mT-s)\left[\|\nabla u(mT)-\nabla u(s)\|_{L^2(\Omega)}^2\right]ds \\
=\ & \frac{C_0}{2\varepsilon_1(\rho+1)^2}\int_\Omega\left[\int_0^{mT} g(mT-s)\,|\nabla u(mT)-\nabla u(s)|^2\,ds\right]dx \\
=\ & \frac{C_0}{2\varepsilon_1(\rho+1)^2}\int_\Omega (g\circ\nabla u)(mT)\,dx, \qquad (3.9)
\end{aligned}$$

where C_0 comes from the embedding $H^1 \hookrightarrow L^{2(\rho+1)}$. From (3.8) and (3.9, we get

$$\frac{1}{(\rho+1)}\left(|u_t|^\rho u_t(mT), \int_0^{mT} g(mT-s)(u(mT)-u(s))\,ds\right)_\Omega$$
$$\leq \frac{\varepsilon_1\left[(\rho+2)E(0)\right]^{\frac{\rho}{(\rho+2)}}}{2}\left(\int_0^{mT} g(s)\,ds\right)\|u_t(mT)\|_{\rho+2}^{\rho+2} + \frac{C_0}{2\varepsilon_1(\rho+1)^2}\int_\Omega (g\circ\nabla u)(mT)\,dx, \tag{3.10}$$

which together with (2.2) implies that

$$\|u_t(mT)\|_{\rho+2}^{\rho+2} \leq (\rho+2)E(mT), \int_\Omega (g\circ\nabla u)(mT)\,dx \leq 2E(mT), \tag{3.11}$$

Substituting (3.11) into (3.10), we get that

$$\frac{1}{(\rho+1)}\left(|u_t|^\rho u_t(mT), \int_0^{mT} g(mT-s)(u(mT)-u(s))\,ds\right)_\Omega$$
$$\leq \frac{\varepsilon_1\left[(\rho+2)E(0)\right]^{\frac{\rho}{(\rho+2)}}}{2}\|g\|_{L^1(0,\infty)}(\rho+2)E(mT) + \frac{C_0}{\varepsilon_1(\rho+1)^2}E(mT)$$
$$= \left\{\frac{\varepsilon_1\left[(\rho+2)E(0)\right]^{\frac{\rho}{(\rho+2)}}}{2}\|g\|_{L^1(0,\infty)}(\rho+2) + \frac{C_0}{\varepsilon_1(\rho+1)^2}\right\}E(mT).$$

It follows that

$$|J_1| \leq C_1\left[E((n+1)T)+E(nT)\right], \tag{3.12}$$

where

$$C_1 := \left\{\frac{\varepsilon_1\left[(\rho+2)E(0)\right]^{\frac{\rho}{(\rho+2)}}}{2}\|g\|_{L^1(0,\infty)}(\rho+2) + \frac{C_0}{\varepsilon_1(\rho+1)^2}\right\} > 0.$$

Next, we estimate $|J_2|$. Using the Holder's inequality and the Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_2}{2}\right)$, we get

$$\begin{aligned}|J_2| &\leq \frac{1}{\rho+1}\int_{nT}^{(n+1)T}\|u_t(t)\|_{\rho+2}^{\rho+1}\left\|\int_0^t g'(t-s)(u(t)-u(s))\,ds\right\|_{\rho+2} dt\\
&\leq \varepsilon_2\int_{nT}^{(n+1)T}\|u_t(t)\|_{\rho+2}^{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho}\,dt\\
&\quad + \frac{1}{4\varepsilon_2(\rho+1)^2}\int_{nT}^{(n+1)T}\left\|\int_0^t g'(t-s)(u(t)-u(s))\,ds\right\|_{\rho+2}^2 dt. \end{aligned} \tag{3.13}$$

Using embedding $H^1 \hookrightarrow L^{2(\rho+1)}$, and (2.1), we get

$$\begin{aligned}&\int_{nT}^{(n+1)T}\left\|\int_0^t g'(t-s)(u(t)-u(s))\,ds\right\|_{\rho+2}^2 dt\\
&\leq C_0\int_{nT}^{(n+1)T}\left\|\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|_{L^2(\Omega)}^2 dt\\
&\leq -C_0 g(0)\int_{nT}^{(n+1)T}\int_\Omega (g'\circ\nabla u)(t)\,dx\,dt. \end{aligned} \tag{3.14}$$

