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SOME FIXED POINT RESULTS IN (p,q)-METRIC SPACES

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Abstract. In this paper, we introduce a (p,q)-metric space, which is a generalization of the ordinary metric and S-metric spaces. We equip them with a Hausdorff topology and give some of their fundamental properties. Some of well known fixed point theory results are extended to these new spaces.

Keywords. Contractive mapping; Fixed point; (p,q)-metric space.

1. Introduction

Since the publication of the Banach's contraction principle in 1922 [1], the fixed point theory in metric spaces has received much attention due to its applications in many areas of pure, applied mathematics and applied sciences. In 2012, Sedghi, Shobe and Aliouche [2] introduced the concept of S-metric spaces as a new environment for the extension of this theory. Indeed, these new spaces are attracting and continue to attract the interest of authors. This allows to establish several efficient and important S -metric versions of classical results like: Banach's contraction principle in its different forms [2, 3], coincidence theorem [4], coupled fixed point theorem [5] and other results in topology and fixed point theory [6, 7, 8, 9]. In 2016 Abdellaoui and Dahmani [10] generalized the S -metric space to the S^* -metric space and gave the corresponding versions of all results obtained in [2]. In this paper, we are interesting in the extension of several fixed point results to new "generalized metric spaces". These new structures (see definitions below) are called (p,q)-metric spaces. They coincide with S -metric spaces if $(p,q)=(2,1)$ and more generally with S^* -metric spaces if $q=1$. The present paper is organized as follows. We start by defining (p,q)-metric spaces, giving some of their fundamental properties and equipping them with a Hausdorff topology. Under some complementary conditions on the (p,q)-metric S, we investigate the continuity, sequential continuity, compactness, sequential compactness and some convergence results in such spaces. The second section is devoted to the (p,q) -metric versions of the Banach's contraction principle and some of its

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derivatives. In the third and last section, we investigate the existence and the uniqueness of fixed points for a certain class of mappings called $\alpha-\psi$-contractive mappings [8]. In particular, if we set successively $p=q=1$ and $p\geq 2, q=1$. We retrieve well known results and formulas for an ordinary, S and S^* -metrics.

2. On the topology of (p,q)-metric spaces

Definition 2.1. Let X be a nonempty set, p and q -two non-zero positive integers. X is said to be a (p,q) -metric space if there exists a mapping $S: X^{p+q} \longrightarrow [0,+\infty[$, which called a (p,q) -metric on X such that, for all $(x_1,x_2,...,x_{p+q}) \in X^{p+q}$ and $(a_1,a_2,...,a_q) \in X^q$,

(s1) $S(x_1,x_2,...,x_{p+q})=0$ if and only if, $x_1=x_2=...=x_{p+q}$;

(s2) $S(x_1,x_2,...,x_{p+q}) \leq \sum_{i=1}^{q}\sum_{j=1}^{p+q} S([x_j]_p,[a_i]_q)$

where, $[x_j]_p = (\underbrace{x_j,...,x_j}_{p})$, $[a_i]_q = (\underbrace{a_i,...,a_i}_{q})$.

Example 2.2. The cases $(p,q)=(2,1)$ and $p\geq 2, q=1$ have been studied and well exemplified in [2, 10]. For $p+q=4$, one can construct (p,q) -metrics as follows. Let (X,d) be a metric space. Then mappings

$$S_1(x,y,z,t) = d(x,t)+d(y,t)+d(z,t); \tag{2.1}$$

$$S_2(x,y,z,t) = d(x,y)+d(x,z)+d(z,t); \tag{2.2}$$

$$S_3(x,y,z,t) = d(x,y)+d(x,z)+d(x,t); \tag{2.3}$$

define respectively $(3,1)$, $(2,2)$ and $(1,3)$ -metrics on X.

Indeed, condition (s1) is obvious for both three mappings. Let us for example prove the condition (s2) for the mapping S2. So, let x,y,z,t,a,b be arbitrary elements of X. Then,

$$\begin{aligned}
S_2(x,y,z,t) &\leq d(x,a)+d(a,y)+d(x,b)+d(b,z)+d(z,a)+d(a,t)\\
&= S_2(x,x,a,a)+S_2(x,x,b,b)+S_2(y,y,a,a)+S_2(z,z,a,a)\\
&\quad +S_2(z,z,b,b)+S_2(t,t,a,a)\\
&\leq S_2(x,x,a,a)+S_2(y,y,a,a)+S_2(z,z,a,a)+S_2(t,t,a,a)\\
&\quad +S_2(x,x,b,b)+S_2(y,y,b,b)+S_2(z,z,b,b)+S_2(t,t,b,b).
\end{aligned}$$

Remark 2.3. In the previous examples, one can consider different ordinary metrics in X. For example, if d_1 is the discrete metric in X, then for any other metric d_2 in X, one obtains a new $(1,3)$-metric in X by setting

$$S_4(x,y,z,t) = d_1(x,y)+d_2(x,z)+d_2(x,t). \tag{2.4}$$

Example 2.4. Let X be a nonempty set, $n\geq 2$ -a positive integer and consider mapping

$$S_5(x_1,...,x_n) = \begin{cases} 0, & if\ x_1=x_2=...=x_n,\\ \\ 1, & otherwise. \end{cases} \tag{2.5}$$

Obviously, S_5 is a (p,q) -metric space for all $p \geq 1,\ q \geq 1$ satisfying the condition $p+q=n$.

Proposition 2.5. *If X is a (p,q) -metric space, then, for all $a,x,y \in X$,*

$$S\left([x]_p,[y]_q\right) \leq pqS\left([x]_p,[a]_q\right)+q^2S\left([y]_p,[a]_q\right); \tag{2.6}$$

$$q^{-2}S\left([y]_p,[x]_q\right) \leq S\left([x]_p,[y]_q\right) \leq q^2S\left([y]_p,[x]_q\right). \tag{2.7}$$

In particular, for $q=1$,

$$\left|S\left([x]_p,a\right)-S\left([y]_p,a\right)\right| \leq pS\left([y]_p,x\right). \tag{2.8}$$

Proof. (2.6) and (2.7) follow directly from the definition of the (p,q) -metric space. Moreover, it follows from (2.7) that

$$q=1 \Longrightarrow S\left([x]_p,y\right)=S\left([y]_p,x\right) \ \ \forall x,y \in X. \tag{2.9}$$

It follows that

$$S\left([x]_p,a\right)-S\left([y]_p,a\right) \leq pS\left([x]_p,y\right)+S\left([y]_p,a\right)-S\left([y]_p,a\right)=pS\left([x]_p,y\right).$$

Similarly, one has

$$S\left([y]_p,a\right)-S\left([x]_p,a\right) \leq pS\left([y]_p,x\right)+S\left([x]_p,a\right)-S\left([x]_p,a\right)=pS\left([y]_p,x\right).$$

Hence,

$$-pS\left([x]_p,y\right) \leq S\left([x]_p,a\right)-S\left([y]_p,a\right) \leq pS\left([y]_p,x\right)=pS\left([x]_p,y\right).$$

□

Remark 2.6. It follows from (2.9) that if $(p,q)=(1,1)$, then S is an ordinary metric.

