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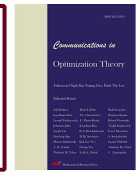

BLOW-UP OF SOLUTIONS FOR ELASTIC MEMBRANE EQUATIONS WITH FRACTIONAL BOUNDARY DAMPING

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Abstract. By writing the elastic membrane equation with the boundary condition of the fractional derivative type as an augmented system, we study the blow up of solutions with nonpositive initial energy combined with a positive initial energy.

Keywords. Elastic membrane equation; Balakrishnan-Taylor damping; Fractional boundary dissipation; Blow up; Life span.

1. Introduction

Partial differential equations with a fractional derivatives have attracted the attention of many researchers in mathematical, biological and physical fields; see, e.g., [1–3]. In the recent years, they have been widely applied in electronics , relaxation vibrations and viscoelasticity etc; see, e.g., [4, 5].

In this paper, we consider the following Kirchhoff equation with Balakrishnan-Taylor damping, fractional boundary condition and source terms:

$$\begin{cases} u_{tt} - \left(\xi_0 + \xi_1 \|\nabla u\|_2^2 + \xi_2(\nabla u, \nabla u_t)\right)\Delta u = |u|^{p-1}u, & x \in \Omega, t > 0, \\ \left(\xi_0 + \xi_1 \|\nabla u\|_2^2 + \xi_2(\nabla u, \nabla u_t)\right)\frac{\partial u}{\partial \nu} = -b\partial_t^{\alpha,\eta} u, & x \in \Gamma_0, t > 0, \\ u = 0, & x \in \Gamma_1, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a regular and bounded domain in $\mathbb{R}^n$, $(n \geq 1)$ with smooth boundary $\partial\Omega$ such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$ and Γ_0, Γ_1 have positive measure, $\partial\nu$ denotes the unit outer normal

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and $(.,.)$ the inner product with its corresponding norm$\|.\|_2$. The functions $u(x,t)$ is the plate transverse displacement. The viscoelastic structural damping terms $\xi_0+\xi_1\|\nabla u\|_2^2+\xi_2(\nabla u,\nabla u_t)$ is the nonlinear stiffness of the membrane. ξ_0,ξ_1,ξ_2 and b are positive constants. The initial data (u_0,u_1) are given functions. From the physical point of view, problem (1.1) is related to the panel flutter equation and to the spillover problem. The notation $\partial_t^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative (see [6] and [7]) defined by the following formula:

$$\partial_t^{\alpha,\eta}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}e^{-\eta(t-s)}u_s(s)ds,\quad \eta\geq 0,$$

where Γ is the usual Euler gamma function and $0<\alpha<1$. We recall some results related to Kirchhoff equation with Balakrishnan-Taylor damping

$$u_{tt}-\left(\xi_0+\xi_1\|\nabla u\|_2^2+\xi_2(\nabla u,\nabla u_t)\right)\Delta u+\int_0^t h(t-s)\Delta u ds=|u|^p u.$$

In [8], Tatar and Zaraï considered the above equation and proved the global existence and polynomial decay of the problem. Exponential decay and blow up of solution to the problem were established in Tatar and Zaraï [9]. Park [10] studied the homogeneous case, and established a general decay result of the problem without imposing the usual relation between the relaxation function h and its derivative. Recently, Ha [11] proved a general decay result of energy without imposing any restrictive growth assumption on the damping term and weakening the usual assumptions on the relaxation function. Balakrishnan-Taylor damping $\xi_2(\nabla u,\nabla u_t)$, was initially proposed by Balakrishnan and Taylor in 1989 [12] and Bass and Zes [13]. For more results concerning Kirchhoff equation with Balakrishnan-Taylor damping, one can refer to Clark [14], Ha [11, 15], Tatar and Zaraï [16–18], Wu [19–21] and You [22]. Since very little attention has been paid to boundary condition of fractional derivative type with source term, motivated by above scenario, we prove under suitable conditions on the initial data that the nonlinear source of polynomial type is able to force solutions to blow-up in finite time. Here, three different cases on the sign of the initial energy are considered. In [23], by redescribing the fractional boundary condition by means of a suitable diffusion equation, we can transform problem (1.1) into an augmented model which can be easily tackled.

