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A SOLUTION FOR THE MINIMAX PROBLEM VIA FIXED POINT THEORY IN COMPLETE ORDERED LOCALLY CONVEX SPACES

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Abstract. In this paper, we establish a type of minimax problem for multivalued mappings in ordered locally convex spaces under mild conditions, which can be viewed as another version of the famous Ky Fan minimax theorem.

Keywords. Fixed point; Measure of noncompactness; Multivalued mapping; Ordered locally convex spaces; Game theory

1. Introduction

The minimax problems have been studied extensively in the literature. These problems are important because the minimax theorem offers a wide array of applications. Minimax theorems of real-valued functions have been discussed since the 1960s; see, e.g., [1], [2], [3], [4] and references therein. In recent years, based on the development of the game theory, the optimization theory, and economics, a great deal of research has been devoted to the study of the minimax theorem [5], [6], [7]. A lot of research has been devoted to the study of the existence of a fixed points of single-valued and multivalued mappings in ordered Banach spaces [8], [9], and in complete locally convex spaces [10], [11].

In this paper, we establish some fixed point theorems of multivalued mappings a complete ordered locally convex spaces under weaker assumptions. It is well known that partial order plays an important role in optimization theory. The optimization problems in the previous references, which were studied in the partial order induced by a closed cone such as the subsets of space, were compact convex or closed convex. But in some situations, the subsets of space are order convex. Our fixed point results can be seen as generalizations of [8, Theorem 2.1.1]

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and [4]. By using our fixed point results, we derive some Minimax theorems of condensing multivalued mapping in the case where the subsets of space are order convex.

2. Notations and preliminaries

Let E be a real vector space. A cone K in E is a subset of E with $K+K\subset K$, $\alpha K\subset K$ for all $\alpha\geq 0$, and $K\cap(-K)=\{0\}$. As usual, E will be ordered by the (partial) order relation

$$x\leq y\Leftrightarrow y-x\in K$$

and the cone K will be denoted by E^+. E is said to be an ordered topological vector space if E is an ordered vector space equipped with a linear topology for which the positive cone E^+ is closed. For two vectors $x,y\in E$, the order interval $[x,y]$ is the set defined by

$$[x,y]=\{z\in E: x\leq z\leq y\}.$$

Note that if $x\nleq y$, then $[x,y]=\phi$.

A cone E^+ of an ordered topological vector space E is said to be normal whenever the topology of E has a base at zero consisting of order convex sets. If the topology of E is also locally convex, then E is said to be an ordered locally convex space. In this case, the topology of E has a base at zero consisting of open, circled, convex, and order convex neighborhoods. $\overline{conv}(A)$ denotes the closed convex hull of A.

The following two lemmas will be useful in the proofs of our results.

Lemma 2.1 ([12, Lemma 2.3]). *If E is an ordered topological vector space, then E is Hausdorff and the order intervals of E are closed.*

Lemma 2.2 ([12, Lemma 2.22 and Theorem 2.23]). *If the cone E^+ of an ordered topological vector space (E,τ) is normal, then the following assertions hold:*

(1) every order interval is $\tau-$bounded,

(2) for every two nets (x_α), $(y_\alpha)\subset E$, (with the same index set I) satisfy $0\leq x_\alpha\leq y_\alpha$ for each α and $y_\alpha\overset{\tau}{\to}0$ imply $x_\alpha\overset{\tau}{\to}0$.

Let E be an ordered locally convex space whose topology is defined by a family $\mathscr{P}$ of continuous semi-norms on E, $\mathscr{B}$ the family of all bounded subsets of E, and Φ the space of all functions $\varphi:\mathscr{P}\to\mathbb{R}^+$ with the usual partial ordering $\varphi_1\leq\varphi_2$ if $\varphi_1(p)\leq\varphi_2(p)$ for all $p\in\mathscr{P}$. The measure of noncompactness on E is the function $\alpha:\mathscr{B}\to\Phi$ such that, for every $B\in\mathscr{B}$, $\alpha(B)$ is the function from $\mathscr{P}$ into $\mathbb{R}^+$ defined by

$$\alpha(B)(p)=\inf\{d>0:\sup\{p(x-y):x,y\in B_i\}\leq d\ \ \forall i\},$$

where the infimum is taken on all subsets B_i such that B is finite union of B_i. Properties of measure of noncompactness in locally convex spaces were presented in [10, Proposition 1.4].