Let X be a (p,q) -metric space. For $a \in X$ and $r \succ 0$, we define the open ball $B_{p,q}(a,r)$ centered at a and having radius $r \succ 0$ as follows:

$$B_{p,q}(a,r)=\left\{x \in X : S\left([x]_p,[a]_q\right) \prec r\right\}.$$

As mentioned in [2, 10], if $q=1$, then the set of all open balls forms a basis of topology in X. Indeed, it is clear that every $x \in X$ is contained on an open ball. Moreover, if $B_{p,q}(a_1,r_1)$ and $B_{p,q}(a_2,r_2)$ are two open balls with a common element a, then one obtains from (2.6) by letting $q=1$ that

$$0 \prec r \prec \min_{k=1,\,2}\left\{\frac{r_k-S([a]_p,\ a_k)}{p}\right\} \Longrightarrow B_{p,q}(a,r) \subset B_{p,q}(a_1,r_1)\bigcap B_{p,q}(a_2,r_2). \tag{2.10}$$

Using (2.6), we obtain that the analogue of (2.10) in the case that $q \succ 1$ is

$$0 \prec r \prec \min_{k=1,\,2}\left\{\frac{r_k-S([a]_p,\ [a_k]_q)}{pq}\right\} \Longrightarrow B_{p,q}(a,r) \subset B_{p,q}(a_1,r_1)\bigcap B_{p,q}(a_2,r_2). \tag{2.11}$$

Clearly, this last formula is meaningless if $S([a]_p,\ [a_1]_q) \geq q^{-4}r_1$ or $S([a]_p,\ [a_2]_q) \geq q^{-4}r_2$. This proves that in the case $q \succ 1$, one cannot automatically claim that the set of all open balls forms a basis of topology. To overcome this, we next introduce the concept of the regular (p,q)-metric.

Definition 2.7. A (p,q)-metric S on X is said to be regular if the set of all open balls constitutes a basis of the topology. In this case, the corresponding topology $\tau(S)$ is formed by all sum (finite or infinite) of open balls.

Obviously, all open balls are in $\tau(S)$. Moreover, if a and b are two distinct elements of X, then

$$0 \prec R \prec \frac{S\left([a]_p, [b]_q\right)}{2\left(q^4+pq^3\right)} \Longrightarrow B_{p,q}(a,R) \bigcap B_{p,q}(b,R) = \emptyset.$$

This implies that the topology $\tau(S)$ is Hausdorff. Note that the (p,q) S- metrics defined in the examples 2.2, 2.3, 2.4 and 2.5 are all regular. Indeed, one has

(1) $S = S_2 \Longrightarrow B_{2,2}(a,r) = B_d(a,r)$,
(2) $S = S_3 \Longrightarrow B_{1,3}(a,r) = B_d(a,\frac{r}{3})$,
(3) $S = S_4 \Longrightarrow B_{1,3}(a,r) = \begin{cases} B_d(a,\frac{r}{2}) & (r \leq 1) \\ B_d(a,\frac{r-1}{2}) & (r \succ 1) \end{cases}$,
(4) $S = S_5 \Longrightarrow B_{p,q}(a,r) = B_d(a,r)$,

where $B_d(a,r) = \{x \in X : d(x,a) \prec r\}$.

Definition 2.8. Let $\{x_n\}$ be a sequence in X. It is said to be $\tau(S)$-convergent to $x \in X$ (written $x = \tau(S).\lim_{n\longrightarrow+\infty} x_n$) if the sequence $\{x_n\}$ is eventually contained in every neighborhood V_x. That is, there exists $n_x \in \mathbb{N}$ such $x_n \in V_x$ for all $n \succ n_x$.

Definition 2.9. A sequence $\{x_n\} \subset X$ is said to be S-convergent to $x \in X$ (written $x = S.\lim_{n\longrightarrow+\infty} x_n$) if the real sequence $\{S(x_n,x)\}$ converges to 0.

Proposition 2.10. *In a (p,q)-metric space, every $\tau(S)$-convergent sequence is S-convergent. Moreover, if the (p,q)-metric S is regular, then the converse is true.*

Proof. Suppose that $\{x_n\}$ is a $\tau(S)$-convergent sequence to x. Since $B_{p,q}(x,\varepsilon)$ is a neighborhood of x for every $\varepsilon \succ 0$, we have that there exists $n_{x,\varepsilon} \in \mathbb{N}$ such that $x_n \in B_{p,q}(x,\varepsilon)$ whenever $n \succ n_{x,\varepsilon}$. This means that $S(x_n,x) \prec \varepsilon$ for all $n \succ n_{x,\varepsilon}$. Suppose now that S is regular and let $\{x_n\}$ be an S-convergent sequence to x . If V_x is any open neighborhood of x, then there exists $r_x \succ 0$ such that $x \in B_{p,q}(x,r_x) \subset V_x$. Since $\{x_n\}$ is S-convergent to x, there exists $n_x \in \mathbb{N}$ such that $x_n \in B_{p,q}(x,r_x)$ for all $n \succ n_x$. This implies that $\{x_n\}$ is eventually contained in V_x. In other words, the sequence $\{x_n\}$ is $\tau(S)$ convergent to x. □

In the following proposition, we summarize fundamental properties of sequences in (p,q) -metric spaces.

Proposition 2.11. *Let (X,S) be a (p,q) -metric space with S regular. Then,*

(p1) *any convergent sequence has only one limit;*
(p2) *any convergent sequence is S-bounded, that is, there exists an open ball which contains all elements of this sequence;*
(p3) *if the sequence $\{x_n\}$ is S-bounded, then, for every $b \in X$, the real sequence $S\left([x_n]_p, [b]_q\right)$ contains at least one convergent sub-sequence;*
(p4) *a sub-sequence of a given sequence $\{x_n\}$ converges to x if and only if every open ball centered on x contains an infinity elements of $\{x_n\}$.*

Proof. Since S is regular, we have that there is an equivalence between the S-convergence and the $\tau(S)$-convergence. Property (p1) follows from the fact that $(X,\tau(S))$ is a Hausdorff topological space. Property (p2) follows from the boundedness of every convergent real sequence.

