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# A NEW CHARACTERIZATION OF ABSOLUTE SUMMABILITY FACTORS

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**Abstract.** By $(A,B)$, we denote the set of all sequences $\lambda=(\lambda_n)$ such that $\sum x_n\lambda_n$ is summable $B$ whenever $\sum x_n$ is summable $A$, where $A$ and $B$ are two methods of summability. In this paper, we characterize the sets $\left(\left|\overline{N},p_n,u_n\right|_k,|\Phi|\right)$, $\left(|\Phi,u_n|_k,\left|\overline{N},p_n\right|\right)$ and $\left(\left|\overline{N},p_n\right|,|\Phi,u_n|_k\right)$, $\left(|\Phi|,\left|\overline{N},p_n,u_n\right|_k\right)$ for $1\leq k<\infty$.

**Keywords.** Sequence spaces; Absolute Riesz summability; Absolute Euler toitent summability; Summability factors

## 1. Introduction

If a sequence $\lambda$ has the property that $\sum x_n\lambda_n$ is summable $B$ whenever $\sum x_n$ is summable $A$, where $A$ and $B$ are two methods of summability, then we say that $\lambda$ is a summability factor of type $(A,B)$ and we write $\lambda\in(A,B)$. In the special case when $\lambda=e$, $e\in(A,B)$ gives the comparisons of these methods, i.e. $A\subset B$, where $e=(1,1,\dots)$. In the literature, there are several papers concerning absolute summability factors of infinite series. For some of them, we refer to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

In [11], Sarıgöl has established the result dealing with summability factor of type $(|C,\alpha|_k, |\overline{N},p_n|)$ for $\alpha>-1$ and $k>1$, which extends some well-known results in [4, 5, 7, 8]. As a generalization of this study, Hazar Güleç and Sarıgöl investigated absolute summability factor relations including Nörlund and weighted means in [12].

In addition to all these classical methods, we consider the Euler totient matrix and introduce a new absolute summability method. We obtain summability factor theorems of the type $\left(\left|\overline{N},p_n,u_n\right|_k,|\Phi|\right)$, $\left(|\Phi,u_n|_k,\left|\overline{N},p_n\right|\right)$ and $\left(\left|\overline{N},p_n\right|,|\Phi,u_n|_k\right)$, $\left(|\Phi|,\left|\overline{N},p_n,u_n\right|_k\right)$ for $1\leq k<\infty$.

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## 2. Preliminaries and Background

Let $\omega$ be the space of all real or complex valued sequences. Any subspace of $\omega$ is called a sequence space. The space of all $k$-absolutely summable sequences, that is,

$$\ell_k = \{x = (x_n) \in \omega : \sum_{n=1}^{\infty} |x_n|^k < \infty\}$$

is an example for classical sequence spaces.

Let $\sum x_v$ be an infinite series with the sequence of partial sums $(s_n)$. Given any infinite matrix $A = (a_{nv})$ with real or complex entries, the $A$-transform of the sequence $s = (s_n)$ is the sequence $A(s) = (A_n(s))$ defined by

$$A_n(s) = \sum_{j=1}^{\infty} a_{nj} s_j$$

provided that the series on the right side are convergent for all $n \in \mathbb{N} = \{1,2,3,...\}$.

A series $\sum x_v$ is said to be summable $|A, u_n|_k$ for $k \geq 1$ if

$$\sum_{n=1}^{\infty} u_n^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty$$

holds ([13]). Here and what follows, $(u_n)$ will be a sequence of positive constants. If $A$ is the Riesz matrix $(\overline{N}, p_n)$, the summability $|A, u_n|_k$ is reduced to $|\overline{N}, p_n, u_n|_k$ ([10]). In particular, for $u_n = \dfrac{P_n}{p_n}$ and $u_n = n$, we get two special absolute summabilities, $|\overline{N}, p_n|_k$ summability introduced in [2] and $|R, p_n|_k$ summability introduced in [14], respectively. By a Riesz matrix, we mean the one, which is such that

$$a_{ni} = \begin{cases} \dfrac{p_i}{P_n} & , \quad 0 \leq i \leq n, \\ 0 & , \quad i > n, \end{cases}$$

where $(p_i)$ is a sequence of positive real constants and $P_n = p_0 + p_1 + ... + p_n \to \infty$ as $n \to \infty$ with $P_n \neq 0, P_{-1} = p_{-1} = 0$. For $(p_n) = e$, each of the two summabilities $|\overline{N}, p_n|_k$ and $|R, p_n|_k$ are the same as $|C, 1|_k$ summability in Flett's notation [15].