In the present paper, we consider problem (1.1). Under a suitable condition on the initial data, we give a several results concerning the blow up results to problem (1.1) for both positive and nonpositive initial energy. The paper is organized as follows. In Section 2, we present the preliminaries and some lemmas. In Section 3, we prove the blow-up of solutions on different cases of the sing of the initial energy.

2. Preliminaries

In this section, we provide some materials for the proof of our results.

Lemma 2.1. *[23] Let μ be the function defined by*

$$\mu(\xi):=|\xi|^{\frac{(2\alpha-1)}{2}},\quad \xi\in\mathbb{R},\quad 0<\alpha<1.$$

Then the relationship between the "input" U and the "output" O of the system

$$\begin{cases} \partial_t \phi(\xi,t)+(\xi^2+\eta)\phi(\xi,t)-U(x,t)\mu(\xi)=0, \quad \xi\in\mathbb{R}, t>0, \eta\geq 0, \\ \phi(\xi,0)=0, \\ O(t):=(\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\phi(\xi,t)\mu(\xi)d\xi \end{cases} \tag{2.1}$$

is given by

$$O := I^{1-\alpha,\eta}U,$$

where

$$I^{\alpha,\eta}u(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\eta(t-s)}u(s)ds.$$

Definition 2.2. A solution u of (P) is called a blow-up solution if there exists a finite time $T^*>0$ such that

$$\lim_{t\to T^{*-}} \left(\|\nabla u(t)\|_2^2\right)^{-1} = 0. \tag{2.2}$$

Lemma 2.3. *[24] Let $\delta>0$ and $B\in C^2(0,\infty)$ be a nonnegative function satisfying*

$$B''(t)-4(\delta+1)B'(t)+4(\delta+1)B(t)\geq 0.$$

If

$$B'(0)>r_2B(0)+K_0, \text{ with } r_2 := 2(\delta+1)-2\sqrt{(\delta+1)\delta},$$

then

$$B'(t)>K_0 \text{ for } t>0, \text{where } K_0 \text{ is a constant}.$$

Lemma 2.4. *[24] If J is a nonincreasing function on $[t_0,\infty)$ and satisfies the differential inequality*

$$J'(t)^2 \geq a+bJ(t)^{2+\frac{1}{\delta}}, \quad t\geq t_0, \tag{2.3}$$

where $a>0$, $b\in\mathbb{R}$, then there exists a finite time T^* such that

$$\lim_{t\to T^{*-}} J(t)=0. \tag{2.4}$$

and T^* is such that:
(i) *If $b<0$, then*

$$T^*\leq t_0+\frac{1}{\sqrt{-b}}\ln\frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}}-J(t_0)}.$$

(ii) *If $b=0$, then*

$$T^*\leq t_0+\frac{J(t_0)}{J'(t_0)}.$$

(iii) *If $b>0$, then*

$$T^*\leq\frac{J(t_0)}{\sqrt{a}}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left(1 - [1 + cJ(t_0)]^{\frac{1}{2\delta}}\right),$$

where $c = \left(\frac{b}{a}\right)^{\delta/(2+\delta)}$.

3. Blow up of solutions

In this section, we use the method in [24] to consider the property of bolwing up of the solution of problem (1.1). By using Lemma 2.1, system (1.1) can be rewritten as :

$$\begin{cases} u_{tt} - \left(\xi_0 + \xi_1 \|\nabla u\|_2^2 + \xi_2 (\nabla u, \nabla u_t)\right) \Delta u = |u|^{p-1} u, \\ \partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) - u_t(x, t)\mu(\xi) = 0, \\ \left(\xi_0 + \xi_1 \|\nabla u\|_2^2 + \xi_2 (\nabla u, \nabla u_t)\right) \frac{\partial u}{\partial \nu} = -b_1 \int_{-\infty}^{+\infty} \phi(\xi, t)\mu(\xi) d\xi, \\ u = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \\ \phi(\xi, 0) = 0, \end{cases} \tag{3.1}$$

where $b_1 = (\pi)^{-1} \sin(\alpha\pi) b$. The energy functional is then given by:

$$\begin{aligned} E(t) := & \frac{1}{2}\|u_t\|_2^2 + \frac{\xi_0}{2}\|\nabla u\|_2^2 + \frac{\xi_1}{4}\|\nabla u\|_2^4 \\ & - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{b_1}{2}\int_{\Gamma_0}\int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho. \end{aligned} \tag{3.2}$$