To prove (p3), we suppose that there exists $a \in X$ and $r \succ 0$ such that $S([x_n]_p, [a]_q) \prec r$ for all $n \in \mathbb{N}$. So, if b is any element of X, then, for all $n \in \mathbb{N}$,

$$S\left([x_n]_p, [b]_q\right) \leq pqS\left([x_n]_p, [a]_q\right) + q^2 S\left([b]_p, [a]_q\right) \prec pqr + q^2 S\left([b]_p, [a]_q\right).$$

This means that, for every $b \in X$, the real sequence $\left\{S\left([x_n]_p, [b]_q\right)\right\}$ is bounded. So, it admits a convergent sub-sequence of the form $\left\{S\left([x_{n_k}]_p, [b]_q\right)\right\}$. Suppose now that the sequence $\{x_n\}$ contains a sub-sequence $\{x_{n_k}\}$, which converges to x. So, we have that $\left\{S\left([x_{n_k}]_p, [x]_q\right)\right\}$ is a convergent to 0 sub-sequence of the real positive sequence $\left\{S\left([x_n]_p, [x]_q\right)\right\}$. As it is known from the elementary real analysis that this is equivalent to that every open real interval centered on 0 contains an infinity of elements of the sequence $\left\{S\left([x_n]_p, [x]_q\right)\right\}$. This is exactly what the property (p4) means. □

It is known [2, 10] that if $\{x_n\}$ and $\{y_n\}$ are two sequence in a $(p,1)$ -metric space, which converge to x and y respectively, then

$$\lim_{n \longrightarrow +\infty} S\left([x_n]_p, y_n\right) = S\left([x]_p, y\right). \tag{2.12}$$

For the case $q \succ 1$, we have the following result

Proposition 2.12. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a (p,q) -metric space X. Assume that $\{x_n\}$ converges to x and $\{y_n\}$ converges to y. Then there are a sub-sequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sub-sequence $\{y_{n_k}\}$ of $\{y_n\}$, such that*

$$q^{-4} S\left([x]_p, [y]_q\right) \leq \lim_{n_k \longrightarrow +\infty} S\left([x_{n_k}]_p, [y_{n_k}]_q\right) \leq q^4 S\left([x]_p, [y]_q\right). \tag{2.13}$$

Proof. Let $\varepsilon \succ 0$. Since sequences $\{x_n\}$ and $\{y_n\}$ are convergent to x and y respectively, then there exists $n_\varepsilon \in \mathbb{N}$ such that

$$n \succ n_\varepsilon \Longrightarrow \begin{cases} S\left([x_n]_p, [x]_q\right) \prec \frac{\varepsilon}{2pq}, \\ \\ S\left([y_n]_p, [y]_q\right) \prec \frac{\varepsilon}{2pq^3}. \end{cases}$$

So, using 2.6, we have, for $n \succ n_\varepsilon$

$$\begin{aligned} S\left([x_n]_p, [y_n]_q\right) &\leq pqS\left([x_n]_p, [x]_q\right) + q^2 S\left([y_n]_p, [x]_q\right) \\ &\leq pqS\left([x_n]_p, [x]_q\right) + pq^3 S\left([y_n]_p, [y]_q\right) + q^4 S\left([x]_p, [y]_q\right) \\ &\prec pq.\frac{\varepsilon}{2pq} + pq^3.\frac{\varepsilon}{2pq^3} + q^4 S\left([x]_p, [y]_q\right) \\ &= q^4 S\left([x]_p, [y]_q\right) + \varepsilon. \end{aligned}$$

By the same way, we have, for $n \succ n_\varepsilon$,

$$\begin{aligned} q^{-4} S\left([x]_p, [y]_q\right) &\leq pq^{-3} S\left([x]_p, [x_n]_q\right) + q^{-2} S\left([y]_p, [x_n]_q\right) \\ &\leq pqS\left([x_n]_p, [x]_q\right) + pq^3 S\left([y_n]_p, [y]_q\right) + S\left([x_n]_p, [y_n]_q\right) \\ &\prec pq.\frac{\varepsilon}{2pq} + pq^3.\frac{\varepsilon}{2pq^3} + S\left([x_n]_p, [y_n]_q\right) \\ &= S\left([x_n]_p, [y_n]_q\right) + \varepsilon. \end{aligned}$$

Hence, we conclude that

$$q^{-4}S\left([x]_p,[y]_q\right)-\varepsilon\prec S\left([x_n]_p,[y_n]_q\right)\prec q^4S\left([x]_p,[y]_q\right)+\varepsilon \;\; \forall n\succ n_\varepsilon. \tag{2.14}$$

This implies that $\left\{a_n=S\left([x_n]_p,[y_n]_q\right)\right\}_{n\succ n_\varepsilon}$ is bounded in $\mathbb{R}$. Thus, it admits a convergent sub-sequence $\left\{a_{n_k}=S\left([x_{n_k}]_p,[y_{n_k}]_q\right)\right\}$, which satisfies the inequality:

$$q^{-4}S\left([x]_p,[y]_q\right)-\varepsilon\prec \lim_{n_k\longrightarrow+\infty}S\left([x_{n_k}]_p,[y_{n_k}]_q\right)\prec q^4S\left([x]_p,[y]_q\right)+\varepsilon. \tag{2.15}$$

Letting $\varepsilon\longrightarrow 0$ in (2.15), we obtain (2.12). □

Definition 2.13. Let (X,S) be a (p,q)-metric space with the S regular. We say that S satisfies the simultaneous convergence property if the sequence $\left\{S([x_n]_p,\ [y_n]_q)\right\}$ S-converges to $S([x]_p,\ [y]_q)$, whenever the sequences $\{x_n\}$ and $\{y_n\}$ converge to x and y respectively. It reads

$$\left\{\lim_{n\longrightarrow+\infty}x_n=x \quad\text{and}\quad \lim_{n\longrightarrow+\infty}y_n=y\right\}\Longrightarrow \lim_{n\longrightarrow+\infty}S([x_n]_p,[y_n]_q)=S([x]_p,[y]_q). \tag{2.16}$$

Remark 2.14. As noted above and proved in [2, 3, 10], every $(p,1)$-metric satisfies the simultaneous convergence property. It is obvious to check that all the S-metrics defined in examples 2.2, 2.3, 2.4 and 2.5 have the simultaneous convergence property.

Definition 2.15. Let (X,S) be a (p,q) -metric space with S regular. A sequence $\{x_n\}$ in a X is said to be of Cauchy if,

$$\forall\varepsilon\succ 0,\ \exists n_\varepsilon\in\mathbb{N}: n,m\succ n_\varepsilon\Longrightarrow S\left([x_n]_p,[x_m]_q\right)\prec\varepsilon$$

or equivalently,

$$\lim_{n,m\longrightarrow+\infty}S\left([x_n]_p,[x_m]_q\right)=0.$$

Note that every convergent sequence is a Cauchy one and every Cauchy sequence is S-bounded.

Definition 2.16. A (p,q) -metric space (X,S) with regular S is said to be complete if every Cauchy sequence in X converges to an element of X.