For a given $m \in \mathbb{N}$ with $m > 1$, Euler totient function $\varphi$ is defined as the number of positive integers less than $m$ that are relatively prime to $m$ and $\varphi(1) = 1$. If two numbers $m$ and $n$ are coprime, then $\varphi(mn) = \varphi(m)\varphi(n)$. Also, the relation $m = \sum_{d|m} \varphi(d)$ holds. By the Euler totient matrix $\Phi = (\phi_{ni})$, we mean the one, which is such that

$$\phi_{ni} = \begin{cases} \frac{\varphi(i)}{n} & , \quad \text{if } i \mid n \\ 0 & , \quad \text{if } i \nmid n. \end{cases}$$

Schoenberg [16] proved that this matrix is regular and defined that a sequence $(x_n)$ of real numbers is $\varphi-$convergent to $\xi \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{d|n} \varphi(d) x_d = \xi.$$

This regular matrix is called as Euler totient matrix operator in [17] and some new sequence spaces were introduced by using this matrix. Here we note that if we take $A$ as Euler totient matrix, then we obtain the absolute summability method $|\Phi,u_n|_k$. If we take $u_n=1$ for all values $n$ and $k=1$ then, we will denote $|\Phi,1|_1=$ as $|\Phi|$.

For any given $m\in\mathbb{N}$ with $m>1$, Möbius function $\mu$ is defined as

$$\mu(m)=\begin{cases}(-1)^r, & \text{if } m=p_1p_2...p_r, \text{ where } p_1,p_2,...,p_r \text{ are} \\ & \text{non-equivalent prime numbers} \\ 0, & \text{if } p^2\mid m \text{ for some prime number } p\end{cases}$$

and $\mu(1)=1$. If two numbers $m$ and $n$ are coprime, then $\mu(mn)=\mu(m)\mu(n)$. Also, the relation $\sum_{d|m}\mu(d)=0$ holds except for $m=1$.

## 3. Summability Factor Relations

In this section, given an arbitrary positive sequence $(p_n)$, we characterize the sets $\left(|\Phi,u_n|_k,\left|\overline{N},p_n\right|\right)$, $\left(\left|\overline{N},p_n,u_n\right|_k,|\Phi|\right)$, $\left(\left|\overline{N},p_n\right|,|\Phi,u_n|_k\right)$, $\left(|\Phi|,\left|\overline{N},p_n,u_n\right|_k\right)$ for $1<k<\infty$ and $\left(\left|\overline{N},p_n\right|,|\Phi|\right)$, $\left(|\Phi|,\left|\overline{N},p_n\right|\right)$. Also we give some new results as special cases.

In order to prove our main results, we need the following lemmas. Note that we use the notation $\ell$ instead of $\ell_1$.

**Lemma 3.1.** [11] *Let $1<k<\infty$. Then, $A\in(\ell_k,\ell)$ if and only if*

$$\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{k^*}<\infty$$

*holds, here and what follows, $k^*$ denotes the conjugate of $k$, i.e., $1/k+1/k^*=1$, and $1/k^*=0$ for $k=1$."*

**Lemma 3.2.** [18] *Let $1\leq k<\infty$. Then, $A\in(\ell,\ell_k)$ if and only if*

$$\sup_v\sum_{n=0}^{\infty}|a_{nv}|^k<\infty$$

*holds.*

**Theorem 3.3.** *Let $1<k<\infty$. A sequence $\lambda\in\left(|\Phi,u_n|_k,\left|\overline{N},p_n\right|\right)$ if and only if*

$$\sum_{i=1}^{\infty}\left(\sum_{n=i}^{\infty}\left|\frac{p_nu_i^{-1/k^*}}{P_nP_{n-1}}\sum_{j=i}^{n}P_{j-1}\lambda_j\sum_{r=i}^{j}\delta_{jr}\right|\right)^{k^*}<\infty,$$