An operator $T:Q\subset E\to E$ is said to be countably condensing if $T(Q)$ is bounded and if, for any countably bounded set A of Q with $\alpha(A)(p)>0$,

$$\alpha(T(A))(p)<\alpha(A)(p)$$

Definition 2.3. Let E be a complete ordered locally convex space with a normal cone E^+. An element $x\in E$ is said to be a fixed point of a multivalued mapping $T:E\to 2^E$ if $x\in T(x)$.

Definition 2.4. Let E be a complete ordered locally convex space with a normal cone E^+. Let $A,B \in 2^E$. Then $A \leq B$ means $a \leq b$ for all $a \in A$ and $b \in B$. A map $T : E \to 2^E$ is said to be isotone nondecreasing if, for $x,y \in E$ and $x \leq y$, $Tx \leq Ty$. A map $T : E \to 2^E$ is said to be isotone nonincreasing if, for $x,y \in E$ and $x \leq y$, $Tx \geq Ty$.

Definition 2.5. Let E be a complete ordered locally convex space with a normal cone E^+. Two maps $S,T : E \to 2^E$ are said to be weakly isotone increasing if, for any $x \in E$, $Sx \leq Ty$ for all $y \in Sx$ and $Tx \leq Sy$ for all $y \in Tx$. S and T are called weakly isotone decreasing if, for any $x \in E$, $Sx \geq Ty$ for all $y \in Sx$ and $Tx \geq Sy$ for all $y \in Tx$. Also two mappings S and T are called weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing.

Definition 2.6. Let E be a complete ordered locally convex space with a normal cone E^+, and let $Q \subset E$. An operator $T : Q \to 2^Q$ is said to be countably condensing if $T(Q)$ is bounded and if, for any countably bounded set A of Q with $\alpha(A)(p) > 0$,

$$\alpha(T(A))(p) < \alpha(A)(p)$$

with $T(A) = \cup_{x \in A} Tx$.

3. Main results

The following results generalize the results of [8] and [4] in complete ordered locally convex spaces, and we add another results whith weak conditions.

Lemma 3.1. *Let E be an ordered topological vector space with a normal cone E^+. Then a monotone net $(u_\alpha) \subset E$ is convergent if and only if it has a weakly convergent subnet.*

Proof. The "only if" part is obvious. For the " if " part, we assume that $(u_\alpha)_{\alpha \in (\alpha)}$ is nondecreasing and let $(u_{\alpha_i})_{i \in (i)} \subset (u_\alpha)$ be a subnet such that $u_{\alpha_i} \to u$ weakly for some $u \in E$, where (α) stands for the indexed set of the net (u_α). Let $\beta \in (\alpha)$ be fixed. For each $\alpha \geq \beta$, let $i_0 \in (i)$ such that $\alpha_{i_0} \geq \alpha$. Thus, for each $i \geq i_0$,

$$u_\beta \leq u_\alpha \leq u_{\alpha_i}. \tag{3.1}$$

Thus, since $u_{\alpha_i} \to u$ weakly and the cone E^+ is weakly closed (being a closed and convex set), we see that $u_\beta \leq u$ for each $\beta \in (\alpha)$. Thus, it follows from [12, Lemma 2.28] that $\lim u_{\alpha_i} = u$.

Now, let $V \in V(0)$ be arbitrary. Since E^+ is normal, we may assume that V is an order convex set. Let $j \in (i)$ such that $u - u_{\alpha_i} \in V$ for each $i \geq j$. If $\beta \geq \alpha_j$, then $0 \leq u - u_\beta \leq u - u_{\alpha_j}$. Hence $u - u_\beta \in V$. That is, $\lim u_\beta = u$ as required. The desired conclusion is proved similarly when (u_α) is nonincreasing. □

In the following theorem, we use the notion of a closed (have closed graph) mapping in the case of a locally convex space. A Hausdorff locally convex space is regular, [13, see Chapter VI, Section 1].

