Commun. Optim. Theory 2020 (2020), Article ID 14 https://doi.org/10.23952/cot.2020.14

MATHRES

Communications in Optimization Theory

Available online at http://cot.mathres.org

MIXED EQUILIBRIUM AND MULTIPLE-SET SPLIT FEASIBILITY PROBLEMS FOR ASYMPTOTICALLY K-STRICTLY PSEUDONONSPREADING MAPPINGS

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Abstract. Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H and let $T : C \to C$ be an asymptotically k-strictly pseudononspreading mapping. We show that the set of fixed points of T is closed and convex, and $I - T$ is demiclosed at 0. In addition, weak and strong convergence theorems for mixed equilibrium and multiple-set split feasibility problems are established in H.

Keywords. Asymptotically $k-$strictly pseudononspreading mappings; Multiple-set split feasibility problem; Weak convergence; Strong convergence; Iterative algorithm.

1. Introduction

Let H_1 and H_2 be two Hilbert spaces. The split feasibility problem (SFP) in finite-dimensional spaces was first introduced by Censor and Elfving [1] for modelling inverse problems which came from phase retrievals and in medical image reconstruction. The SFP has various important applications in several disciplines; see, e.g., [2, 3, 4, 5, 6, 7, 8, 9] and the references therein.

Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S_i : H_1 \to H_1$ and $T_i : H_2 \to H_2$, $i = 1, 2, \ldots, N$ be two finite families of mappings such that $K := \bigcap_{i=1}^N F(S_i)$ and $Q := \bigcap_{i=1}^N F(T_i)$, where $F(S_i) \neq \emptyset$ and $F(T_i) \neq \emptyset$ are the sets of fixed points of S_i and T_i, respectively. The multiple set split feasibility problem (MSSFP) is

$$to find\ x^* \in \bigcap_{i=1}^{N} F(S_i)\ such\ that\ Ax^* \in \bigcap_{i=1}^{N} F(T_i). \tag{1.1}$$

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Received March 16, 2020; Accepted May 15, 2020.

We use Γ in the sequel to denote the set of solutions of MSSFP (1.1), that is,

$$\Gamma = \{x^* \in \bigcap_{i=1}^{N} F(S_i) : Ax^* \in \bigcap_{i=1}^{N} F(T_i)\}. \tag{1.2}$$

Let H be real Hilbert space and C be a convex and closed subset of H. Recall that $T : C \to C$ is said to be nonspreading [10] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \; \forall \, x, y \in C.$$

It is easily to see that the above inequality is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \; \forall \, x, y \in C.$$

Recall that $T : C \to C$ is said to be k-strictly pseudo-nonspreading [11] if there exists $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle, \; \forall \, x, y \in D(T).$$

It was shown in [11] that the class of nonspreading mappings is a proper subclass of the class of k-strictly pseudononspreading mappings. In [11], they also showed that the class of k-strictly pseudononspreading mappings and the important class of *λ-strictly pseudocontractive mappings* (i.e., a mapping $T : C \to C$ is said to be λ-strictly pseudocontractive mappings if there exists a constant $\lambda \in [0, 1)$ such that $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda\|x - Tx - (y - Ty)\|^2$, $\forall x, y \in C$.) are independent.

Recently, Ma and Wang [6] and Quan and Chang [7] studied the class of *asymptotically k-strictly pseudononspreading mappings* in real Hilbert spaces. Following Osilike and Isiogugu [11], they called a mapping $T : C \to C$ asymptotically k-strictly pseudononspreading if there exist $k \in [0, 1)$ and a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\forall n \geq 1, \; x, y \in C$,

$$\|T^n x - T^n y\|^2 \leq k_n\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 + 2\langle x - T^n x, y - T^n y\rangle. \tag{1.3}$$

This class of mappings properly contains the class of *asymptotically nonspreading mappings* studied by Naraghirad [12] and Phuengrattana [13]. Naraghirad [12] and Phuengrattana [13] called a mapping $T : C \to C$ asymptotically nonspreading if

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + 2\langle x - T^n x, y - T^n y\rangle, \; \forall n \geq 1, \; x, y \in C.$$

Both Ma and Wang [6] and Quan and Chang [7] assumed that every k-strictly pseudononspreading mapping is also asymptotically k-strictly pseudononspreading. This appears to be an oversight as we shall establish later in our discussion. While Quan and Chang [7] required that an asymptotically k-strictly pseudononspreading mapping $T : C \to C$ is, in addition, continuous to prove that $F(T)$ is closed and convex and that $(I - T)$ is demiclosed at zero. Ma and Wang [6] imposed the continuity to prove that $F(T)$ is closed and convex, and a stronger condition of uniformly Lipschitzian to prove that $(I - T)$ is demiclosed at zero. They both proved weak and strong convergence theorems for mixed equilibrium and multiple-set split feasibility problems for uniformly Lipschitzian asymptotically k-strictly pseudononspreading self-mappings in real Hilbert spaces.

