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A CLASS OF PLANAR POLYNOMIAL VECTOR FIELD WITH EXPLICIT NON-ALGEBRAIC LIMIT CYCLES

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Abstract. With the help of the Bernoulli equation, we establish a new class of planar polynomial vector field of the form:

$$x' = -y(x^2+y^2)^l + xR_{2l}(x,y) + xS_m(x,\ y),$$
$$y' = x(x^2+y^2)^l + yR_{2l}(x,\ y) + yS_m(x,\ y),$$

where R_{2l}, S_m are homogeneous polynomials of degrees $2l$ and m, respectively with $l < m$. We prove that this class of differential systems has at most one explicit limit cycle. We obtain our result by giving a general class of differential systems of degree five with explicit non algebraic limit cycles.

Keywords. Non-algebraic limit cycle; Polynomial vector field; Stability.

1. Introduction and the statement of the main results

In the qualitative theory of planar polynomial differential systems, the most important and difficult problem is the determination of its limit cycle.

We recall that a limit cycle of a differential system is an isolated periodic solution in the set of all periodic solutions of this system. If this isolated periodic solution is contained in an algebraic curve of the plane, we say that it is algebraic, otherwise, it is called non-algebraic. In general, the orbits of polynomial differential systems are contained in analytic curves, which are not algebraic.

To distinguish the algebricity from the non algebricity of a limit cycle, usually, it is not easy. Thus, Odani [1] in 1995 proved that the limit cycle of the van der Pol differential system exhibited in 1926, (see [2]) is non-algebraic but its limit cycle is not known explicitly.

Recently, several examples of polynomial differential systems with explicit non-algebraic limit cycle were given. Gasull, Giacomini and Torregrosa in [3] gave the first example of

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polynomial differential system of degree 5 with explicit non-algebraic limit cycles. Later, Al-Dosary [4] gave a similar polynomial differential system of degree 5 exhibiting an explicit non-algebraic limit cycle. In [5], the authors gave a polynomial differential system of degree 9. Simultaneously, two explicit limit cycles, one is algebraic and the other one is non-algebraic. In 2012, Benterki and Llibre [6] gave an example of polynomial differential systems of degree 3 with explicit non-algebraic limit cycles.

The main goal of this paper is to present a new class of polynomial differential systems and to prove that this class has an explicit non–algebraic limit cycles.

We consider our new class defined as follows

$$\begin{aligned} x' &= -y(x^2+y^2)^l + xR_{2l}(x,\ y) + xS_m(x,\ y), \\ y' &= x(x^2+y^2)^l + yR_{2l}(x,\ y) + yS_m(x,\ y), \end{aligned} \tag{1.1}$$

where $R_{2l}(x,\ y)$ and $S_m(x,\ y)$ are real polynomials in the variables x and y of degrees m, $2l$, respectively with $l < m$.

In order to present our results, we need to define the invariant algebraic curve of a differential system.

Definition 1.1. We consider the polynomial differential system

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1.2}$$

where the dot denotes the derivatives of the functions with respect the time t. The algebraic solution of this system is a real complex polynomial $F(x,y)$ satisfying the equation

$$P(x,\ y)\frac{\partial F(x,\ y)}{\partial x} + Q(x,\ y)\frac{\partial F(x,\ y)}{\partial y} = F(x,\ y)K(x,\ y), \tag{1.3}$$

such that ($K(x,\ y)$ is called the cofactor of $F(x,\ y)$) the degree of the cofactor is one less than the degree of the vector field.

Our first main result is the following.

Theorem 1.2. *The polynomial vector field* (1.1) *has at most one hyperbolic limit cycle, and in polar coordinates it writes as*

$$r = \left(\exp\left[(2l-m)\int_0^{\theta} f(s)ds \right] \left[u_0 + (2l-m)\int_0^{\theta} g(s)\exp\left(-(2l-m)\int_0^{s} f(u)du \right) ds \right] \right)^{\dfrac{1}{2l-m}},$$

where
$$f(\theta) = R_{2l}(\cos\theta, \sin\theta), \quad g(\theta) = S_m(\cos\theta, \sin\theta), \quad u_0 = \frac{AB}{1-A},$$
$$A = \exp\left((2l-m)\int_0^{2\pi} f(s)ds \right), \quad B = (2l-m)\int_0^{2\pi} g(s)\exp\left(-(2l-m)\int_0^{s} f(u)du \right) ds.$$