Lemma 3.1. *Let (u, ϕ) be a regular solution of problem (3.1). Then, the energy functional defined by (3.2) satisfies*

$$\frac{dE(t)}{dt} = -b_1 \int_{\Gamma_0}\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2 d\xi d\rho - \frac{\xi_2}{4}\left(\frac{d}{dt}\|\nabla u\|_2^2\right)^2 \leq 0. \tag{3.3}$$

Proof. Multiplying the first equation in (3.1) by u_t, integrating over Ω and using integration by parts, we get

$$\frac{1}{2}\|u_t\|_2^2 - \left(\xi_0 + \xi_1\|\nabla u\|_2^2 + \xi_2(\nabla u, \nabla u_t)\right)\int_\Omega \Delta u u_t dx = \int_\Omega |u|^{p-1} u u_t dx.$$

Then

$$\begin{aligned} & \frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2 + \frac{\xi_0}{2}\|\nabla u\|_2^2 + \frac{\xi_1}{4}\|\nabla u\|_2^4 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\right] \\ & + b_1 \int_{\Gamma_0} u_t(x, t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t) d\xi d\rho + \frac{\xi_2}{4}\left(\frac{d}{dt}\|\nabla u\|_2^2\right)^2 = 0. \end{aligned} \tag{3.4}$$

Now multiplying the second equation in (3.1) by $b_1\phi$ and integrating over $\Gamma_0 \times (-\infty, +\infty)$, we get

$$\begin{aligned} & \frac{b_1}{2}\frac{d}{dt}\int_{\Gamma_0}\int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho + b_1 \int_{\Gamma_0}\int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, t)|^2 d\xi d\rho \\ & - b_1 \int_{\Gamma_0} u_t(x, t) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t) d\xi d\rho = 0. \end{aligned} \tag{3.5}$$

By combining (3.2), (3.4) and (3.5), we get (3.3). The Lemma is proved. □

Remark 3.2. After integration of (3.3) over (0,t), we get

$$E(t) = E(0) - b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, s)|^2 d\xi d\rho ds \\ - \frac{\xi_2}{4} \int_0^t \left(\frac{d}{ds} \|\nabla u\|_2^2 \right)^2 ds. \tag{3.6}$$

Now, we define

$$H(t) = \|u\|_2^2 + \frac{\xi_2}{2} \int_0^t \|\nabla u\|_2^4 ds \\ + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) \left(\int_0^s \phi(\xi, z) dz \right)^2 d\xi d\rho ds. \tag{3.7}$$

Lemma 3.3. *Let (u, ϕ) be a regular solution of problem (3.1). Then*

$$\int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) \phi(\xi, t) \int_0^t \phi(\xi, s) ds d\xi d\rho = \\ \int_{\Gamma_0} u(x,t) \int_{-\infty}^{+\infty} \phi(\xi, t) \mu(\xi) d\xi d\rho - \int_{\Gamma_0} \int_{-\infty}^{+\infty} |\phi(\xi, t)|^2 d\xi d\rho. \tag{3.8}$$

Proof. Using the second equation in (3.1), we obtain

$$(\xi^2 + \eta)\phi(\xi, t) = u_t(x,t)\mu(\xi) - \partial_t \phi(\xi, t). \tag{3.9}$$

Then, integrate the last equality over [0, t] to get

$$\int_0^t (\xi^2 + \eta)\phi(\xi, s) ds = u(x,t)\mu(\xi) - \phi(\xi, t). \tag{3.10}$$

Multiplying (3.10) by ϕ and integrating over $\Gamma_0 \times (-\infty, +\infty)$, we obtain (3.8). □

Lemma 3.4. *If $p > 3$, then*

$$H''(t) - (p+3)\|u_t\|_2^2 \geq -2(p+1)E(0) + \frac{\xi_2(p+1)}{2} \int_0^t \left(\frac{d}{ds} \|\nabla u\|_2^2 \right)^2 ds \\ + 2(p+1) b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta)|\phi(\xi, s)|^2 d\xi d\rho ds. \tag{3.11}$$