It is our purpose in this paper to show that the class of strictly pseudononspreading mappings and the class of asymptotically strictly pseudononspreading mappings are independent. Let

$T : C \subset H \to C$ be asymptotically k-strictly pseudononspreading, we establish that $F(T)$ is closed and convex and prove the demiclosedness property of $(I - T)$ at zero without any further continuity assumption on T. We then prove weak and strong convergence results for mixed equilibrium and multiple-set split feasibility problems without further continuity assumptions on the operator T.

2. Preliminaries

Definition 2.1. Let E be a real Banach space and C a nonempty convex closed subset of E. A mapping $T : C \to C$ is said to be *semicompact* if, for any bounded sequence $\{x_n\} \subset C$ with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x^* \in C$.

Recall that a space E is said to have the Opial property if, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \forall\, y \in E \text{ with } y \neq x. \tag{2.1}$$

It is well known that every Hilbert space satisfies the Opial condition.

Definition 2.2. Let H be a Hilbert space. A mapping $T : H \to H$ is said to be demiclosed at p if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$ and the sequence $\{T(x_n)\}$ converges strongly to p, $Tx^* = p$.

For all $x, y \in H$, we have the following.

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \ t \in [0,1] \tag{2.2}$$

and

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.3}$$

If $\{x_n\}_{n=1}^{\infty}$ is a sequence in H, which converges weakly to z, then

$$\limsup_{n\to\infty} ||x_n - y||^2 = \limsup_{n\to\infty} ||x_n - z||^2 + ||z - y||^2, \ \forall y \in H. \tag{2.4}$$

Lemma 2.3. [14, 15] *Let $\{a_n\}$ and $\{\alpha_n\}$ be the sequences of nonnegative number satisfying*

$$a_{n+1} \leq (1 + \alpha_n)a_n, \ \forall\, n \geq 1, \tag{2.5}$$

with $\sum_{n=1}^{\infty} \alpha_n < \infty$. Then (i) $\lim_{n\to\infty} a_n$ exists; (ii) if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

3. Main results

We begin this section with the following examples.

Example 3.1. (Strictly pseudononspreading mapping, which is not asymptotically strictly pseudononspreading) Let $\mathfrak{R}$ denote the reals with the usual norm and define $T : \mathfrak{R} \to \mathfrak{R}$ by

$$Tx = \begin{cases} -x, \ x \in (-\infty, 0], \\ -2x, \ x \in (0, \infty). \end{cases}$$

We show that

$$|\, Tx - Ty\,|^2 \leq |\, x - y\,|^2 + \frac{3}{4}\,|\, x - Tx - (y - Ty)\,|^2 + 2\langle x - Tx, y - Ty\rangle, \ \forall x, y \in \mathfrak{R}.$$

Observe that, for all $x,y \in (-\infty,0]$,

$$| Tx-Ty |^2 = | x-y |^2 .$$

Furthermore,

$$\begin{aligned} &| x-y |^2 + \frac{3}{4} | x-Tx-(y-Ty) |^2 + 2\langle x-Tx, y-Ty\rangle \\ &= | x-y |^2 + 3 | x-y |^2 + 8xy \geq | x-y |^2 = | Tx-Ty |^2 . \end{aligned}$$

For all $x,y \in (0,\infty)$, we obtain $| Tx-Ty |^2 = 4 | x-y |^2$, $| x-Tx-(y-Ty) |^2 = 9 | x-y |^2$, and $2\langle x-Tx, y-Ty\rangle = 18xy$. Thus

$$\begin{aligned} &| x-y |^2 + \frac{3}{4} | x-Tx-(y-Ty) |^2 + 2\langle x-Tx, y-Ty\rangle \\ &= | x-y |^2 + \frac{27}{4} | x-y |^2 + 18xy \\ &= 4 | x-y |^2 + \frac{15}{4} | x-y |^2 + 18xy \geq 4 | x-y |^2 = | Tx-Ty |^2 . \end{aligned}$$

Furthermore, for all $x \in (-\infty,0]$ and $y \in (0,\infty)$, we obtain

$$| Tx-Ty |^2 = | x-2y |^2 = | x-y |^2 + 3y^2 - 2xy,$$

$$\frac{3}{4} | x-Tx-(y-Ty) |^2 = 3 | x-y |^2 + \frac{3}{4}y^2 - 3xy + 3y^2,$$

and $2\langle x-Tx, y-Ty\rangle = 12xy$. Thus

$$\begin{aligned} &| x-y |^2 + \frac{3}{4} | x-Tx-(y-Ty) |^2 + 2\langle x-Tx, y-Ty\rangle \\ &= | x-y |^2 + 3 | x-y |^2 + \frac{3}{4}y^2 - 3xy + 3y^2 + 12xy \\ &= | x-y |^2 + 3y^2 - 2xy + 3(x^2+y^2+2xy) + \frac{3}{4}y^2 - xy \\ &\geq | x-y |^2 + 3y^2 - 2xy = | Tx-Ty |^2 . \end{aligned}$$

We conclude that T is $\frac{3}{4}$-strictly pseudononspreading.