*where*

$$\delta_{jr}=\begin{cases}\dfrac{\mu\left(\frac{j}{r}\right)r}{\varphi(j)}, & r\mid j, \\ -\dfrac{\mu\left(\frac{j-1}{r}\right)r}{\varphi(j-1)}, & r\mid j-1, \\ \dfrac{\mu(j)}{\varphi(j)}-\dfrac{\mu(j-1)}{\varphi(j-1)}, & r=1, \\ 0, & \textit{otherwise.}\end{cases} \tag{3.1}$$

*Proof.* Let $(t_n)$ and $(\tilde{t}_n)$ denote the $n$th Euler totient mean of the series $\sum x_n$ and the $n$th Riesz mean of the series $\sum \lambda_n x_n$, respectively. Since $(s_n)$ is the sequence of partial sums of infinite series $\sum x_n$, we can write the $n$th Euler totient mean $(t_n)$ of the sequence $(s_n)$ by

$$t_n = \sum_{i=1}^{\infty} \phi_{ni} s_i = \sum_{i=1}^{n} \phi_{ni} \sum_{j=1}^{i} x_j = \sum_{j=1}^{n} x_j \sum_{\substack{i=j \\ i|n}}^{n} \frac{\varphi(i)}{n}$$

and

$$\tilde{t}_n = \frac{1}{P_n} \sum_{j=0}^{n} \left(P_n - P_{j-1}\right) x_j \lambda_j.$$

Then we define the sequences $(T_n)$ and $\left(\tilde{T}_n\right)$ as

$$\begin{aligned} T_n &= u_n^{1/k^*} \left(t_n - t_{n-1}\right) \\ &= u_n^{1/k^*} \left[\sum_{j=1}^{n-1} x_j \left(\sum_{\substack{i=j \\ i|n}}^{n} \frac{\varphi(i)}{n} - \sum_{\substack{i=j \\ i|n-1}}^{n-1} \frac{\varphi(i)}{n-1}\right) + x_n \frac{\varphi(n)}{n}\right] \ (n \geq 2)\,, \ T_1 = u_1^{1/k^*} x_1 \end{aligned} \tag{3.2}$$

and

$$\tilde{T}_n = \tilde{t}_n - \tilde{t}_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{j=1}^{n} P_{j-1} x_j \lambda_j (n \geq 1), \ \tilde{T}_0 = x_0 \lambda_0.$$

By virtue of (3.2), if we write $y_n = \sum_{j=1}^{n} u_j^{-1/k^*} T_j$, then

$$\begin{aligned} x_n &= \sum_{j|n} \frac{\mu\left(\frac{n}{j}\right) j}{\varphi(n)} y_j - \sum_{j|n-1} \frac{\mu\left(\frac{n-1}{j}\right) j}{\varphi(n-1)} y_j = \sum_{r=1}^{n} \delta_{nr} y_r \\ &= \sum_{r=1}^{n} \delta_{nr} \sum_{j=1}^{r} u_j^{-1/k^*} T_j = \sum_{j=1}^{n} \left(\sum_{r=j}^{n} \delta_{nr} u_j^{-1/k^*}\right) T_j, \end{aligned} \tag{3.3}$$

where $\delta = (\delta_{nr})$ is as in (3.1). Then, using (3.3), we get, for $n \geq 1$,

$$\begin{aligned} \tilde{T}_n &= \frac{p_n}{P_n P_{n-1}} \sum_{j=1}^{n} P_{j-1} x_j \lambda_j = \frac{p_n}{P_n P_{n-1}} \sum_{j=1}^{n} P_{j-1} \lambda_j \sum_{i=1}^{j} \left(\sum_{r=i}^{j} \delta_{jr} u_i^{-1/k^*}\right) T_i \\ &= \frac{p_n}{P_n P_{n-1}} \sum_{i=1}^{n} \sum_{j=i}^{n} P_{j-1} \lambda_j \left(\sum_{r=i}^{j} \delta_{jr} u_i^{-1/k^*}\right) T_i = \sum_{i=1}^{n} f_{ni} T_i, \end{aligned}$$

where the matrix $F = (f_{ni})$ is defined by

$$f_{ni} = \begin{cases} \dfrac{p_n u_i^{-1/k^*}}{P_n P_{n-1}} \sum_{j=i}^{n} P_{j-1} \lambda_j \sum_{r=i}^{j} \delta_{jr}, & 1 \leq i \leq n, \\ 0, & i > n. \end{cases}$$