Theorem 3.2. *Let E be a complete ordered locally convex space with a normal cone E^+. Let Ω be an order convex subset of E. Let $u_0, v_0 \in \Omega$, $u_0 \leq v_0$ and let $T : \Omega \to 2^\Omega$ be a monotone closed and isotone nondecreasing mappings such that T fixes the interval $[u_0, v_0]$. Suppose that T is condensing. Then, T has a minimal fixed point u and a maximal fixed point v in Ω.*

Proof. Consider the sequences (u_n) and (v_n) defined by:

$$u_n \in Tu_{n-1}, \quad v_n \in Tv_{n-1}, \quad n \in \mathbb{N}. \tag{3.2}$$

Since T is isotone nondecreasing, we fixes the interval $[u_0, v_0]$. It follows from (3.2) that

$$u_0 \leq u_1 \leq .... \leq u_n \leq ..... \leq v_n \leq ... \leq v_1 \leq v_0. \tag{3.3}$$

Observe that $[u_0, v_0] \subset \Omega$ because Ω is a order convex subset of E. Letting $A = \{u_0, u_1, ...\}$, we have $A = \{u_0\} \cup T(A)$ and the set A is bounded since T is condensing (so in particular $T(\Omega)$ is bounded). So, $\overline{A}$ is compact by [14, p 89], and $\{u_n\}$ has a convergent subnet, which converges to $u \in [u_0, v_0]$. By (3.3), $\{u_n\}$ is nondecreasing. It follows from lemma 3.1 that the original sequence $\{u_n\}$ converges to $u \in [u_0, v_0] \subset \Omega$. Also, we have

$$u = \lim_{n\to\infty} u_n \text{ and } u_n \in Tu_{n-1}$$

Since T is monotone closed mapping, we have $u \in Tu$. Similarly, we can prove that $\{v_n\}$ converges to some $v \in E$ and $v \in Tv$.

Finally, we prove that u and v are the maximal and minimal fixed points of T in $[u_0, v_0] \subset \Omega$. Indeed, let $x \in [u_0, v_0]$ and $x \in Tx$. Since T is nondecreasing, we have $u_n \leq x \leq v_n$. Taking limit $n \to \infty$, we obtain $u \leq x \leq v$. □

Theorem 3.3. *[15] Let E be a complete locally convex space. Let $C \subset E$ be a nonempty compact convex subset, and let $T : C \to 2^C$ be a monotone closed such that $T(x)$ is nonempty closed and convex for every $x \in C$. Then T has a fixed point $u \in C$.*

Theorem 3.4. *Let E be a complete locally convex space. Let $C \subset E$ be a nonempty closed, bounded and convex subset, and let $T : C \to 2^C$ be a monotone closed such that $T(x)$ is nonempty closed and convex for every $x \in C$, and $\exists k \in [0, 1[: \alpha(T(\Omega))(p) \leq k\alpha(\Omega)(p)$,for all $\Omega \subseteq K$. Then, T has a fixed point $u \in C$.*

Proof. Let $\Omega_0 = C$. By induction, we show $\Omega_n = \overline{conv}(T(\Omega_{n-1}))$, for $n \in \mathbb{N}$. Note that $\Omega_n \subset \Omega_{n-1}$ and $\alpha(\Omega_n)(p) \leq k^n \alpha(\Omega_0)(p)$. Indeed, $\Omega_1 \subset \Omega_0$, and the properties of α give

$$\alpha(\Omega_1)(p) = \alpha(\overline{conv}(T(\Omega_0)))(p) = \alpha(T(\Omega_0))(p) \leq k\alpha(\Omega_0)(p).$$