Next we show that T is not asymptotically strictly pseudononspreading. Consider $x=0$, and $y \in (0,\infty)$. Then $T^{2n}x = 0$ and $T^{2n}y = 2^n y$. For arbitrary sequence $\{k_n\} \subseteq [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$, there exists a positive integer N_0 such that $k_n \leq 2$, $\forall n \geq N_0$. Hence

$$\langle T^{2n}x - T^{2n}y, x-y\rangle = 2^n y^2 = 2^n | x-y |^2 \geq 2 | x-y |^2 > k_n | x-y |^2, \ \forall n \geq N_0.$$

Example 3.2. (Asymptotically strictly pseudononspreading mapping, which is not strictly pseudononspreading) Let $H = \ell^2(\Re) = \{x = \{x_j\}_{j=1}^{\infty} : x_j \in \Re, \forall j \text{ and } \sum_{j=1}^{\infty} |x_j|^2 < \infty\}$ and let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real numbers such that $a_2 > 0$, $0 < a_j < 1$, $j \neq 2$, and $\lim_{n\to\infty} \prod_{j=2}^{\infty} a_j = \frac{1}{2}$. Let

$$C = \{x \in H : \max\{\|x\|, \|z\|\} \leq 1\},$$

where

$$z = (\sqrt{a_2}x_1, \sqrt{a_2}x_2, x_3, x_4, x_5, \ldots).$$

Define $T : C \to C$ by

$$Tx = T(x_1, x_2, x_3, \dots) = (0, x_1^2, a_2x_2, a_3x_3, \dots).$$

Then T is $\frac{1}{2}-$strictly asymptotically pseudononspreading. For all $x, y \in C$, we have

$$\begin{aligned}
\|T^nx - T^ny\|^2 &\leq [2\prod_{j=2}^{n} a_j]^2\|x-y\|^2 \\
&= [2\prod_{j=2}^{n} a_j]^2\|x-y\|^2 + \frac{1}{2}\|x - T^nx - (y - T^ny)\|^2 \\
&\quad +2\langle x - T^nx, y - T^ny\rangle - [\frac{1}{2}\|x - T^nx - (y - T^ny)\|^2 \\
&\quad +2\langle x - T^nx, y - T^ny\rangle] \\
&= [2\prod_{j=2}^{n} a_j]^2\|x-y\|^2 + \frac{1}{2}\|x - T^nx - (y - T^ny)\|^2 \\
&\quad +2\langle x - T^nx, y - T^ny\rangle - \frac{1}{2}\|x - T^nx - (y - T^ny)\|^2 \\
&\quad +\|x - T^nx - (y - T^ny)\|^2 - \|x - T^nx\|^2 - \|y - T^ny\|^2 \\
&= [2\prod_{j=2}^{n} a_j]^2\|x-y\|^2 + \frac{1}{2}\|x - T^nx - (y - T^ny)\|^2 \\
&\quad +2\langle x - T^nx, y - T^ny\rangle + \frac{1}{2}\|x - T^nx - (y - T^ny)\|^2 \\
&\quad -\|x - T^nx\|^2 - \|y - T^ny\|^2 \\
&\leq [2\prod_{j=2}^{n} a_j]^2\|x-y\|^2 + \frac{1}{2}\|x - T^nx - (y - T^ny)\|^2 \\
&\quad +2\langle x - T^nx, y - T^ny\rangle.
\end{aligned}$$

Hence T is asymptotically $\frac{1}{2}-$strictly pseudononspreading with $k_1 = 2$, $k_n = [2\prod_{j=2}^{n} a_j]^2$, $n \geq 2$; $k_n \subseteq [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$.

To verify that T is not strictly pseudononspreading, we consider the vectors $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, \dots)$ and $y = (0, 0, 0, \dots)$. Let $\{a_j\}_{j=1}^{\infty}$ be a real sequence satisfying the conditions in the Example with $a_2 = \frac{8}{3}$. Then $x, y \in C$, $\|x - y\|^2 = \frac{1}{3}$ and $Tx = (0, \frac{1}{9}, \frac{a_2}{3}, \frac{a_3}{3}, 0, 0, 0, \dots), Ty = (0, 0, 0, \dots)$. Furthermore,

$$\|Tx - Ty\|^2 = \|(0, \frac{1}{9}, \frac{a_2}{3}, \frac{a_3}{3}, 0, 0, 0, \dots)\|^2 = \frac{1}{81} + \frac{a_2^2}{9} + \frac{a_3^2}{9} = \frac{65}{81} + \frac{a_3^2}{9},$$

$$x - Tx = (\frac{1}{3}, \frac{2}{9}, \frac{(1 - a_2)}{3}, \frac{-a_3}{3}, 0, 0, 0, \dots);\ y - Ty = (0, 0, 0, \dots) \text{ and}$$

$$\|x - Tx - (y - Ty)\|^2 = \frac{1}{9} + \frac{4}{81} + \frac{(1 - a_2)^2}{9} + \frac{a_3^2}{9}.$$