Proof. From (3.7), we have

$$H'(t) = 2 \int_\Omega u u_t dx + \frac{\xi_2}{2} \|\nabla u\|_2^4 \\ + 2b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) \phi(\xi, s) \int_0^s \phi(\xi, z) dz d\xi d\rho ds, \tag{3.12}$$

and

$$H''(t) = 2\|u_t\|_2^2 + 2 \int_\Omega u u_{tt} dx + 2\xi_2 (\nabla u, \nabla u_t) \|\nabla u\|_2^2 \\ + 2b_1 \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2 + \eta) \phi(\xi, t) \int_0^t \phi(\xi, s) ds d\xi d\rho. \tag{3.13}$$

Employing the divergence theorem and Lamma 3.3 , we get

$$\begin{aligned} H''(t) =&(p+3)\|u_t\|_2^2+\xi_0(p-1)\|\nabla u\|_2^2+\xi_1(\frac{p+1}{2}-2)\|\nabla u\|_2^4 \\ &+b_1(p-1)\int_{\Gamma_0}\int_{-\infty}^{+\infty}|\phi(\xi,t)|^2 d\xi d\rho-2(p+1)E(0) \\ &+2(p+1)b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ &+\frac{\xi_2(p+1)}{2}\int_0^t\left(\frac{d}{ds}\|\nabla u\|_2^2\right)^2 ds. \end{aligned} \tag{3.14}$$

Since $p>3$, we have that (3.11) holds. □

Lemma 3.5. *Assume that $p>3$ holds and that either one the following conditions is satisfied*
(i)$E(0)<0$,
(ii) $E(0)=0$, and

$$H'(0)>\frac{\xi_2}{2}\|\nabla u_0\|_2^4. \tag{3.15}$$

(iii)$E(0)>0$, and

$$H'(0)>r\left[H(0)+k_0\right]+\frac{\xi_2}{2}\|\nabla u_0\|_2^4, \tag{3.16}$$

where

$$r=\frac{(p+1)-\sqrt{(p+1)^2-2(p+1)}}{2},$$

and

$$k_0=\frac{\xi_2}{2}\|\nabla u_0\|_2^4+2E(0). \tag{3.17}$$

Then $H^{'}(t)>\frac{\xi_2}{2}\|\nabla u_0\|_2^4$, for $t>t_0$, for $t>t_0$ where

$$t^*=\max\left\{0,\frac{2H'(0)-\xi_2\|\nabla u_0\|_2^4}{4(p+1)E(0)}\right\}, \tag{3.18}$$

where $t_0=t^*$ in case (i), and $t_0=0$ in case (ii) and (iii)

Proof. If $E(0)<0$, then we have from (3.11) that

$$H^{''}(t)\geq -2(p+1)E(0),$$

which gives

$$H^{'}(t)\geq H^{'}(0)-2(p+1)E(0)t,$$

Thus, we obtain

$$H^{'}(t)>\frac{\xi_2}{2}\|\nabla u_0\|_2^4, \quad t\geq t^*$$

where t^* is defined in (3.18).

(ii) If $E(0)=0$, then we obtain from (3.11) that

$$H''(t)\geq 0, \quad t\geq 0.$$

Furthermore, if (3.15) holds, then

$$H'(t) > \frac{\xi_2}{2}\|\nabla u_0\|_2^4, \quad t > 0.$$

(iii) For the case that $E(0) > 0$, we first note that

$$2\int_0^t \|\nabla u\|_2^2 \frac{d}{dt}\|\nabla u\|_2^2 ds = \|\nabla u\|_2^4 - \|\nabla u_0\|_2^4. \tag{3.19}$$

By Young's inequality, we have

$$\|\nabla u\|_2^4 \le \|\nabla u_0\|_2^4 + \int_0^t \|\nabla u\|_2^4 ds + \int_0^t \left(\frac{d}{dt}\|\nabla u\|_2^2\right)^2 ds. \tag{3.20}$$