Then, $\sum \lambda_n x_n$ is summable $\left|\overline{N}, p_n\right|$ whenever $\sum x_n$ is summable $|\Phi, u_n|_k$ if and only if $\tilde{T} = \left(\tilde{T}_n\right) \in \ell$ whenever $T = (T_n) \in \ell_k$, or equivalently, the matrix $F = (f_{ni})$ maps $\ell_k$ into $\ell$ , i.e., $F \in (\ell_k, \ell)$. Thus, it follows from Lemma 3.1 that $F \in (\ell_k, \ell)$ iff

$$\sum_{i=1}^{\infty} \left( \sum_{n=i}^{\infty} \left| \frac{p_n u_i^{-1/k^*}}{P_n P_{n-1}} \sum_{j=i}^{n} P_{j-1} \lambda_j \sum_{r=i}^{j} \delta_{jr} \right| \right)^{k^*} < \infty.$$

This completes the proof. □

Note that $e \in \left(|\Phi, u_n|_k, \left|\overline{N}, p_n\right|\right)$ leads us to a comparison of summability fields of methods $|\Phi, u_n|_k$ and $\left|\overline{N}, p_n\right|$, where $e = (1, 1, \ldots)$. That is, the following result gives us necessary and sufficient condition for $|\Phi, u_n|_k \subset \left|\overline{N}, p_n\right|$.

**Corollary 3.4.** *Let $1 < k < \infty$. Then, $e \in \left(|\Phi, u_n|_k, \left|\overline{N}, p_n\right|\right)$ if and only if*

$$\sum_{i=1}^{\infty} \left( \sum_{n=i}^{\infty} \left| \frac{p_n u_i^{-1/k^*}}{P_n P_{n-1}} \sum_{j=i}^{n} P_{j-1} \sum_{r=i}^{j} \delta_{jr} \right| \right)^{k^*} < \infty,$$

*where $\delta = \left(\delta_{jr}\right)$ is defined in (3.1).*

**Theorem 3.5.** *Let $1 < k < \infty$. A sequence $\lambda \in \left(\left|\overline{N}, p_n, u_n\right|_k, |\Phi|\right)$ if and only if*

$$\sum_{j=1}^{\infty} \left( \sum_{n=j}^{\infty} \left| b_{nj} \right| \right)^{k^*} < \infty,$$

*where*

$$b_{nj} = \begin{cases} \dfrac{u_j^{-1/k^*}}{p_j} \left(\lambda_j \Omega_{nj} P_j - \Omega_{n,j+1} P_{j-1} \lambda_{j+1}\right), & 1 \le j \le n-1, \\ \dfrac{\lambda_n \Omega_{nn} u_n^{-1/k^*} P_n}{p_n}, & j = n, \\ 0, & j > n \end{cases} \tag{3.4}$$

*and*

$$\Omega_{nj} = \begin{cases} \sum\limits_{\substack{i=j \\ i|n}}^{n} \frac{\varphi(i)}{n} - \sum\limits_{\substack{i=j \\ i|n-1}}^{n-1} \frac{\varphi(i)}{n-1}, & 1 \le j \le n-1, \\ \dfrac{\varphi(n)}{n}, & j = n, \\ 0, & j > n. \end{cases} \tag{3.5}$$

*Proof.* Let $(\tilde{t}_n)$ and $(t_n)$ denote the $n$th Riesz mean of the series $\sum x_n$ and the $n$th Euler totient mean of the series $\sum \lambda_n x_n$, respectively. Since $(\tilde{s}_n)$ is the sequence of partial sums of infinite series $\sum \lambda_n x_n$, then we can write the sequences $(\tilde{t}_n)$ and $(t_n)$ by

$$\tilde{t}_n = \frac{1}{P_n} \sum_{j=0}^{n} \left(P_n - P_{j-1}\right) x_j$$

and

$$t_n=\sum_{i=1}^{\infty}\phi_{ni}\tilde{s}_i=\sum_{i=1}^{n}\phi_{ni}\sum_{j=1}^{i}x_j\lambda_j=\sum_{j=1}^{n}x_j\lambda_j\sum_{\substack{i=j\\ i|n}}^{n}\frac{\varphi(i)}{n},$$