Thus, for any $\beta \in [0,1)$, we have

$$\begin{aligned} \|x-y\|^2+\beta\|x-Tx-(y-Ty)\|^2 &< \|x-y\|^2+\|x-Tx-(y-Ty)\|^2 \\ &= \frac{1}{3}+\frac{1}{9}+\frac{4}{81}+\frac{(1-a_2)^2}{9}+\frac{a_3^2}{9} \\ &= \frac{65}{81}+\frac{a_3^2}{9}=\|Tx-Ty\|^2. \end{aligned}$$

Example 3.3. (Asymptotically k-strictly pseudononspreading, which is not asymptotically nonspreading) Let $\mathbb{R}$ denote the reals with the usual norm. Let $T:\mathbb{R}\to\mathbb{R}$ be defined for each $x\in\mathbb{R}$ by

$$Tx=\begin{cases} x, & x\in(-\infty,0), \\ -2x, & x\in[0,\infty). \end{cases}$$

For $n>1$, we have

$$T^nx=\begin{cases} x, & x\in(-\infty,0), \\ -2x, & x\in[0,\infty). \end{cases}$$

Then T is asymptotically k-strictly pseudononspreading.

Proof. Clearly, $F(T)=(-\infty,0]$. To show that T is asymptotically k-strictly pseudononspreading, we observe that $\forall\, k\in[0,1)$ and $\{k_n\}_{n=1}^{\infty}\subseteq[1,\infty)$, if $x,y\in(-\infty,0)$, then

$$\begin{aligned} |T^nx-T^ny|^2 &= |x-y|^2+k|x-T^nx-(y-T^ny)|^2+2\langle x-T^nx,y-T^ny\rangle \\ &\leq k_n|x-y|^2+k|x-T^nx-(y-T^ny)|^2+2\langle x-T^nx,y-T^ny\rangle, \end{aligned}$$

since $|T^nx-T^ny|^2=|x-y|^2$, and

$$k|x-T^nx-(y-T^ny)|^2=2\langle x-T^nx,y-T^ny\rangle=0.$$

For all $x,y\in[0,\infty)$, we have $|T^nx-T^ny|^2=4|x-y|^2$, $|x-T^nx-(y-T^ny)|^2=9|x-y|^2$ and $2\langle x-T^nx,y-T^ny\rangle=18xy\geq 0$. Thus

$$\begin{aligned} |T^nx-T^ny|^2=4|x-y|^2 &= |x-y|^2+\frac{1}{3}|x-T^nx-(y-T^ny)|^2 \\ &\leq |x-y|^2+\frac{1}{3}|x-T^nx-(y-T^ny)|^2 \\ &\quad +2\langle x-T^nx,(y-T^ny)\rangle. \\ &\leq k_n|x-y|^2+\frac{1}{3}|x-T^nx-(y-T^ny)|^2 \\ &\quad +2\langle x-T^nx,(y-T^ny)\rangle. \end{aligned}$$

Finally, for all $x\in(-\infty,0)$ and $y\in[0,\infty)$, we have

$$|T^nx-T^ny|^2=|x+2y|^2=x^2+4xy+4y^2,$$

$$2\langle x-T^nx,y-T^ny\rangle=0,$$

and

$$\frac{1}{3}|x-T^nx-(y-T^ny)|^2=3y^2.$$

Hence

$$\begin{aligned}
&|x-y|^2+\frac{1}{3}|x-T^nx-(y-T^ny)|^2+2\langle x-T^nx,(y-T^ny)\rangle\\
&=x^2-2xy+4y^2\\
&=x^2+4xy+4y^2-6xy\\
&\geq x^2+4xy+4y^2 \text{ (since } -6xy\geq 0)\\
&=(x+2y)^2=|x+2y|^2=|T^nx-T^ny|^2.
\end{aligned}$$

Hence, for all $x,y\in\mathbb{R}$, we have

$$|T^nx-T^ny|^2\leq k_n|x-y|^2+\frac{1}{3}|x-T^nx-(y-T^ny)|^2+2\langle x-T^nx,y-T^ny\rangle.$$