Remark 2.17. The (p,q) -metric spaces defined in Examples 2.1, 2.2 and 2.3 are S-complete if and only if the initial metric space (X,d) is complete.

Definition 2.18. Let (X_i,S_i), $i=1,2$ be two (p_i,q_i)-metric spaces with S_i regular. A self-mapping T from X_1 into X_2 is said to have a limit $b\in X_2$ at the point $a\in X_1$ (written: $\lim_{x\longrightarrow a}f(x)=b$) if

$$\forall\varepsilon\succ 0,\ \exists\delta=\delta(a,\varepsilon)\succ 0: S_1([x]_p,[a]_q)\prec\delta\Longrightarrow S_2([Tx]_p,[b]_q)\prec\varepsilon.$$

If $b=Ta$, we say that T is continuous at a.

As in the case of the ordinary metric, we have an equivalence between the continuity and the sequential continuity.

Proposition 2.19. *Let (X,S) be a (p,q)-metric space with S regular. A self-mapping T on X is continuous at $x\in X$ if and only if, for every sequence $\{x_n\}$ converging to x, the sequence $\{Tx_n\}$ converges to Tx.*

Proof. The proof is a simple adaption of the proof in the ordinary metric case. Indeed, we suppose that T is continuous at $a \in X_1$ and let $\{x_n\}$ be a convergent to a sequence. Then,

$$\forall \varepsilon \succ 0,\ \exists \delta = \delta(a,\varepsilon) \succ 0 : S_1([x]_p,[a]_q) \prec \delta \Longrightarrow S_2([Tx]_p,[Ta]_q) \prec \varepsilon.$$

and

$$\exists N_\delta \in \mathbb{N} : n \succ N_\delta \Longrightarrow S_1([x_n]_p,\ [a]_q) \prec \delta.$$

Thus,

$$\forall \varepsilon \succ 0,\ \exists N_\varepsilon = N_\delta : n \succ N_\varepsilon \Longrightarrow S_2([Tx_n]_p,\ [Ta]_q) \prec \varepsilon.$$

In other words, the sequence $\{Tx_n\}$ converges to Ta. Conversely, suppose that T is sequentially continuous but not continuous at $a \in X$. In this case, there exists $\varepsilon \succ 0$ and a sequence $\{x_n\} \subset X$ such that

$$S_1([x_n]_p,[a]_q) \prec \frac{1}{n+1} \quad \text{and} \quad S_2([Tx_n]_p,\ [Ta]_q) \succ \varepsilon, \quad \forall n \in \mathbb{N}.$$

Thus, we have two contradictory statements: the sequence $\{x_n\}$ converges to a but the sequence $\{Tx_n\}$ does not converge to Ta. □

We end this section with the two following useful results on the compactness in (p,q)-metric spaces and its link with the completeness.

Theorem 2.20. *A compact (p,q) -metric space (X, S) with S regular is sequentially compact. That is, every sequence in X contains a convergent sub-sequence.*

Proof. Without lose of the generality, it can be assumed that the set $Y = \{x_n : n \in \mathbb{N}\}$ is infinite (It means that the sequence takes an infinity of different values in X). Suppose now that X is compact but non sequentially compact. In this case, there exists a sequence $\{x_n\}$ in X, which does not contain any convergent sub-sequence. According to the point (p4) of the proposition 2.11,

$$\forall x \in X,\ \exists r_x \succ 0 : Card\left(B_{p,q}(x,r_x)\bigcap\{x_n, n \in \mathbb{N}\}\right) \prec +\infty.$$

Since X is compact, there is a finite number $a_1,...,a_n$ of elements of X such that

$$X = \bigcup_{k=1}^{n} B_{p,q}(a_k, r_{a_k}).$$

By construction each of the open balls $B_{p,q}(a_k, r_{a_k})$ and their union X contains only a finite number of elements of Y. In other words, we have the following meaningless conclusion: there are elements of the sequence $\{x_n\}$ wich are not in X. Thus, compactness implies sequential compactness. □

Corollary 2.21. *A compact (p,q) -metric space (X,S) with S regular is complete.*

Proof. We have to prove that every Cauchy sequence in X converges. For this, let $\{x_n\}$ be a Cauchy sequence in the compact (p,q) -metric space X. Thus,

$$\forall \varepsilon \succ 0,\ \exists n_1(\varepsilon) \in \mathbb{N} : m \succ n_1(\varepsilon),\ n \succ n_1(\varepsilon) \Longrightarrow S\left([x_n]_p,[x_m]_q\right) \prec \frac{\varepsilon}{2pq}.$$

According to the previous theorem, there exists a sub-sequence $\{x_{f(n)}\}$, which converges to a certain $x \in X$. So,

$$\exists\, n_2(\varepsilon) \in \mathbb{N} : n \succ n_2(\varepsilon) \Longrightarrow S\left([x_{f(n)}]_p, [x]_q\right) \prec \frac{\varepsilon}{2q^4}.$$

Therefore, for $n \succ n(\varepsilon) = \max(n_1(\varepsilon), n_2(\varepsilon))$, one has

$$\begin{aligned} S\left([x_n]_p, [x]_q\right) &\leq pqS\left([x_n]_p, [x_{f(n)}]_q\right) + q^4 S\left([x_{f(n)}]_p, [x]_q\right) \\ &\prec pq\frac{\varepsilon}{2pq} + q^4\frac{\varepsilon}{2q^4} = \varepsilon. \end{aligned}$$

Thus, the Cauchy sequence $\{x_n\}$ converges to x. □

3. Banach's contraction principle in a (p,q)-metric space

In the following, all (p,q)-metrics S will be supposed to be regular. As proved above, this implies that there is an equivalence between the $\tau(S)$ convergence and the S convergence. There is also an equivalence between the continuity and the sequential continuity.

Definition 3.1. Let (X,S) be a (p,q) -metric space. A self-mapping T on X is called a k -contraction (or k-self-contraction) if there exists $0 \prec k \prec 1$ such that

$$S\left([Tx]_p, [Ty]_q\right) \leq kS\left([x]_p, [y]_q\right) \quad \forall x, y \in X.$$

It is clear that a k -contraction on X is a continuous mapping. Indeed, for every $\varepsilon \succ 0$,

$$S\left([x]_p, [y]_q\right) \prec \frac{\varepsilon}{k} \Longrightarrow S\left([Tx]_p, [Ty]_q\right) \prec \varepsilon.$$

The following result is the (p,q) -metric version of the Banach's contraction principle.