respectively. Now, we define the sequences $(\tilde{T}_n)$ and $(T_n)$ as

$$\tilde{T}_n=u_n^{1/k^*}(\tilde{t}_n-\tilde{t}_{n-1})=\frac{u_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{j=1}^{n}P_{j-1}x_j, n\geq 1 \text{ and } \tilde{T}_0=x_0 \tag{3.6}$$

and

$$\begin{aligned} T_n &= t_n-t_{n-1} \\ &= \left[\sum_{j=1}^{n-1}x_j\lambda_j\left(\sum_{\substack{i=j\\ i|n}}^{n}\frac{\varphi(i)}{n}-\sum_{\substack{i=j\\ i|n-1}}^{n-1}\frac{\varphi(i)}{n-1}\right)+x_n\lambda_n\frac{\varphi(n)}{n}\right], n\geq 2 \text{ and } T_1=\lambda_1x_1. \end{aligned} \tag{3.7}$$

By virtue of (3.6), we write

$$x_n=\frac{u_n^{-1/k^*}P_n}{p_n}\tilde{T}_n-\frac{u_{n-1}^{-1/k^*}P_{n-2}}{p_{n-1}}\tilde{T}_{n-1},\ n\geq 1,\ \text{and } x_0=\tilde{T}_0. \tag{3.8}$$

Also we can write (3.7) by

$$\begin{aligned} T_n &= \sum_{j=1}^{n-1}x_j\lambda_j\left(\sum_{\substack{i=j\\ i|n}}^{n}\frac{\varphi(i)}{n}-\sum_{\substack{i=j\\ i|n-1}}^{n-1}\frac{\varphi(i)}{n-1}\right)+x_n\lambda_n\frac{\varphi(n)}{n} \\ &= \sum_{j=1}^{n-1}x_j\lambda_j\Omega_{nj}+x_n\lambda_n\Omega_{nn}, \end{aligned}$$

where $\Omega_{nj}$ is defined by (3.5). Then, using (3.8), we get, for $n\geq 1$,

$$\begin{aligned} T_n &= \sum_{j=1}^{n-1}x_j\lambda_j\Omega_{nj}+x_n\lambda_n\Omega_{nn} \\ &= \sum_{j=1}^{n-1}\lambda_j\Omega_{nj}\left(\frac{u_j^{-1/k^*}P_j}{p_j}\tilde{T}_j-\frac{u_{j-1}^{-1/k^*}P_{j-2}}{p_{j-1}}\tilde{T}_{j-1}\right) \\ &\quad +\lambda_n\Omega_{nn}\left(\frac{u_n^{-1/k^*}P_n}{p_n}\tilde{T}_n-\frac{u_{n-1}^{-1/k^*}P_{n-2}}{p_{n-1}}\tilde{T}_{n-1}\right) \\ &= \lambda_n\Omega_{nn}\frac{u_n^{-1/k^*}P_n}{p_n}\tilde{T}_n+\sum_{j=1}^{n-1}\frac{u_j^{-1/k^*}}{p_j}\left(\lambda_j\Omega_{nj}P_j-\Omega_{n,j+1}P_{j-1}\lambda_{j+1}\right)\tilde{T}_j \\ &= \sum_{j=1}^{n}b_{nj}\tilde{T}_j \end{aligned}$$

where $B=\left(b_{nj}\right)$ is as in (3.4). Then, $\sum\lambda_n x_n$ is summable $|\Phi|$ whenever $\sum x_n$ is summable $\left|\bar{N},p_n,u_n\right|_k$ if and only if $T=(T_n)\in\ell$ whenever $\tilde{T}=\left(\tilde{T}_n\right)\in\ell_k$, or equivalently, the matrix $B=\left(b_{nj}\right)$ maps $\ell_k$ into $\ell$ , i.e., $B\in(\ell_k,\ell)$. Thus, it follows from Lemma 3.1 that $B\in(\ell_k,\ell)$ iff

$$\sum_{j=1}^{\infty}\left(\sum_{n=j}^{\infty}\left|b_{nj}\right|\right)^{k^*}<\infty,$$

which is asserted. □

**Corollary 3.6.** *Let $1<k<\infty$. Then, $e\in\left(\left|\bar{N},p_n,u_n\right|_k,|\Phi|\right)$ if and only if*

$$\sum_{j=1}^{\infty}\left(\sum_{n=j}^{\infty}\left|b_{nj}\right|\right)^{k^*}<\infty,$$