Next, to prove that T is not asymptotically nonspreading in the terminology of Naraghirad [12] and Phuengrattana [13], we consider $x=4$ and $y=\frac{1}{4}$. Then

$$|x-y|^2+2\langle x-T^nx,y-T^ny\rangle=32.0625<56.25=|T^nx-T^ny|^2.$$

In general, if $\{k_n\}\subseteq[1,\infty)$ is such that $\lim_{n\to\infty}k_n=1$, then there exists a positive integer N_0 such that $1\leq k_n\leq 2$, $\forall n\geq N_0$. Thus, for any $n\geq N_0$, we have

$$\begin{aligned}
&k_n|x-y|^2+2\langle x-T^nx,y-T^ny\rangle\\
&\leq 2|x-y|^2+2\langle x-T^nx,y-T^ny\rangle=46.125<56.25=|T^nx-T^ny|^2.
\end{aligned}$$

Thus T is not asymptotically nonspreading in the more general sense of the special case for which $k=0$ in our definition of asymptotically k-strictly pseudononspreading. □

We remark here that Example 3.2 is motivated by [16, Example 2] and [11, Example 3.3]. Next, we establish the following important properties for asymptotically $k-$strictly pseudononspreading mappings, which play crucial roles in the proof of our convergence results.

Lemma 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space and let $T:C\to C$ be an asymptotically $k-$strictly pseudononspreading mapping with a sequence $\{k_n\}\subseteq[1,\infty)$. Then, for all $x,y\in C$ and $n\in\mathbb{N}$,*

$$\|T^nx-T^ny\| \leq \frac{\sqrt{k_n}+\sqrt{k}}{1-\sqrt{k}}\|x-y\|+\frac{1}{1-\sqrt{k}}\sqrt{2|\langle x-T^nx,y-T^ny\rangle|}. \tag{3.1}$$

Proof. From (1.3), we have

$$\begin{aligned}\|T^n x - T^n y\|^2 &\leq k_n\|x-y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 \\ &\quad +2\langle x - T^n x, y - T^n y\rangle \\ &\leq \left(\sqrt{k_n}\|x-y\| + \sqrt{k}\|x - T^n x - (y - T^n y)\|\right)^2 \\ &\quad +2\langle x - T^n x, y - T^n y\rangle \\ &\leq \left(\sqrt{k_n}\|x-y\| + \sqrt{k}\|x - T^n x - (y - T^n y)\|\right)^2 \\ &\quad +2|\langle x - T^n x, y - T^n y\rangle| \\ &\leq \Big[\sqrt{k_n}\|x-y\| + \sqrt{k}\|x - T^n x - (y - T^n y)\| \\ &\quad +\sqrt{2}\sqrt{|\langle x - T^n x, y - T^n y\rangle|}\Big]^2. \qquad (3.2)\end{aligned}$$

Hence

$$\begin{aligned}\|T^n x - T^n y\| &\leq \sqrt{k_n}\|x-y\| + \sqrt{k}\|x - T^n x - (y - T^n y)\| \\ &\quad +\sqrt{2|\langle x - T^n x, y - T^n y\rangle|} \\ &\leq \sqrt{k_n}\|x-y\| + \sqrt{k}\|x-y\| + \sqrt{k}\|T^n x - T^n y\| \\ &\quad +\sqrt{2|\langle x - T^n x, y - T^n y\rangle|}, \qquad (3.3)\end{aligned}$$

and (3.1) follows. □

Proposition 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space and let $T : C \to C$ be an asymptotically $k-$strictly pseudononspreading mapping. Then $F(T)$ is closed and convex.*

Proof. If $F(T) = \emptyset$, then it is clear. Next, we suppose $F(T) \neq \emptyset$ and let $\{p_n\}_{n=1}^{\infty} \subseteq F(T)$ be an arbitrary sequence in $F(T)$ converging to p. We prove that $p \in F(T)$. Using Lemma 3.4, we obtain

$$\begin{aligned}\|p - Tp\| &\leq \|p - p_n\| + \|p_n - Tp\| \\ &\leq \|p - p_n\| + \frac{\sqrt{k_1}+\sqrt{k}}{1-\sqrt{k}}\|p_n - p\| \\ &\quad +\frac{1}{1-\sqrt{k}}\sqrt{2|\langle p_n - Tp_n, p - Tp\rangle|}.\end{aligned}$$

Hence

$$0 \leq \|p - Tp\| \leq \|p_n - p\| + \frac{\sqrt{k_1}+\sqrt{k}}{1-\sqrt{k}}\|p_n - p\|. \qquad (3.4)$$

Letting $n \to \infty$ in (3.4), we obtain

$$Tp = p.$$

To prove that $F(T)$ is convex, we fix $p_1, p_2 \in F(T)$, and set $p = \lambda p_1 + (1-\lambda)p_2$, where $\lambda \in [0,1]$. We prove that $p \in F(T)$. As in the proof of Lemma 2.2 of [6], we obtain

$$\begin{aligned} \|p - Tp\| &\leq \|p - T^n p\| + \|T^n p - Tp\| \\ &= \|p - T^n p\| + \|T(T^{n-1}p) - Tp\| \\ &\leq \|p - T^n p\| + \frac{\sqrt{k_1} + \sqrt{k}}{1-\sqrt{k}}\|p - T^{n-1}p\| \\ &\quad + \frac{1}{1-\sqrt{k}}\sqrt{2|\langle T^{n-1}p - T^n p, p - Tp\rangle|}. \end{aligned} \tag{3.5}$$