A combination of (3.12) and (3.20) shows that

$$\begin{aligned} H'(t) \le\ & \|u\|_2^2 + \|u_t\|_2^2 + \frac{\xi_2}{2}\|\nabla u_0\|_2^4 + \frac{\xi_2}{2}\int_0^t \|\nabla u\|_2^4 ds \\ & + \frac{\xi_2}{2}\int_0^t \left(\frac{d}{dt}\|\nabla u\|_2^2\right)^2 ds + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds \\ & + b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho. \end{aligned} \tag{3.21}$$

By (3.7), (3.11) and (3.21), we get

$$H''(t) - (p+1)H'(t) + (p+1)\{H(t)+k_0\} \ge 0, \tag{3.22}$$

where k_0 is defined in (3.17). Now let

$$B(t) = H(t) + k_0.$$

Then $B(t)$ satisfies

$$B''(t) - (p+1)B'(t) + (p+1)B(t) \ge 0. \tag{3.23}$$

Using Lemma 2.3 in (3.23) and (3.16), we get

$$H'(t) > \frac{\xi_2}{2}\|\nabla u_0\|_2^4, \quad t \ge 0.$$

□

Theorem 3.6. *Assume that $p > 3$. Then the solution (u,ϕ) blows up in finite time T^* in the sense of (2.2) and T^* is such that:*
(i) If $E(0) < 0$, then

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)}. \tag{3.24}$$

Furthemore, if $J(t_0) < \min\left\{1, \sqrt{\frac{a}{-b'}}\right\}$, then we have

$$T^* \le t_0 + \frac{1}{\sqrt{-b'}} \ln \frac{\sqrt{\frac{a}{-b'}}}{\sqrt{\frac{a}{-b'}} - J(t_0)}. \tag{3.25}$$

(ii) If $E(0)=0$ and (3.15) holds, then

$$T^{*}\leq -\frac{J(0)}{J'(0)} \tag{3.26}$$

or

$$T^{*}\leq \frac{J(0)}{J'(0)}. \tag{3.27}$$

(iii) If

$$0<E(0)<\frac{(p-1)\left[H'(t_0)-\frac{\xi_2}{2}\|\nabla u\|_2^4\right]^2 J(t_0)^{\frac{1}{\gamma_1}}}{4(p+1)}$$

and (3.16) holds, then

$$T^{*}\leq \frac{J(0)}{\sqrt{a}} \tag{3.28}$$

or

$$T^{*}\leq 2^{\frac{3\gamma_1+1}{2\gamma_1}}\frac{\gamma_1 c}{\sqrt{a}}\{1-[1+cJ(0)]^{\frac{-1}{2\gamma_1}}\}, \tag{3.29}$$

where $c=(\frac{b'}{a})^{2+\frac{1}{\gamma_1}}$, $\gamma_1=\frac{p-3}{4}$ and J(t), a and b' are given in (3.30), (3.41) and (3.42) respectively.

Proof. Let

$$J(t)=\left\{H(t)+(T-t)\frac{\xi_2}{2}\|\nabla u_0\|_2^4\right\}^{-\gamma_1}. \tag{3.30}$$

Differentiating $J(t)$ twice, we obtain

$$J^{'}(t)=-\gamma_1 J(t)^{1+\frac{1}{\gamma_1}}\left\{H'(t)-\frac{\xi_2}{2}\|\nabla u_0\|_2^4\right\}, \tag{3.31}$$

and

$$J^{''}(t)=-\gamma_1 J(t)^{1+\frac{2}{\gamma_1}}Q(t), \tag{3.32}$$

where

$$\begin{aligned} Q(t)=&H^{''}(t)\left\{H(t)+(T-t)\frac{\xi_2}{2}\|\nabla u_0\|_2^4\right\}\\ &-(1+\gamma_1)\left\{H'(t)-\frac{\xi_2}{2}\|\nabla u_0\|_2^4\right\}^2. \end{aligned} \tag{3.33}$$

It follows from (3.12) that

$$\begin{aligned} H'(t)-\frac{\xi_2}{2}\|\nabla u_0\|_2^4=2\int_\Omega u u_t dx+\xi_2\int_0^t\|\nabla u\|_2^2\frac{d}{ds}\|\nabla u\|_2^2 ds\\ +2b_1\int_0^t\int_{\Gamma_0}\int_{-\infty}^{+\infty}(\xi^2+\eta)\phi(\xi,s)\int_0^s\phi(\xi,z)dzd\xi d\rho ds. \end{aligned} \tag{3.34}$$