So $(*)$ is obvious for $n = 1$. Suppose that this relations is valid to order $n > 1$. Then,

$$\Omega_{n+1} = \overline{conv}(T(\Omega_n)) \subset \overline{conv}(T(\Omega_{n-1})) = \Omega_n,$$

and

$$\alpha(\Omega_{n+1})(p) = \alpha(\overline{conv}(T(\Omega_n)))(p) = \alpha(T(\Omega_n))(p) \leq k\alpha(\Omega_n)(p) \leq k^{n+1}\alpha(\Omega_0)(p).$$

So, $\lim_{n\to\infty} \alpha(\Omega_n)(p) = 0$, donc $\Omega = \cap_{n\geq 0}\Omega_n$ is non-empty and compact [10, see Lemma 2.4], and Ω is convex. On the other hand, $T(\Omega_n) \subset T(\Omega_{n+1}) \subset \overline{conv}(T(\Omega_{n-1})) = \Omega_n, n \geq 1$. So, $T : \Omega \to 2^\Omega$. By using theorem 3.3, we have that T has a fixed point in $\Omega \subset C$. □

Theorem 3.5. *Let E be a complete locally convex space. Let $C \subset E$ be a nonempty closed, bounded and convex subset, and let $T : C \to 2^C$ be a condensing and monotone closed mapping such that $T(x)$ is nonempty closed and convex for every $x \in C$. Then, T has a fixed point $u \in C$.*

Proof. Let $k \in [0,1[$ and $S = kT$. So, for all $\Omega \subseteq K, \alpha(S(\Omega))(p) = k\alpha(T(\Omega))(p) \leq k\alpha(\Omega)(p)$. Then, by using Theorem 3.4, we have that S has a fixed point $x \in Sx = kTx$. We pose $k = k_n \to 1$ if $n \to \infty$. So, $x_n \in k_nTx_n$. Note that $\{x_n - Tx_n\} = (1-k_n)\{Tx_n\}$ is relatively compact. Now,

$$\begin{aligned} \alpha(x_n)(p) &= \alpha((x_n - Tx_n) + T(x_n))(p) \\ &\leq \alpha((x_n - Tx_n))(p) + \alpha(T(x_n))(p) \\ &= \alpha(T(x_n))(p) \\ &< \alpha(x_n)(p). \end{aligned}$$

So, $\alpha(x_n)(p) = 0$. By using [14, p 89], we have $\{x_n\}$ has a convergent subnet which converges to $z \in C$. As T is monotone closed, we have $z \in Tz$. This completes the proof. □

Theorem 3.6. *Let E be a complete ordered locally convex space with a normal cone E^+. Let Ω be an order convex subset of E. Let $u_0, v_0 \in \Omega$, $u_0 \leq v_0$ and let $T : \Omega \to 2^\Omega$ be a monotone closed and isotone nonincreasing mappings such that T^2 fixes the interval $[u_0, v_0]$. Suppose that T is condensing and $T(x)$ is nonempty closed and convex for every $x \in C$. Then, T has a fixed point in Ω.*

Proof. Note that T is condensing and monotone closed, so is T^2. We also have that T^2 is nondecreasing and fixes the interval $[u_0, v_0]$. Then, from 3.3, T^2 has a minimal fixed point u and a maximal fixed point v in Ω. It is easy to see that Tu and Tv are likwise a fixed point of T^2. Therefore, we have

$$u \leq Tv \leq Tu \leq v$$

Now, if $x \in [u, v]$, then

$$u \leq Tv \leq Tx \leq Tu \leq v$$

It follows that T fixes the interval $[u, v]$. Then $T[u, v]$ is bounded. Now, because the cone E^+ is normal, we have that the interval $[u, v]$ is a convex, closed, and bounded subset of E. By using Theorem 3.5, it follows that T has a fixed point in $[u, v] \subset \Omega$. □