*where*

$$b_{nj}=\begin{cases}\dfrac{u_j^{-1/k^*}}{p_j}\left(\Omega_{nj}P_j-\Omega_{n,j+1}P_{j-1}\right), & 1\leq j\leq n-1,\\ \dfrac{\Omega_{nn}u_n^{-1/k^*}P_n}{p_n}, & j=n,\\ 0, & j>n,\end{cases}$$

*where $\Omega_{nj}$ is defined by (3.5).*

**Theorem 3.7.** *Let $1<k<\infty$. The sequence $\lambda\in\left(|\Phi|,\left|\bar{N},p_n,u_n\right|_k\right)$ if and only if*

$$\sup_i\sum_{n=i}^{\infty}\left|\frac{u_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{j=i}^{n}P_{j-1}\lambda_j\sum_{r=i}^{j}\delta_{jr}\right|^k<\infty,$$

*where*

$$\delta_{jr}=\begin{cases}\dfrac{\mu\left(\frac{j}{r}\right)r}{\varphi(j)}, & r\mid j\\ -\dfrac{\mu\left(\frac{j-1}{r}\right)r}{\varphi(j-1)}, & r\mid j-1\\ \dfrac{\mu(j)}{\varphi(j)}-\dfrac{\mu(j-1)}{\varphi(j-1)}, & r=1\\ 0, & \textit{otherwise.}\end{cases}\tag{3.9}$$

*Proof.* Let $(s_n)$ be the sequence of partial sums of the infinite series $\sum x_n$. Consider the sequences $T=(T_n)$ and $\tilde{T}=\left(\tilde{T}_n\right)$ defined as

$$\begin{aligned}T_n &= t_n-t_{n-1} \qquad (3.10)\\ &= \left[\sum_{j=1}^{n-1}x_j\left(\sum_{\substack{i=j\\ i\mid n}}^{n}\frac{\varphi(i)}{n}-\sum_{\substack{i=j\\ i\mid n-1}}^{n-1}\frac{\varphi(i)}{n-1}\right)+x_n\frac{\varphi(n)}{n}\right](n\geq 2),\ T_1=x_1\end{aligned}$$

and

$$\tilde{T}_n = u_n^{1/k^*}\left(\tilde{t}_n - \tilde{t}_{n-1}\right) = \frac{u_n^{1/k^*} p_n}{P_n P_{n-1}} \sum_{j=1}^{n} P_{j-1} x_j \lambda_j \; (n \geq 1),$$

where

$$t_n = \sum_{j=1}^{n} x_j \sum_{\substack{i=j \\ i|n}}^{n} \frac{\varphi(i)}{n} \text{ and } \tilde{t}_n = \frac{1}{P_n} \sum_{j=0}^{n} \left(P_n - P_{j-1}\right) x_j \lambda_j .$$

Let $y_n = \sum_{j=1}^{n} T_j$. By using (3.10), we obtain

$$\begin{aligned} x_n &= \sum_{j|n} \frac{\mu\left(\frac{n}{j}\right) j}{\varphi(n)} y_j - \sum_{j|n-1} \frac{\mu\left(\frac{n-1}{j}\right) j}{\varphi(n-1)} y_j = \sum_{r=1}^{n} \delta_{nr} y_r \\ &= \sum_{r=1}^{n} \delta_{nr} \sum_{j=1}^{r} T_j = \sum_{j=1}^{n} \left( \sum_{r=j}^{n} \delta_{nr} \right) T_j , \end{aligned} \tag{3.11}$$

where $\delta = (\delta_{nr})$ is as in (3.9). It follows from (3.11) that

$$\begin{aligned} \tilde{T}_n &= \frac{u_n^{1/k^*} p_n}{P_n P_{n-1}} \sum_{j=1}^{n} P_{j-1} x_j \lambda_j = \frac{u_n^{1/k^*} p_n}{P_n P_{n-1}} \sum_{j=1}^{n} P_{j-1} \lambda_j \sum_{i=1}^{j} \left( \sum_{r=i}^{j} \delta_{jr} \right) T_i \\ &= \frac{u_n^{1/k^*} p_n}{P_n P_{n-1}} \sum_{i=1}^{n} \sum_{j=i}^{n} P_{j-1} \lambda_j \left( \sum_{r=i}^{j} \delta_{jr} \right) T_i = \sum_{i=1}^{n} f_{ni} T_i \; (n \geq 1), \end{aligned}$$

where the matrix $F = (f_{ni})$ is defined by

$$f_{ni} = \begin{cases} \dfrac{u_n^{1/k^*} p_n}{P_n P_{n-1}} \sum\limits_{j=i}^{n} P_{j-1} \lambda_j \sum\limits_{r=i}^{j} \delta_{jr}, & 1 \leq i \leq n \\ 0, & i > n. \end{cases}$$