Since $a_n = \|p - T^n p\| \to 0$ *as* $n \to \infty$, we have $\|T^{n-1}p - T^n p\| \leq \|T^{n-1}p - p\| + \|p - T^n p\| \to 0$ as $n \to \infty$. Hence, letting $n \to \infty$ in (3.5), we obtain $p = Tp$. □

Proposition 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space and let $T : C \to C$ be an asymptotically $k-$strictly pseudononspreading mapping with $F(T) \neq \emptyset$. Then $(I - T)$ is demiclosed at* 0*, (i.e., if $\{x_n\}$ is any sequence in C which converges weakly to p, and $\|x_n - Tx_n\| \to 0$, then $p = Tp$).*

Proof. Let $\{x_n\}$ be a sequence in C, which converges weakly to p and $\{x_n - Tx_n\}$ converges strongly to 0. We prove that $p \in F(T)$. Since $\{x_n\}$ converges weakly, it is bounded. For each $x \in H$, define a mapping $f : H \to [0, \infty)$ by

$$f(x) := \limsup_{n\to\infty} \|x_n - x\|^2. \tag{3.6}$$

It is clear that $\{Tx_n\}$ is bounded. For any arbitrary but fixed integer $m \geq 1$, and for $z \in F(T)$, we use Lemma 3.4 to obtain

$$\begin{aligned} \|T^m x_n\| &\leq \|T^m x_n - T^m z\| + \|T^m z\| \\ &\leq \frac{\sqrt{k_m} + \sqrt{k}}{1-\sqrt{k}}\|x_n - z\| \\ &\quad + \frac{1}{1-\sqrt{k}}\sqrt{2|\langle x_n - T^m x_n, z - T^m z\rangle|} + \|T^m z\| \\ &\leq \frac{\sqrt{k_m} + \sqrt{k}}{1-\sqrt{k}}\|x_n - z\| + \|z\|. \end{aligned}$$

Thus, $\{T^m x_n\}$ is bounded.

Next, we prove that $\|x_n - T^m x_n\| \to 0$ *as* $n \to \infty$. Note that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Suppose now that $\|x_n - T^m x_n\| \to 0$ as $n \to \infty$. We prove $\|x_n - T^{m+1}x_n\| \to 0$ as $n \to \infty$. Using Lemma 3.4, we obtain

$$\begin{aligned} \|x_n - T^{m+1}x_n\| &\leq \|x_n - T^m x_n\| + \|T^m x_n - T^{m+1}x_n\| \\ &= \|x_n - T^m x_n\| + \|T^m x_n - T^m(Tx_n)\| \\ &\leq \|x_n - T^m x_n\| + \frac{\sqrt{k_m} + \sqrt{k}}{1-\sqrt{k}}\|x_n - Tx_n\| \\ &\quad + \frac{1}{1-\sqrt{k}}\sqrt{2|\langle x_n - T^m x_n, Tx_n - T^{m+1}x_n\rangle|}. \end{aligned}$$

Since $\{Tx_n - T^{m+1}x_n\}$ is bounded, (i.e. $\|Tx_n - T^{m+1}x_n\| \le \|Tx_n - x_n\| + \|x_n - T^{m+1}x_n\|$) and $\lim_{n\to\infty}\|x_n - Tx_n\| = \lim_{n\to\infty}\|x_n - T^m x_n\| = 0$, we have that $\lim_{n\to\infty}\|x_n - T^{m+1}x_n\| = 0$. Hence, we conclude that $\|x_n - T^m x_n\| \to 0$ as $n\to\infty$ for all $m \ge 1$. Now, using (2.4), we obtain

$$f(T^m p) = f(p) + \|T^m p - p\|^2, \tag{3.7}$$

and it follows from the proof of Lemma 2.4 of [6] that

$$\lim_{m\to\infty}\|p - T^m p\| = 0.$$

We now use Lemma 3.4 to obtain

$$\begin{aligned} 0 \le \|p - Tp\| &\le \|p - T^m p\| + \|T^m p - Tp\| \\ &= \|p - T^m p\| + \|T(T^{m-1}p) - Tp\| \\ &\le \|p - T^m p\| + \frac{\sqrt{k_1}+\sqrt{k}}{1-\sqrt{k}}\|p - T^{m-1}p\| \\ &\quad + \frac{1}{1-\sqrt{k}}\sqrt{2|\langle T^{m-1}p - T^m p, p - Tp\rangle|}. \end{aligned} \tag{3.8}$$

Since $a_m = \|p - T^m p\|$, and $\lim_{m\to\infty} a_m = 0$, we have $\lim_{m\to\infty} a_{m-1} = \lim_{m\to\infty}\|p - T^{m-1}p\| = 0$. Hence, $\|T^{m-1}p - T^m p\| \le \|T^{m-1}p - p\| + \|p - T^m p\| \to 0$, as $n\to\infty$. Thus, letting $m\to\infty$ in (3.8) yields $Tp = p$. □

Using Lemma 3.4, Propositions 3.5 and 3.6, we obtain the following results.