Theorem 3.2. *Let X be a complete (p,q) -metric space and T be a k self contraction on X with $0 \prec kq^4 \prec 1$. Then, equation $T(x) = x$ admits a unique solution $u \in X$. Moreover, for every $x \in X$, sequence*

$$x_0 = x,\ x_1 = T(x_0),\ \ldots,\ x_{n+1} = T(x_n),\ \ldots$$

converges to u and for every natural number n there exists an infinite and strictly increasing sequence $\{m_k\}$ of natural numbers such that

$$\lim_{m_k \longrightarrow +\infty} S\left([T^n(x)]_p, [T^{m_k}(x)]_q\right) \leq pq\frac{k^n}{1-kq^4} S\left([x]_p, [T(x)]_q\right), \tag{3.1}$$

where T^n is the n-th iterative of T.

Proof. To prove the uniqueness, let us suppose that there are two different elements $a \neq b$ such that $T(a) = a$ and $T(b) = b$. Then, we get the following contradiction

$$0 \leq S\left([a]_p, [b]_q\right) = S\left([T(a)]_p, [T(b)]_q\right) \leq kS\left([a]_p, [b]_q\right) \prec S\left([a]_p, [b]_q\right).$$

For the existence, let x be an arbitrary element of X and consider the sequence:

$$x_0 = x,\ x_1 = T(x), \ldots,\ x_{n+1} = T(x_n), \ldots\ \ n = 0, 1, 2, \ldots$$

Clearly,

$$S\left([x_n]_p,[x_{n+1}]_q\right)\leq k^n S\left([x_0]_p,[x_1]_q\right),\quad n=1,2,\ldots \tag{3.2}$$

Using (2.6) and (2.7), one can see that, for $m\succ n$,

$$\begin{aligned}
S\left([x_n]_p,[x_m]_q\right) &\leq pqS\left([x_n]_p,[x_{n+1}]_q\right)+q^4S\left([x_{n+1}]_p,[x_m]_q\right)\\
&\leq pq\left\{S\left([x_n]_p,[x_{n+1}]_q\right)+q^4S\left([x_{n+1}]_p,[x_{n+2}]_q\right)\right\}\\
&\quad +q^8S\left([x_{n+2}]_p,[x_m]_q\right)\\
&\quad \vdots\\
&\leq pq\sum_{i=n}^{m-2}q^{4(i-n)}S\left([x_i]_p,[x_{i+1}]_q\right)+q^{4(m-n-1)}S\left([x_{m-1}]_p,[x_m]_q\right)\\
&\prec pq\sum_{i=n}^{m-2}q^{4(i-n)}k^iS\left([x_0]_p,[x_1]_q\right)+pqq^{4(m-n-1)}k^{m-1}S\left([x_0]_p,[x_1]_q\right)\\
&= pqk^n\sum_{i=0}^{m-n-1}q^{4i}k^iS\left([x_0]_p,[x_1]_q\right)\\
&\leq pq\frac{k^n}{1-kq^4}S\left([x_0]_p,[x_1]_q\right).
\end{aligned}$$

Therefore,

$$m\succ n\Longrightarrow S\left([x_n]_p,[x_m]_q\right)\prec pq\frac{k^n}{1-kq^4}S\left([x_0]_p,[x_1]_q\right). \tag{3.3}$$

(3.3) shows that sequence $\{x_n\}$ is of Cauchy so that it converges to an element $u\in X$. Using the continuity of T, we get

$$u=\lim_{n\longrightarrow+\infty}x_n=\lim_{n\longrightarrow+\infty}T\left(x_{n-1}\right)=T\left(\lim_{n\longrightarrow+\infty}x_{n-1}\right)=T\left(u\right).$$

Let $n\in\mathbb{N}$ be fixed. Consider the constant sequence $\{y_m=x_n\}$ and the sequence $\{z_m=x_m\}$. Clearly,

$$\lim_{m\longrightarrow+\infty}y_m=x_n\quad\text{and}\quad\lim_{m\longrightarrow+\infty}z_m=u.$$

As seen in the proof of Proposition 2.12, there exists a sub-sequence $\{z_{m_k}=x_{m_k}\}$ such that $\lim_{m_k\longrightarrow+\infty}S\left([x_n],x_{m_k}\right)$ exists. Obviously, this limit satisfies (3.3). In other words,

$$\lim_{m_k\longrightarrow+\infty}S\left([T^n(x)]_p,[T^{m_k}(x)]_q\right)\leq pq\frac{k^n}{1-kq^4}S\left([x]_p,[T(x)]_q\right).$$

□

Corollary 3.3. *If we suppose in the theorem 3.2 that the (p,q)-metric S satisfies the simultaneous convergent property (see Definition 2.13), then equation $T(x)=x$ admits a unique solution $u\in X$. Moreover, for every $x\in X$, sequence*

$$x_0=x,\ x_1=T(x_0),\ \ldots,\ x_{n+1}=T(x_n)\ \ldots$$

converges to u and for every natural number n,

$$S\left([T^n(x)]_p, u\right) \leq p\frac{k^n}{1-k}S\left([x]_p, T(x)\right). \tag{3.4}$$

Proof. We need to prove only the relation 3.4. To obtain this formula, it suffices to remark that since $\{y_m = x_n\}$ converges to x_n and $\{z_m = x_m\}$ converges to the fixed point u. Using the simultaneous convergence property and taking into account 3.3, one has

$$S\left([T^n(x)]_p, u\right) = \lim_{m_k \longrightarrow +\infty} S\left([T^n(x)]_p, [T^{m_k}(x)]_q\right) \leq pq\frac{k^n}{1-kq^4}S\left([x]_p, [T(x)]_q\right).$$

This completes the proof. □

Theorem 3.4. *Let (X,S) be a compact (p,q) -metric space and T be a self mapping on X such that*

$$S\left([T(x)]_p, [T(y)]_q\right) \prec S\left([x]_p, [y]_q\right) \quad \forall x,y \in X, \;\; x \neq y. \tag{3.5}$$

Assume that S satisfies the simultaneous convergence property. Then, equation $T(x) = x$ has a unique solution in X.

Proof. Consider the function $x \in X \longmapsto g(x) = S([x]_p,\ [Tx]_q)$. If $\{x_n\}$ is any convergent sequence to x, we obtain from the continuity of T and the simultaneous convergence property that

$$\lim_{n\longrightarrow+\infty} g(x_n) = \lim_{n\longrightarrow+\infty} S([x_n]_p,\ [Tx_n]_q) = S([\lim_{n\longrightarrow+\infty} x_n]_p,\ [\lim_{n\longrightarrow+\infty} Tx_n]_q) = S([x]_p,\ [Tx]_q).$$

Thus, g is a positive continuous function from the compact X into $[0,\ +\infty[$ regarded as a $(1,1)$-metric space with $S = |.|$. So, there exists $x_0 \in X$ such that $g(x_0) = \min_{x\in X} g(x)$. We claim that x_0 is a fixed point for T. Indeed, the contrary leads to the following contradiction:

$$g(x_0) \leq g(T(x_0)) = S\left([T(x_0)]_p, [T(T(x_0))]_q\right) \prec S(x_0, T(x_0)) = g(x_0).$$

The proof of the uniqueness of the fixed point is similar to Theorem 3.2. □

Remark 3.5. Compactness of X in the previous theorem is essential. Indeed, let $X = [1, +\infty[$ and $S(x,y,z,t) = |x-y| + |x-z| + |z-t|$. (X,S) is a complete but not compact $(2,2)$ -metric space (for example, sequence $x_n = \{n\}$ does not contain any convergent in X sub-sequence). Moreover, function $T(x) = e^{-x}$ is an $\frac{1}{e}$ -contraction in X:

$$S(Tx,Tx,Ty,Ty) = |Tx - Ty| \prec \frac{1}{e}|x-y| = \frac{1}{e}S(x,x,y,y).$$

However, there is no fixed point since $T(x) \prec x$, $\forall x \in [1, +\infty[$.