Substituting (3.8) and (3.14) into (3.13), we get

$$\begin{aligned} |J_2| &\leq \varepsilon_2 \left[(\rho+2) E(0)\right]^{\frac{\rho}{(\rho+2)}} \int_{nT}^{(n+1)T} \|u_t(t)\|_{\rho+2}^{\rho+2} dt \\ &\quad - \frac{C_0 g(0)}{4\varepsilon_2 (\rho+1)^2} \int_{nT}^{(n+1)T} \int_\Omega \left(g' \circ \nabla u\right)(t)\, dxdt. \end{aligned} \tag{3.15}$$

Next, we estimate $|J_3|$. Using the Young's inequality (for $\varepsilon = \varepsilon_3$), $\|\nabla u(t)\|_{L^2(\Omega)}^2 \leq 2l^{-1}E(0)$, the Cauchy-Schwarz inequality and (2.1), we get

$$\begin{aligned} |J_3| &\leq \frac{\varepsilon_3}{2} \int_{nT}^{(n+1)T} \left(\xi_0 + \xi_1 \|\nabla u\|_{L^2(\Omega)}^2\right)^2 \|\nabla u(t)\|_{L^2(\Omega)}^2 dt \\ &+ \frac{1}{2\varepsilon_3} \int_{nT}^{(n+1)T} \left\| \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|_{L^2(\Omega)}^2 dt \\ &\leq \frac{\varepsilon_3}{2} \left(\xi_0 + \xi_1 2l^{-1}E(0)\right)^2 \int_{nT}^{(n+1)T} \|\nabla u(t)\|_{L^2(\Omega)}^2 dt \\ &+ \frac{1}{2\varepsilon_3} \|g\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (g \circ \nabla u)(t)\, dxdt. \end{aligned} \tag{3.16}$$

Now, we estimate $|J_4|$. Using the Cauchy-Schwarz inequality and (2.1), we get

$$\begin{aligned} J_4 &= \int_{nT}^{(n+1)T} \left\| \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|_{L^2(\Omega)}^2 dt \\ &\leq \|g\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s) \|\nabla u(t) - \nabla u(s)\|_{L^2(\Omega)}^2 dsdt \\ &= \|g\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (g \circ \nabla u)(t)\, dxdt. \end{aligned} \tag{3.17}$$

Using the Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_4}{2}\right)$, the Cauchy-Schwarz inequality and (2.1), we get

$$\begin{aligned} |J_5| &\leq \varepsilon_4 \int_{nT}^{(n+1)T} \left\| \int_0^t g(t-s) \nabla u(t)\, ds \right\|_{L^2(\Omega)}^2 dt \\ &\quad + \frac{1}{4\varepsilon_4} \int_{nT}^{(n+1)T} \left\| \int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|_{L^2(\Omega)}^2 dt \\ &\leq \varepsilon_4 \|g\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s) \|\nabla u(t)\|_{L^2(\Omega)}^2 dsdt \\ &\quad + \frac{1}{4\varepsilon_4} \|g\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t g(t-s) \|\nabla u(t) - \nabla u(s)\|_{L^2(\Omega)}^2 dsdt \\ &= \varepsilon_4 \|g\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \left(\int_0^t g(s)\, ds\right) \|\nabla u(t)\|_{L^2(\Omega)}^2 dt \\ &\quad + \frac{1}{4\varepsilon_4} \|g\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (g \circ \nabla u)(t)\, dxdt. \end{aligned} \tag{3.18}$$

Substituting (3.15)-(3.18) into (3.6) and using (3.12), we find from $\|g\|_{L^1(0,\infty)} < \xi_0$ write

$$
\begin{aligned}
&\frac{1}{\rho+1}\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\int_\Omega |u_t|^{\rho+2}\,dxdt\\
&\le C_1\left[E((n+1)T)+E(nT)\right]+\varepsilon_2\left[(\rho+2)E(0)\right]^{\frac{\rho}{(\rho+2)}}\int_{nT}^{(n+1)T}\|u_t(t)\|_{\rho+2}^{\rho+2}dt\\
&-\frac{C_0 g(0)}{4\varepsilon_2(\rho+1)^2}\int_{nT}^{(n+1)T}\int_\Omega\left(g'\circ\nabla u\right)(t)\,dxdt\\
&+\frac{\varepsilon_3}{2}\left(\xi_0+\xi_1 2l^{-1}E(0)\right)^2\int_{nT}^{(n+1)T}\|\nabla u(t)\|_{L^2(\Omega)}^2 dt\\
&+\frac{1}{2\varepsilon_3}\xi_0\int_{nT}^{(n+1)T}\int_\Omega (g\circ\nabla u)(t)\,dxdt+\xi_0\int_{nT}^{(n+1)T}\int_\Omega (g\circ\nabla u)(t)\,dxdt\\
&+\varepsilon_4\xi_0\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|_{L^2(\Omega)}^2 dt+\frac{1}{4\varepsilon_4}\xi_0\int_{nT}^{(n+1)T}\int_\Omega (g\circ\nabla u)(t)\,dxdt.
\end{aligned}
\tag{3.19}
$$