2. The Proof of Theorem 1.2

The polynomial vector field (1.1) in polar coordinates becomes

$$\begin{aligned} \dot{r} &= f(\theta) r^{2l+1} + g(\theta) r^{m+1}, \\ \dot{\theta} &= r^{2l}. \end{aligned} \tag{2.1}$$

Where

$$f(\theta) = R_{2l}(\cos\theta, \sin\theta),$$
$$g(\theta) = S_m(\cos\theta, \sin\theta).$$

Taking as independent variable the coordinate θ, this differential system writes

$$\frac{dr}{d\theta} = f(\theta) r + g(\theta) r^{m-2l+1},$$

which is a bernoulli equation. By introducing the standard change of variables $u = r^{2l-m}$, we obtain the linear differential equation

$$\frac{du}{d\theta} = (2l-m) f(\theta) u + (2l-m) g(\theta). \tag{2.2}$$

Notice that the polynomial vector field (1.1) has a periodic orbit if and only if Equation (2.2) has a strictly positive $2\pi-$ periodic solution.

The solution of (2.2) with initial condition $u(0) = u_0$ is:

$$u(\theta,\ u_0) = \left(e^{(2l-m)\int_0^\theta f(s)ds} \left(u_0 + (2l-m) \int_0^\theta g(s) e^{\left(-(2l-m)\int_0^s f(u)du\right)} ds \right) \right)^{\dfrac{1}{2l-m}}. \tag{2.3}$$

It is easy to check that the solution $u(\theta; u_0)$ of differential equation (2.2) is such that $u(\theta; u_0) = u_0$ is $A(u_0 + B) = u_0$, $u_0 = \dfrac{AB}{1-A}$.

For hyperbolicity of system (1.1), we consider the Poincaré return map $u(2\pi; u_0) = P(u_0)$.

Then $P(u_0) = \exp\left[(2l-m) \int_0^{2\pi} f(s)ds \right] \neq 1$, for all u_0. We conclude that the limit cycle is hyperbolic.

Remark 2.1. We can write equation (2.3) in the form $F(r,\ \theta) = r^{m-2l} u(\theta,\ u_0) - 1$. The function $F(r,\ \theta)$ can be algebraic or not in Cartesian coordinates depending in the concert system considered. We know that if we have form (1.2), then equation (1.3) can be extended to non-algebraic function F; and in this case the cofactor K is not necessarily a polynomial.

Independently of the algebraicity of F, its corresponding cofactor satisfying (1.3) is always the polynomial $K(x,y) = (m-2l) S_m(x,y)$.

Lemma 2.2. *The polynomial vector field* (1.1) *has* $F(x,\ y) = -(x^2+y^2)^{l+1}$ *as an algebraic solution with cofactor*

$$\begin{aligned} K(x,y) &= (2l+2) S_m + div(-y(x^2+y^2)^l + x R_{2l}, x(x^2+y^2)^l + y R_{2l}), \\ &= 2(l+1) \left[S_m(x,y) + R_{2l}(x,y) \right]. \end{aligned}$$

Remark 2.3. $F(x;\ y)$ is formed by a product of complex invariant straight line through he origin.

Proof. We have the equation (1.3), which is equivalent to

$$\begin{aligned} & P(x,y)\frac{\partial F(x,y)}{\partial x}+Q(x,y)\frac{\partial F(x,y)}{\partial y} \\ = & -2(l+1)x(x^2+y^2)^l\left[-y(x^2+y^2)^l+x(R_{2l}+S_m)\right], \\ & -2(l+1)y(x^2+y^2)^l\left[x(x^2+y^2)^l+y(R_{2l}+S_m)\right], \\ = & -(x^2+y^2)^{l+1}2(l+1)[S_m(x,y)+R_{2l}(x,y)]. \end{aligned}$$

We use the Euler's formula

$$2l.R_{2l}(x,y)=x\frac{\partial R_{2l}}{\partial x}+y\frac{\partial R_{2l}}{\partial y},$$

$$m.S_m(x,y)=x\frac{\partial S_m}{\partial x}+y\frac{\partial S_m}{\partial y}.$$

It follows that

$$\begin{aligned} & -(x^2+y^2)^{l+1}2(l+1)[S_m(x,y)+R_{2l}(x,y)] \\ & =\left[2(l+1)S_m+div(-y(x^2+y^2)^l+xR_{2l},x(x^2+y^2)^l+yR_{2l})\right]F(x,y). \end{aligned}$$

□

Now we give an integrating factor for polynomial vector field system (1.1).