Theorem 3.7. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $S_i : H_1 \to H_1$, $i = 1,2,\ldots,N$ be asymptotically ξ_i-strictly pseudononspreading mapping with $\{\beta_n^{(i)}\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(\beta_n^{(i)} - 1) < \infty$ and let $T_i : H_2 \to H_2$, $i = 1,2,\ldots,N$ be asymptotically k_i-strictly pseudononspreading mapping with $\{k_n^{(i)}\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$. Let $\beta_n = \max_{1\le i\le N}\{\beta_n^{(i)}\}$, $k_n = \max_{1\le i\le N}\{k_n^{(i)}\}$, $K := \bigcap_{i=1}^N F(S_i)$ and $Q := \bigcap_{i=1}^N F(T_i)$, where $F(S_i) \ne \emptyset$ and $F(T_i) \ne \emptyset$ are the sets of fixed points of S_i and T_i, respectively. For any arbitrary $x_1 \in H_1$, the sequence $\{x_n\}_{n=1}^{\infty}$ is given by*

$$\begin{cases} u_n = x_n + \gamma A^*[T^n_{n(modN)} - I]Ax_n, & n \ge 1, \\ x_{n+1} = (1-\alpha_n)u_n + \alpha_n S^n_{n(modN)}u_n, & n \ge 1, \end{cases} \tag{3.9}$$

where γ is a constant and $\gamma \in (0, \frac{1-k}{\|A\|^2})$, $k = \max\{k_1,k_2,k_3,\ldots,k_N\}$ and $\{\alpha_n\}$ is sequence in $(0, 1-\xi]$ with $\xi = \max\{\xi_1,\xi_2,\xi_3,\ldots,\xi_N\}$. Let $\Gamma = \{x \in H_1 : x \in K \text{ and } Ax \in Q\} \ne \emptyset$. Then, the sequence $\{x_n\}$ generated from x_1 by (3.9) converges weakly to a point in Γ. If in addition S_i is semicompact, then $\{x_n\}$ and $\{u_n\}$ converges strongly to a point x^* in Γ.

Proof. Following the proof in Theorem 3.1 of [6, 7], we obtain the following.

(i) For each $p \in \Gamma$, $\lim_{n\to\infty}\|x_n - p\|$ and $\lim_{n\to\infty}\|u_n - p\|$ exist and $\lim_{n\to\infty}\|x_n - p\| = \lim_{n\to\infty}\|u_n - p\|$.

(ii)

$$\lim_{n\to\infty}\|S^n_{n(modN)}u_n - u_n\| = 0; \tag{3.10}$$

$$\lim_{n\to\infty}\|(T^n_{n(modN)} - I)Ax_n\| = 0. \tag{3.11}$$

(iii)

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \tag{3.12}$$

and

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.13}$$

We now use Lemma 3.4 to prove (without the uniformly Lipschitzian assumptions) that

$$\lim_{n\to\infty} \|u_n - S_{n(modN)}u_n\| = 0, \tag{3.14}$$

$$\lim_{n\to\infty} \|Ax_n - T_{n(modN)}Ax_n\| = 0. \tag{3.15}$$

It suffices to prove that, for all $j = 1, 2, \ldots, N$,

$$\lim_{i\to\infty} \|u_{iN+j} - S_j u_{iN+j}\| = 0 \text{ and } \lim_{i\to\infty} \|Ax_{iN+j} - T_j Ax_{iN+j}\| = 0. \tag{3.16}$$

It follows from (3.10) that

$$\lim_{i\to\infty} \|u_{iN+j} - S_j^{iN+j} u_{iN+j}\| = 0, \ \forall j = 1, 2, \ldots, N. \tag{3.17}$$

Furthermore, for all $j = 1, 2, \ldots, N$, we obtain

$$\begin{aligned} \|S_j^{iN+j-1}u_{iN+j-1} - S_j^{iN+j}u_{iN+j}\| \leq\ & \|S_j^{iN+j-1}u_{iN+j-1} - u_{iN+j-1}\| \\ & + \|u_{iN+j-1} - u_{iN+j}\| \\ & + \|u_{iN+j} - S_j^{iN+j}u_{iN+j}\| \to 0 \text{ as } n \to \infty; \end{aligned} \tag{3.18}$$