The local version of theorem 3.2 is as follows.

Theorem 3.6. *Let (X,S) be a compact (p,q)-metric space with S satisfies the simultaneous convergence property, T a self mapping on X and b an arbitrary element of X. Assume that there exist $r \succ 0$ and a constant $0 \prec k \prec q^{-4}$ such that T defines a k -contraction from the closure $\overline{B_{p,q}(b,r)}$ of $B_{p,q}(b,r)$ into X and*

$$S\left([b]_p, [T(b)]_q\right) \prec (1-kq^4)\frac{r}{pq^3}.$$

Then, equation $T(x)=x$ has a unique solution in $\overline{B_{p,q}(b,r)}$.

Proof. Note that $(\overline{B_{p,q}(b,r)},S)$ is a complete (p,q)-metric space (as a closed subspace of the complete space (X,S)). Moreover, there exists r_0 such that $0\prec r_0\prec r$ and

$$S\left([b]_p,[T(b)]_q\right)\prec(1-kq^4)\frac{r_0}{pq^3}.$$

Obviously, $(\overline{B_{p,q}(b,r_0)}\subseteq\overline{B_{p,q}(b,r)}$. Moreover, $(\overline{B_{p,q}(b,r_0)},S)$ is a complete (p,q)-metric space. If $x\in\overline{B_{p,q}(b,r_0)}$ then, $S([x]_p,[b]_q)\leq r_0$. Indeed, as in the ordinary metric case, it is easy to check that

$$x\in\overline{B_{p,q}(b,r_0)}\Longrightarrow x=\lim_{n\longrightarrow+\infty}x_n\ \text{ with }\ S([x_n]_p,[b]_q)\prec r_0,\ \ \forall n.$$

So, using the simultaneous convergence property, we have

$$S([x]_p,[b]_q)=\lim_{n\longrightarrow+\infty}S([x_n]_p,[b]_q)\leq r_0.$$

Let us now prove that $T\left(\overline{B_{p,q}(b,r_0)}\right)\subseteq\overline{B_{p,q}(b,r_0)}$. Indeed, for any $x\in\overline{B_{p,q}(b,r_0)}$, one has

$$\begin{aligned}S\left([T(x)]_p,[b]_q\right)&\leq q^2S\left([b]_p,[T(x)]_q\right)\leq pq^3S\left([b]_p,[T(b)]_q\right)+q^4S\left([T(x)]_p,[T(b)]_q\right)\\&\leq pq^3S\left([b]_p,[T(b)]_q\right)+kq^4S\left([x]_p,[b]_q\right).\\&\prec pq^3(1-kq^4)\frac{r_0}{pq^3}+kq^4r_0=r_0.\end{aligned}$$

Thus, all conditions of Theorem 3.2 are satisfied by T regarded as a mapping on $\overline{B_{p,q}(b,r_0)}$. Consequently, equation $T(x)=x$ admits a unique solution u in $\overline{B_{p,q}(b,r_0)}$. We claim that u is the unique solution of this equation in $\overline{B_{p,q}(b,r)}$. Indeed, if v is another solution belonging to $\overline{B_{p,q}(b,r)}$, then

$$S\left([u]_p,[v]_q\right)=S\left([T(u)]_p,[T(v)]_q\right)\leq kS\left([u]_p,[v]_q\right)\prec S\left([u]_p,[v]_q\right),$$

which is a contradiction. □

4. α-ψ-CONTRACTIVE TYPE MAPPINGS

In this section, we seek both the existence and the uniqueness of fixed points for a certain class of mappings in a (p,q) -metric space. Mappings under study are not contractive, however, they satisfy a certain contractive condition, which is explained in Definition 4.4. The method used, in the following details, was improved after being impressed by [8]. As in the previous section, all considered (p,q)-metrics are supposed regular.

Definition 4.1. For every integer $q\succ 0$, denote by Ψ_q the class of non-decreasing functions $\psi:[0,+[\longrightarrow[0,+[$ such that $\sum_{n=1}^{+\infty}q^{4n}\psi^n(t)\prec+\infty$ for each $t\succ 0$, where ψ^n is the $n-th$ iterate of ψ.

Lemma 4.2. [11] *For every function $\psi:[0,+[\longrightarrow[0,+[$ the following holds: If ψ is non decreasing, then $\lim_{n\longrightarrow+\infty}\psi^n(t)=0,\ \ \forall t\succ 0\Longrightarrow\psi(t)\prec t,\forall t\succ 0$.*

Corollary 4.3. *If $\psi\in\Psi_q$, then ψ is continuous at 0 and $\psi(0)=0$.*

Proof. Note that

$$\psi \in \Psi_q \Longrightarrow \sum_{n=1}^{+\infty} q^{4n}\psi^n(t) \prec +\infty \Longrightarrow \lim_{n\longrightarrow+\infty} q^{4n}\psi^n(t)=0 \ \ \forall t\geq 0.$$

It follows from Lemma 4.2 that $\psi(t)\prec q^{-4}t,\ \ \forall t\succ 0$. So,

$$0\leq \lim_{t\longrightarrow 0^+}\psi(t)\leq \lim_{t\longrightarrow 0^+} q^{-4}t=0.$$

Suppose now that $\psi(0)=a\succ 0$. Since ψ is non decreasing, we have the contradiction:

$$a=\psi(0)\leq\psi(a)\prec q^{-4}a\prec a.$$

Finally, we have proved that $\psi(0)=0$ and $\lim_{t\longrightarrow 0^+}\psi(t)=0$. This is exactly the desired result. □

Definition 4.4. Let X be a (p,q)-metric space. A self-mapping $T:X\longrightarrow X$ is said to be $\alpha-\psi$-contractive if there exists $\alpha : X^{p+q}\longrightarrow [0,+[$ and $\psi\in\Psi_q$ such that

$$\alpha([x]_p,[y]_q)S([Tx]_p,[Ty]_q)\leq \psi(S([x]_p,[y]_q))\ \ \forall x,y\in X.$$