Since $g(0) > 0$, we can select a points $t_1 < T$ with t_1 close to zero such that for all $t \ge t_1 \int_0^t g(s)\,ds \ge t_1 g(t_1) := c_0$. Then (3.19) is equivalent to

$$
\begin{aligned}
&\left\{\frac{1}{\rho+1}t_1 g(t_1)-\varepsilon_2\left[(\rho+2)E(0)\right]^{\frac{\rho}{(\rho+2)}}\right\}\int_{nT}^{(n+1)T}\|u_t(t)\|_{\rho+2}^{\rho+2}dt\\
&\le C_1\left[E((n+1)T)+E(nT)\right]-\frac{C_0 g(0)}{4\varepsilon_2(\rho+1)^2}\int_{nT}^{(n+1)T}\int_\Omega\left(g'\circ\nabla u\right)(t)\,dxdt\\
&+\frac{\varepsilon_3}{2}\left(\xi_0+\xi_1 2l^{-1}E(0)\right)^2\int_{nT}^{(n+1)T}\|\nabla u(t)\|_{L^2(\Omega)}^2 dt\\
&+\xi_0\left\{\frac{1}{2\varepsilon_3}+1+\frac{1}{4\varepsilon_4}\right\}\int_{nT}^{(n+1)T}\int_\Omega (g\circ\nabla u)(t)\,dxdt\\
&+\varepsilon_4\xi_0\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|_{L^2(\Omega)}^2 dt.
\end{aligned}
\tag{3.20}
$$

Now, multiplying (1.1) by u and integrating over $\Omega\times(nT,(n+1)T)$, we infer that

$$
\begin{aligned}
&\int_{nT}^{(n+1)T}\left(|u_t(t)|^\rho u_{tt}(t),u(t)\right)_\Omega dt\\
&-\int_{nT}^{(n+1)T}\left(\left(\xi_0+\xi_1\|\nabla u(t)\|_{L^2(\Omega)}^2\right)\Delta u(t),u(t)\right)_\Omega dt\\
&+\int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)\Delta u(s)\,ds,u(t)\right)_\Omega dt=0.
\end{aligned}
\tag{3.21}
$$

Using $|u_t(t)|^\rho u_{tt}(t)u(t)=\frac{1}{\rho+1}\frac{d}{dt}\left\{|u_t(t)|^\rho u_t(t)u(t)\right\}-\frac{1}{\rho+1}|u_t(t)|^{\rho+2}$, we get

$$
\begin{aligned}
&\int_{nT}^{(n+1)T}\left(|u_t(t)|^\rho u_{tt}(t),u(t)\right)_\Omega dt\\
&=\frac{1}{\rho+1}\left(|u_t(t)|^\rho u_t(t),u(t)\right)_\Omega\Big|_{nT}^{(n+1)T}-\frac{1}{\rho+1}\int_{nT}^{(n+1)T}\|u_t(t)\|_{\rho+2}^{\rho+2}dt.
\end{aligned}
\tag{3.22}
$$

Hence,

$$
\begin{aligned}
& -\int_{nT}^{(n+1)T}\left(\left(\xi_0+\xi_1\|\nabla u(t)\|_{L^2(\Omega)}^2\right)\Delta u(t),u(t)\right)_\Omega dt \\
= & \int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u(t)\|_{L^2(\Omega)}^2\right)(\nabla u(t),\nabla u(t))_\Omega dt \\
= & \int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u(t)\|_{L^2(\Omega)}^2\right)\|\nabla u(t)\|_{L^2(\Omega)}^2 dt
\end{aligned} \tag{3.23}
$$

and

$$
\begin{aligned}
& \int_{nT}^{(n+1)T}\left(\int_0^t g(t-s)\Delta u(s)\,ds,u(t)\right)_\Omega dt \\
= & -\int_{nT}^{(n+1)T}\int_0^t g(t-s)(\nabla u(s),\nabla u(t))_\Omega\,dsdt \\
= & \int_{nT}^{(n+1)T}\int_0^t g(t-s)(\nabla u(t)-\nabla u(s),\nabla u(t))_\Omega\,dsdt \\
& -\int_{nT}^{(n+1)T}\int_0^t g(t-s)(\nabla u(t),\nabla u(t))_\Omega\,dsdt \\
= & \int_{nT}^{(n+1)T}\int_0^t g(t-s)(\nabla u(t)-\nabla u(s),\nabla u(t))_\Omega\,dsdt \\
& -\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|_{L^2(\Omega)}^2 dt.
\end{aligned} \tag{3.24}
$$