4. THE APPLICATION TO GAME THEORY

In this section, we give an application of Theorem 3.6 to game theory with a new method and new hypothesis. A game is a triple (A, B, K), where A, B are nonempty ordred sets ([5, page 326]), whose elements are called strategies, and $K : A \times B \to \mathbb{R}$ is the gain function. There are two players, α and β, and $K(x, y)$ represents the gain of the player α when he chooses the strategy $x \in A$ and the player β chooses the strategy $y \in B$. The quantity $-K(x, y)$ represents the gain of the player β in the same situation. The target of the player α is to maximize his gain when the player β chooses a strategy that is the worst for α, that is, to choose $x_0 \in A$ such that

$$\inf_{y \in B} K(x_0, y) = \max_{x \in A} \inf_{y \in B} K(x, y).$$

Similarly, the player β chooses $y_0 \in B$ such that

$$\sup_{x \in A} K(x, y_0) = \min_{y \in B} \sup_{x \in A} K(x, y).$$

It follows

$$\sup_{x\in A}\inf_{y\in B}K(x,y)=\inf_{y\in B}K(x_0,y)\leq K(x_0,y_0)\leq \sup_{x\in A}K(x,y_0)\leq \inf_{y\in B}\sup_{x\in A}K(x,y). \tag{4.1}$$

Note that, in general,

$$\sup_{x\in A}\inf_{y\in B}K(x,y)\leq \inf_{y\in B}\sup_{x\in A}K(x,y). \tag{4.2}$$

If the equality holds in (4.1), then, we have from (4.2) that

$$\sup_{x\in A}\inf_{y\in B}K(x,y)=K(x_0,y_0)=\inf_{y\in B}\sup_{x\in A}K(x,y). \tag{4.3}$$

The common value in (4.3) is called the value of the game, $(x_0,y_0)\in A\times B$ a solution of the game and x_0 and y_0 winning strategies. To prove the existence of a solution of a game, we have to prove equality (4.2). For more details on game theory and minimax theorems, we refer to the books of Carl-Heikkilä [5] and Aubin [6]. Let $(X,\leq_X)$ and $(Y,\leq_Y)$ be a complete ordered locally convex spaces. Consider in the product space $(X\times Y,\ll)$ the following partial orders. For any $A_1\times B_1, A_2\times B_2\in 2^{X\times Y}$, we denote

$$A_1\times B_1\ll A_2\times B_2 \text{ iff } A_1\leq_X A_2 \text{ and } B_1\leq_Y B_2$$

With

$$A_1\leq_X A_2 \text{ iff } x_1\leq_X x_2,\quad \forall (x_1,x_2)\in A_1\times A_2,\quad (1)$$

It is easy to see that if X^+ is a normal cone in X and Y^+ is a normal cone in Y, then $(X^+\times Y^+)$ is also a normal cone of the product ordered topological space $X\times Y$. In this section, $\leq$ and $<$ mean the total order relation of $\mathbb{R}$. For $A\subset X, B\subset Y$, $\pi_X: A\times B\to A$ denotes the first projection mapping, i.e., $\pi_X(x,y)=x$, $\forall (x,y)\in A\times B$. And for a subset $A\subset X, B\subset Y$, $\pi_Y: A\times B\to B$ denotes the second projection mapping, i.e., $\pi_Y(x,y)=y$, $\forall (x,y)\in A\times B$. The projection mapping π_X is continuous (see, [16, see Chapter IV, p 81]). We set

$$\phi(x)=\min_{y\in B}K(x,y)=\min K(x\times B), x\in A$$

and

$$\psi(y)=\max_{x\in A}K(x,y)=\max K(A\times y), y\in B$$

We exclude the trivial case, and assume that the sets

$$\{x\in A: K(x,y)=\psi(y)\} \text{ and } \{y\in B: K(x,y)=\phi(x)\}$$

are nonempty. We set

$$N_y=\{x\in A: K(x,y)=\psi(y)\} \text{ and } M_x=\{y\in B: K(x,y)=\phi(x)\}$$

and

$$N_{D'}=\cup_{y\in D'}N_y,\;\; M_D=\cup_{x\in D}M_x$$

for any set $(D\times D')$ of $A\times B$

In the sequel, we consider the measure of noncompactness $\alpha^{\times}(p)$ on a product of locally convex spaces; see [17].