Lemma 2.4. *Consider planar vector field* (1.1) *and define*

$$V(x,y)=-(r^{m-2l}u(\theta,u_0)-1)(x^2+y^2)^{l+1},$$

where $H(\theta,\ u_0)$ *is the function given in* (2.3) *and* $u_0=a_0$ *is the value for which this function is* 2π *-periodic. Then, whenever it is defined,* $1/V(x;y)$ *is an integrating factor of the system and we call* $V(x,\ y)$ *an inverse integrating factor.*

Proof. We know that if F_1 and F_2 are two solutions of system (1.2), with cofactors K_1 and K_2, respectively, then

$$div\left(\frac{(P,Q)}{F_1F_2}\right)=\frac{1}{F_1F_2}(div(P,Q)-(K_1+K_2)).$$

Take $F_1(x,y)=-(x^2+y^2)^{l+1}$ and $F_2(x,y)=r^{m-2l}H(\theta,\ u_0)-1$. Note that the associated cofactors are $2(l+1)[S_m(x,\ y)+R_{2l}(x,\ y)]$, and $K_2(x,y)=(m-2l)S_m(x,\ y)$, respectively.

On the other way, taking the vector field associated to system (1.1), we get

$$\begin{aligned} div(P,Q) &= 2(S_m+R_{2l})+x\frac{\partial R_{2l}}{\partial x}+y\frac{\partial R_{2l}}{\partial y}+x\frac{\partial S_m}{\partial x}+y\frac{\partial S_m}{\partial y}, \\ &= (m+2)S_m+2(l+1)R_{2l}, \\ &= 2(l+1)(S_m+R_{2l})+(m-2l)S_m, \\ &= K_1+K_2. \end{aligned}$$

Gathering all the above results, we get $div\left(\frac{(P,Q)}{F_1F_2}\right)=0$. This completes the proof. □

3. Applications

Consider planar polynomial vector field (1.1). We extend the case to include systems with $l=1$, $m=4$. In this case, the equation (1.1) becomes

$$\begin{aligned}\dot{x} &= -y(x^2+y^2)+x(ax^2+bxy+cy^2)+x(\alpha x^4+\beta x^2y^2+\lambda y^4),\\ \dot{y} &= x(x^2+y^2)+y(ax^2+bxy+cy^2)+y(\alpha x^4+\beta x^2y^2+\lambda y^4),\end{aligned} \tag{3.1}$$

3.1. Algebraic solutions.

Lemma 3.1. *The planar vector field (3.1) has $F_1(x,\,y)=-(x^2+y^2)^2$ as an algebraic solution with cofactor*

$$K_1(x,y)=4\left[ax^2+bxy+cy^2+\alpha x^4+\beta x^2y^2+\lambda y^4\right].$$

Proof. From Lemma 2.2, we can conclude the desired result. □

Lemma 3.2. *Consider the planar vector field (3.1), and let $F_2(x,\,y)=0$ be the invariant algebraic curve with cofactor*

$$k_2(x,y)=a_{00}+a_{20}x^2+a_{11}xy+a_{02}y^2+r(\alpha x^4+\beta x^2y^2+\lambda y^4).$$

If

$$\begin{cases} a_{00}=0,\\ \dfrac{a_{11}}{b} \text{ is integer,}\\ r-\dfrac{a_{11}}{b} \text{ is even,}\end{cases}$$

then vector field (3.1) has $F_2(x,\,y)=0$ as a second algebraic curve.