By substituting (3.22)-(3.24) into (3.21), we obtain

$$
\begin{aligned}
& \frac{1}{\rho+1}\left(|u_t(t)|^\rho u_t(t),u(t)\right)_\Omega\Big|_{nT}^{(n+1)T}-\frac{1}{\rho+1}\int_{nT}^{(n+1)T}\|u_t(t)\|_{\rho+2}^{\rho+2}dt \\
& +\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u(t)\|_{L^2(\Omega)}^2\right)\|\nabla u(t)\|_{L^2(\Omega)}^2 dt \\
& +\int_{nT}^{(n+1)T}\int_0^t g(t-s)(\nabla u(t)-\nabla u(s),\nabla u(t))_\Omega\,dsdt \\
& -\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|_{L^2(\Omega)}^2 dt=0.
\end{aligned} \tag{3.25}
$$

We next estimate

$$
\begin{aligned}
I_1 \quad : & = \frac{1}{\rho+1}\left(|u_t|^\rho u_t(t),u(t)\right)_\Omega\Big|_{nT}^{(n+1)T} \\
: & = \frac{1}{\rho+1}\left(|u_t|^\rho u_t((n+1)T),u((n+1)T)\right)_\Omega-\frac{1}{\rho+1}\left(|u_t|^\rho u_t(nT),u(nT)\right)_\Omega.
\end{aligned}
$$

Using the Holder's inequality, the Young's inequality (for $\varepsilon=\varepsilon_5$), $\|u_t(t)\|_{\rho+2}^{\rho}\leq[(\rho+2)E(0)]^{\frac{\rho}{(\rho+2)}}$, embedding $H^1\hookrightarrow L^{2(\rho+1)}$, $\|u_t(t)\|_{\rho+2}^{\rho+2}\leq(\rho+2)E(t)$ and $\|\nabla u(t)\|_{L^2(\Omega)}^2\leq 2l^{-1}E(t)$, we get

$$\begin{aligned}
\frac{1}{\rho+1}\left(|u_t(t)|^{\rho}u_t(t),u(t)\right)_{\Omega} &\leq \frac{1}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+1}\|u(t)\|_{\rho+2}\\
&\leq \frac{\varepsilon_5}{2(\rho+1)^2}\|u_t(t)\|_{\rho+2}^{2(\rho+1)}+\frac{1}{2\varepsilon_5}\|u(t)\|_{\rho+2}^2\\
&\leq \frac{\varepsilon_5}{2(\rho+1)^2}[(\rho+2)E(0)]^{\frac{\rho}{(\rho+2)}}\|u_t(t)\|_{\rho+2}^{\rho+2}+\frac{C_0}{2\varepsilon_5}\|\nabla u(t)\|_{L^2(\Omega)}^2\\
&\leq \frac{\varepsilon_5}{2(\rho+1)^2}[(\rho+2)E(0)]^{\frac{\rho}{(\rho+2)}}(\rho+2)E(t)+\frac{C_0}{\varepsilon_5}l^{-1}E(t)\\
&= C_2E(t),
\end{aligned}$$

where$C_2:=\left\{\frac{\varepsilon_5}{2(\rho+1)^2}[(\rho+2)E(0)]^{\frac{\rho}{(\rho+2)}}(\rho+2)+\frac{C_0}{\varepsilon_5}l^{-1}\right\}>0$. It follows that

$$|I_1|\leq C_2\{E((n+1)T)+E(nT)\}. \tag{3.26}$$

To estimate $I_2:=\int_{nT}^{(n+1)T}\int_0^t g(t-s)(\nabla u(t)-\nabla u(s),\nabla u(t))_{\Omega}\,dsdt$, we find from the Young's inequality $\left(\text{for } \varepsilon=\frac{\varepsilon_6}{2}\right)$ and (2.1) that

$$\begin{aligned}
|I_2| &\leq \frac{1}{4\varepsilon_6}\int_{nT}^{(n+1)T}\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|_{L^2(\Omega)}^2\,dsdt\\
&\quad+\varepsilon_6\int_{nT}^{(n+1)T}\int_0^t g(t-s)\|\nabla u(t)\|_{L^2(\Omega)}^2\,dsdt\\
&= \frac{1}{4\varepsilon_6}\int_{nT}^{(n+1)T}\int_{\Omega}\left[\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|^2\,ds\right]dxdt\\
&\quad+\varepsilon_6\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|_{L^2(\Omega)}^2\,dt\\
&= \frac{1}{4\varepsilon_6}\int_{nT}^{(n+1)T}\int_{\Omega}(g\circ\nabla u)(t)\,dxdt\\
&\quad+\varepsilon_6\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|_{L^2(\Omega)}^2\,dt.
\end{aligned}\tag{3.27}$$