$$\begin{aligned} & \|S_j^{iN+j-1}u_{iN+j} - S_j^{iN+j-1}u_{iN+j-1}\| \\ & \leq \frac{\sqrt{k_{iN+j-1}} + \sqrt{k}}{1 - \sqrt{k}} \|u_{iN+j} - u_{iN+j-1}\| \\ & + \frac{1}{1 - \sqrt{k}} \times \sqrt{2|\langle u_{iN+j} - S_j^{iN+j-1}u_{iN+j}, u_{iN+j-1} - S_j^{iN+j-1}u_{iN+j-1}\rangle|} \\ & \to 0 \text{ as } i \to \infty; \end{aligned} \tag{3.19}$$

$$\begin{aligned} \|u_{iN+j} - S_j^{iN+j-1}u_{iN+j}\| \leq\ & \|u_{iN+j} - S_j^{iN+j-1}u_{iN+j-1}\| \\ & + \|S_j^{iN+j-1}u_{iN+j-1} - S_j^{iN+j-1}u_{iN+j}\| \\ \leq\ & \|u_{iN+j} - u_{iN+j-1}\| + \|u_{iN+j-1} - S_j^{iN+j-1}u_{iN+j-1}\| \\ & + \|S_j^{iN+j-1}u_{iN+j-1} - S_j^{iN+j-1}u_{iN+j}\| \\ & \to 0 \text{ as } i \to \infty; \end{aligned} \tag{3.20}$$

$$\begin{aligned} \|S_j^{iN+j-1}u_{iN+j} - S_j^{iN+j}u_{iN+j}\| \leq\ & \|S_j^{iN+j-1}u_{iN+j} - S_j^{iN+j-1}u_{iN+j-1}\| \\ & + \|S_j^{iN+j-1}u_{iN+j-1} - S_j^{iN+j}u_{iN+j}\| \\ & \to 0 \text{ as } i \to \infty. \end{aligned} \tag{3.21}$$

From (3.10), (3.17)-(3.21), we have that

$$\begin{aligned}
& \|u_{iN+j} - S_j u_{iN+j}\| \\
\leq \; & \|u_{iN+j} - S_j^{iN+j} u_{iN+j}\| + \|S_j^{iN+j} u_{iN+j} - S_j u_{iN+j}\| \\
\leq \; & \|u_{iN+j} - S_j^{iN+j} u_{iN+j}\| + \frac{\sqrt{k_1}+\sqrt{k}}{1-\sqrt{k}} \|u_{iN+j} - S_j^{iN+j-1} u_{iN+j}\| \\
& + \frac{1}{1-\sqrt{k}} \times \sqrt{2|\langle u_{iN+j} - S_j u_{iN+j}, S_j^{iN+j-1} u_{iN+j} - S_j^{iN+j} u_{iN+j}\rangle|} \\
& \to 0 \text{ as } i \to \infty. \qquad (3.22)
\end{aligned}$$

Similarly, we obtain $\lim_{i\to\infty} \|Ax_{iN+j} - T_j A x_{iN+j}\| = 0$. The rest of the proof follows from [6, 7] and this completes the proof. □

Remark 3.8. The associated results in [6, 7] can be extended to our more general class of mappings by using Lemma 3.4 and method of Theorem 3.7.

The following is an example of discontinuous asymptotically k-strictly pseudononspreading mapping, which is covered by our results but for which the results of [6, 7] do not.

Example 3.9. Let $\Re$ denote the reals with the usual norm and define a mapping $T : \Re \to \Re$ by

$$Tx = \begin{cases} 0, \; x \in (-\infty, 2], \\ 1, \; x \in (2, \infty). \end{cases}$$

For every $x, y \in (-\infty, 2]$, and $\beta \in [0,1)$, we have

$$\begin{aligned}
& |x-y|^2 + \beta\, |x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\
& = x^2 + y^2 + \beta\, |x - Tx - (y - Ty)|^2 \\
& \geq 0 = |Tx - Ty|^2 .
\end{aligned}$$

Furthermore, for every $x, y \in (2, \infty)$, we have

$$\begin{aligned}
& |x-y|^2 + \beta\, |x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\
& = |x-y|^2 + \beta\, |x - Tx - (y - Ty)|^2 \\
& \quad + 2(x-1)(y-1) \geq 0 = |Tx - Ty|^2 .
\end{aligned}$$

For every $x \in (-\infty, 2], y \in (2, \infty)$, we have

$$\begin{aligned}
& |x-y|^2 + \beta\, |x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \\
& = x^2 + y^2 - 2xy + \beta\, |x - Tx - (y - Ty)|^2 + 2x(y-1) \\
& = x^2 - 2x + y^2 + \beta\, |x - Tx - (y - Ty)|^2 \\
& = (x-1)^2 + y^2 - 1 + \beta\, |x - Tx - (y - Ty)|^2 \\
& \geq y^2 - 1 \geq 3 > 1 = |Tx - Ty|^2 .
\end{aligned}$$

T is strictly pseudononspreading. Since $T^n x = 0$ for all $x \in \Re$, $n \geq 2$, we have that T is strictly asymptotically pseudononspreading.

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