Theorem 4.1. *Let $X_1, X_2, ..., X_n$ be a complete locally convex spaces. Assume that $\alpha_1, \alpha_2, ..., \alpha_n$ be the measure of noncompactness in $X_1, X_2, ..., X_n$ respectively, Suppose that $F : ([0,+\infty[)^n \to [0,+\infty[$ is*

(1) convexe.
(2) $F(x_1, x_2, ..., x_n) = 0$ if and only if $x_i = 0$ for $i = 1,2,...,n$.

Then, for each $D \in \mathscr{B}(X_1 \times X_2 \times ... \times X_n)$,

$$\alpha^{\times}(D)(p) = F(\alpha_1(D_1)(p), \alpha_2(D_2)(p), ..., \alpha_n(D_n)(p)).$$

defines a measure of noncompactness in $X_1 \times X_2 \times ... \times X_n$, where $D_1, D_2, ..., D_n$ denote the natural projection of D into X_i for $i = 1,...,n$.

Theorem 4.2. *Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be a complete ordered locally convex spaces, and let $A \subset X$ and $B \subset Y$ nonempty order convex sets. Let $(u_0, v_0), (u_1, v_1) \in A \times B$ such that $(u_0, v_0) \ll (u_1, v_1)$. Suppose that*

(1) the functions $K : A \times B \to \mathbb{R}, \psi$ and ϕ are continuous;
(2) $\forall x, x' \in A$ with $x \leq_X x' \Rightarrow M_x \geq_Y M_{x'}$, and
$\forall y, y' \in B$ with $y \leq_Y y' \Rightarrow N_y \geq_X N_{y'}$;
(3) $\forall (D \times D') \subset \mathscr{B}(X \times Y)$:
$F(\alpha_1(N_{D'})(p), \alpha_2(M_D)(p)) < F(\alpha_1(D)(p), \alpha_2(D')(p))$;
(4) $\exists (w_0, w_1) \in N_{v_0} \times M_{u_0}$ such that $\{(u_0, v_0)\} \ll N_{w_1} \times M_{w_0}$
and $\exists (w'_0, w'_1) \in N_{v_1} \times M_{u_1}$ such that $N_{w'_1} \times M_{w'_0} \ll \{(u_1, v_1)\}$.

Then,

$$\min_{y \in B} \max_{x \in A} K(x,y) = \max_{x \in A} \min_{y \in B} K(x,y) \tag{4.4}$$

and the game (A, B, K) has a solution.

Proof. We pose $C = A \times B$ and $c = (x, y)$. The product set C is ordred convex (product of two ordred convex). Therefore, the following two mapping can be defined by:

$$\begin{aligned} T : C &\to 2^C, \\ c &\mapsto N_y \times M_x, \end{aligned}$$

where $c = (x,y) \in A \times B$. First, we show that T have the closed graph. Indeed, Let $\{(x_\alpha, y_\alpha)\}$ be a net in C such that $(x_\alpha, y_\alpha) \to (x,y) \in C$. Let $\{(u_\alpha, v_\alpha)\}$ be a net such that $(u_\alpha, v_\alpha) \in T(x_\alpha, y_\alpha)$, and $(u_\alpha, v_\alpha) \to (u, v)$. We then show that $(u, v) \in T(x, y)$. Note that

$$\begin{aligned} (u_\alpha, v_\alpha) \in T(x_\alpha, y_\alpha) \quad &\Leftrightarrow \quad (u_\alpha, v_\alpha) \in N_{y_\alpha} \times M_{x_\alpha} \\ &\Leftrightarrow \quad K(u_\alpha, y_\alpha) = \psi(y_\alpha) \text{ and } K(x_\alpha, v_\alpha) = \phi(x_\alpha). \end{aligned}$$

Since K and ϕ are continuous, for $\alpha \in I$, we have that

$$K(u,y) = \psi(y) \text{ and } K(x,v) = \phi(x)$$

So, $(u, v) \in T(x, y)$, which implies that T has a closed graph.