Proof. We know that the algebraic solution given by Lemma 2.2 is $F_1(x,y)=-(x^2+y^2)^2$ with cofactor $K_1(x,y)=4\left[ax^2+bxy+cy^2+\alpha x^4+\beta x^2y^2+\lambda y^4\right]$. This curve coincides with the two complex lines $y=\pm ix$. In [6], the first step ($k=0$) of the method applied to each one of the two complex lines, $y=\pm ix$ will give enough restrictions to prove that the two algebraics solutions of system (3.1) are the ones described in Lemma 2.2 and Lemma 3.1. Suppose that the differential system has an other real or complex algebraic solution F_2 and that it does not contain any of the given two lines as a factor. By using Lemma 2.6 in [3], it is not restrictive to assume that F_2 is real and that its cofactor is an even function, i.e., $K(-x;-y)=K(x;y)$. Since the degree of the vector field (3.1) is 5, we know that the degree of $K_2(x;y)$ is at most 4. By the above restrictions on $K(x;y)$ and by using also Lemma 2.5 in [3], we can write it as the real polynomial $K_2(x,y)=a_{00}+a_{20}x^2+a_{11}xy+a_{02}y^2+r(\alpha x^4+\beta x^2y^2+\lambda y^4)$, where r is the degree of the corresponding algebraic curve $F_2(x;y)=0$. We apply the first step of the method, and take $k=0$. By considering the cases $\alpha(x)=\pm ix$, we obtain

$$\begin{aligned}\int \frac{K_2(x,\alpha(x))}{P(x,\alpha(x))}dx &= \int \frac{K_2(x,\pm ix)}{P(x,\pm ix)}dx = \int \frac{K_0(x)}{P_0(x)}dx,\\ &= \int \frac{a_{00}+(a_{20}-a_{02}\pm ia_{11})x^2+r(\alpha+\lambda-\beta)x^4}{(a-c\pm ib)x^3+(\alpha+\lambda-\beta)x^5}dx,\\ &= \int \frac{a_{00}+Ax^2+rBx^4}{Dx^3+Bx^5}dx,\end{aligned}$$

such that

$A=a_{20}-a_{02}\pm ia_{11}$, $B=\alpha+\lambda-\beta$, $D=a-c\pm ib$. Then

$$\int \frac{K_2(x,\alpha(x))}{P(x,\alpha(x))}dx=\int \left[\begin{array}{c}\frac{a_{00}}{D}\frac{1}{x^3}+\left(r+\frac{B}{D^2}\left(a_{00}-\frac{A}{B}D\right)\right)\frac{Bx}{Bx^2+D}-\\ \frac{B}{D^2}\left(a_{00}-\frac{A}{B}D\right)\frac{1}{x}\end{array}\right]dx$$

$$=\frac{1}{2}\left(\begin{array}{c} r+\frac{(\alpha+\lambda-\beta)a_{00}}{(a-c\pm ib)^2}-\\ \frac{(a_{20}-a_{02})(a-c)+a_{11}b\pm i(a_{11}(a-c)-b(a_{20}-a_{02}))}{(a-c)^2+b^2.}\end{array}\right)\ln|Bx^2+D|$$

$$-\left(\frac{a_{00}B}{D^2}-\frac{(a_{20}-a_{02})(a-c)+a_{11}b\pm i(a_{11}(a-c)-b(a_{20}-a_{02}))}{(a-c)^2+b^2.}\right)\ln|x|-$$

$$\frac{a_{00}}{2D}\frac{1}{x^2}.$$

To forcing $F(x,\alpha(x))=F(x,\pm ix)=C_0\exp\left(\int \frac{K_0(x)}{P_0(x)}dx\right)$ to be a polynomial, with C_0, an arbitrary constant, we obtain the necessary conditions $a_{00}=0$, $\frac{a_{11}}{b}$ is integer *and* $r-\frac{a_{11}}{b}$ is even. □

Lemma 3.3. *If* $(a+c)>0$ *and*

$$\int_0^{2\pi}\left((\alpha+\lambda-\beta)\cos^4 s+(\beta-2\lambda)\cos^2 s+\lambda).e^{((a+c)s-\frac{(c-a)\sin 2s}{2}+b\sin^2 s)}\right)ds>0,$$

then vector field (3.1) *has exactly one limit cycle. This limit cycle is non-algebraic.*