Then, $\sum \lambda_n x_n$ is summable $\left|\overline{N}, p_n, u_n\right|_k$ whenever $\sum x_n$ is summable $|\Phi|$ if and only if $\tilde{T} = \left(\tilde{T}_n\right) \in \ell_k$ whenever $T = (T_n) \in \ell$, or equivalently, the matrix $F = (f_{ni})$ maps $\ell$ into $\ell_k$ , i.e., $F \in (\ell, \ell_k)$. Thus, from Lemma 3.2, we conclude that

$$\sup_i \sum_{n=i}^{\infty} \left| \frac{u_n^{1/k^*} p_n}{P_n P_{n-1}} \sum_{j=i}^{n} P_{j-1} \lambda_j \sum_{r=i}^{j} \delta_{jr} \right|^k < \infty .$$

This completes the proof. □

**Corollary 3.8.** *The sequence* $\lambda \in \left( |\Phi| , \left|\overline{N}, p_n\right| \right)$ *if and only if*

$$\sup_i \sum_{n=i}^{\infty} \left| \frac{p_n}{P_n P_{n-1}} \sum_{j=i}^{n} P_{j-1} \lambda_j \sum_{r=i}^{j} \delta_{jr} \right| < \infty ,$$

*where* $\delta_{jr}$ *is as in* (3.9).

**Corollary 3.9.** *Let* $1<k<\infty$. *Then,* $e\in\left(|\Phi|,\left|\overline{N},p_n,u_n\right|_k\right)$ *if and only if*

$$\sup_i\sum_{n=i}^{\infty}\left|\frac{u_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{j=i}^{n}P_{j-1}\sum_{r=i}^{j}\delta_{jr}\right|^k<\infty,$$

*where* $\delta=\left(\delta_{jr}\right)$ *is as in* (3.9).

**Theorem 3.10.** *Let* $1<k<\infty$. *The sequence* $\lambda\in\left(\left|\overline{N},p_n\right|,\left|\Phi,u_n\right|_k\right)$ *if and only if*

$$\sup_j\sum_{n=j}^{\infty}\left|b_{nj}\right|^k<\infty,$$

*where*

$$b_{nj}=\begin{cases}u_n^{1/k^*}\dfrac{\lambda_j\Omega_{nj}P_j-\Omega_{n,j+1}P_{j-1}\lambda_{j+1}}{p_j}, & 1\leq j\leq n-1,\\ u_n^{1/k^*}\dfrac{\lambda_n\Omega_{nn}P_n}{p_n}, & j=n,\\ 0, & j>n\end{cases}\tag{3.12}$$

*and*

$$\Omega_{nj}=\begin{cases}\sum\limits_{\substack{i=j\\ i|n}}^{n}\frac{\varphi(i)}{n}-\sum\limits_{\substack{i=j\\ i|n-1}}^{n-1}\frac{\varphi(i)}{n-1}, & 1\leq j\leq n-1,\\ \dfrac{\varphi(n)}{n}, & j=n,\\ 0, & j>n.\end{cases}\tag{3.13}$$

*Proof.* Let $(\tilde{s}_n)$ be the sequence of partial sums of the infinite series $\sum\lambda_nx_n$. Consider the sequences $(\tilde{T}_n)$ and $(T_n)$ defined as

$$\tilde{T}_n=\tilde{t}_n-\tilde{t}_{n-1}=\frac{p_n}{P_nP_{n-1}}\sum_{j=1}^{n}P_{j-1}x_j\ (n\geq 1),\ \ \tilde{T}_0=x_0\tag{3.14}$$