Substituting (3.26) and (3.27) into (3.25), we can write

$$\begin{aligned}
&-\frac{1}{(\rho+1)}\int_{nT}^{(n+1)T}\|u_t(t)\|_{\rho+2}^{\rho+2}\,dt+\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u(t)\|_{L^2(\Omega)}^2\right)\|\nabla u(t)\|_{L^2(\Omega)}^2\,dt\\
\leq\ & C_2\{E((n+1)T)+E(nT)\}+\frac{1}{4\varepsilon_6}\int_{nT}^{(n+1)T}\int_{\Omega}(g\circ\nabla u)(t)\,dxdt\\
&+(\varepsilon_6+1)\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|_{L^2(\Omega)}^2\,dt.
\end{aligned}\tag{3.28}$$

Multiplying (3.20) by γ_1 and multiplying (3.28) by γ_2, we get

$$\begin{aligned}
&\left[\gamma_1\left\{\frac{t_1 g(t_1)}{(\rho+1)} - \varepsilon_2\left[(\rho+2)E(0)\right]^{\frac{\rho}{(\rho+2)}}\right\} - \frac{\gamma_2}{(\rho+1)}\right]\int_{nT}^{(n+1)T}\|u_t(t)\|_{\rho+2}^{\rho+2}dt \\
&+\gamma_2\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u(t)\|^2_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}dt \\
\leq\ & \{\gamma_1 C_1+\gamma_2 C_2\}\left[E((n+1)T)+E(nT)\right] \\
&-\gamma_1\frac{C_0 g(0)}{4\varepsilon_2(\rho+1)^2}\int_{nT}^{(n+1)T}\int_\Omega\left(g'\circ\nabla u\right)(t)\,dxdt \\
&+\left\{\gamma_1\frac{\varepsilon_3}{2}\left(\xi_0+\xi_1 2l^{-1}E(0)\right)^2\right\}\int_{nT}^{(n+1)T}\|\nabla u(t)\|^2_{L^2(\Omega)}dt \\
&+\left\{\gamma_1\xi_0\left\{\frac{1}{2\varepsilon_3}+1+\frac{1}{4\varepsilon_4}\right\}+\frac{\gamma_2}{4\varepsilon_6}\right\}\int_{nT}^{(n+1)T}\int_\Omega(g\circ\nabla u)(t)\,dxdt \\
&+\{\gamma_1\varepsilon_4\xi_0+\gamma_2(\varepsilon_6+1)\}\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}dt. \qquad (3.29)
\end{aligned}$$

Let

$$\begin{cases}
\varepsilon_2 := \dfrac{t_1 g(t_1)}{2(\rho+1)\left[(\rho+2)E(0)\right]^{\frac{\rho}{(\rho+2)}}} > 0, \\[2ex]
\varepsilon_3 := \dfrac{6\varepsilon\xi_0}{\gamma_1\left(\xi_0+\xi_1 2l^{-1}E(0)\right)^2} > 0, \\[2ex]
\varepsilon_4 := \dfrac{\varepsilon}{\gamma_1\xi_0} > 0, \\[2ex]
\varepsilon_6 := 2\varepsilon > 0,
\end{cases} \qquad (3.30)$$

and

$$\begin{cases}
\gamma_2 := 1, \\[1ex]
\gamma_1 := \dfrac{2(\rho+2)}{t_1 g(t_1)} > 0.
\end{cases} \qquad (3.31)$$

Substituting (3.30) and (3.31) into (3.29), we have

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\|u_t(t)\|_{\rho+2}^{\rho+2}dt+\xi_0\int_{nT}^{(n+1)T}\|\nabla u(t)\|^2_{L^2(\Omega)}dt \\
&+\xi_1\int_{nT}^{(n+1)T}\|\nabla u(t)\|^4_{L^2(\Omega)}dt \\
\leq\ & C_3\left[E((n+1)T)+E(nT)\right]-C_4\int_{nT}^{(n+1)T}\int_\Omega\left(g'\circ\nabla u\right)(t)\,dxdt \\
&+3\varepsilon\xi_0\int_{nT}^{(n+1)T}\|\nabla u(t)\|^2_{L^2(\Omega)}dt+C_5\int_{nT}^{(n+1)T}\int_\Omega(g\circ\nabla u)(t)\,dxdt \\
&+(3\varepsilon+1)\int_{nT}^{(n+1)T}\left(\int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}dt, \qquad (3.32)
\end{aligned}$$