Example 4.5. Let (X,S) a (p,q)-metric space defined in Example 2.5, that is,

$$S\left(x_1,...x_{p+q}\right)=\begin{cases} 0, & if\ x_1=...=x_{p+q},\\ 1, & otherwise.\end{cases}$$

Set $\alpha\left(x_1,...x_{p+q}\right)=\frac{S\left(x_1,...x_{p+q}\right)}{2q^4},\quad \psi(t)=\frac{t}{2q^4},\ \ t\geq 0$. Obviously,

$$\sum_{k=1}^{+\infty} q^{4n}\psi^n(t)=\sum_{k=1}^{+\infty} q^{4n}\frac{t}{2^n q^{4n}}=\sum_{k=1}^{+\infty}\frac{t}{2^n}=t,\ \ \forall t\geq 0.$$

Thus, $\psi\in\Psi_q$. Moreover, if T is a self mapping on X, then, for all x,y in X,

$$\alpha([x]_p,[y]_q)S([Tx]_p,[Ty]_q)=\frac{S([x]_p,[y]_q)}{2q^4}S([Tx]_p,[Ty]_q)\leq\frac{S([x]_p,[y]_q)}{2q^4}\leq\psi(S([x]_p,[y]_q)).$$

So, every self-mapping T on X is α-ψ-contractive.

Example 4.6. Let (X,S) be a (p,q) -metric space. Define functions $\psi(t)=kt^2,\ \ 0\leq t\prec 1$, where k is a constant satisfying the condition $kq^4\prec 1$ and $\alpha(x_1,...x_{p+q})=\ln(1+S(x_1,...x_{p+q}))$. Then, every k-self-contraction on X is α-ψ-contractive. Indeed, $kq^4\prec 1$ implies $\psi\in\Psi_q$. Moreover, if T is a k self-contraction on X, then, for all x,y in X,

$$\begin{aligned}\alpha([x]_p,[y]_q)S([Tx]_p,[Ty]_q) &= \ln(1+S([x]_p,[y]_q))S([Tx]_p,[Ty]_q)\leq S([x]_p,[y]_q)S([Tx]_p,[Ty]_q)\\ &\leq k[S([x]_p,[y]_q)]^2=\psi(S([x]_p,[y]_q)).\end{aligned}$$

Definition 4.7. A self mapping $T:X\longrightarrow X$ is said to be α-admissible if

$$\alpha(x_1,...,x_{p+q})\geq 1\Longrightarrow\alpha(Tx_1,...,Tx_{p+q})\geq 1,\ \ \forall x_1,...,x_{p+q}\in X.$$

Remark 4.8. The self-contraction T in Example 4.6 is α-admissible if and only if, for all $x_1,...,x_{p+q}\in X$,

$$S(x_1,...,x_{p+q})\geq e-1\Longrightarrow S(Tx_1,...,Tx_{p+q})\geq e-1.$$

Theorem 4.9. *Let (X,S) be a complete (p,q)-metric space and T an α-ψ-contractive self mapping on X, where $\psi \in \Psi_q$. Assume that*

(i) *T is α -admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha\left([x_0]_p, [Tx_0]_q\right) \geq 1$;*
(iii) *T is continuous.*

Then, T has a fixed point in X.

Proof. Consider the sequence

$$\{x_n = T^n x_0, \quad n \geq 1\}.$$

Since T is α -admissible and $\alpha\left([x_0]_p, [Tx_0]_q\right) \geq 1$, we have

$$\alpha\left([x_n]_p, [Tx_{n+1}]_q\right) \geq 1, \quad \forall n.$$

Moreover, it follows from the α-ψ-contractivity of T that, for all $n \geq 1$,

$$S([Tx_{n-1}]_p, [Tx_n]_q) \leq \alpha([x_{n-1}]_p, [x_n]_q) S([Tx_{n-1}]_p, [Tx_n]_q) \leq \psi(S([x_{n-1}]_p, [x_n]_q)).$$

So, by induction on $n \geq 1$, we have

$$S\left([x_n]_p, [x_{n+1}]_q\right) = S\left([Tx_{n-1}]_p, [Tx_n]_q\right) \leq \psi^n\left(S\left([x_0]_p, [Tx_0]_q\right)\right).$$

Fixing $\varepsilon \succ 0$, taking $n(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\varepsilon)} q^{4n} \psi^n\left(S\left([x_0]_p, [Tx_0]_q\right)\right) \prec \frac{\varepsilon}{pq}$$

and using (2.6), we get for all integers $m \succ n \succ n(\varepsilon)$ that

$$\begin{aligned}
S\left([x_n]_p, [x_m]_q\right) &\leq pqS\left([x_n]_p, [x_{n+1}]_q\right) + q^4 S\left([x_{n+1}]_p, [x_m]_q\right) \\
&\leq pq\left\{S\left([x_n]_p, [x_{n+1}]_q\right) + q^4 S\left([x_{n+1}]_p, [x_{n+2}]_q\right)\right\} \\
&\quad +(q^4)^2 S\left([x_{n+1}]_p, [x_m]_q\right) \\
&\ \ \vdots \\
&\leq pq \sum_{i=n}^{m-2} q^{4(i-n)} S\left([x_i]_p, [x_{i+1}]_q\right) + q^{4(m-n-1)} S\left([x_{m-1}]_p, [x_m]_q\right) \\
&\leq pq \sum_{i=n}^{m-2} q^{4i} S\left([x_i]_p, [x_{i+1}]_q\right) + pq^{4m} S\left([x_{m-1}]_p, [x_m]_q\right) \\
&= pq \sum_{i=n}^{m-1} q^{4i} S\left([x_i]_p, [x_{i+1}]_q\right) \leq pq \sum_{i=n}^{m-1} q^{4i} \psi^i\left(S\left([x_0]_p, [Tx_0]_q\right)\right) \\
&\leq pq \sum_{i \geq n(\varepsilon)} q^{4i} \psi^i\left(S\left([x_0]_p, [Tx_0]_q\right)\right) \prec pq\frac{\varepsilon}{pq} = \varepsilon.
\end{aligned}$$

Therefore, $\{x_n = T^n x_0\}$ is a Cauchy sequence in the complete (p,q) -metric space X. Thus, it converges to a certain element $u \in X$. Using the continuity of the mapping T, we have

$$u = \lim_{n \longrightarrow +\infty} x_{n+1} = \lim_{n \longrightarrow +\infty} Tx_n = T\left(\lim_{n \longrightarrow +\infty} x_n\right) = Tu.$$

Finally, u is a fixed point for the mapping T. □

Corollary 4.10. *Let T be a continuous self mapping in a complete (p,q)-metric space X. Assume that T is α admissible and $\alpha([x_0]_p,[Tx_0]_q) \geq 1$ for a certain $x_0 \in X$. Suppose also that there exists a constant $L \in \left[0, q^{-4}\right[$ such that, for all x and y in X,*

$$\alpha\left([x]_p,[y]_q\right) S\left([Tx]_p,[Ty]_q\right) \leq LS\left([x]_p,[Ty]_q\right). \tag{4.1}$$

Then, T has a fixed point in X.