and

$$\begin{aligned}T_n&=u_n^{1/k^*}\left(t_n-t_{n-1}\right)\\ &=u_n^{1/k^*}\left[\sum_{j=1}^{n-1}x_j\lambda_j\left(\sum_{\substack{i=j\\ i|n}}^{n}\frac{\varphi(i)}{n}-\sum_{\substack{i=j\\ i|n-1}}^{n-1}\frac{\varphi(i)}{n-1}\right)+x_n\lambda_n\frac{\varphi(n)}{n}\right]\ (n\geq 2),\ \ T_1=u_1^{1/k^*}\lambda_1x_1,\end{aligned}$$

where

$$\tilde{t}_n=\frac{1}{P_n}\sum_{j=0}^{n}\left(P_n-P_{j-1}\right)x_j\ \text{ and }\ t_n=\sum_{j=1}^{n}x_j\lambda_j\sum_{\substack{i=j\\ i|n}}^{n}\frac{\varphi(i)}{n}.$$

By using (3.14), we obtain

$$x_n=\frac{P_n}{p_n}\tilde{T}_n-\frac{P_{n-2}}{p_{n-1}}\tilde{T}_{n-1}\ (n\geq 1),\ \ x_0=\tilde{T}_0.\tag{3.15}$$

It follows from (3.15) that

$$\begin{aligned} T_n &= u_n^{1/k^*}\left[\sum_{j=1}^{n-1} x_j\lambda_j\Omega_{nj} + x_n\lambda_n\Omega_{nn}\right] \\ &= u_n^{1/k^*}\left[\sum_{j=1}^{n-1} \lambda_j\Omega_{nj}\left(\frac{P_j}{p_j}\tilde{T}_j - \frac{P_{j-2}}{p_{j-1}}\tilde{T}_{j-1}\right) + \lambda_n\Omega_{nn}\left(\frac{P_n}{p_n}\tilde{T}_n - \frac{P_{n-2}}{p_{n-1}}\tilde{T}_{n-1}\right)\right] \\ &= u_n^{1/k^*}\left[\sum_{j=1}^{n-1} \frac{\lambda_j\Omega_{nj}P_j - \Omega_{n,j+1}P_{j-1}\lambda_{j+1}}{p_j}\tilde{T}_j + \lambda_n\Omega_{nn}\frac{P_n}{p_n}\tilde{T}_n\right] \\ &= \sum_{j=1}^{n} b_{nj}\tilde{T}_j \ \ (n \geq 1), \end{aligned}$$

where $b_{nj}$ and $\Omega_{nj}$ are as in (3.12) and (3.13), respectively.

Then, $\sum \lambda_n x_n$ is summable $|\Phi, u_n|_k$ whenever $\sum x_n$ is summable $\left|\overline{N}, p_n\right|$ if and only if $T = (T_n) \in \ell_k$ whenever $\tilde{T} = (\tilde{T}_n) \in \ell$, or equivalently, $B = (b_{nj}) \in (\ell, \ell_k)$. Thus, from Lemma 3.2, we conclude that

$$\sup_j \sum_{n=j}^{\infty} \left|b_{nj}\right|^k < \infty.$$

□

**Corollary 3.11.** *The sequence $\lambda \in \left(\left|\overline{N}, p_n\right|, |\Phi|\right)$ if and only if*

$$\sup_j \sum_{n=j}^{\infty} \left|b_{nj}\right| < \infty,$$

*where*

$$b_{nj} = \begin{cases} \dfrac{\lambda_j\Omega_{nj}P_j - \Omega_{n,j+1}P_{j-1}\lambda_{j+1}}{p_j}, & 1 \leq j \leq n-1, \\ \dfrac{\lambda_n\Omega_{nn}P_n}{p_n}, & j = n, \\ 0, & j > n \end{cases}$$

*and $\Omega_{nj}$ is as in (3.13).*

**Corollary 3.12.** *Let $1 < k < \infty$. Then, $e \in \left(\left|\overline{N}, p_n\right|, |\Phi, u_n|_k\right)$ if and only if*

$$\sup_j \sum_{n=j}^{\infty} \left|b_{nj}\right|^k < \infty,$$

*where*

$$b_{nj} = \begin{cases} u_n^{1/k^*}\dfrac{\Omega_{nj}P_j - \Omega_{n,j+1}P_{j-1}}{p_j}, & 1 \leq j \leq n-1, \\ u_n^{1/k^*}\dfrac{\Omega_{nn}P_n}{p_n}, & j = n, \\ 0, & j > n \end{cases}$$

*where $\Omega_{nj}$ is defined by (3.13).*

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