Proof. Vector field (3.1) can be written in polar coordinates $(r,\ \theta)$ defined by $x=r\cos\theta$, $y=r\sin\theta$ as

$$\dot r=(a\cos^2\theta+c\sin^2\theta+b\cos\theta\sin\theta)r^3+(\alpha\cos^4\theta+\lambda\sin^4\theta+\beta\cos^2\theta\sin^2\theta)r^5,$$

$$\dot\theta=r^2.$$

This system can be written as follows

$$\frac{dr}{d\theta}=(a\cos^2\theta+c\sin^2\theta+b\cos\theta\sin\theta)r+(\alpha\cos^4\theta+\lambda\sin^4\theta+\beta\cos^2\theta\sin^2\theta)r^3, \quad (3.2)$$

which is a Bernouli equation. By introducing the change of variables $u = \frac{1}{r^2} \rightarrow \frac{dr}{d\theta} = \frac{-du}{2u\sqrt{u}d\theta}$, we obtain the linear differential equation
$\frac{du}{d\theta} = -2(a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta)u - 2(\alpha\cos^4\theta + \lambda\sin^4\theta + \beta\cos^2\theta\sin^2\theta)$. Then

$$\frac{du}{d\theta} = f(\theta)u + g(\theta),$$

where

$$\begin{aligned} f(\theta) &= -2(a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta), \\ g(\theta) &= -2(\alpha\cos^4\theta + \lambda\sin^4\theta + \beta\cos^2\theta\sin^2\theta). \end{aligned}$$

We now that the vector field (3.1) has a periodic solution if and only if (3.2) has a strictly positive $2\pi-$periodic solution. The solution satisfying the initial condition $u(0) = u_0$ is given by:

$$\begin{aligned} u(\theta, u_0) &= \left[\exp\left(\int_0^\theta f(s)ds\right)\right]\left[u_0 + \int_0^\theta \left(g(s)\exp\left(-\int_0^s f(t)dt\right)\right)ds\right], \\ &= e^{\left(-(a+c)\theta + \frac{(c-a)\sin 2\theta}{2} - b\sin^2\theta\right)}\left[u_0 + \int_0^\theta \left(g(s)e^{\left((a+c)s - \frac{(c-a)\sin 2s}{2} + b\sin^2 s\right)}\right)ds\right], \end{aligned}$$

Such that $g(s) = -2((\alpha+\lambda-\beta)\cos^4 s + (\beta - 2\lambda)\cos^2 s + \lambda)$.

The condition of the periodic solution of period 2π starting at $u = u_0 > 0$ is given by the equation $u(2\pi) = u(0)$, which implies

$$\begin{aligned} u_0 = \; & \frac{e^{-2(a+c)\pi}}{1 - e^{-2(a+c)\pi}}\int_0^{2\pi} -2\Big(((\alpha+\lambda-\beta)\cos^4 s + (\beta-2\lambda)\cos^2 s \\ & +\lambda)e^{\left((a+c)s - \frac{(c-a)\sin 2s}{2} + b\sin^2 s\right)}\Big)ds. \end{aligned}$$

Recall that $u_0 = r_0^2$, where $r_0 > 0$ is the intersection of the periodic solution with the positive x-axis. So, the existence of such r_0 and consequently the existence of periodic solution need u_0 to be strictly positive and this is already assumed in the hypothesis of Lemma 3.3. We have

$$\int_0^{2\pi} f(s)ds = -2(a+c)\pi < 0.$$

Then

$$\frac{e^{-2(a+c)\pi}}{1 - e^{-2(a+c)\pi}} > 0.$$

On the other hand, we have from the hypothesis of the lemma that

$$\int_0^{2\pi} -2\left((\alpha+\lambda-\beta)\cos^4 s + (\beta-2\lambda)\cos^2 s + \lambda).e^{\left((a+c)s - \frac{(c-a)\sin 2s}{2} + b\sin^2 s\right)}\right)ds > 0,$$

which concludes that $u_0 > 0$. Then the $2\pi-$ periodic solution of equation (3.1) does exist.

In order to prove that this periodic solution is an isolated periodic orbit, it is enough to show that the poincaré return map $u(2\pi, u_0)$ has

$$0 \neq \frac{du(2\pi, u_0)}{du_0} = e^{-2(a+c)\pi} < 1,$$

for all u_0.

This limit cycle is no-algebraic, and if the assumptions of Lemma 3.1 are satisfied, then the system admits only two algebraic solutions. This completes the proof. □

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