Now, we show T is countably condensing map. Indeed, Let $Q \subseteq C$ a countable and bounded set. Using the hypothesis (3), we have

$$\begin{aligned}
\alpha^{\times}(T(Q))(p) &= \alpha^{\times}(\cup_{c\in Q}Tc)(p) \\
&= \alpha^{\times}(\cup_{(x,y)\in Q}N_y \times M_x)(p) \\
&= \alpha^{\times}(\cup_{(x,y)\in Q}N_{\pi_B(x,y)} \times M_{\pi_A(x,y)})(p) \\
&= \alpha^{\times}(N_{\pi_B(Q)} \times M_{\pi_A(Q)})(p) \\
&= F(\alpha_1(N_{\pi_B(Q)})(p), \alpha_2(M_{\pi_A(Q)})(p)) \\
&< F(\alpha_1(\pi_A(Q))(p), \alpha_2(\pi_B(Q))(p)) \\
&= \alpha^{\times}(Q)(p).
\end{aligned}$$

This shows that T is a countably condensing on C. Using hypothesis (2), it is clear to see that T is nonincreasing. Finally, from $w = (w_0, w_1), w' = (w'_0, w'_1), u = (u_0, v_0), v = (u_1, v_1)$, we have

$$\begin{aligned}
\exists (w_0, w_1) \in N_{v_0} \times M_{u_0} : \{(u_0, v_0)\} \ll N_{w_1} \times M_{w_0} &\Leftrightarrow \exists w \in Tu, \{u\} \ll Tw \\
&\Leftrightarrow \{u\} \ll \bigcup_{w\in Tu} Tw \\
&\Leftrightarrow \{u\} \ll T^2 u,
\end{aligned}$$

which implies that

$$\begin{aligned}
\exists (w'_0, w'_1) \in N_{v_1} \times M_{u_1} : N_{w'_1} \times M_{w'_0} \ll \{(u_1, v_1)\} &\Leftrightarrow \exists w' \in Tv, Tw' \ll \{v\} \\
&\Leftrightarrow \bigcup_{w'\in Tv} Tw \ll \{v\} \\
&\Leftrightarrow T^2 v \ll \{v\}
\end{aligned}$$

Since T^2 is nondecreasing, we, therefore, have that T^2 fixes the interval $[u, v]$. Thus, by using theorem 3.6, we have that T has a fixed point $c^{\star} = (x^{\star}, y^{\star})$. So, $c^{\star} \in Tc^{\star} = N_{y^{\star}} \times M_{x^{\star}}$. In other words,

$$x^{\star} \in N_{y^{\star}} \Leftrightarrow K(x^{\star}, y^{\star}) = \max_{x\in A} K(x, y^{\star}) \geq \inf_{y\in B} \max_{x\in A} K(x, y),$$

and

$$y^{\star} \in M_{x^{\star}} \Leftrightarrow K(x^{\star}, y^{\star}) = \min_{y\in B} K(x^{\star}, y) \leq \sup_{x\in A} \min_{y\in B} K(x, y)$$

Taking into account these last two inequalities and (4.2), we get

$$K(x^{\star}, y^{\star}) \leq \sup_{y\in B} \min_{x\in A} K(x, y) \leq \inf_{x\in A} \max_{y\in B} K(x, y) \leq K(x^{\star}, y^{\star}),$$

which implies that

$$\max_{x\in A} \min_{y\in B} K(x, y) = K(x^{\star}, y^{\star}) = \min_{y\in B} \max_{x\in A} K(x, y).$$

This completes the proof. □

Corollary 4.3. *Under the same assumptions of Theorem 4.2, we now consider that* $\exists(x_0,y_0),(x_1,y_1) \in X\times Y$ *such that* $(x_0,y_0) \ll (x_1,y_1)$ *and the functions* $K : X\times Y \to \mathbb{R}$. *Then,*

$$\min_{y\in Y}\max_{x\in X} K(x,y) = \max_{x\in X}\min_{y\in Y} K(x,y) \tag{4.5}$$

and the game (X,Y,K) *has a solution.*

Proof. Since $[x_0,x_1]$ and $[y_0,y_1]$ are two ordred convex sets in X and Y, respectively, we apply the previous theorem for $A = [x_0,x_1]$ and $B = [y_0,y_1]$ to obtain the desired conclusion. □

We give another application of Theorem 3.6, which is to result of J.von Neumann [18] (see also [7]).