Proof. This is a direct application of the previous theorem with $\psi(t) = Lt$, $\forall t \geq 0$. ☐

Example 4.11. Let $X = [0,+\infty[$ and $S(x,y,z,t) = |x-y|+|x-z|+|z-t|$ for all $x,y,z,t \in X$. As mentioned above equipped with S, X becomes a $(2,2)$-complete metric space (see Example 2.2). Now, define a mapping $T : X \longrightarrow X$ by $Tx = \frac{x+1}{3.2^4}$ and a mapping $\alpha : X^4 \longrightarrow [0,+\infty[$ by

$$\alpha(x,y,z,t) = \begin{cases} e^{\max\{y,\ t\}-\max\{x,\ z\}} & if \quad \max\{y,t\} \geq \max\{x,z\} \\ 0 \quad otherwise \end{cases}$$

It is not difficult to check that mapping T is α-admissible. Moreover, for all $x,y \in X$, $\alpha(x,x,y,y) = 1$ and

$$\begin{aligned} \alpha(x,x,y,y)\,S(Tx,Tx,Ty,Ty) &= S(Tx,Tx,Ty,Ty) = |Tx-Ty| \\ &= \frac{1}{3.2^4}|x-y| = \frac{1}{3.2^4}S(x,x,y,y) = \psi(S(x,x,y,y)), \end{aligned}$$

where $\psi(t) = \frac{1}{3.2^4}t$, $\forall t \geq 0$ belongs to the class Ψ_2. Thus, all conditions of Theorem 4.9 are satisfied. So, T has a fixed point. Note that in this case, $u = \frac{1}{3.2^4-1}$ is the unique fixed point.

The following result shows that the condition of continuity for T in previous theorem can be replaced by a complementary condition on the mapping α.

Theorem 4.12. *Let X be a complete (p,q)-metric space and T an α-ψ-contractive self mapping on X, where $\psi \in \Psi_q$. Assume that*

(i) *T is α -admissible;*
(ii) *there exists $x_0 \in X$ such that: $\alpha\left([x_0]_p,[Tx_0]_q\right) \geq 1$;*
(iii) *for every convergent sequence $\{x_n\}$ of elements of X,*

$$\alpha\left([x_n]_p,[x_{n+1}]_q\right) \geq 1,\ \ \forall n \in \mathbb{N} \Longrightarrow \alpha\left([x_n]_p, \lim_{m\longrightarrow+\infty}[x_m]_q\right) \geq 1,\ \ \forall n \in \mathbb{N}$$

Then, T has a fixed point in X.

Proof. Let us define the convergent sequence $\{x_n\}$ as in Theorem 4.9 and let $u = \lim_{n\longrightarrow+\infty} x_n$. We know by the proof of Theorem 4.9 that

$$\alpha\left([x_n]_p,[x_{n+1}]_q\right) \geq 1,\ \ \forall n \in \mathbb{N}.$$

By assumption (iii) of this theorem, we have

$$\alpha\left([x_n]_p,[u]_q\right) \geq 1,\ \ \forall n \in \mathbb{N}.$$

Using 2.6 and 2.7, we obtain that, for all $n\in\mathbb{N}$,

$$\begin{aligned} S\left([Tu]_p,[u]_q\right) &\leq pqS\left([Tu]_p,[Tx_n]_q\right)+q^2S\left([u]_p,[Tx_n]_q\right)\\ &\leq pq^3S\left([Tx_n]_p,[Tu]_q\right)+q^2S\left([u]_p,[Tx_n]_q\right)\\ &\leq pq^3\alpha\left([x_n]_p,[u]_q\right)S\left([Tx_n]_p,[Tu]_q\right)+q^2S\left([u]_p,[Tx_n]_q\right)\\ &\leq pq^3\psi\left(S\left([x_n]_p,[u]_q\right)\right)+q^2S\left([u]_p,[x_{n+1}]_q\right). \end{aligned}$$

Taking the limit as $n\longrightarrow+\infty$ and using the continuity of $\psi\in\Psi_q$ at 0 (see Corollary 4.3), we find

$$0\leq S\left([Tu]_p,[u]_q\right)\leq pq^3\lim_{n\longrightarrow+\infty}\psi\left(S\left([x_n]_p,[u]_q\right)\right)=pq^3\lim_{t\longrightarrow0^+}\psi(t)=0.$$

Finally, we deduce that u is a fixed point for the self-mapping T. □

Both Theorems 4.9 and 4.12 give sufficient conditions only for the existence of fixed points. In the following, we give a sufficient criterion for the uniqueness of fixed points.

Proposition 4.13. *Let X be a complete (p,q)-metric space and let $T:X\longrightarrow X$ be an α-admissible and α-ψ-contractive self-mapping with $\psi\in\Psi_q$. Suppose that u and v are two fixed points of T and assume that there exists $w\in X$ such that*

$$\alpha\left([u]_p,[w]_q\right)\geq1\quad\text{and}\quad\alpha\left([v]_p,[w]_q\right)\geq1.$$

Then, $u=v$.

Proof. Since T is α-admissible, we have

$$\alpha\left([u]_p,[T^nw]_q\right)=\alpha\left([T^nu]_p,[T^nw]_q\right)\geq1$$

and

$$\alpha\left([v]_p,[T^nw]_q\right)=\alpha\left([T^nv]_p,[T^nw]_q\right)\geq1.$$

Thus,

$$\begin{aligned} S\left([u]_p,[T^{n+1}w]_q\right) &= S\left([Tu]_p,[TT^nw]_q\right)\leq\alpha\left([u]_p,[T^nw]_q\right)S\left([Tu]_p,[TT^nw]_q\right)\\ &\leq \psi\left(S\left([u]_p,[T^nw]_q\right)\right). \end{aligned}$$

By induction on n, we have

$$S\left([u]_p,[T^{n+1}w]_q\right)\leq\psi^{n+1}S\left([u]_p,w]_q\right)\leq q^{4(n+1)}\psi^{n+1}(S\left([u]_p,w]_q\right)).$$

Therefore,

$$0\leq\lim_{n\longrightarrow+\infty}S\left([u]_p,[T^{n+1}w]_q\right)\leq\lim_{n\longrightarrow+\infty}q^{4(n+1)}\psi^{n+1}(S\left([u]_p,w]_q\right))=0.$$

In other words, we have

$$\lim_{n\longrightarrow+\infty}T^nw=u.$$

Similarly,

$$\lim_{n\longrightarrow+\infty}T^nw=v.$$

It implies that $u=v$. □

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