where

$$\begin{cases} C_3 := \left\{ \left(\dfrac{2(\rho+2)}{t_1 g(t_1)} \right) C_1 + C_2 \right\} > 0, \\ C_4 := \dfrac{[(\rho+2)E(0)]^{\frac{\rho}{(\rho+2)}} (\rho+2) C_0 g(0)}{[t_1 g(t_1)]^2 (\rho+1)} > 0, \\ C_5 := \left\{ \dfrac{2(\rho+2)}{t_1 g(t_1)} \xi_0 \left\{ \dfrac{2(\rho+2)}{t_1 g(t_1)} \left(\dfrac{\left(\xi_0 + \xi_1 2 l^{-1} E(0)\right)^2}{12\varepsilon\xi_0} + \dfrac{\xi_0}{4\varepsilon} \right) + 1 \right\} + \dfrac{1}{8\varepsilon} \right\} > 0. \end{cases}$$

Adding and subtracting in (3.32) the term

$$-\int_{nT}^{(n+1)T} \int_\Omega \left(\int_0^t g(s)\,ds \right) |\nabla u(t)|^2 \,dxdt \quad \text{and} \quad \int_{nT}^{(n+1)T} \int_\Omega a(x) (g \circ \nabla u)(t)\,dxdt,$$

we obtain

$$\begin{aligned} & (1-3\varepsilon) \int_{nT}^{(n+1)T} \int_\Omega \left(\xi_0 - \int_0^t g(s)\,ds \right) |\nabla u(t)|^2 \,dxdt \\ & + \int_{nT}^{(n+1)T} \|u_t(t)\|_{\rho+2}^{\rho+2} dt + \xi_1 \int_{nT}^{(n+1)T} \|\nabla u(t)\|_{L^2(\Omega)}^4 dt \\ & + \int_{nT}^{(n+1)T} \int_\Omega (g \circ \nabla u)(t)\,dxdt \\ \leq \; & C_3 [E((n+1)T) + E(nT)] + C_5 \int_{nT}^{(n+1)T} \int_\Omega (g \circ \nabla u)(t)\,dxdt \\ & + C_5 \int_{nT}^{(n+1)T} \int_\Omega k_1 \left(-g' \circ \nabla u \right)(t)\,dxdt, \end{aligned} \tag{3.33}$$

where$k_1 := \dfrac{C_4}{C_5} > 0$. Choosing ε sufficiently small, $k_1 > 0$ and T large enough and using

$$\begin{aligned} & \alpha_1 \left\{ \|u_t(t)\|_{\rho+2}^{\rho+2} + \|\nabla u(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^4 + \int_\Omega (g \circ \nabla u)(t)\,dx \right\} \\ \leq \; & E(t) \leq \alpha_2 \left\{ \|u_t(t)\|_{\rho+2}^{\rho+2} + \|\nabla u(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^4 + \int_\Omega (g \circ \nabla u)(t)\,dx \right\} \end{aligned}$$

where

$$\alpha_1 := \min\left\{ \frac{1}{\rho+2}, \frac{l}{2}, \frac{\xi_1}{4} \right\} > 0, \; \alpha_2 := \max\left\{ \frac{\xi_0}{2}, \frac{\xi_1}{4}, \frac{1}{2} \right\} > 0,$$

we get

$$\begin{aligned} \int_{nT}^{(n+1)T} E(t)\,dt \leq \; & C_6 [E((n+1)T) + E(nT)] + C_7 \int_{nT}^{(n+1)T} \int_\Omega (g \circ \nabla u)(t)\,dxdt \\ & + C_7 \int_{nT}^{(n+1)T} \int_\Omega k_1 (-g' \circ \nabla u)(t)\,dxdt, \end{aligned} \tag{3.34}$$

where

$$C_6 := \frac{\alpha_2 C_3}{\min\{1, (1-3\varepsilon) l, \xi_1\}} > 0, \; C_7 := \frac{\alpha_2 C_5}{\min\{1, (1-3\varepsilon) l, \xi_1\}} > 0.$$

In the last step, we need to relate the viscoelastic energy to the viscoelastic damping. In the case when the relaxation function obeys a linear equation, this relation is straightforward and is expressed by a suitable multiplication. However, in the case of general decays, additional arguments are used. Here, we follow [22]. From the (**A2**) made on the viscoelastic kernel g and [22, Lemma 4], we obtain

$$(g\circ\nabla u)(t)\leq \hat{H}_{\alpha}^{-1}(-g'\circ\nabla u)(t),\quad t\in[nT,(n+1)T],\tag{3.35}$$

where $\hat{H}_{\alpha}$ is a rescaling of H_{α} with

$$H_{\alpha}(s)=\alpha s^{1-\frac{1}{\alpha}}H\left(s^{\frac{1}{\alpha}}\right),$$

and $\alpha\in(0,1)$ is such that

$$\sup_{t>0}\int_{0}^{t}g^{1-\alpha}(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|^{2}ds<\infty.$$