Theorem 4.4. *Let* $(X,\leq_X)$ *and* $(Y,\leq_Y)$ *be a complete ordered locally convex spaces and* $A\subset X, B\subset Y$ *nonempty ordred convex sets. For* $Q,Q'\subset A\times B$, *we pose*

$$N_y = \{x\in A : (x,y)\in Q\} \text{ and } M_x = \{y\in B : (x,y)\in Q'\}$$

and suppose that

(1) the sets Q,Q' *are closed;*
(2) $\forall x,x'\in A$ *with* $x\leq_X x' \Rightarrow M_x \geq_Y M_{x'}$, *and*
$\forall y,y'\in B$ *with* $y\leq_Y y' \Rightarrow N_y \geq_X N_{y'}$;
(3) $\forall(D\times D')\subset\mathscr{B}(X\times Y)$:
$F(\alpha_1(N_{D'})(p),\alpha_2(M_D)(p)) < F(\alpha_1(D)(p),\alpha_2(D')(p))$;
(4) $\exists(w_0,w_1)\in N_{v_0}\times M_{u_0}$ *such that* $\{(u_0,v_0)\}\ll N_{w_1}\times M_{w_0}$
and $\exists(w'_0,w'_1)\in N_{v_1}\times M_{u_1}$ *such that* $N_{w'_1}\times M_{w'_0}\ll\{(u_1,v_1)\}$.

Then, $Q\cap Q'\neq\emptyset$

Proof. Set $C = A\times B$ and $c = (x,y)$. The product set C is ordred convex (product of two ordred convex). Then the following two mapping can be defined by

$$\begin{aligned} T : C &\to 2^C, \\ c &\mapsto N_y\times M_x, \end{aligned}$$

where $c = (x,y)\in A\times B$. First, we will show that T have the closed graph. Indeed, Let $\{(x_\alpha,y_\alpha)\}$ be a net in C such that $(x_\alpha,y_\alpha)\to(x,y)\in C$. Let $\{(u_\alpha,v_\alpha)\}$ be a net such that $(u_\alpha,v_\alpha)\in T(x_\alpha,y_\alpha)$, and $(u_\alpha,v_\alpha)\to(u,v)$. We next show that $(u,v)\in T(x,y)$. Observe that

$$\begin{aligned} (u_\alpha,v_\alpha)\in T(x_\alpha,y_\alpha) &\Leftrightarrow (u_\alpha,v_\alpha)\in N_{y_\alpha}\times M_{x_\alpha} \\ &\Leftrightarrow u_\alpha\in N_{y_\alpha} \text{ and } v_\alpha\in M_{x_\alpha} \\ &\Leftrightarrow (u_\alpha,y_\alpha)\in Q \text{ and } (x_\alpha,v_\alpha)\in Q'. \end{aligned}$$

Since Q,Q' are closed, for $\alpha\in I$, we have that

$$(u,y)\in Q \text{ and } (x,v)\in Q'$$

So, $(u,v)\in T(x,y)$, which implies that T has a closed graph. We prove the rest by the same method of theorem 4.2. Thus, T has a fixed point $c^\star = (x^\star,y^\star)$. So, we have $c^\star\in Tc^\star$, that is,

$$(x^\star,y^\star)\in N_{y^\star}\times M_{x^\star} \Leftrightarrow x^\star\in N_{y^\star} \text{ and } y^\star\in M_{x^\star},$$

which implies

$$(x^\star, y^\star) \in Q \cap Q'.$$

This completes the proof. □

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