Hence, taking (3.7), (3.11), (3.33), and (3.34) into account, we obtain

$$Q(t)\geq -2(p+1)E(0)J(t)^{\frac{-1}{\gamma_1}}+(p+1)\left\{\mathbf{AC}-\mathbf{B}^2\right\}, \tag{3.35}$$

where

$$\mathbf{A} = \|u\|_2^2 + \tfrac{\xi_2}{2}\int_0^t \|\nabla u\|_2^4 ds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left(\int_0^s \phi(\xi,z)dz\right)^2 d\xi d\rho ds,$$

$$\mathbf{B} = \int_\Omega u u_t dx + \tfrac{\xi_2}{2}\int_0^t \|\nabla u\|_2^2 \tfrac{d}{ds}\|\nabla u\|_2^2 ds + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\phi(\xi,s)\int_0^s \phi(\xi,z)dz d\xi d\rho ds,$$

$$\mathbf{C} = \|u_t\|_2^2 + b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)|\phi(\xi,s)|^2 d\xi d\rho ds + \tfrac{\xi_2}{2}\int_0^t \left(\tfrac{d}{ds}\|\nabla u\|_2^2\right)^2 ds.$$

Thus, we obtain Now we observe that, for all $\rho_1 \in IR$ and $t > 0$,

$$\begin{aligned} \mathbf{A}\rho_1^2 + 2\mathbf{B}\rho_1 + \mathbf{C} &= [\rho_1\|u\|_2 + \|u_t\|_2]^2 \\ &+ b_1 \int_0^t \int_{\Gamma_0} \int_{-\infty}^{+\infty} (\xi^2+\eta)\left[\rho_1 \int_0^s \phi(\xi,z)dz + |\phi(\xi,s)|\right]^2 d\xi d\rho ds \\ &+ \frac{\xi_2}{2}\int_0^t \left[\rho_1\|\nabla u\|_2^2 + \left(\frac{d}{ds}\|\nabla u\|_2^2\right) ds\right]^2. \end{aligned} \tag{3.36}$$

It is easy to see that

$$\mathbf{A}\rho_1^2 + 2\mathbf{B}\rho_1 + \mathbf{C} \geq 0,$$

and

$$\mathbf{B}^2 - \mathbf{AC} \leq 0. \tag{3.37}$$

Hence, taking (3.35) and (3.37) into account, we get

$$Q(t) \geq -2(p+1)E(0)J(t)^{\frac{-1}{\gamma_1}}, \quad t \geq t_0. \tag{3.38}$$

Therefore, by (3.32), and (3.38), we get

$$J''(t) \leq \frac{(p+1)(p-3)}{4} E(0) J(t)^{1+\frac{1}{\gamma_1}}, \quad t \geq t_0. \tag{3.39}$$

From Lemma 3.5, $J'(t) < 0$ for $t \geq t_0$. Multiplying (3.39) by $J'(t)$ and integrating it from t_0 to t, we have

$$J'(t)^2 \geq a + b' J(t)^{2+\frac{1}{\gamma_1}}, \tag{3.40}$$

where

$$\begin{aligned} a = \Bigg[&\frac{(p-3)^2}{16}\left(H'(t_0) - \frac{\xi_2}{2}\|\nabla u_0\|_2^4\right)^2 \\ &- \frac{(p-3)^2(p+1)}{4(p-1)} E(0) J(t_0)^{\frac{-1}{\gamma_1}}\Bigg] J(t_0)^{2+\frac{2}{\gamma_1}}, \end{aligned} \tag{3.41}$$

and

$$b' = \frac{(p-3)^2(p+1)}{4(p-1)} E(0). \tag{3.42}$$

Then by Lemma 2.4 the proof of theorem is completed.

Hence, there exists a finite time T such that $\lim_{t\to T^{*-}} J(t) = 0$ and the upper bounds of T^* are estimated according to the sign of $E(0)$ (see Lemma 2.4). □

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