From $(\mathbf{A2})$, it is clear that $\alpha\geq\alpha_0$. The main point, however, is that the argument can be reiterated (based on [22, Lemma 4] with α=1). This allows us to replace H_{α}, the function in (3.35), by the original function $\hat{H}$, which is a rescaling of $H(s)$. This means $\hat{H}=cH\left(\frac{C}{s}\right)$ for some c, $C>0$. Now, using (3.34) and (3.35), we deduce that

$$\begin{aligned}\int_{nT}^{(n+1)T}E(t)\,dt &\leq C_6\left[E((n+1)T)+E(nT)\right]\\ &\quad+C_7\int_{nT}^{(n+1)T}\int_{\Omega}\left[\hat{H}^{-1}+k_1\right](-g'\circ\nabla u)(t)\,dxdt.\end{aligned}\tag{3.36}$$

Next, we employ the following version of the Jensen's inequality. We use (3.36) in order to bring the functions H in front of the integrals. Let us denote $\alpha_0:=meas(\Omega)$. We note that the function $\hat{H}^{-1}+k_1$ is concave. Let $F^{-1}=\hat{H}^{-1}+k_1$, $f(x)=(-g'\circ\nabla u)(t)$, $h(x)=T$, $h_0=T\alpha_0$ and $h_0^{-1}=\alpha_0^{-1}T^{-1}$. Thus

$$\begin{aligned}&\int_{nT}^{(n+1)T}\int_{\Omega}\left[\hat{H}^{-1}+k_1\right]\left(-g'\circ\nabla u\right)(t)\,dxdt\\ \leq\ &\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_{\Omega}\left(-g'\circ\nabla u\right)(t)\,dxdt\right].\end{aligned}\tag{3.37}$$

On the other hand, we can write

$$\begin{aligned}E((n+1)T)-E(nT) &= \frac{1}{2}\int_{nT}^{(n+1)T}\int_{\Omega}\left\{\left(g'\circ\nabla u\right)(t)-g(t)\left|\nabla u(t)\right|^{2}\right\}dxdt\\ &= -\int_{nT}^{(n+1)T}D(t)\,dt,\end{aligned}$$

where

$$D(t):=\frac{1}{2}\int_{\Omega}\left\{\left(-g'\circ\nabla u\right)(t)+g(t)\left|\nabla u(t)\right|^{2}\right\}dx.$$

Substituting (3.37) into (3.36) and using

$$E(nT)=E((n+1)T)+\int_{nT}^{(n+1)T}D(t)\,dt,\tag{3.38}$$

we get

$$\begin{aligned}\int_{nT}^{(n+1)T} E(t)\,dt &\leq C_6\left\{2E((n+1)T)+\int_{nT}^{(n+1)T} D(t)\,dt\right\}\\ &\quad +C_7\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_{\Omega}\left(-g'\circ\nabla u\right)(t)\,dxdt\right].\end{aligned}$$

In view of

$$\int_{nT}^{(n+1)T}\int_{\Omega}\left(-g'\circ\nabla u\right)(t)\,dxdt \leq 2\int_{nT}^{(n+1)T} D(t)\,dt,$$

we get

$$\begin{aligned}\int_{nT}^{(n+1)T} E(t)\,dt &\leq C_6\left\{2E((n+1)T)+\int_{nT}^{(n+1)T} D(t)\,dt\right\}\\ &\quad +2C_7\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T} D(t)\,dt\right]\\ &= 2C_6E((n+1)T)+C_6\int_{nT}^{(n+1)T} D(t)\,dt\\ &\quad +2C_7\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T} D(t)\,dt\right]\\ &\leq 2C_6E((n+1)T)+C_8\left[\hat{H}^{-1}+k_2\right]\left[\int_{nT}^{(n+1)T} D(t)\,dt\right]\\ &= 2C_6E((n+1)T)+C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right], \end{aligned} \tag{3.39}$$

where

$$\begin{cases} C_8 := \max\{2C_7,1\} > 0,\\ \tilde{H} := \left[\hat{H}^{-1}+k_2\right]^{-1},\\ k_2 := (C_6+2C_7k_1) > 0.\end{cases} \tag{3.40}$$

In particular, integrating $\frac{d}{dt}\{E(t)\} \leq 0$ from t to $(n+1)T$ yields

$$E((n+1)T) \leq E(t) \quad \text{for all } (n+1)T \geq t. \tag{3.41}$$

This implies that

$$\begin{aligned}\int_{nT}^{(n+1)T} E(t)\,dt &\geq \int_{nT}^{(n+1)T} E((n+1)T)\,dt\\ &= \int_{nT}^{(n+1)T} dt E((n+1)T)\\ &= TE((n+1)T).\end{aligned} \tag{3.42}$$

Substituting (3.42) into (3.39), we get

$$TE((n+1)T) \leq 2C_6E((n+1)T)+C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right],$$

and then

$$(T-2C_6)E((n+1)T) \leq C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right],$$

where C_6 is a positive constant. Note that

$$E\left((n+1)T\right) \leq C_9 \tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right],$$

where

$$C_9 := \frac{C_8}{(T-2C_6)} > 0. \tag{3.43}$$

This gives that

$$\tilde{H}\left(C_9^{-1}E\left((n+1)T\right)\right) \leq \int_{nT}^{(n+1)T} D(t)\,dt, \tag{3.44}$$

Substituting (3.38) into (3.44), we obtain that

$$\tilde{H}\left(C_9^{-1}E\left((n+1)T\right)\right) \leq E(nT) - E\left((n+1)T\right),$$

which implies that

$$E\left((n+1)T\right) + \tilde{H}\left(C_9^{-1}E\left((n+1)T\right)\right) \leq E(nT), \quad n = 1,2,3....$$

This completes the proof. □

Lemma 3.2. *Let p be a positive, increasing function such that $p(0) = 0$. Define an increasing function q by $q(x) \equiv x - (I+p)^{-1}(x)$. Consider a sequence F_n of positive numbers which satisfies*

$$F_{m+1} + p\left(F_{m+1}\right) \leq F_m. \tag{3.45}$$

Then $F_m \leq S(m)$, where $S(t)$ is a solution of the differential equation

$$\frac{d}{dt}\{S(t)\} + q\left(S(t)\right) = 0, \quad S(0) = F_0. \tag{3.46}$$

Moreover, if $p(x) > 0$ for $\forall x > 0$, then $\lim_{t\longrightarrow\infty} S(t) = 0$.

Proof. Assume $F_m \leq S(m)$ and prove that $F_{m+1} \leq S(m+1)$. Inequality (3.45) is equivalent to$(I+p)F_{m+1} \leq F_m$, From the facts that $(I+p)^{-1}$ is monotone increasing, $F_{m+1} \leq (I+p)^{-1}F_m$, and $(I+p)^{-1}F_m = (I-q)F_m$, we get

$$F_{m+1} \leq (I-q)F_m = F_m - q(F_m). \tag{3.47}$$

On the other hand, since q is an increasing function, the solution $S(t)$ of equation (3.46) is described by a nonlinear contraction. In particular, integrating $\frac{d}{dt}\{S(t)\} \leq 0$ from m to τ yields$S(\tau) \leq S(m)$ for all $t \geq \tau$. Integrating equation (3.46) from m to $(m+1)$ yields

$$S(m+1) - S(m) + \int_{m}^{m+1} q\left(S(\tau)\right) d\tau = 0.$$

Since q is increasing, we obtain for all $m \leq \tau \leq m+1$

$$\begin{aligned}\int_{m}^{m+1} q\left(S(\tau)\right) d\tau &\leq \int_{m}^{m+1} q\left(S(m)\right) d\tau \\ &= q\left(S(m)\right)\int_{m}^{m+1} d\tau \\ &= q\left(S(m)\right).\end{aligned}$$

It follows that $-\int_m^{m+1} q(S(\tau))\,d\tau \geq -q(S(m))$, for all $m \leq \tau \leq m+1$. Using the inductive assumption $F_m \leq S(m)$, we get $S(m+1) \geq S(m) - q(S(m)) = (I-q)S(m) \geq (I-q)F_m = F_m - q(F(m))$. This further yields $S(m+1) \geq F_{m+1}$. This completes the proof. □

Theorem 3.3. *Assume that* $(\mathbf{A1})-(\mathbf{A4})$ *ar the place. Then there exist positive constants* c_1, c_2 *and* T_0 *such that the solution of problem (1.1)-(1.3) satisfies* $E(t) \leq s(t)$, *where* $s(t)$ *verifies the ODE*

$$s_t + \hat{H}(s) = 0,\ s(0) = E(0),\ t \geq T_0 > 0,$$

with $\hat{H}(s) = c_1 H(c_2 s)$.

Proof. Letting $F_m \equiv E(mt)$ and $F_0 \equiv E(0)$, we have $E(mT) \leq S(m)$, $m = 0,1,2,3....$ Setting $t = mT + \tau$ and recalling the evolution property gives

$$E(t) \leq E(mT) \leq S(m) \leq S\left(\frac{t-\tau}{T}\right) \leq S\left(\frac{t}{T} - 1\right).$$

This completes the proof. □

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